

Evaluation of a learning trajectory for length in the early years

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Abstract Measurement is a critical component of mathematics education, but research on the learning and teaching of measurement is limited, especially compared to topics such as number and operations. To contribute to the establishment of a research base for instruction in measurement, we evaluated and refined a previously developed learning trajectory in early length measurement, focusing on the developmental progressions that provide cognitive accounts of the development of children’s strategic and conceptual knowledge of measure. Findings generally supported the developmental progression, in that children reliably moved through the levels of thinking in that progression. For example, they passed through a level in which they measured length by placing multiple units or

attempting to iterate a unit, sometimes leaving gaps between units. However, findings also suggested several refinements to the developmental progression, including the nature and placement of indirect length comparison in the developmental progression and the role of vocabulary, which was an important facilitator of learning for some, but not all, children.

Keywords Measurement · Length · Learning trajectory · Developmental progressions · Hierarchic interactionism

1 Introduction

Measurement is a critical component of mathematics education, but research on the learning and teaching of measurement is limited, especially compared to topics such as number and operations. To build a description of children’s learning of measurement that is as detailed as the field has developed for these other topics, we evaluated and refined the early childhood portion of a learning trajectory (LT) in the domain of length measurement, especially the developmental progression that provides cognitive accounts of the development of children’s conceptual and strategic knowledge of measure. Although previous evaluations of the developmental progression have provided empirical support, none have closely examined the cognitive components of the levels of thinking and none have examined the actual development of children longitudinally. Such validation and refinement are significant, because valid, reliable, elaborate developmental progressions constitute infrastructures for improving assessment tools, instructional strategies, curricular materials, and professional development models (National Research Council, 2009; Sarama & Clements, 2011).

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Measurement is not only a principal real-world application of mathematics, but also forms a critical part of the foundation for quantitative reasoning, including arithmetic, ratio, proportion, and relations among variables (Gravemeijer, 1999; Sarama & Clements, 2009). The indications are, however, that such foundation usually is not laid in the USA, as measurement is not taught well. In international comparisons, US children were weakest in measurement (Ginsburg, Cooke, Leinwand, Noell, & Pollock, 2005). Thus, research that improves curriculum, assessment, and teaching is needed.

2 Theoretical framework and original length learning trajectory

One approach to addressing challenges in learning and teaching mathematics is the creation of an LT, which can serve as the core of research projects, curricula, and professional development (Clements & Sarama, 2004; Simon, 1995; Smith, Wiser, Anderson, & Krajcik, 2006). We define learning trajectories (LTs) as “descriptions of children’s thinking and learning in a specific mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children’s achievement of specific goals in that mathematical domain” (Clements & Sarama, 2004, p. 83). By illuminating potential developmental progressions, LTs bring coherence and consistency to the three main components of education: assessment, professional development, and instruction. That is, an LT describes how concepts and skills develop over time, and this helps provide one set of guidelines for testing, teacher training, and teaching, facilitating connections among these activities.

2.1 The hierarchic interactionism framework

We view LTs through a theoretical lens termed *hierarchic interactionism* (Sarama & Clements, 2009). “Hierarchic” indicates the influence and interaction of domain-general and domain-specific cognitive components and the interactions of innate competencies, internal resources, and experience. Children progress through domain-specific levels of understanding in ways (LTs) that can be characterized by specific mental objects and actions (i.e., both concept and process) that build hierarchically on previous levels. Hierarchic interactionism’s LTs are built upon, and thus share characteristics with, earlier research programs, but differ in significant ways (for a complete analysis, see Sarama & Clements, 2011).

LTs owe much to these previous efforts, which have progressed from accumulations of connections (Thorndike, 1922) to increasingly sophisticated and complex views of cognition and learning (Gagné, 1965/1970; Piaget, Inhelder, & Szeminska, 1960; Resnick & Ford, 1981). However, these applications of cognitive theory to educational sequences tended to feature simple sequences based on accretion of numerous facts and skills. LTs include such hierarchies (each level builds hierarchically on the concepts and procedures of previous levels), but are not as limited to sequences of skills or “logically” determined prerequisite pieces of knowledge. LTs describe children’s *levels of thinking*, not just their ability to correctly respond to a mathematics question; they cannot be summarized by stating a mathematical definition, concept, or rule (cf. Gagné, 1965/1970). Levels of thinking describe *how* children think about a topic and *why*—including the cognitive actions on objects that constitute that thinking.

Further, the ramifications for instruction from earlier theories were often based on transmission views, which hold that these facts and skills are presented and then passively absorbed. In comparison, LTs have an interactionist view of pedagogy. Finally, note that terms, such as “learning progressions” are used (more often in science education) somewhat ambiguously, sometimes indicating only developmental progressions, and at other times, also suggesting a sequence of instructional activities. One advantage of the LT construct is the connection of these two aspects.

Hierarchic interactionism theory postulates that various models and types of thinking grow in tandem to a degree, but a critical mass of ideas from each level must be constructed before thinking, characteristic of the subsequent level, becomes ascendant in the child’s mental actions and behavior (see also Clements, Battista, & Sarama, 2001). This often involves “fall back” to prior levels of thinking under increasingly complex demands, as typified in findings about children developing sophistication of units (and units of units) for measuring perimeter (Barrett, Clements, Klanderma, Pennisi, & Polaki, 2006). With experience, the level of thinking becomes robust, and progressions follow a predictable pattern of learning activity: first, there are “sensory-concrete” levels, wherein perceptual concrete supports are necessary and reasoning proceeds through limited cases; second, verbally based generalizations follow as the child abstracts ideas beyond the immediate sense; third, there are “integrated concrete” understandings that rely on internalized, mental representations that serve as mental models for operations or abstractions (Clements & Sarama, 2009; Sarama & Clements, 2009). We also see concepts and skills proceeding in tandem, with benefit accruing where concepts are established to give context and foundation to the skills.

2.2 Original length learning trajectory

Previous publications described how we used hierarchic interactionism theory to review and synthesize existing research and create an LT for length measurement for the *Building Blocks* project (Clements & Sarama, 2007). The review and LT have been described in detail elsewhere (Clements & Sarama, 2009; Sarama & Clements, 2009); space constraints permit only brief examples here. For instance, Piagetians defined the concept of length measurement as the synthesis of sub-division and change of position, which involves taking one part out of the whole and iterating that unit along the whole (Piaget et al., 1960). They posited that young children did not possess understanding of length, based on interviews in which children appeared to believe that placing an object between two positions changed the distance between these positions (Miller & Baillargeon, 1990). However, children 3.5–5 years of age appear to understand these concepts in some settings (Bartsch & Wellman, 1988). Similarly, the Piagetians argued that meaningful measurement and indirect comparison using transitive reasoning is impossible until children conserve length. However, children do not necessarily need to develop conservation before they can learn some measurement ideas (Clements, 1999; Hiebert, 1981; Petitto, 1990). Thus, children have an intuitive understanding on which to base reasoning about distance and length, but that reasoning develops over years, possibly requiring specific educational experiences to build concepts and skills that enable children to (1) align endpoints (Piaget et al., 1960), (2) use a third object and transitivity to compare the length of two objects that cannot be compared directly (Hiebert, 1981), (3) place length units along objects to find their lengths and associate higher counts with longer objects (Hiebert, 1981; Stephan, Bowers, Cobb, & Gravemeijer, 2003), (4) understand the need for equal-length units (Ellis, Siegler, & Van Voorhis, 2003), (5) use rulers accurately and meaningfully (Lehrer, 2003; Stephan et al., 2003), and (6) make inferences about the relative size of objects (e.g., if the number of units are the same, but the units are different, the total size is different) (Nunes & Bryant, 1996).

We conducted qualitative (Sarama & Clements, 2009) and quantitative (Rasch, see Clements, Sarama, & Liu, 2008; Sarama & Clements, 2009; Szilagyi, Sarama, & Clements, 2010) studies of a first draft of the trajectory, which led to some refinements (e.g., combining two contiguous levels). A synthesis of these and other studies formed the foundation for the developmental progression of *Building Blocks*' LT for length, summarized in Table 1 (for full details, see Clements & Sarama, 2009; Sarama & Clements, 2009, Chapter 7 in each book).

Although our previous studies supported this LT, there were two major weaknesses. First, none examined the actual development of children longitudinally. Cross-sectional analyses are suggestive of, but cannot validate, developmental sequences; e.g., they cannot address whether some children “skip” certain levels in a sequence. As another example, LTs can include “subtrajectories”—strands within the topic (e.g., the counting LT includes both verbal and object counting, which are closely related, but can develop somewhat independently, such as when children learn verbal counting before learning one-to-one correspondence). We suspected that the non-numerical strategies (e.g., directly or indirectly comparing objects) might constitute such a subtrajectory.

The second major weakness of previous studies was that they could not examine cognitive components (the “actions on objects” in Table 1) of the developmental progression's levels of thinking. Therefore, the present study utilized a teaching experiment (TE) methodology, which, although limited the number of participants, had the potential to ameliorate both these weaknesses (Steffe et al., 2000). The TE was designed to evaluate the early childhood section of the LT¹ longitudinally, including close observations of children's strategies and other behavioral indicators of children's thinking processes. We asked whether levels of the original developmental progression provided a valid description of young children's acquisition of concepts and strategies related to length measurement, and whether any levels could be altered to more accurately and completely describe young children's learning of length measurement.

3 Methods

3.1 Participants

Participants were from a small, urban school in New York State. All children who returned permission forms were given a measurement competencies assessment. There were between 18 and 22 participants each year in whole class observations and TEs. We then, in consultation with teachers, selected a cohort of eight pre-K children from this group varying in SES, cultural background, gender, and achievement, to study longitudinally for 4 years. Two children moved out of the district after the first year and a third child after the second year. Each time a child left the cohort, another was selected to replace him/her, using the

¹ This study is part of a larger project to validate and refine measurement learning trajectories. Later developmental levels are investigated in connected studies (e.g., see Barrett et al., 2011 and Barrett et al., this issue).

Table 1 The developmental progression and mental actions on objects for the Learning Trajectory for Length Measurement (from Sarama & Clements, 2009, pp. 289–291)

Developmental progression	Actions on objects
<p>Pre-length quantity recognizer</p> <p>Does not identify length as attribute</p> <p>“This is long. Everything straight is long. If it’s not straight, it can’t be long.”</p>	<p>Action schemes^a, both physical/kinesthetic and visual, implicitly trace linear extents—initially not, and later only partially, connected to explicit vocabulary</p>
<p>Length quantity recognizer</p> <p>Identifies length/distance as attribute using appropriate vocabulary. May understand length as an absolute descriptor (e.g., all adults are tall), but not as comparative (e.g., one person is taller than another)</p> <p>“I’m tall, see?”</p>	<p>Action schemes are connected to length vocabulary and used to compare lengths. However, salient differences at one end of the objects can be substituted for a scan, leading to inaccuracies if the other endpoints are not aligned</p>
<p>Length direct comparer</p> <p>Physically aligns two objects to determine which is longer or if they are of the same length</p> <p>Stands two sticks up next to each other on a table and says, “This one’s bigger.”</p>	<p>The scheme addresses length as a linear extent from endpoint to endpoint of a path. With perceptual support, objects can be mentally, then physically, slid and rotated into alignment and their endpoints compared</p>
<p>Indirect length comparer</p> <p>Compares the length of two objects by representing them with a third object</p> <p>Compares the length of two objects with a piece of string</p>	<p>A mental image of a particular length can be built, maintained, and manipulated. With perceptual support, such images can be compared. For some, explicit transitive reasoning may be applied to the images or their symbolic representations</p>
<p>Note: at this level, children are also aware of the process of measurement; however, when asked to measure, may assign a length by guessing or moving along a length while counting (without equal length units) or use a ruler (but lacking in concepts and/or skill)</p> <p>Moves finger along a line segment, saying 10, 20, 30, 31, 32</p>	<p>If asked to measure, a counting scheme operates on an intuitive unit of spatial extent or amount of movement, directing movements along a length while counting. The sensory-concrete mental actions require the perceptual support of the object to be measured</p>
<p>End-to-end length measurer</p> <p>Lays units end to end. May not recognize the need for equal-length units or to avoid gaps. The ability to apply resulting measures to comparison situations develops late in this level. Can “read off” measures from a ruler, but only accurately with substantial guidance</p> <p>Lays 9-in. cubes in a line beside a book to measure how long it is</p>	<p>An implicit concept that lengths can be composed as repetitions of shorter lengths underlies the scheme (that must overcome previous schemes, which use continuous mental processes to evaluate continuous extents, and thus are more easily instantiated)</p>
<p>Length unit relater and repeater</p> <p>Measures by repeated use of a unit (but initially may not be precise). Relates size and number of units</p> <p>“If you measure with cm, not inches, you’ll need more, because each one is smaller.”</p>	<p>Actions schemes include the ability to iterate a mental unit along a perceptually available object. The image of each placement can be maintained while the physical unit is moved to the next iterative position. With the support of a perceptual context, schemes can predict that fewer larger units will be required to measure an object’s length. These action schemes allow the application of counting all addition schemes to be applied to measures</p>
<p>Iterates a single unit to measure. Recognizes that different units will result in different measures and that identical units should be used. Uses rulers with minimal guidance</p>	
<p>Length measurer</p> <p>Measures, knowing need for identical units, relationship between different units, partitions of unit, zero point on rulers, and accumulation of distance.</p> <p>Begins to estimate:</p> <p>“I used a meter stick three times, then there was a little left over. So, I lined it up from 0 and found 14 centimeters. So, it’s 3 meters, 14 centimeters in all.”</p>	<p>The length scheme has additional hierarchical components, including the ability to simultaneously image and conceive of an object’s length as a total extent and a composition of units. This scheme adds constraints on the use of equal-length units and, with rulers, on the use of a zero point. Units themselves can be partitioned, allowing accurate use of units and subordinate units</p>

^a We use “scheme” to mean a cognitive unit with three components: a structure that recognizes a situation, a process that acts on that situations, and a result (Sarama & Clements, 2009; Steffe, Thompson, & Glasersfeld, 2000)

same process as selecting children initially, to maintain a cohort of eight for data collection. Thus, in this report, we focus on the cohort’s measurement thinking and learning

from pre-K to Grade 1, with a particular focus on the five children for whom we have 3 years of complete longitudinal data (recall footnote 1).

3.2 Data sources

We conducted repeated cycles of clinical interviews, beginning with the initial assessment administered to all children. Tasks for that assessment included the measurement items from a previously developed and validated interview-based assessment (for validation, see Clements et al., 2008; Clements, Sarama, & Wolfe, 2011) as well as tasks from previous empirical studies (Szilagy et al., 2010). Subsequent clinical interviews included both modifications of these same tasks and newly created tasks designed to elicit specific strategies and responses at particular levels of the developmental progression (see <http://childrensmeasurement.org> for an appendix to this article describing tasks in detail).

To further evaluate the LT, we conducted TEs with individual focal children, as well as whole class teaching experiments (CTEs). Each TE was informed by all previous observations; we directed our focus toward clearer understanding of children's behaviors and length understanding. To ensure this research was grounded within classroom instruction and teacher practices, classroom instructional sequences were co-designed and implemented with collaborating teachers. We also observed any lesson or sequence of lessons taught by the classroom teacher that included substantial attention to measurement. Thus, data sources included video records of the initial interview assessment, clinical interviews, and TEs, including teacher-led instruction on measurement-related science and math lessons, written classroom observations, researchers' journals from assessments and TEs, and classroom teachers' reflective journals. This integration of individual and classroom foci was intended to address the naturalistic setting of children's experiential learning from multiple perspectives, including viewing learning as a psycho-cognitive and a socio-cognitive phenomenon (Fischer & Bidell, 2006). The researchers reviewed each data source to identify behaviors specific to length understanding. These behaviors were coded, then compared to the length LT to identify the level at which a child demonstrated knowledge or highlight gaps and/or inconsistencies in the length LT. As consistent behaviors were evidenced, revisions of the LT were considered and checked against new data.

4 Results

4.1 Year 1: pre-K

Initial interviews and observations revealed that children already identified length as an attribute. All children demonstrated behaviors consistent with the *Length Quantity Recognizer* level. For example, when given a foam

rectangle with a slot cut out of it and a smaller, rectangular piece that fit width-wise in the slot, but was too short, Robert quickly placed the piece in the slot. When asked if it fit well, Robert said, "No, it's small." When asked about the length of a pencil, Robert traced his finger along the length and said, "Big." Alice used "long" to describe pictures of a tower, a snake and a worm, consistently using this term for objects that extended further along one dimension. We take this as indicating evidence of the Length Quantity Recognizer level, because these children quantified objects in terms of length but were unable to directly compare the lengths of two objects. Some also demonstrated behaviors at the *Length Direct Comparer* level. For example, when Edith was asked which of four popsicle sticks was of the same length as a green strip, she placed the strip in turn next to each stick, saying, "Not this one...not this one...not this one...This one!"

The next 5 weeks of the school year involved teacher-initiated activities intended to challenge children's thinking about length (e.g., various activities in which children were given sets of sticks to describe, compare, and order by length; comparing blocks of the same length but laid in different orientations; cutting a string as long as the child's arm and using it to compare with the length of other objects). During these activities and the associated instruction, all children demonstrated behaviors indicating movement toward the Length Direct Comparer level. Consider Marina's responses to questions about a piece of string she had cut to be as long as her arm.

Researcher: Is the pencil shorter or longer than your arm?

Marina: (placing the string next to the pencil) No, it's short.

Researcher: How about checking Spiderman? Is he shorter or longer than your arm?

Marina: (holding the string up next to a doll) Longer. [Note: Doll was significantly shorter than the string.]

Researcher: So which one is longer? Spiderman or your arm?

Marina: Spiderman.

Researcher: Spiderman is longer?

Marina: Yes.

Researcher: So this Spiderman is longer than your arm? (Asks child to stretch out her arm and lays Spiderman on child's arm.)

Marina: Yes.

Marina was able to say that objects were short or longer (although her first use of the term "short" rather than "shorter" may indicate the persistent influence of early dichotomous classifications), but did not demonstrate consistently an ability to compare objects of different lengths. From this we interpret that her understanding of

direct comparison was emergent but incomplete, whereas other children (as in Lia's vignette, above) could directly compare objects by length.

At these early levels, we found that for some children limited vocabulary was a barrier to accurate identification of their length understanding. In an attempt to isolate whether children were limited only by vocabulary or truly lacked understanding, the researcher/teacher and children began to establish a set of common gestures and phrases to help identify more specific, detailed aspects of length magnitude. For example, during an initial interview, when asked, "What do you know that is big?" Robert said, "Giants are big." The researcher asked, "Are they big this way?" (demonstrating 'tall' with her hands), "or are they big this way?" (demonstrating 'wide' with her hands), "or some other way?" Robert demonstrated with his hands that he was referring to height. After such questioning, children's description of size changed. For example, a short time later Robert interrupted a description of a short boy by saying, "When it is tall this way (holding his hands apart at the same level, palms facing each other) it's wide, but when turning this way (moving his hands so one was above the other) it's big." Robert's use of gestures gave a better indication of his length understanding than his limited vocabulary allowed him to express.

In contrast, in a spring TE, when children were shown several pictures on a computer, Alice was able to use standard vocabulary as well as gestures to indicate her length understanding:

Researcher: Do you remember any words we use?

Alice: Yeah, tall and short.

Researcher: We *did* use tall and short.

Alice: And long (holding her hands far apart, palms facing each other) and short (moving her hands close together, palms facing).

Researcher: Good, as I show you the pictures, tell me how you would describe each. Tell me about that mouse.

Alice: It's shorter (placing her hands close together, one above and one below the mouse).

Researcher: Shorter than what?

Alice: The cat (moving her hands to the cat and spreading them farther apart).

Researcher: (showing another picture) Tell me about the dog.

Alice: (placing her hands, palms facing each other, on each side of the dog) He's bigger than the mouse.

Researcher: What do you mean by bigger?

Alice: He's very wide (spreading her hands far apart).

Researcher: Is there anything else about him other than being very wide?

Alice: His legs are very big...very *long* (tracing legs with her finger).

Alice not only used vocabulary and gestures to describe length, but also began using them to compare length. We interpret this behavior as an indication that she was a Length Quantity Recognizer moving toward length direct comparison.

The same day, Lia described a picture of a dog as "big and fat" using her hands to show that "fat" meant wide and "big" meant long; she seemed satisfied describing the objects and did not shift to comparing objects. Gail showed evidence of comparison, but struggled with vocabulary, naming different length sticks, "shortest, really shortest, medium and medium little." Robert used more precise vocabulary to describe lengths of string, saying, "The string is longer than...the ruler." Thus, some children were transitioning to direct comparison, while others showed only Length Quantity Recognizer behaviors.

Consistent with the LT, by late spring of the pre-K year, all children had progressed from Length Quantity Recognizer to Length Direct Comparer. Whereas previously, children had simply guessed which of two objects was longer or shorter, they then directly compared the lengths of two objects, physically aligning the ends of these objects and visually ascertaining which object was longer. For example, when asked to sort five sticks of different lengths, Marina was able to select the longest and the shortest immediately, and then continued sorting them correctly by comparing them in pairs, a task she could not do the preceding fall.

During a whole class CTE that spring, we observed similar development. When two boys were asked to compare blocks, they kept the blocks next to each other saying, "This is bigger, see?" This reinforced data from individual sessions with our focus children, where all demonstrated evidence of direct comparison; in the CTE, 90% of the class could directly compare two objects when asked.

During another CTE session, children were asked, "Would the desk fit through the door?" Nearly all children suggested direct comparison of the desk and door—pushing the desk to the door to "see" if it would fit. The teacher then suggested using an object to measure the desk; each child chose an object (e.g., a paint brush, a book, a plate, a doll, a block, etc.), but no one was sure of what to do with it. One child suggested blocks; the teacher guided the discussion toward unit iteration and then indirect comparison. However, none of the children suggested indirect comparison outright; they merely attempted to follow the teacher's directions as she guided them toward it. At the end of pre-K, children were unable to transition from direct comparison to indirect comparison, even when they could directly compare with ease.

In our initial interviews, no child was able to consistently measure objects end to end, or iterate a unit along a length. By the final TE of Year 1, Alice and Lia

demonstrated evidence of end-to-end measurement; when asked to measure the length of a rectangle, they laid out and counted 1-in. unit strips. All other focus children showed growth toward this level, but not consistent evidence of it, e.g., laying out some strips, but not counting or relating them to a length. Additionally, although children were presented tasks in which the potential to iterate a unit existed, none demonstrated evidence of this ability.

4.2 Year 2: kindergarten

Our initial TE verified that children, now kindergartners, remained able to directly compare lengths but that their end-to-end and iteration skills had not developed further. Lia, who had been able to measure end to end the previous spring, could not complete a task requiring the same skill.

In the following TE, when asked to compare two objects that could not be manipulated physically, all but one of the children compared the objects mentally rather than using a third object to compare indirectly. In the following example, Edith utilized a train track as a third object to determine which of two thrown balls was closer to a target.

Researcher: Which ball is closer? Do you think the train track can help us on this one? (Researcher produces a piece of train track.)

Edith (Lays the train track along each distance. The end of the train track extends past the researcher's ball.) Yours is short because that is big.

Researcher: What was big?

Edith: This one is big (indicating her distance) and this one is shorter (indicating the researcher's distance).

In contrast, Alice was able to use a third object to compare indirectly in one situation, but in another appeared to use a mental (visual) comparison, rather than indirect comparison, when comparing two lengths of track glued in different orientations (shown in Fig. 1).

Alice first compared two train tracks glued to a piece of paper: one oriented vertically and the other horizontally.

Researcher: (indicating the tracks in the left pane of Fig. 1) How about this? Is one longer, shorter, or they are they both the same length?

Alice: This one is longer (indicates the longer one).

Researcher: How do you know that?

Alice: Because this one looks... (uses her hands to show the length of each track by placing one hand at each end).

Researcher: Can you use this to check? (hands Alice another piece of track)

Alice: (lays the third piece of track next to the shorter track) It's the same size.

Researcher: How about this one? (indicating the longer track).

Alice: (moves the third track next to it) No.

Researcher: So which one's longer?

(Alice points to the longer track).

Second, Alice compared two lengths of track arranged at an angle to one another (the right pane of Fig. 1).

Researcher: What do you think? Is one longer or shorter, or are they the same length?

Alice: The same length.

Researcher: Why do you say that?

Alice: Because if they were different lengths they wouldn't make a housetop.

Researcher: They wouldn't make a housetop if they were different lengths; how come?

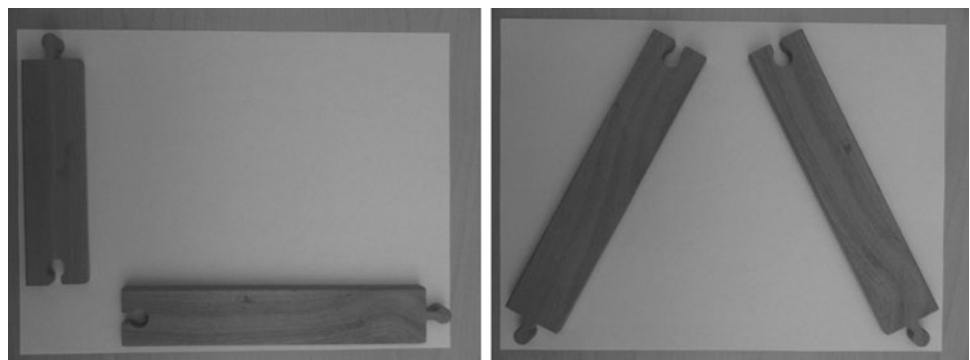
Alice: Because this one would be like here (placing her hand partway up the right-hand track) and this one would be all the way up here (placing her hand at the top of the left-hand track) and then it wouldn't look right.

Researcher: Can you use this to check? (gives Alice a third piece of train track).

Alice: (places the third train track between the two) I can't really fit it inside.

Even after being guided to use indirect comparison with explicit transitive inference (within the task illustrated in the left pane of Fig. 1), Alice did not show signs that she went beyond visual comparison, nor did she confirm her

Fig. 1 Two orientations of straight paths for indirect comparison



imagistic comparison using a third object in the situation shown in the right pane of Fig. 1. However, she did show that she could compare the lengths visually. Six of eight children were unable to use a third object to compare indirectly, even when presented with an object and suggestions for use. However, like Alice, these children could mentally compare objects in certain situations. We took this as evidence that children were comparing objects directly, but mentally, rather than comparing indirectly.

In contrast, children became facile *End-to-End Length Measurers*, learning to measure objects with a physical unit during measurement tasks in TEs. In general, they were able to lay units end to end (without spaces or overlaps between units), count the number of units, and then state correctly the total length measured in terms of that unit. For example, in a November TE, when asked to measure the space between two objects, Edith laid blocks end to end (although not always in the same orientation); she also laid blocks end to end around the perimeter of a rectangle and counted them to measure the perimeter. Likewise, in late spring, when asked to measure the length of a desk, Marina laid rulers end to end along the edge of the desk, using the whole ruler as the counting unit; during the same TE, she laid tiles end to end along the length of a rectangle to measure its length. In the following TE, Robert also lined up and counted ruler units end to end across a hypothetical kitchen floor to measure its length, making no reference to the markings on the rulers. In February, Lia was asked to lay out inch units and compare them to the numbers on the ruler. This helped her reconnect to end-to-end measurement, but not to make a firm connection to the ruler (e.g., she counted six units and noted, “They go up to 6” on the ruler;” she never measured with the ruler per se). Thus, none of the children consistently used a ruler to measure length, despite repeated introduction in various TEs; it was simply another object to lay down end to end or a tool with numbers to be counted. Table 2 lists the dates children demonstrated consistent evidence of end-to-end measurement and indirect comparison.

Table 2 Development of children’s length understanding through kindergarten: children with complete longitudinal data

Child	End to end	Indirect length comparison
Alice	April 2008 (pre-K)	November 2008
Lia	April 2008 (pre-K) ^a , February 2009	February 2009
Edith	November 2008	February 2009
Robert	May 2009	April 2009
Marina	February 2009	June 2009

^a Lia initially demonstrated end-to-end measurement during pre-K, but was unable to demonstrate it again until spring of kindergarten

Exit interviews showed that children consistently used end-to-end measurement; however, if not given enough units to lay the length of the object, all had difficulty measuring. One task was to measure one side of a desk. All students were successful if given enough connecting cubes to reach from one end to the other. However, when using a ruler for the same task, Edith aligned it with the left edge of the desk, but then simply counted the number marks from 1 through 12. Because the ruler did not cover the whole length of the desk, she did not know what to do with the empty space. Robert correctly aligned the origin of the ruler with the left edge of the desk and counted from 1 to 12. He then slid the ruler to the right, aligning the right edge of the ruler with the right edge of the desk and re-counted from 1 to 12. He did not address the empty space either time. Both Marina and Lia aligned the 12” end of the ruler with the edge of the desk and counted from 0 to 12. We take this as evidence that children were established End-to-End Length Measurers, but not yet Length Unit Relater and Repeaters. They could not iterate to fill an empty space, or correctly use a ruler other than to count numbers. They appeared to need to fill a space with physical units to consider it “measured.”

4.3 Year 3: first grade

During entrance interviews at the beginning of first grade, children used end-to-end measurement consistently. For example, when asked to measure the length of a desk, Edith laid down two rulers end to end and counted out the marks on the rulers as she had done in her exit interview the previous spring. Similarly, when asked to measure the length of a rectangle, Robert laid down inch tiles along the length and counted them.

Evidence of indirect comparison also continued to appear in the children’s first-grade year. Children demonstrated different developmental paths to the non-numerical procedure of indirect comparison with irregular patterns of progress and regression. For example, Edith had demonstrated indirect comparison in four different TEs between February and April of the second year (e.g., using a third track to compare the lengths of two others or to determine which of two thrown beanbags was closest to a target), but began to favor end-to-end measurement within the first two TEs in October of the third year. Lia and Marina, on the other hand, utilized indirect length comparison well into the third year (alongside end-to-end measurement) before beginning to favor end-to-end measurement during TEs conducted in December and January. Robert used one or the other, but not both, in different episodes.

Children also continued to use mental imagery to compare objects’ lengths, as seen in kindergarten. For example, in one fall TE, Marina visually inspected the

perpendicular sides of a desk and then pointed to them, saying, “This one’s longer; this one’s shorter.” We assume such cases illustrate a transitional strategy of mentally moving an image of one object to coincide with the other. Although this is perhaps closer to direct than indirect comparison, for about half of the children, the use of such mental imagery in comparison tasks was followed closely by their explicit use of an indirect comparison strategy. In another TE that fall, children were asked to compare two line segments in two different optical illusions. Optical illusions often cause us to think that lines are different lengths even when they are not, thus making imagistic comparisons difficult or impossible and necessitating measurement to verify. Children were shown the images in Fig. 2. When asked to compare the segments, all children initially made mental comparisons. When asked to verify, five of the eight children reasoned indirectly, using a blue rod in the first activity and a ruler in the second activity to compare two line segments; the remaining three children did not use a third object in either activity.

By mid-year, five of eight focal children were able to measure objects longer than a ruler or for which there were not enough objects to lay end to end. Children appeared to maintain a mental image of the unit being used, iterating it to fill the empty space. We initially interpreted this visual iteration as an indicator that children might be using competencies from the *Length Unit Relater and Repeater* level. However, focused observations showed that such iteration is more consistent with End-to-End Length Measurer, in which children mentally place objects end to

end to measure length (often missing the constraint of equal-length units). For example, Marina demonstrated End-to-End Length Measurer (albeit with actions that were harbingers of Length Unit Relater and Repeater) as she visually iterated a few units mentally in attempting to determine which of several paths was longest. When asked to check, she placed her finger at the beginning of the path counting, “One,” then hopped her finger counting, “Two...Three...” However, the hop distance was inconsistent; after three hops, she seemed to get lost, reaching for a ruler. Placing the ruler along the path, Marina pointed at and counted out the units. Marina counted movements along a path; we interpret the lack of equal-length intervals to mean that that she was attempting to apply, but could not hold, a mental unit throughout. The ruler served as a scaffold, providing units she could count.

We also found disconfirming evidence for our initial hypothesis regarding such visual iteration during a February 2010 CTE conducted with the entire first-grade class. At the time of the CTE, three children consistently had demonstrated the ability to iterate during individual TEs; thus, we had identified their thinking at Length Unit Relater and Repeater. Three other children had demonstrated inconsistent iteration skills, seeming to understand the need for iteration, but struggling to maintain equal-sized units. Given a single unit, these children counted as they moved a unit across the space, but did not move the unit with a consistent increment. The remaining two children were unable to measure if not given enough units to lay end to end. During the CTE, children were tasked with measuring the dimensions of the classroom. All children gathered enough objects to lay end to end across the room, without showing concern that those objects must be of the same unit. The next day, when measuring the dimensions of their desks, children were given 20 connecting cubes, enough to lay end to end along a little more than half of the desk. Some children spread out the cubes to make them reach from one edge to the other, leaving space between the cubes. Other children laid their 20 connecting cubes end to end, starting at one edge of the desk, then attempted to visually estimate the cubes needed to fill the leftover space. No child approached the task using iteration; we take this as evidence that children were still End-to-End Length Measurer.

The observations from this CTE also led us to explore in depth children’s use of a ruler to measure length. During the individual TEs, our focal children had demonstrated correct placement of the ruler and the ability to give a correct measure by observing the alignment of the ruler and the endpoint of the object (e.g., “five inches”). We initially took this as an understanding of unit and evidence of Length Unit Relater and Repeater. However, during the fourth day of the CTE, we tested the strength of this

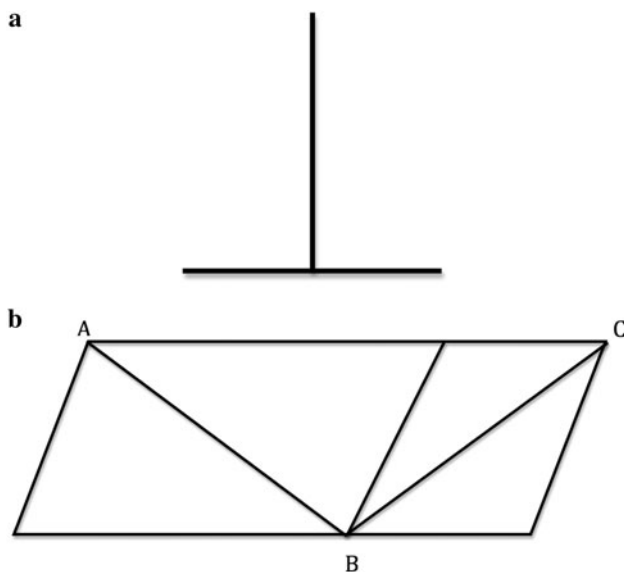


Fig. 2 Figures used in tasks—**a** upside down T shape and **b** parallelogram within which two different line segments (AB and BC) were drawn

understanding. All the first-grade children created their own rulers on a paper strip, marking off lengths of five connecting cubes and individual “connecting cube units.” Given objects to measure, most of the children could align rulers correctly and read the correct number for length; given several objects, they could even locate one that was a given length by aligning it with their rulers. However, when asked how many of the connecting cubes would equal the length of the object, many of the children just provided guesses, showing no connection to their “ruler” measures.

We followed up the CTE with a TE in March specifically designed to probe counting versus space understanding. In the TE, children were given a ruler and inch tiles. They were asked, “How many inches long is the ruler?” Every child took inch tiles, laid them end to end along the ruler, and counted out a total of 12, consistent with understanding at End-to-End Measurer. The researcher then pointed at the ‘5’ on the ruler and asked, “How many inch tiles would be the same length as this number on the ruler?” Of our eight focal children, only Edith and Alice could answer correctly without counting tiles. With guidance, all but two of the children began to grasp the idea that the numbers on the ruler and the number of inch tiles meant the same thing. However, when measuring other objects, these children were still reluctant to say that if an object was 8 in. tiles long, it would align with the ‘8’ on the ruler; they still needed the tiles for verification. We interpreted this as evidence that the idea of a unit was not fully established in children’s minds, even though they could use a ruler to produce a correct number of inches for a length. Thus, by the end of the first-grade year, only Edith and Alice had progressed to the Length Unit Relater and Repeater level.

4.4 Children’s development, pre-K to Grade 1

One of our goals was to identify and trace the dominant level of length understanding demonstrated by each child during each TE. The graphs in Fig. 3 trace these levels for the five children who participated all 3 years (the other three had incomplete, but consistent data). Children generally progressed through the original LT, from Length Quantity Recognizer to Length Direct Comparer, then to End-to-End Length Measurer and Length Unit Relater and Repeater. One exception to smooth upward growth was Lia, who demonstrated end-to-end measurement at the end of kindergarten, but had regressed to the Length Direct Comparer level at the beginning of first grade. Lia quickly and consistently moved ahead to end-to-end thinking after the first TE in September 2008, so we attribute the fallback to a lack of engagement in measurement activities during the summer.

5 Discussion and implications

Although measurement is clearly important in mathematics and science, US elementary and middle school students often lack conceptual competence for measurement. Recent comparison studies show that US students perform far below the mean on measurement and geometry items (Ginsburg et al., 2005). This research was designed to evaluate and refine an LT for length measurement in the early years that could provide a foundation for curriculum decisions and professional development (Clements & Sarama, 2004; Simon, 1995; Smith et al., 2006). The LT had been supported by evidence, but all studies were cross-sectional. Although the TE methodology used in the present study limits the number of participants and thus presents a caveat regarding generalizability, it provides two missing perspectives—a longitudinal account of development and an examination of the cognitive components of the levels of thinking.

Our findings are generally consistent with the original *Building Blocks* LT. Given that this original LT was based on a review of research available at that time (Clements & Sarama, 2009; Sarama & Clements, 2009), such findings replicate the previous research and provide additional empirical support for the LT described in Table 1. For example, they indicate that development of length concepts and skills is a slow and extended process, but that young children do have the ability to understand and learn these ideas (Bartsch & Wellman, 1988; Ellis et al., 2003; Hiebert, 1981; Miller & Baillargeon, 1990; Nunes & Bryant, 1996). They reliably move through the levels posited; for example, most pass into and through the End-to-End Length Measurer level, in which they place multiple units or begin to iterate a unit but leave gaps between units (Lehrer, 2003).

The graphs in Fig. 3 summarize children’s development and indicate predominately development through these levels as hypothesized. This should not be misconstrued as indicating that children’s behaviors were always consistent with a single level. Consistent with the theory of hierarchic interactionism, although children work predominately in one level of an LT, they employ concepts above this level in some contexts (and, of course, each level hierarchically builds on previous levels, so strategies from previous levels are often instantiated). For example, Fig. 4 provides more detail regarding Lia’s length understanding, as demonstrated across the first 3 years. We identified a dominant level of thinking (highlighted in gray) demonstrated by the child during each TE, but also she employed strategies below or above that level in various tasks.

Findings also suggest two open questions concerning, and one refinement of, the original LT. The first open question involves the LT’s assumption (sometimes implicit

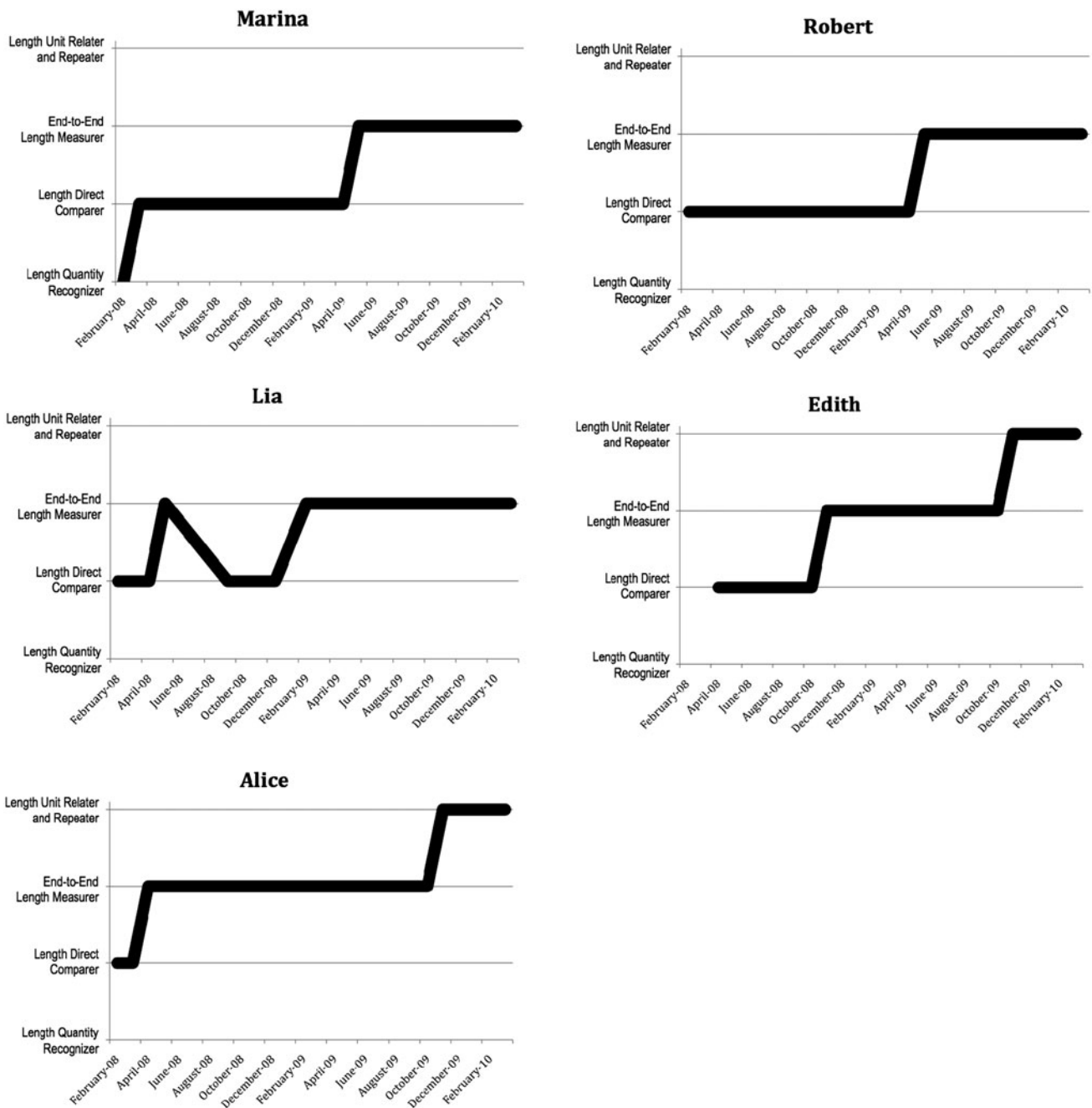


Fig. 3 Dominant levels of length measurement thinking demonstrated

in the original LT, but explicitly discussed when we initiated this study) that vocabulary developments may be important, even necessary, for children to construct higher levels of length understanding, especially the first level, Length Quantity Recognizer (where vocabulary was explicitly stated in the original LT). This expectation was realized for most children, whose development and appropriation of length vocabulary scaffolded their learning (Vygotsky, 1934/1986). However, at least some children’s responses to tasks appeared to be based on

theorems-in-action (Vergnaud, 1982) without verbal mediation. This absence of vocabulary in processing measurement tasks did not appear to hinder children’s development of new solution strategies or (at least some initial) new levels of thinking. As with other findings, a caveat is the small number of cases; therefore, this issue remains one for future research.

A refinement concerned the roles and placement of the Indirect Length Comparer level. The difference between the non-measurement (non-numerical) processes of direct

Fig. 4 Levels of length understanding demonstrated by one child across the TEs

	Length Quantity Recognizer	Length Direct Comparer	Indirect Length Comparer	End-To-End Length Measurer	Length Unit Relater and Repeater	Length Measurer
02/12/08	✓	✓		✗	✗	
04/09/08	✓	✓	✗			
05/14/08		✓	(✓)	(✓)		
05/21/08	✓	✓	(✓)	✓	✗	
10/20/08	✓	✓				
12/01/08		✓	(✓)			
02/25/09		✓	✓	✓		
04/07/09		✓	✓	✓		
04/21/09		✓		✓		
04/23/09				✓		
04/28/09				✓	((✓))	
05/06/09		✓		✓		
05/07/09				✓		
10/26/09		✓	(✓)	✓		
11/20/09		✓	✓	✓	(✓)	
12/11/09		✓	✓	✓	(✓)	
12/18/09		✓		✓	(✓)	((✓))
02/26/10		✓		✓	(✓)	
03/05/10		✓		✓	((✓))	
03/26/10		✓		✓	✗	✗
05/24/10				✓	(✓)	✗

Key: ✓ – Displays complete competency
 (✓) – Displays partial competency or completes a task with minimal guidance
 ((✓)) – Displays minimal competence or has significant guidance or teaching
 ✗ – When presented with a specific task, does not display any evidence of level

and indirect comparison and the numerical measurement processes has long been recognized (e.g., Hiebert, 1981; Piaget et al., 1960), although different learning trajectories consider them either separate or distinct but related strategies (Barrett & Battista, 2011). Based on several previous studies (Sarama & Clements, 2009), including a Rasch analysis (Sarama & Clements, 2009), the original length LT predicted that they were closely related and that children achieved the Indirect Length Comparer level of thinking just prior to the End-to-End Measurer level. Supporting the first prediction, all children learned direct comparison before other non-numerical or numerical processes. The actions on objects of the Direct Comparer level (such as alignment and comparison of the endpoints of lengths) were hierarchically related to all subsequent length development, as supported by the data. Although children made progress on Indirect Length Comparer competencies, that progress was not complete or reliable (other studies did not investigate the competence in numerous contexts, as we did here). Further, children reliably laid units end to end to measure an object before they reliably used indirect comparison with explicit transitive inference. In summary, we are planning additional studies to investigate whether

indirect comparison might develop as direct comparison competencies (both physical and mental imagery based) increase in strength and, possibly, in parallel with measurement strategies. Our present hypothesis, a refinement to the original LT, is diagrammed in Fig. 5.

In a related finding, children's use of mental imagery to conduct length comparisons initially suggested that such behaviors might signify thinking at the Indirect Length Comparer level. Subsequent observations revealed not only disconfirming cases, but also a distinct interpretation. That is, children's behaviors indicated that such comparisons were more consistent with a mental application of thinking at Length Direct Comparer. In this type of direct comparison, children formed a mental image of the first object's length and then transformed that image, superposing it on the perception of the first object. Thus, such mental imagery may serve as a transition to, and a catalyst for, the development of indirect comparison with explicit transitive reasoning.

We believe that these results are consistent with our understanding of reasoning about early, perceptual comparisons of discrete quantities (see Sarama & Clements, 2009, Chapter 4), a hypothesis that is also open to further

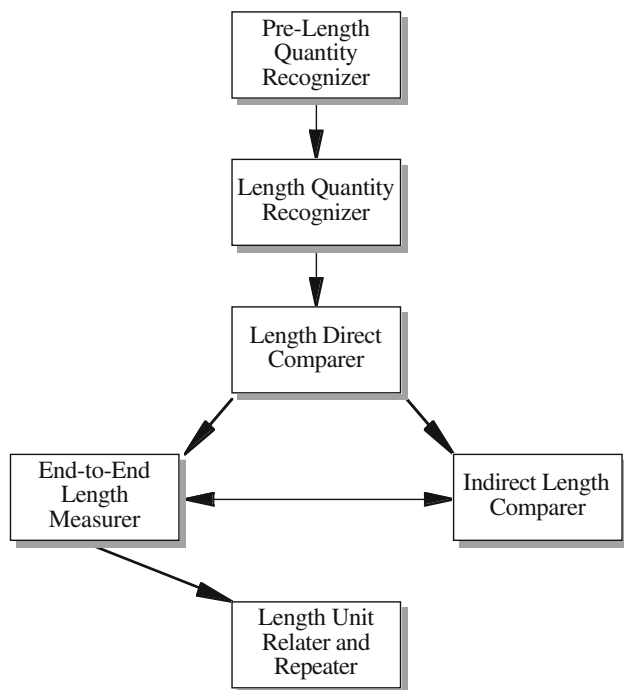


Fig. 5 A revised relationship between the hypothesized levels of thinking

research. That is, when asked to compare two sets of objects, children often initially use only perceptual cues. Later, they understand the need to count the sets and do so, but if their perceptions contradict the results of counting (e.g., 5 small blue blocks and 4 large red blocks), they fall back to the perceptual cues for their final answer (“more red”). Later in development, they are able to use the cardinal result of counting to ascertain the more numerous set. Likewise, the children in this study fall back to perceptual cues to compare lengths when they are unable to physically compare them (directly or with a third object) or to measure with multiple copies of the unit (that completely “fill the space”). (Alternatively, some use geometric arguments, such as Alice’s “roof on the house” symmetry-based argument, rather than measuring, but still relying on visually based perceptions.) They understand how to use multiple copies of a physical unit to measure, but are unable to iterate a single unit and accumulate these iterations. That is, accumulation of distance is the understanding that as one iterates a unit along the length of an object and count the iteration, the number words signify the space covered by units counted up to that point (Petitto, 1990). This signification is not sufficiently robust and salient for a child at this level. Children in the US may take an extended time to accumulate distance with understanding, given the emphasis of US curriculum and culture on counting discrete units (without explicit awareness that units are involved).

Children also used mental imagery in measurement tasks. We initially hypothesized this to indicate behaviors from the Length Unit Relater and Repeater level, in an internalized and therefore sophisticated manner. However, focused subsequent observations showed that such iteration is an unsophisticated instantiation of End-to-End Length Measurer behaviors. Thus, the role of mental imagery in children’s development of length measurement is a second open question.

We are planning future studies to evaluate these and other hypotheses. Given the small number of children in our studies, findings and modifications to the research-based original LT should be considered with caution, and, especially as our modifications were determined post hoc, should be replicated in future research. Further evaluation, refinement, and extension (to subsequent levels and thus grades) of the length LT should allow the generation of a new research-based approach to teaching children measurement. We believe that these learning trajectories, both the levels of development and the activities that successfully help children move ahead in their development, will be a significant contribution to helping children build a strong foundation in measurement.

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