

Five people are randomly assigned seats in a room which has 15 seats. Find the probability that no two people are seated next to each other if

- a) the seats are arranged in a row
- b) the seats are arranged in a circle.

Solution:

Part a) Let A be the collection of all 5 element subsets of $\{1, \dots, 11\}$ and B the collection of all 5 element subsets of $\{1, \dots, 15\}$ with no two consecutive numbers. We show that A and B have the same size, i.e. the same number of members. Consider $\{a_1, a_2, a_3, a_4, a_5\}$ in A . Then the 5 element set $\{a_1, a_2 + 1, a_3 + 2, a_4 + 3, a_5 + 4\}$ is in B , and it is clear that every set in B arises exactly in this fashion. (Formally, this association gives a one-to-one correspondence between A and B .)

The size of A is the number of combinations of 11 objects taken 5 at a time, which is $11!/(5! \cdot 6!)$. The number of ways that 5 people can sit in 15 seats is the number of combinations of 15 objects taken 5 at a time, which is $15!/(5! \cdot 10!)$. The probability that no two people are seated next to each other is the ratio of these two numbers, which is easily computed (even by hand) as $2/13$, or about 15.4%.

Part b) Assume the chairs are numbered $1, \dots, 15$. By symmetry, we may assume that some specific person sits in chair 1 and that chair 15 is next to chair 1 in the circle. If no two people can be seated next to each other, we must distribute the remaining 4 people in the 12 chairs numbered $3, \dots, 14$ in such a way that none of these people are seated next to each other. A similar argument as in part (a) shows that this can be done in $9!/(4! \cdot 5!)$ ways. The total number of ways the remaining 4 people can be distributed in the remaining 14 chairs (without regard to where they are sitting) is $14!/(4! \cdot 10!)$. The probability that no two people are seated next to each other is the ratio of these two numbers, which is $18/143$ or about 12.6%.