Algebra Preliminary Examination

Department of Mathematics, University of Denver

Winter 2013 (January 14, 2013)

Name:

Instructions:

• The duration of the exam is 4 hours.
• The exam has three parts, each part consisting of four problems.
• Each problem is worth 10 points.
• All problems from part 1 and the best 6 problems from parts 2 and 3 (combined) will determine your score.
• A score of 70% guarantees a pass.

Points:

| Problem 1.1 | /10 |
| Problem 1.2 | /10 |
| Problem 1.3 | /10 |
| Problem 1.4 | /10 |
| Problem 2.1 | /10 |
| Problem 2.2 | /10 |
| Problem 2.3 | /10 |
| Problem 2.4 | /10 |
| Problem 3.1 | /10 |
| Problem 3.2 | /10 |
| Problem 3.3 | /10 |
| Problem 3.4 | /10 |

Total points: Percentage: Passed: Yes No
Problem 1.1: [2.5 points each]
(i) Prove that \((\mathbb{Z}[x], +)\) is isomorphic to \((\mathbb{Q}^+, \cdot)\).
(ii) Find an example of a nontrivial group \(G\) and a nontrivial normal subgroup \(N\) of \(G\) such that \(G/N\) is isomorphic to \(G\).
(iii) Prove that a group \(G\) is abelian if and only if the map \(f : G \to G\) defined by \(x \mapsto x^2\) is a homomorphism.
(iv) Prove that if \(G\) is a finite group and the map \(f : G \to G\), \(x \mapsto x^2\) is surjective, then \(G\) has odd order.

Problem 1.2: [5 points each] Let \(R\) be a commutative ring.
(i) Prove that the set \(\eta(R) = \{x \in R \mid x^n = 0 \text{ for some } n \in \mathbb{Z}\}\) is an ideal of \(R\).
(ii) Prove that \(\eta(R/\eta(R)) = 0\).

Problem 1.3: [10 points] Prove that no rational root of \(x^{15} - 4x^7 + 2x^5 - 6\) can be equal to an expression involving rational numbers, the four fundamental arithmetical operations, square roots and cube roots (applied in any successive order).

Problem 1.4: [2, 4, 4 points] Given an integral domain \(R\), let \(T\) be the set of all upper triangular matrices of the form \(\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}\), where \(x, y \in R\) and \(x \neq 0\), and let \(D\) be the set of all diagonal matrices in \(T\). For matrices with entries in \(R\), we define
\[
\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \otimes \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_2b_2 \\ a_3b_3 & a_4b_4 \end{bmatrix}.
\]

Also, we write \(\oplus\) for regular matrix multiplication.
(i) Show that \((T, \oplus)\) is an abelian group and that \(D\) is a (normal) subgroup of it.
(ii) Show that \(\otimes\) is well-defined on \(T/D\) and that \((T/D, \oplus, \otimes)\) is a field.
(iii) Show that \(T/D\) is isomorphic to the field of quotients of \(R\).
Part 2: Group Theory

Problem 2.1: [3, 7 points] Note that $2013 = 3 \cdot 11 \cdot 61$.

(i) Show that there is a unique abelian group of order 2013 up to isomorphism.
(ii) Show that there exist at least two groups of order 2013 up to isomorphism.

Problem 2.2: [4, 6 points] Let $G$ be a finite group.

(i) If $S_a = \{gag^{-1} | g \in G\}$ is a conjugacy class, show that $|S_a|$ divides $|G|$.
(ii) Now assume $G$ is a finite $p$-group for some prime $p$. Show that $|Z(G)| > 1$, where $Z(G)$ denotes the center of $G$.

Problem 2.3: [10 points] Find all normal subgroups of $S_n$ for $n \geq 5$.

Problem 2.4: [10 points] Let $G$ be a finite group with the property that every maximal subgroup is normal. Prove that $G$ is nilpotent. (Hint: It is sufficient to show that every Sylow subgroup is normal.)
Problem 3.1: [5 points each]
(i) Assume that for all sequences \((p_i)_{i \in \mathbb{N}}\) in an integral domain \(D\), if \(p_i | p_{i+1}\) for all \(i \in \mathbb{N}\), then there is a \(j \in \mathbb{N}\) such that \(p_{j+1} | p_j\). Prove that every non-zero non-unit element of \(D\) can be written as a product of irreducibles.
(ii) Show that an element \(p\) of an integral domain \(D\) is irreducible iff \((p)\) is maximal (proper) among all principal ideals of \(D\).

Problem 3.2: [10 points] Show that every ideal of a ring \(R\) is finitely generated (namely it is generated by a finite set) iff every sequence \(I_1 \subset I_2 \subset \ldots\) of ideals of \(R\) (each properly contained in the next) has finite length.

Problem 3.3: [2 points each] Give an example for each of the following, if possible:
(i) A non-commutative domain that is not a division ring.
(ii) A unique factorization domain that is not a principal ideal domain.
(iii) An infinite domain of non-zero characteristic.
(iv) A domain of order 10.
(v) A field of order 8.

Problem 3.4: [10 points] Let \(F \subseteq E \subseteq K\) be fields with \(K\) a normal extension of \(F\). Show that \(K\) is a normal extension of \(E\) and give an example to show that \(E\) need not be a normal extension of \(F\).