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CHAPTER 3

LEARNING AND TEACHING GEOMETRY WITH COMPUTERS IN THE ELEMENTARY AND MIDDLE SCHOOL

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With their graphical as well as mathematical capabilities, computers seem well-poised to facilitate elementary and middle school students' learning of geometry and their development of spatial sense. Students can see and construct multiple examples of geometric objects, apply transformations to these objects, and connect various representations of geometric concepts while using computers. In this chapter, we review research addressing such technological potentialities for several categories of computer environments: turtle geometry, dynamical geometry software, computer manipulatives, and other software approaches, such as computer-assisted instruction and games. For each environment, we review research on learning and teaching specific geometric topics, including shape, measurement, and transformations. We considered all studies located through extensive computer and library searches. We address themes that cut across these categories and conclude with implications for instruction and
curriculum development. We begin with a brief description of each of the environments.

**COMPUTER ENVIRONMENTS**

Historically, one of the first geometric computer environments available was turtle graphics, usually as a component of the Logo programming language. With Logo, students direct the movements of an on-screen turtle to draw geometric paths and shapes. For example, students could create a rectangle by entering the command “repeat 2 [fd 20 rt 90 fd 50 rt 90],” which means run the list of commands twice, starting with forward 20 steps, turn right 90°, and so forth. Certain turtle geometry environments (e.g., Clements & Meredith, 1994; Clements & Sarama, 1996) include enhanced capabilities, such as performing geometric motions and other transformations on shapes, such as slide 30 10 (which performs a slide, or translation, of a figure, increasing its x-coordinates by 30 and its y-coordinates by 10).

Dynamical geometry software (DGS) environments allow students to alter the original objects by moving components, such as vertices and edges, to different locations on the screen. As the original objects are modified, the results of all constructions and transformations applied to those objects are updated immediately on the screen. The students also can measure lengths, angles, and areas of objects on the screen and then can observe how the measures are affected as the object is altered dynamically. The Geometer’s Sketchpad (Jackiw, 1991, 2002) and Cabri Geometry II (IMAG-CNRS Université Joseph Fourier, 1998) are two common examples of DGS environments.

Static construction environments, such as Geometric Supposer (Schwartz & Yerushalmy, 1986/2000), were available prior to the existence of DGS environments. These environments typically allow students to construct geometric objects on the computer, to apply common Euclidean constructions, and to make measurements based on the constructions. However, static environments do not allow students to manipulate directly the original objects and to observe immediately the effects of the manipulations. Instead, the environment can apply the students’ previously recorded constructions to a similar object.

Similar to the DGS environments, another type of computer environment involves computer manipulatives and tools or procedures that act on and transform these manipulatives. For example, students might manipulate software versions of pattern blocks, acting on them with tools that might include geometric motions and dilations. Frequently, such
computer manipulatives are developed for younger students, and attempt to closely mirror physical objects and actions.

Computer-assisted instruction (CAI) includes examples of software designed to assist individuals' learning of specific concepts via tutorial or drill-and-practice strategies. Another strategy for learning specific concepts is the computer game.

**TURTLE GEOMETRY**

The earliest and most extensively researched computer environment for learning geometry is turtle geometry, an approach to exploring geometry by directing a turtle, or on-screen pointer, via various versions of the Logo programming language or related programs. Given the constant state of change in the field of computers in education, one might question the perceived relevance of turtle graphics. There are at least three reasons that this research is significant. First, various versions of Logo are still being sold (and shareware versions downloaded) and used in classrooms; also, numerous alternate forms of turtle geometry exist (e.g., “Maps and Movement,” Education Development Center, 1993). Second, students' work in Logo environments has much to teach us about learning in other similar, yet distinct, computer and non-computer environments that are not necessarily computational. Further, the research has implications for the development of theories about the learning and teaching of mathematics.

**Why Turtle Geometry?**

Action—physical and mental—is deemed important to the learning of geometry by most major theoretical perspectives. Piaget and Inhelder (1967) claimed that a child’s representation of space is not a perceptual “reading off” of the child’s spatial environment, but is the result of prior active manipulation of that environment.

Action is also important to both learning and teaching according to the theory of Pierre and Dina van Hiele. This theory posits that students progress through levels of thought in geometry (van Hiele, 1986). At Level 0, students do not reliably distinguish circles, triangles, and squares from nonexemplars of those classes and appear to be unable to form reliable mental images of these shapes (this level has been added to the original formulation and has received empirical support, see Clements, Swaminathan, Hannibal, & Sarama, 1999). Level 1 is the visual level, in which students can only recognize shapes as wholes and cannot form
mental images of them. A given figure is a rectangle, for example, because "it looks like a door." At Level 2, the descriptive/analytic level, students recognize and characterize shapes by their properties. For instance, a student might think of a square as a figure that has four equal sides and four right angles. At Level 3, the abstract/relation level, students can form abstract definitions, distinguish between necessary and sufficient sets of conditions for a concept, and understand and sometimes even provide logical arguments in the geometric domain. They can classify figures hierarchically (by ordering their properties) and give informal arguments to justify their classifications (e.g., a square is identified as a rhombus because it can be thought of as a "rhombus with some extra properties"). At Level 4, students can establish theorems within an axiomatic system.

In this theory, progress is dependent upon instruction more than age. Teachers can "reduce" subject matter to a lower level, leading to rote memorization, but students cannot bypass levels and achieve understanding. The latter requires working through certain "phases" of instruction. The van Hiele theory also includes a model of teaching that progresses through five phases in moving students from one level of thinking to the next. In Phase 1, Information, the teacher places ideas at the student's disposal. In Phase 2, Guided Orientation, students are actively engaged in exploring objects (e.g., folding, measuring) so as to encounter the principal connections of the network of conceptual relations that is to be formed. In Phase 3, Explicitation, students are guided to become explicitly aware of their geometric conceptualizations, describe these conceptualizations in their own language, and learn traditional mathematical language. In Phase 4, Free Orientation, students solve problems whose solution requires the synthesis and utilization of those concepts and relations. In Phase 5, Integration, teachers encourage students to reflect on and consolidate their geometric knowledge, with an increased emphasis on the use of mathematical structures as a framework for consolidation and, eventually, place these consolidated ideas in the structural organization of formal mathematics. At the completion of Phase 5, a new level of thought is attained for the topic. Only in the Explicitation and Integration phases is the learner's intention sharply directed.

Instruction Phase 2 centers around students' manipulation of objects. Research indicates that computer environments, including most of the types discussed in this chapter, can facilitate that type of manipulation. As a Logo example, consider how turtle geometry activities might be used to encourage students to progress to Levels 2 (descriptive/analytic) and 3 (abstract/relation) in the van Hiele hierarchy. For instance, with the concept of rectangle, students initially are able only to identify visually presented examples, a Level 1 (visual) activity. Using Logo, however,
students can be asked to construct a sequence of commands (a procedure) to draw a rectangle (see the rectangle procedure at the upper right of Figure 3.1). This "allows, or obliges, the child to externalize intuitive expectations. When the intuition is translated into a program it becomes more obtrusive and more accessible to reflection" (Papert, 1980a, p. 145). That is, in constructing a rectangle procedure as part of instructional Phase 2, the students must analyze the visual aspects of the rectangle and reflect on how its component parts are put together, an activity that encourages Level 2 thinking. Furthermore, if asked during a Phase 4 activity to design a rectangle procedure that takes the length and width as inputs, students must write a type of definition for a rectangle, one that the computer understands. Students thus begin to build intuitive knowledge about the concept of defining a rectangle, knowledge that later can be synthesized in instructional Phase 5 and eventually integrated and formalized into an abstract definition—a Level 3 activity. Asking students whether a square or a parallelogram can be drawn by their rectangle procedure (if given the proper inputs) encourages students to think about the ways in which they begin to consider the structure and properties of the shape in coherent and logical ways, another Level 3 activity.

Figure 3.1. Using coordinate and differential geometry commands to create a design of congruent rectangles in Geo-Logo.
Early empirical findings on Logo were ambiguous (Clements, 1985). Reviews concluded that there were conflicting results about the effects of Logo on overall mathematics achievement. Experiments by Logo’s developers generated positive reports (Papert, Watt, diSessa, & Weir, 1979). In the United Kingdom, low-achieving 11-year-old boys with 2 years of directed Logo programming experience improved to perform at the same level as a control group on one general mathematics test but fell behind the control group on another test (Howe, O’Shea, & Plane, 1980). Other studies showed little positive effect on mathematics achievement (Akdog, 1985; Pea, 1983), although there were promising results for geometric concepts (Lehrer & Smith, 1986; Noss, 1987).

More recent reviews generally have been positive (Clements & Sarama, 1997; Yelland, 1995), as illustrated by the following quotation from McCoy (1996):

> Logo programming, particularly turtle graphics at the elementary level, is clearly an effective medium for providing mathematics experiences ... when students are able to experiment with mathematics in varied representations, active involvement becomes the basis for their understanding. This is particularly true in geometry. (p. 443)

The Logo Geometry project conducted a major evaluation of the Logo Geometry (LG) curriculum, including 1,624 kindergarten to Grade 7 U.S. students and their teachers (Clements, Battista, & Sarama, 2001). The curriculum is described in detail elsewhere (Battista & Clements, 1991; Clements et al., 2001) so only a brief summary is provided here.

The curriculum is divided into three strands: Paths, Shapes, and Motions. The concept of path is explicitly taught and used as an organizing idea for beginning geometric concepts. Students walk paths and discuss their movements, then construct similar paths in Logo. This includes lessons about grids and basic turtle geometry commands, Logo procedures, turn measure, and debugging. In the fifth and final lesson, students apply some of their previously learned skills in a problem-solving environment, and learn about the process of “undoing” (i.e., finding the inverse for) a sequence of actions. They are asked to write procedures that move the turtle from one point on a scene (depicted on the computer screen) to another (one of three restaurants) and then return the turtle to the starting point along the same path. The students must find a pattern for returning to the starting point (undoing the original commands) so that they can bring the turtle home even if the destination is off the screen (which requires replacing the last command with its inverse; e.g., LT 90 for RT 90).

Once students firmly grasp the concept of path, the curriculum asks them to think about special paths such as squares and triangles. The goal
of this second strand of the curriculum is to have students view these shapes as paths and thus begin analyzing the shapes in terms of their constituent components and properties. The sequence of lesson topics is squares, rectangles, equilateral triangles, regular polygons, classifying angles, interior angles of a polygon, parallel lines and parallelograms, and classification of quadrilaterals.

The goal of the third and final strand is for students to develop concepts in motion (transformational) geometry. Fundamental to this strand are the ideas that there are an infinite number of figures congruent to a given figure and that these figures may be related by a combination of geometric motions (i.e., isometries of the plane). Lesson topics include introduction to symmetry, mirror images and symmetric figures, introduction to geometric motions, planar motions with Logo tools, spatial visualization and prediction of the effects of these motions, congruence and motions, symmetry and motions, and motions as flips. Most teachers separated the strands by one or more months across a school year.

The evaluation included a wide assortment of research techniques: pre- and posttesting with paper and pencil, interviews, classroom observations, and case studies (Clements et al., 2001). Across grades K-6, LG students scored significantly higher than control students on a general geometry achievement test, making about double the gains of the control groups. These are especially significant because the assessment was a paper-and-pencil test that did not allow the experimental group access to the computer environments in which they had learned and because the curriculum is a relatively short intervention, lasting only 6 weeks.

Research across the various projects and studies has focused on the following concepts: plane figures, especially students' levels of geometric thinking about those figures; measurement; and motion geometry, congruence, and symmetry. We discuss each of these in turn.

Shape

Logo experience appears to encourage students to view and describe geometric objects in terms of the actions or procedures used to construct them (Clements & Battista, 1989; Clements et al., 2001). When asked to describe geometric shapes, students with Logo experience proffer not only more statements overall, but also more statements that explicitly mention components and geometric properties of shapes, an indication of Level 2 (descriptive/analytic) thinking (Assaf, 1986; Clements & Battista, 1989, 1990; Lehrer & Smith, 1986). Guided Logo experience appears to enhance significantly students' concepts of plane figures (Butler & Close, 1989; Clements, 1987). In one study, students with guided
Logo experience were able to apply their knowledge of geometry better than did a comparison group, but there was no difference between the groups in their knowledge of basic geometric facts. The researchers concluded that the use of Logo influenced the way in which students mentally represented their knowledge of geometric concepts (Lehrer, Randle, & Sancilio, 1989).

Results of the Logo Geometry project provide support for the hypothesis that LG is effective in developing schemes for basic geometric figures that improve performance on van Hiele-based assessments. A close analysis of specific responses showed that LG students developed rule-based, conceptual knowledge of shapes—for example, using verbally mediated properties and ideas, rather than only visual images, to classify shapes. Further, they performed substantially better than control students on hierarchical classification, and on a related competence of attributing statements of geometric properties, especially in the intermediate grades. Most students were just beginning to progress to Level 3 in the van Hiele hierarchy for the two-dimensional shapes that were assessed. Many students held multiple conceptions, some of which were evoked and applied only in certain situations. One episode the researchers analyzed illustrates the role turtle geometry played (Clements et al., 2001). Fifth-grader Jonathan had just successfully made and analyzed several rectangle procedures. He then used his “general” rectangle procedure in conjunction with a “turn” command to draw a tilted rectangle (labeled 4 in Figure 3.2), and is now attempting to make a nonrectangular parallelogram (labeled 7).

Teacher: Could you use different inputs, or is it just impossible?
Jonathan: Maybe, if you used different inputs. [Jonathan types in a new initial turn. He stares at the picture of the parallelogram on the activity sheet. He examines his Logo code for a while.] No, you can’t. Because the lines are slanted, instead of a rectangle going like that. [He traces a rectangle over the parallelogram.]

Teacher: Yes, but this one’s slanted [indicates the tilted rectangle, labeled 4, that Jonathan had successfully drawn with the Logo procedure].
Jonathan: Yeah, but the lines are slanted. This one’s still in the size [shape] of a rectangle. This one [parallelogram]—the thing’s slanted. This thing [rectangle] ain’t slanted. It looks slanted, but if you put it back [shows a turn by gesturing, meaning to turn it so that the sides are vertical and horizontal] it wouldn’t be slanted. Any way you move this [the parallelogram], it wouldn’t be a rectangle. The lines are slanty to each other. So, there’s no way.
Figure 3.2. Shapes that students are to attempt to reproduce with a "rectangle" procedure.

When Jonathan examined his code rather than rerunning it again, we believe he "ran through the procedure definition in his head," which contributed to his emerging sense of the properties of shapes and of his certainty that a given shape was not a member of the class of rectangles.

It has been noted that social and affective variables were important in distinguishing differences in performance that were gender based in Logo contexts (Hoyle & Sutherland, 1989; Yelland, 1994a, 1994b). Yet it is important to note that such studies did not qualitatively distinguish between the differences, but rather described the various ways in which boys and girls approached and deployed strategies in computer-based spatially oriented tasks. What is evident is that we need to be aware that such differences may exist and be sure that we value both approaches. At the current time it often seems as if traditional views concerning expertise with computers have elevated characteristics that are more compatible with masculine performance and interactions, and in doing so have often considered the performance of females as being deficient. There is now a
range of empirical evidence to reveal that gender differences are artifacts of task and measures of performance (Ching, Kafai, & Marshall, 2002; Edwards, 1991; Yelland, 2002).

Thus, appropriate use of Logo can help elementary and middle school students to analyze these figures and how their components are put together. Logo environments and tasks provide students with opportunities to analyze and reflect on the properties of two-dimensional shapes, at least within the Logo setting. Also, the need to build Logo procedures allows students to develop understandable, implicit definitions of these shapes. These experiences facilitate a transition from Levels 0 and 1 (pre-recognition and visual) to Level 2 or 3 (descriptive/analytic or abstract/reational) of geometric thought, at least in the specific domains of two-dimensional shapes and geometric motions (Assaf, 1986; Clements & Battista, 1989; Clements, Sarama, & Battista, 1998; Hughes & Macleod, 1986; Kynigos, 1993; Lehrer & Smith, 1986). This is likely due to Logo’s incorporation, implicitly, of the types of properties that will be developed by Level 1 thinkers explicitly, something that textbooks often fail to do (Battista & Clements, 1988a; Fuys, Geddes, & Tischler, 1988).

**Measurement**

**Angles, Angle Measure, and Turns**

Considering the critical role that turtle rotations play in forming geometric figures (e.g., Jonathan’s formation of rectangle 4 in Figure 3.2), it might be expected that Logo experiences facilitate the development of the geometric concepts of angle and angle measure. Several research studies have reported that, while the Logo experience does not eliminate “errors,” it appears to have a significant, positive effect on students’ ideas about angle. For example, responses of control students in one study were more likely to reflect little knowledge of angle and to use common language such as “a corner” and “a line tilted.” In contrast, the responses of the Logo students indicated more generalized and mathematically oriented conceptualizations,² such as “Two segments that come together at a point. It’s sort of a place where two lines come together” (Clements & Battista, 1989, p. 456).

Several researchers have reported a positive effect of Logo on students’ angle concepts (Browning, 1991; Clements & Battista, 1989; du Boulay, 1986; Frazier, 1987; Kieran, 1986a; Kieran & Hillel, 1990; Olive, Lankeau, & Scally, 1986). For example, students who worked with the LOGO curriculum did not identify angles from a set of figures better than control students did, but did significantly outperform control students in drawing an angle (Clements et al., 2001). Their angle concept appeared richer in
that they were much more likely to draw nonprototypical examples of angles. LG students' descriptions of angles emphasized the concept of rotation, and, to a lesser extent, bending, more than did those of control students, and they were more able to draw "larger" angles. There was no difference between LG and control groups on angle measure estimation, but LG students substantially outperformed the control group on applying knowledge of angle measure in analytical and problem-solving situations.

One micro-genetic study confirmed that students transform physical and mental action into concepts of turn and angle (Clements & Burns, 2000). Students synthesized and integrated two schemes, "turn as body movement" and "turn as number," as originally identified in an earlier study (Clements, Battista, Sarama, & Swaminathan, 1996). They used a process of psychological curtailment in which students gradually replace full rotations of their bodies with smaller rotations of an arm, hand, or finger, and eventually internalized these actions as mental imagery in which, perhaps, an internalized, schematized, projection of a body part rotated.

Logo experiences may foster some misconceptions of angle measure, including considering the amount of rotation along the path (e.g., the exterior angle in a polygon) or the degree of rotation from the vertical (Clements & Battista, 1989). In addition, such experiences do not replace previous misconceptions of angle measure. For example, students' misconceptions about angle measure in computer environments and difficulties coordinating the relationships between the turtle's rotation and the constructed angle have persisted for years, especially if not properly guided by their teachers (Clements, 1987; Cope & Simmons, 1991; Hoyles & Sutherland, 1986; Kieran, 1986a; Kieran, Hillel, & Erhwanger, 1986). In general, however, Logo experience appears to facilitate understanding of angle measure. Logo students' conceptualizations of angle size are more likely to reflect mathematically correct, coherent, and abstract ideas (Clements & Battista, 1989; Findlayson, 1984; Kieran, 1986b; Noss, 1987) and show a progression from van Hiele Level 0 (pre-recognition) to Level 2 (descriptive/analytic) in the span of the treatment (Clements & Battista, 1989). If Logo experiences emphasize the difference between the angle of rotation and the angle formed as the turtle traced a path, misconceptions regarding the measure of rotation and the measure of the angle may be avoided (Clements & Battista, 1989; Clements et al., 2001; Kieran, 1986b). For example, the implementation of LG effectively facilitated students' development of concepts of internal and external angles (Clements et al., 2001). Not all concepts are significantly affected; however, most experiences were of short duration. This is impor-
tant, as benefits often do not emerge until more than a year of Logo experience (Kelly, Kelly, & Miller, 1986-87).

An important note is that tools based on research and fine-tuned through field testing can encourage students' matematization. For example, Turtle Math (Clements & Meredith, 1994) was designed based on six general principles culled from the research, for instance, "encourage the growth of the abstract from the visual" (Clements & Sarama, 1995). However, the specific tools were changed to fit students' needs based on early pilot work. For example, we found that we needed turn tools that measured turns precisely and also simple tools that quickly and clearly illustrated the turtle's heading and gave general benchmarks (rays at 30-degree intervals). Finally, we needed commands (LTF for "left face," equivalent to LT 90) that avoided quantifying turns at all in the first group of lessons. This process resulted in a version of Logo more likely to support students' learning. In one study, Geo-Logo's (Geo-Logo is the same environment as Turtle Math, but with different activities) slow turns supported by the use of lines as representations of rays to emphasize the measure of the turn helped students build dynamic imagery for rotations (Sarama, 1995). The measurement and labeling tools encouraged them to eschew guessing, when they were required to use such mathematical processes as measurement and analysis. Unlike regular Logo, the graphics in the drawing window always precisely reflect the commands in the command center. In this way, the set of commands serve as a proleptic procedure, that is, its structure encouraged students to view and use it as a procedure before they formally defined it as such. This feature helped students encode contrasts between different commands.

Length

There is evidence that Logo experiences affect measurement competencies beyond the measure of rotation and angle, because Logo permits the student to manipulate units and to explore transformations of unit size and number of units without the distracting dexterity demands associated with measuring instruments and physical quantity. For example, young Logo students were more accurate than control students in measure tasks (Campbell, 1987). The control students were more likely to underestimate distances, particularly the longest distances, to have difficulty compensating for the halved unit size, and to underestimate the inverse relationship between unit size and unit numeracy. In a study of third-grade students, the Geo-Logo environment was critical in providing meaningful tasks that helped promote students' growth through three levels of strategies for solving length problems (Clements, Battista, Sarama, Swaminathan, & McMillen, 1997b; Sarama, 1995). Students' integration of number and geometry was especially potent and synergistic in the
Figure 3.3. A missing measures problem with directions, "Write a Logo procedure to draw this figure."

Logo environment. For example, students learned both geometric concepts and arithmetic in solving “missing measure” problems, such as that pictured in Figure 3.3 (Clements & Meredith, 1994; Clements & Sarama, 1996).

Other studies (Yelland, 2002) have revealed that the Geo-Logo environment, when embedded in an investigative curriculum context, is a powerful medium for learning in which students are able to experiment actively with measures of length and angle in dynamic ways that were not possible without the technology. In such contexts students developed understandings about units of measurement that were built up over time and that reflected an ability to recognize the need for consistency in units of measurement as well as an ability to estimate and modify quantities of measure to match varying situations. As an example, in one curriculum unit there was the opportunity to compare the differences in measure of a given distance with varying “step sizes.” That is, two turtles had to move the same distance, but one turtle’s step size was a multiple of the other’s. Students could complete the activities without making the requisite connections between number and distance, but when scaffolded by the teacher they were able to articulate sophisticated levels of reasoning about
the context that illustrated that they were cognizant of the relationship between unit size and measure. In the last in a series of four tasks, Jesse (aged 7 years) was able to predict the measure for a given distance on the basis of his previous experience in the environment. He was trying to determine the distance between two points by comparing steps that were three times bigger than those of his turtle, which had traversed the distance with 8 of its steps. His realization that the numbers could be connected by a factor of 3 was sudden and dramatic:

Jesse: Because see every step is 3 times bigger and it equals 3 ... so and ours is 3 of little ones, that's how, that's how I was working it out.
Teacher: So for every step ...
Jesse: 8 times 3
Teacher: So for every step, there's 3 little ones in it
Jesse: So I reckon it's 28
Teacher: So how could you do it on the calculator?
Jesse: 3 times 8
Teacher: Ok, do you want to try that Jesse?
Girl: 3 times 8 [pause] ... 24 ... he said 28!

The geometric setting provided both motivations and models for thinking about number and arithmetic operations. The motivations included game settings and the desire to create geometric forms. The models included length and rotation as settings for building a strong sense of both numbers and operations on numbers, with measuring and labeling tools supporting such construction. Conversely, the numerical aspects of the measures provided a context in which students had to attend to certain properties of geometric forms. The measures made such properties (e.g., opposite sides equal in length) more concrete and meaningful to the students. For example, students had to keep opposite lengths equal in certain missing length problems. They could use the length label tool to inspect each length. The dynamic links between the two domains of symbolic text and graphics structured in the Geo-Logo environment (e.g., a change in code automatically reflected in a corresponding change in the geometric figure) facilitated students’ construction of connections between their own number and spatial schemes. Finally, Geo-Logo provided feedback that students used to reflect on their own thinking (Yelland, 2002).

For example, in a maze activity that built on previous experiences in a computer maze task with only right angle turns, one third-grade girl was able to use her experience to estimate that a turn was 120 degrees and further extended her intuitive thinking by then recognizing that an
upcoming turn was approximately half of this size and therefore had to be 60 degrees. In a paper-and-pencil task the same girl had not successfully calculated half of 120 in a pretest context. Students also created contexts in which they used mathematical ideas and concepts that they would not have encountered in the traditional mathematics curriculum for their level. For example, in one task they were required to create a robot with component parts of a given perimeter (e.g., eyes of 100 steps). Making a square was relatively easy, because 100 divided by 4 yielded 25 steps for a side. However, one pair wanted their eyes to be equilateral triangles and thus used a calculator to divide 100 by 3 and the resultant number of 33.33 (reoccurring) caused great excitement for them as it was not only a whole line of threes but it had a “dot” which they had not encountered prior to this time. These third-grade students would not normally encounter decimals until year 5 of the mathematics curriculum, but were so fascinated by the dot that they wished to use it for all the parts of their robot and sought to use decimal numbers in each case.

Students exploring geometric concepts in Logo environments are also able to learn more sophisticated ideas about measurement, as well as about directions and coordinates. For example, one microworld allowed the turtle to measure distances and turns relative to previously constructed points on the plane, thus emphasizing non-intrinsic geometry as well as the turtle’s intrinsic geometry (Kynigos, 1992). The turtle facilitated learning this set of ideas as well. For example, students adopted and used the notion of the turtle “putting its nose to look” at some direction rather than “turning this much” (Kynigos, 1992, p. 106). Several studies, then, indicate that enriching the primitives and tools available to students can facilitate their construction of geometric notions and can increase analytical, rather than (only) visual, approaches (Clements & Battista, 1992a; Kynigos, 1992).

Measurement, Arithmetic, and Problem Solving

Work with LG significantly improved primary grade students’ recognition of the relevance of arithmetic processes in the solution of geometric measurement problems, and it improved their ability to apply these processes accurately (Clements et al., 2001). This has been confirmed in several additional studies with the Geo-Logo environment (Clements et al., 1996; Clements et al., 1997b; Masters, 1997; Yelland, Clements, Masters, & Sarama, 1996; Yelland & Masters, 1997).

Students sometimes use the support of the meaningful tasks and computer environment to “extend their reach” intellectually. One third-grade girl, for example, could predict accurately that the “turn will be 60°
because it was half of that last one" (120°) even though she could not calculate half of 120 on an interview or on a test (Masters, 1997). Students in the Masters study also invented new concepts for themselves that were not part of the curriculum, such as decimal numbers.

**Geometric Motions, Symmetry, and Congruence**

Research indicates that Logo experiences can also aid the learning of motion geometry and related ideas such as congruence and symmetry. In one study, students working with a Logo unit on motion geometry made only slow progress beyond van Hiele Level 0 (prerecognition) (A. T. Olson, Kieren, & Ludwig, 1987). There was, however, definite evidence of a beginning awareness of the properties of transformations. In another study, middle school students achieved a working understanding of transformations and used visual feedback to correct overgeneralizations when working in a Logo microworld, with some evidence that this was superior to students' learning without the use of Logo (Edwards, 1991).

Another study specifically investigated the effects of computer and non-computer environments on learning of geometric motions (Johnson-Gentle, Clements, & Battista, 1994). Two treatment groups of fifth-grade students, one of which used specially designed Logo computer environments and one of which used manipulatives and paper and pencil, received eight lessons on geometric motions, identical except for the Logo/non-Logo aspects. Both treatment groups, especially the Logo group, performed at a higher level of geometric thinking than did a non-treatment control group. The Logo group outperformed the non-Logo group on the delayed posttest—the same assessment administered a month after the posttest. Therefore, there was support for the effectiveness of the use of a Logo-based curriculum and for the notion that the Logo-based version enhanced the construction of higher-level conceptualizations of motion geometry. This conclusion was based on examination of the differences between the groups on specific items. The greatest differences were on those items that required students to resist applying intuitive visual thinking and apply analytical thinking in a comprehensive manner.

The need in Logo environments for more complete, precise, and abstract explication may account for students' creation of conceptually richer concepts for motions. That is, in Logo, students have to specify steps to a non-interpretive agent, with thorough specification and detail. The results of these commands can be observed, reflected on, and corrected; the computer serves as an explicative agent. In non-computer manipulative environments, one can make intuitive movements and corrections without explicit awareness of geometric motions. For example,
even young students can move puzzle pieces into place without conscious awareness of the geometric motions that can describe these physical movements. In contrast, with Logo, the use of a computer language makes the motions "more obtrusive and more accessible to reflection," in Papert's words, previously quoted.

Perhaps for similar reasons, Logo can aid students' development of symmetry concepts. Students as young as first grade have been observed using such mathematical notions as symmetry in their Logo work (Kull, 1986). In addition, students through middle school who are involved in Logo (Edwards, 1991; Gallou-Dumiel, 1989; J. K. Olson, 1985) learn symmetry concepts. One student used a specially-designed Logo symmetry micro-world to learn such concepts and effectively transferred her mathematical understandings to a paper-and-pencil problem (Hoyles & Healy, 1997). Similarly, LG microworlds helped students score significantly higher than control students on posttest measures of symmetry, both on tasks asking them to draw all the symmetry lines for given figures and those that asked them to draw the "other half" of a figure to create a symmetric figure. For the latter, there was a tendency for these effects to be particularly strong for young (kindergarten) LG students (Clements et al., 2001). Compared to students using paper and pencil, students using Logo work with more precision and exactness (Gallou-Dumiel, 1989; Johnson-Gentile et al., 1994). For example, writing LG commands for the creation of symmetric figures, testing symmetry by flipping figures via LG commands, and discussing these actions apparently encouraged students to build richer and more general images of symmetric relations and to reflect on the construction of symmetric figures and abstract the properties of symmetry. Students had to abstract and externally represent their actions in a more explicit and precise fashion for the LG activities than they did in other activities such as free-hand drawing of symmetric figures. This is supported by findings that differences between Logo and manipulatives groups existed on those items that required students to resist applying intuitive visual (only) thinking and to apply analytical thinking in a comprehensive manner (Johnson-Gentile et al., 1994). LG students also performed better than control students on congruence items, not so much by identifying whether pairs of figures were congruent as by justifying their answers (Clements et al., 2001). This suggests a movement away from van Hiele Level 1 (visual) thinking toward Level 2 (descriptive/analytic) thinking. Interestingly, the largest difference was for those questions with congruent pairs of figures. This difference suggests that LG helped students more on questions in which describing a difference (e.g., "this one is bigger") could not be used as a springboard to justification, as describing congruency forced students to focus on precise similarities or processes used to establish them. In summary, there is evidence in support of the hypothesis that Logo experiences can help elemen-
Summary and Conclusions

Exploring and programming in Logo environments can help students construct elaborate knowledge networks (rather than mechanical chains of rules and terms) for geometric topics. Empirical research identifies several unique characteristics of Logo that can facilitate students' learning (the following list is taken from Clements et al., 2001).

- The commands and structure of the Logo language can be consistent with geometric symbols and structures in ways that are pedagogically useful. For example, turtle geometry commands such as FD and RT focus students' attention on critical aspects of figures.
- Drawing with Logo's turtle graphics and creating run-able code are meaningful and interesting settings for students; these settings then motivate the use and learning of geometric and other mathematical ideas (Ainley, 1997).
- The turtle world involves measures that are visible, quantifiable, and formalizable, helping to connect spatial and numeric thinking (Clements et al., 2001; Clements et al., 1996; Clements et al., 1997b; Noss & Hoyles, 1992a).
- Logo can encourage the manipulation of screen objects in ways that facilitate students' thinking of them as mathematical objects and thus as representatives of a class. In this way, Logo can evoke more abstract geometric thinking (Clements et al., 2001).
- Logo can provide scaffolding for mathematical analysis; that is, a symbolic representation on the computer can allow the student to erect scaffolding around the solution of a problem, and subsequently attend to only those elements of the solution for which details need to be filled in (Noss & Hoyles, 1992a).
- Logo can help structure students' play to encourage symbolic and mathematical characteristics of exploratory mathematical activity (Hoyles, 1993). Or, as Papert (1995) states, "The computer simply, but very significantly, enlarges the range of opportunities to engage as a bricoleur or bricoleuse in activities with scientific and mathematical content" (p. 145).
DYNAMICAL GEOMETRY SOFTWARE ENVIRONMENTS

Why Dynamical Geometry Software (DGS)?

Research results suggest that elementary and middle school students can improve their understandings of geometry concepts while using DGS such as Cabri Geometry (IMAG-CNRS Universite Joseph Fourier, 1998) and The Geometer's Sketchpad (Jackiw, 1991, 2002). The use of dynamic environments also may help students improve their visualization skills (Dixon, 1997) and their ability to focus on interrelationships of the parts of a geometric shape (Battista & Borrows, 1997). However, it appears that students benefit most from using the environments when they interact with figures that are constructed to retain certain properties. The figures might be developed by students themselves or could be contained in prepared files that are available to students.

Initially, students tend to use the programs as basic drawing packages (Foletta, 1995; Pratt & Ainley, 1997; Vincent, 1998; Vincent & McCrae, 1999). In this case, assembling components, such as points, segments, lines, and circles, produces figures that look like the desired outcome, but the figures usually do not retain the desired characteristics when the components are “dragged” on the screen. Figure 3.4 illustrates the effects of dragging a vertex of a right triangle that was produced using The Geometer's Sketchpad as a drawing program. In addition to drawing, dynamic environments can be used to construct figures that will retain certain properties when various components are dragged on the screen. Figure 3.5 illustrates an example of dragging a vertex of a right triangle that was constructed to remain a right triangle.

![Diagram of a right triangle with angles labeled.](image)

Figure 3.4. A right triangle that was drawn. The right side of the figure shows the result when point A is dragged to the right.
Figure 3.5. A right triangle that was constructed. The right side of the figure shows the result when point A is dragged to the right (and somewhat downward).

**Shape**

Vincent and McCrae (1999) relate the process of drawing, rather than constructing, to van Hiele Level 1 because students at that level are focusing mainly on the visual appearance of shapes. One technique that helps some students move away from drawing and towards constructing in dynamic environments is asking them to produce figures that could not be "messed up" by dragging (Vincent & McCrae, 1999). For example, students could be asked to produce a rectangle that always would be a rectangle regardless of how its components were dragged. While working on these types of tasks, students began to focus on geometric properties of the figures and on relationships between properties of the figures and then used the capabilities of the programs to construct figures so that the properties and relationships would always hold. When working on the rectangle example, students focused on the necessity of having right angles and on having two pairs of opposite sides of equal lengths. Their focus on the necessity of right angles is similar to Jonathan's focus on the angles of a rectangle while working within the Logo Geometry curriculum (see the excerpt with dialogue between Jonathan and the teacher that was presented earlier in this chapter). Vincent and McCrae (1999) associate focusing on properties of the figures with the second van Hiele level (descriptive/analytic) and focusing on relationships within the figures with the third van Hiele level (abstract/relational).
Having students develop conjectures about geometric figures they have drawn is another possible approach to engaging students in thinking at the abstract-relational van Hiele level within a dynamic geometry environment (Hoyles, 2001; Hoyles & Healy, 1999). For example, Hoyles and Healy asked middle-school-aged students “to construct with Cabri [Geometry I] a quadrilateral in which the angle bisectors of two adjacent angles cross at right angles” (p. 107) and to identify and prove properties of the quadrilateral. The students made measurements within Cabri and looked for measurements that remained constant while dragging components of the quadrilateral. From these observations, some of the students deduced properties of the shape and focused on relationships among parts of the shape, similar to the students in the Vincent and McCrae (1999) study. However, the students did have difficulties developing proofs for their conjectures about the properties and relationships.

There also is evidence that students can progress through the van Hiele levels by focusing on the parts of shapes and on relationships among the parts while using prepared figures within a DGS program. Barrista and Borrow (1997) noted that fifth-grade students viewed researcher-constructed geometric shapes (referred to as “shape makers”1) as representations of classes of shapes, such as squares, rectangles, parallelograms, and trapezoids. Manipulating the shape makers in the dynamic environment and then reflecting on their manipulations helped the students to move from thinking about the shapes holistically (indicative of the visual level) to focusing on the relationships among parts of the shapes, including class hierarchies (indicative of descriptive/analytic and abstract/relational levels). In a sense, their manipulations of the shape makers were “physical” actions within a computer environment, as described previously in this chapter. Students also have improved their understanding of properties of shapes and relationships among parts of shapes while using geometry environments such as the Geometric Supposer (Schwartz & Yerushalmy, 1986/2000). For example, eighth-grade students enrolled in a course that integrated the use of Geometric Supposer were less apt to have the common misconception that a triangle’s altitude is always a segment in the interior of the triangle (Yerushalmy, 1991). The students enrolled in the course also developed the ability to use and check many examples while exploring patterns and to form accurate definitions for various geometry concepts.

**Geometric Transformations**

Students also have been able to develop and deepen their understandings about transformation geometry concepts in the presence of DGS environments. Dixon (1997) used The Geometer’s Sketchpad in teaching
four classes of eighth-grade students for approximately 3 weeks. The students used the dynamic environment’s capabilities to apply reflections and rotations to points of the plane and then conjectured about the properties of those types of transformations. Five classes of eighth-grade students formed the control group. They studied reflections and rotations using manipulatives rather than the computer. Students who had access to the dynamic environment outperformed students in the control group on measures involving concepts of reflection and rotation. This suggests that the use of DGS is helpful in improving middle school students’ understandings of reflection and rotation.

A study by Glass (2001) provides additional insights into middle school students’ conceptualizations of translations, reflections, rotations, and compositions of same-type transformations. Each of the eighth-grade students in the study spent approximately five hours working through a series of tasks in an individual interview setting. In contrast to using the software’s menu options, as in Dixon (1997), the students interacted with pre-constructed dynamic representations of the three transformation types in The Geometer’s Sketchpad. For example, the specific translation being represented was determined by the direction and magnitude of a vector on the screen. The students could change the specific translation by changing the direction and/or magnitude of the vector via dragging. Similarly, the students could interact with the reflection representation by dragging the visible line of reflection to change its location in the plane or its inclination. Similarly, they changed the specific rotation being represented by moving the center of rotation or by altering a dial-like controller that determined the angle of rotation. The most common strategy students used was centered on the perceived movement from the location of the pre-image to the location of the image. For example, students identified examples of reflections based on their perception of the image beginning on top of the pre-image and then flipping to its displayed location. The focus on a movement of the pre-image suggests that the student may be utilizing a visual level of understanding of transformation.

One concern about the use of dynamic environments in studying transformation geometry concepts is that students will focus on the dynamic movement rather than on the resulting relationships between pre-image and image pairs. However, middle school students also can focus on end results of a motion, rather than on the motion, in identifying transformation types (Glass, 2001). For example, they focused on the direction the image faced in relation to the pre-image. If the image faced the opposite direction as the pre-image, then the students tentatively identified the represented transformation as a reflection rather than a translation. Further, a few students utilized properties of transformations in their discussion, for example, observing that corresponding vertices of the pre-image
and image were equidistant from the apparent line of reflection. Glass (2001) suggested that students who consistently provided reasons from the motion-based category had conceptualized transformations as processes and may not have had an object view of transformation. In comparison, students who utilized reasons from the end-result and property-based categories were able to think of transformations as objects, as well as processes. It appears that the capability to dynamically link alternative representations of the same transformation type provides an environment in which students may begin to develop an object understanding of transformation.

Summary and Conclusions

Similar to the case with Logo-based environments, research results suggest the use of DGS can be beneficial to students in their development of understandings of geometric shapes and figures. In some situations, middle school students can progress from the first van Hiele level (visual) to the second and third van Hiele levels (descriptive/analytic and abstract/relational) while using the environments to construct figures that retain certain properties (Pratt & Ainley, 1997; Vincent, 1998; Vincent & McCrae, 1999). Students also can deepen their understandings of geometric shapes and figures while working with preconstructed figures (Battista & Borrow, 1997; Olive, 1991).

DGS environments also appear to facilitate middle school students’ learning of transformation geometry concepts (Dixon, 1997). Whether the students are taught how to use the environment’s built in transformation commands or are provided with preconstructed representations of transformations, students can deepen their understandings of translations, reflections, and rotations (Dixon, 1997; Glass, 2001). Further, there is some evidence (Glass, 2001) that the dynamic nature of the environment can influence the development of the students’ understandings.

**COMPUTER MANIPULATIVES**

Computer manipulatives, similar to DGS environments, provide geometric objects and tools that can be used to act on those objects. The objects are usually geometric regions that correspond to physical shape manipulatives, such as tangram pieces. Research indicates that the use of physical manipulatives facilitates the learning of sound representations of geometric concepts (Clements & Battista, 1992b). Use of manipulatives seemed to allow students to try out their ideas, examine and reflect on them, and
modify them. This approach seemed to maintain student interest, assist students in creating definitions and new conjectures, and facilitate students’ insight into new relationships. Unfortunately, in the past, nearly half of the teachers reported that their students used manipulatives less than once a week, or not at all (Driscoll, 1983). There are signs that in kindergarten classrooms use of manipulatives has been higher in recent years (Hausken & Rathbun, 2004); however, the early percentages may not have changed for higher grades over the years, with only 54% and 40% of fifth- and eighth-grade teachers reporting weekly use of manipulatives (Stecher, Barron, Kaganoff, & Goodwin, 1998). Surveys substantiate that use of manipulatives remains low in traditional classrooms (Ross, McDougall, Hogaboam-Gray, & LeSage, 2003). This may be important, because manipulative use for a school year or longer results in significant positive effects, but use of shorter duration often does not (Sowell, 1989). Further, even though most teachers claim they are using reform-oriented methods when they use manipulatives, teachers with a more didactic, compared to cognitive, orientation to teaching taught their students to use the manipulates to represent standard algorithms (Niederhauser & Stoddard, 2001).

Why Computer Manipulatives?

Most practitioners and researchers argue that manipulatives are effective because they are “concrete.” By “concrete,” they probably mean objects that students can grasp with their hands—what we called physical manipulatives. This sensory nature ostensibly makes manipulatives “real,” connected with one’s intuitively meaningful personal self. However, we cannot assume that concepts can be “read off” manipulatives because objects may be manipulated meaningfully without the concepts being illuminated (Clements, 1999). A manipulative’s physicality does not carry the meaning of the mathematical idea. Further, when we speak of concrete understanding, we are not always referring to physical objects. Teachers of later grades expect students to have a concrete understanding that goes beyond manipulatives: Shapes (or other geometric objects) should be “concrete” objects that middle-grade students can manipulate mentally. There are different ways, then, to think about “concrete.” Students have Sensory-Concrete (Clements, 1999) knowledge when they need to use physical material to make sense of an idea. For example, very young students do best with physical cutouts of shapes. Integrated-Concrete knowledge is knowledge that is connected in special ways. Memories of physical experiences, mental images, and ideas are concrete. The root of the word concrete means “to grow together.” Sidewalk concrete is strong due to the
combination of separate particles in an interconnected mass. Integrated-Concrete thinking is strong due to the combination of many separate ideas in an interconnected structure of knowledge (Clements, 1999).

If Integrated-Concrete knowledge is our goal, computers might provide useful representations. They may be just as personally meaningful, and even more manageable, flexible, and extensible than their physical counterparts. For example, one group of young students learned number concepts with a computer felt board environment. They constructed "bean-stick pictures" by selecting and arranging beans, sticks, and number symbols. Compared to a physical bean-stick environment, this computer environment offered equal, and sometimes greater control and flexibility (Char, 1989). In a similar vein, students who used physical and software manipulatives demonstrated a much greater sophistication in classification and logical thinking than did a control group that used physical manipulatives only (J. K. Olson, 1988).

Advantages of Computer Manipulatives for Learning and Teaching Geometry

Sarama and Clements identified two categories of advantages: (a) practical and pedagogical benefits and (b) mathematical and psychological benefits. Illustrations of each follow, taken from the participant observation research with kindergaten-age students (Sarama, Clements, & Yuketic, 1996) using Shapes, a software version of pattern blocks, that extends what students can do with these shapes (see Figure 3.6).

Practical/Pedagogical Benefits

This first group includes advantages that help students in a practical manner or provide pedagogical opportunities for the teacher.

1. Providing another medium, one that can store and retrieve configurations. Computers can provide another medium for building, one in which progressive development can take place day after day (i.e., physical blocks have to be put away most of the time—on the computer, they can be saved and worked on again and again, and there is an infinite supply for all students). This was observed when a group of students were working on a pattern with physical manipulatives (Sarama et al., 1996). They wanted to move it slightly on the rug. Two girls (four hands) tried to keep the design together, but they were unsuccessful. Marisssa told Leah to fix the design. Leah
Shapes is a computer manipulative, a software version of pattern blocks, which extends what students can do with these shapes. Students create as many copies of each shape as they want and use computer tools to move, combine, and duplicate these shapes to make pictures and designs and to solve problems.

Figure 3.6. An example of the Shapes computer manipulatives environment.

tried, but in re-creating the design, she inserted two extra shapes and the pattern was no longer the same. The girls experienced considerable frustration at their inability to get their “old” design back. Had the students been able to save their design, or had they been able to move their design and keep the pieces together, their group project would have continued.

2. Providing a manageable, flexible manipulative. Computer manipulatives can be more manageable for students than their physical counterparts. For example, computer-generated shapes may be designed to snap into position on command, connecting accurately with other shapes or other objects, such as a puzzle. Unlike physical manipulatives, computer manipulatives stay where they are put. While working on the Shapes software, students quickly learned to glue the shapes together and move them as a group when they needed more space to continue their designs.

3. Providing an extensible manipulative. Certain constructions are easier to make with the software than with physical manipulatives. Students were observed making non-equilateral triangles in such an environment by partially occluding shapes with other shapes. For example, they created a 30°-60°-90° triangle by occluding half of an equilateral triangle.
Mathematical/Psychological Benefits

Perhaps the most powerful advantages are that the actions possible with software can embody the processes educators want students to develop and internalize as mental processes. As previously noted, this is not a simple process. It is the student's actions, in interaction with the software and other people that are eventually internalized.

1. Bringing mathematical ideas and processes to conscious awareness. A software program's turn and flip tools can help students explicate these motions. For example, when kindergartner Mitchell (Sarama et al., 1996) worked off-computer, he quickly manipulated the pattern block pieces, resisting answering any questions that were asked as to his intent or his reasons. When working on-computer, he was aware of his actions, in that when asked how many times he turned a particular piece, he said, "Three," without hesitation. Thus, he was becoming explicitly aware of the motions and beginning to quantify them.

2. Changing the very nature of the manipulative. Software's flexibility allows students to explore geometric figures in ways not available with physical shape sets. For example, students can change the size of the computer shapes, altering all shapes or only some.

3. Allowing composition and decomposition processes. Use of software can encourage composition and decomposition of shapes. For example, students can readily "glue" shapes together to make composite units.

Summary

Definitions of what constitutes a "manipulative" may need to be expanded to include computer manipulatives, which, at certain phases of learning, may be more efficacious than their physical counterparts. With both physical and computer manipulatives, educators need to provide meaningful representations in which the objects and actions available to students parallel the mathematical objects (ideas) and actions (processes or algorithms) we wish the students to learn. We then need to guide students to make connections among these representations (Lesh, 1990). We do not yet know what modes of presentation are crucial and what sequence of representations we should use before symbols are introduced (Baroody, 1989; Clements, 1989; Metz, 1995). We should be careful about adhering to the concrete → pictorial → abstract sequence, especially when there is more than one way of thinking about "concrete" (Parmar &
Cawley, 1997). However, we need additional research to show what portion and what sequences of experiences with various media are optimal. Few studies address such issues. Our goal should be to have students connect manipulative models to their intuitive, informal understanding of concepts and to abstract symbols, learn to translate between representations, and reflect on the constraints of the manipulatives that embody the principles of a mathematical system (Thompson & Thompson, 1990). We need more research on the use of only computer manipulatives.

OTHER SOFTWARE APPROACHES

Computer-Assisted Instruction (CAI)

Although there would seem to be a large array of CAI titles on the market there are relatively few that pertain to developing geometric concepts in the elementary school. In a similar way there have been relatively few research findings, especially in the recent past, which sought to elucidate the nature and extent of learning geometric concepts in CAI contexts. Most of the studies have always considered the virtues of CAI versus traditional teaching methods in treatment studies.

Students instructed in geometry with CAI often score significantly higher than those having just classroom instruction, from the elementary years (Austin, 1984; Morris, 1983) to high school (Hannafin & Sullivan, 1996). Consistent with other research, learners may not accurately gauge the amount of instruction that is optimal for them; researchers recommend using a full computer-assisted instruction program rather than letting students choose how much time to spend working with the program (Hannafin & Sullivan, 1996).

Early research included the Stanford Drill and Practice Program and Project Solo based at the University of Pittsburgh (Knight-Burns & Bozeman, 1981). Although each project reported significant gains in mathematics test scores in general, there seemed to be conflicting, and even confusing claims regarding the relative merits of CAI versus traditional forms of instruction in mathematics. Research in the area of CAI effectiveness broadly concerns its effectiveness with respect to five variables: (a) student achievement, (b) student attitude to CAI and subject matter, (c) time savings relative to unit completions/ mastery learning, (d) learning retention, and (e) cost factors. However, it was difficult to attribute success or lack thereof to one specific variable. It also became evident that variables interacted in different ways, sometimes creating effective outcomes. What became apparent was that the use of CAI in teaching contexts supported by an effective teacher who accommodated the technology into
existing teaching practices was the most effective for student learning (Knight-Burns & Bozeman, 1981). This implication is consistent across all of the types of software reviewed here.

McClurg (1992) investigated the spatial skills of students in Years 3 and 4 (9 and 10 years of age). Specifically she was interested in elucidating whether CAI software could influence understandings and performance in tests of spatial ability. She found a significant treatment effect in favor of students using the CAI software involving the rotation of objects and that this effect was evident irrespective of the gender of the student. The results of the study suggested that the rotation element of problem-solving software might have been the primary influence on the observed gains on Figural Classification tasks. McClurg identified three factors that she claimed positively influenced problem solving: visual perception and discrimination, differentiation of opposite oblique segments, and recognition of reflected images in a context characterized by opportunities for experimentation, discussion, and repetition in the CAI contexts.

Lowrie (1998) considered the relationship between two-dimensional and three-dimensional representations in CAI contexts. He investigated the ways in which two students (7 and 8 years of age) engaged in a computer-based problem-solving activity that required a degree of visual spatial reasoning. Lowrie noted that the CAI activity that required the students to build a room onscreen helped the students to construct spatial understandings associated with perspective and representations, but only when the teacher provided scaffolding to the students. Lowrie argued that students could complete computer-based problems that appeared to focus on visual and spatial understandings without their being able to engage in effective mathematical thinking unless a teacher who could foster the mathematical understandings necessary to complete the tasks provided scaffolding.

Other studies have examined the ways in which problem-solving CAI influenced performance as demonstrated in Basic Skills tests which included geometric or spatial understandings (e.g., The Canadian Test of Basic Skills). For example, one study (reported in Fletcher, Hawley, & Piele, 1990) found that a CAI group performed significantly better on a standard test of mathematics achievement than control groups. In another study, when the use of CAI was supplemented with traditional mathematical instruction, gains were significantly higher (Fletcher et al., 1990).

Games

Computer games have been found to be marginally effective at promoting learning of angle estimation skills (Bright, 1985) and effective in facilitating achievement in coordinate geometry (Morris, 1983). Game
design can significantly increase the effectiveness of a program. More
effective than manipulating geometric objects is manipulating screen repre-
sentations of the specific concepts students are to learn (Sedighian &
Sedighian, 1996). For example, rather than dragging a shape to turn it, a
“direct concept manipulation” interface might have students manipulate
a tool for turns, including turn center and amount of rotation. That is,
given that the goal is to learn the concept of motion, the direct concept
manipulation interface includes tools for the motions themselves. Stu-
dents manipulate these tools rather than performing the motions directly
on the shapes. The dynamic tools used by Glass (2001), which were
described in the dynamic geometry section of this chapter, are other
examples of direct concept manipulations for translation, reflection, and
rotation. Manipulating an on-screen mathematical representation of the
transformation being applied to the shape in this way leads to higher
achievement than manipulating the shape directly, especially when paired
with appropriate use of scaffolding, such as gradually removing visual
feedback aids and requiring the use of specific transformations to achieve
some configurations (Sedighian & Sedighian, 1996). Thus, the effective-
ness of computer games depends on software design such as interface
styles and scaffolding, as well as on teacher and student expectations, on
the level of integration with other learning activities, and on the pattern
of use.

Several studies have investigated the effects of computer games on stu-
dents’ spatial thinking. Video games may have a positive effect on spatial
abilities, although short-term experiences may not yield results (Pepin &
Dorval, 1986). In a study of fifth-grade students, spatial performance was
significantly better in boys than in girls during pretest assessment (Sub-
rahmanyam & Greenfield, 1994). Video game practice significantly
improved spatial performance on the posttest assessment for both gen-
ders; there was no significant interaction of gender with experimental
treatment. However, video game practice was more effective for students
who started out with relatively poor spatial skills. This suggests that video
games may be useful in equalizing individual differences in spatial skill
performance, including those associated with gender. In another study,
students already relatively skilled in spatial visualization tended to benefit
more from the low active control (passively watching) than the high active
control condition. Students who were less skilled benefited from the high
active control condition; they apparently learned the strategy of holistic
mental rotation, in which an entire object is visualized as turning (Smith,
1993).

Other studies have investigated the effects of Tetris or similar games
on spatial abilities, with generally positive results (Bright, Usnick, & Will-
jams, 1992). One study, part of a large-scale curriculum development
project funded by the National Science Foundation, investigated the application and development of spatial thinking in an instructional unit on area and motions. The computer application was Tumbling Tetrominoes (Clements, Russell, Tierney, Battista, & Meredith, 1995). Results revealed strong positive effects on spatial abilities and the establishment of spatial-numeric connections; students doubled their scores on a test of spatial thinking after only 2 weeks of instruction (Clements, Battista, Sarama, & Swaminathan, 1997a).

Implications for Pedagogy

Research provides several guidelines for learning and teaching geometry. Because of the larger number of studies in, and longer history of, research involving turtle geometry, investigations of Logo tend to dominate this area, but we attempt to draw implications for all computer environments. Studies have shown that when students’ learning is scaffolded by teachers, students work more effectively than when it is not (Yelland, 2003). In this way, the role of the teacher as facilitator proved to be fundamental and critical to computer-based learning environments. Teachers were able to assist students in focusing on the salient aspects of tasks and to create contexts in which students were able to develop skills in metacognition, which ultimately enhanced students’ concepts of length (Yelland, 2003). Research has also shown (Yelland & Masters, 1997) that when students are scaffolded effectively in the exploration of spatial concepts they can explore powerful ideas at much younger ages. Yelland and Masters (1997) provided data indicating that Australian students as young as 8 years of age created rectangles using variable side lengths and placed congruent rectangles at specified locations in quadrants. Such mathematical concepts are traditionally not introduced until Year 9 in Australia in most mathematics curricula. The students in this research explored these concepts in Year 3 and were able to articulate the ways in which the concepts of variables, points of location, and quadrants facilitated the programming process in the computer microworld.

However, not all research has yielded positive results regarding technology as a medium for learning. First, few studies report that students “master,” or retain and use, the mathematical concepts that are the teachers’ goals for instruction. Second, some studies show no significant differences between Logo and control groups (P. A. Johnson, 1986). Without teacher guidance, mere “exposure” to Logo often yields little learning (Clements & Meredith, 1993). Third, some studies have shown limited transfer. For example, the scores of students from two ninth-grade Logo classes did not differ significantly from those of control students on subse-
quent high school geometry grades and tests (Olive, 1991; Olive et al., 1986). One reason is that students do not always think mathematically, even if the Logo environment invites such thinking (Noss & Hoyles, 1992a). For example, some students rely excessively on visual or spatial cues and avoid analytical work (Hillel & Kieran, 1988). This visual approach is not related to an ability to create visual images but to the role of the visual “data” (i.e., the students’ perceptions) of a geometric figure in determining students’ Logo constructions. Although helpful initially, this approach inhibits students from arriving at mathematical generalizations if overused. Further, there is little reason for students to abandon visual approaches unless teachers present tasks that can only be resolved using an analytical, generalized, mathematical approach. For example, simple “missing lengths” tasks, as described previously, can be solved by visual estimation, but students find that analysis is more efficient, and necessary for more complicated tasks. Finally, dialogue between teachers and students is essential for encouraging predicting, reflecting, and higher-level reasoning (e.g., Clements et al., 2001; Hoyles & Noss, 1987a).

In sum, studies showing the most positive effects involve carefully planned sequences of activities and teacher scaffolding that supports the development of specific mathematical concepts and understandings. Appropriate teacher mediation of students’ work with those activities is necessary for students to construct geometric concepts successfully and to articulate the ways in which these concepts are connected to other conceptual areas of mathematics. This mediation must help students forge links between experiences within computer environments and other experiences and between computer-based procedural knowledge and more traditional conceptual knowledge (Clements & Battista, 1989; Lehrer & Smith, 1986). Care must be taken that such links are not learned by rote (Hoyles & Noss, 1992). Teachers must be involved in planning and overseeing the computer experiences to ensure that students reflect on and understand the mathematical concepts (McCoy, 1996). They need to (a) focus students’ attention on particular aspects of their experience, (b) elude informal language and provide formal mathematical language for the mathematical concepts, (c) suggest paths to pursue, (d) facilitate disequilibrium using computer feedback as a catalyst, and (e) continually connect the ideas developed to those embedded in other contexts. Teachers must provide structure for computer-based tasks and explorations to facilitate desired learning. To accomplish all this, teachers need specifically designed computer activities and environments. It is unclear whether it is necessary to have students plan away from the computer. It has been used successfully in some studies (Clements, 1990). Other authors, however, conclude that asking pupils to plan away from the com-
puter can be unnecessarily and unnaturally restrictive (Hoyle and Sutherland, 1989).

CURRICULA AND SOFTWARE

One finding that is consistent across studies is the critical role played by both the teacher and by the curriculum in which the computer software is embedded. The following list summarizes implications for using computer environments in geometry curricula (adapted from Clements and Battista, 1994; Clements et al., 2001).

- Curricula must find the right level of representation for students. For example, it may not be an efficient use of time to have students write procedures to perform geometric motions, but they can efficiently use tools available in enhanced environments (such as Logo or GeoGebra) to perform geometric motions (Battista and Clements, 1988; Clements and Sarama, 1996; du Boulay, 1986).
- Carefully planned and researched sequences of computer activities lead to greater likelihood of learning (Borer, 1993; Clements et al., 2001).
- Versions of computer environments with tools specifically designed to enhance mathematical activity that are highlighted in the curriculum lead to greater likelihood of mathematical learning than versions without such tools (Clements and Sarama, 1995; Kynigos, 1993; Sarama, 1995).
- The most effective use of computer environments may involve full integration into the mathematics curriculum. For example, too much school mathematics involves exercises devoid of meaning. Computers can provide environments in which students use mathematics meaningfully to achieve their own purposes. They can provide formal symbolizations that students can invoke, manipulate, and understand (Hoyle and Noss, 1987b). Using computer environments in this way can help fulfill the early vision of "teaching students to be mathematicians vs. teaching about mathematics" (Papert, 1980, p. 177).

TECHNOLOGY AND GEOMETRY

There is evidence that computer environments can support learning and teaching in geometry in new and dynamic ways, as well as complementing and enriching traditional strategies. Unfortunately, the results of empiri-
cal studies such as those cited here have had little impact on mathematics curricula around the world. Mathematics in schools occurs mainly on paper, even in settings based on "standards" or reform. It has been demonstrated in this chapter that Logo environments can help students construct elaborate knowledge networks for geometric topics via several unique characteristics that can facilitate students' learning. These characteristics include the following: (a) multiple meaningful representations of geometric objects and processes and tools to link these representations; (b) measures that are visible, quantifiable, and able to be formalized, helping to connect spatial and numeric thinking; and (c) creation of "runnable" code (or scripts) that encourages a procedural view of geometric constructs, as well as the habit of exploring, posing, and solving problems. Similarly, DGS environments can allow for similar representations and encourage students to develop viable geometric concepts, as well as to explore geometric properties, situations, and conjectures. Computer manipulatives similarly have both practical/pedagogical benefits such as being more flexible and extensible, and mathematical/psychological benefits such as bringing mathematical ideas and processes to conscious awareness, changing the very nature of the manipulative, and allowing and encouraging a greater variety of geometric actions, such as composition and decomposition processes. Finally, some computer CAI and games have also been shown to improve spatial and geometric understandings of young students.

Such research then affords the opportunities for educators to consider the unique characteristics of computer environments that are able to provide contexts for new types of learning in mathematics in the information age. These characteristics include the following (adapted from Clements & Battista, 1994; Clements et al., 2001).

- Computers can promote the connection of symbolic with visual representations, thereby supporting the construction of mathematical strategies and ideas from initial intuitions and visual approaches (Clements & Burns, 2000; Clements & Sarama, 1995; Noss & Hoyles, 1992a). Thus, computers can serve as transitional devices from physical movements to more abstract mathematical conceptualizations. Computers also can make mathematics more concrete, while simultaneously supporting students' formalization of actions algebraically as a computer program (Hoyles, 1993).

- Representations students create within software programs can help document their actions, leading to meaningful mathematical symbolization. Students can build a symbolic mathematical language based on this documentation.
- Students' thoughts are captured in these symbolic representations and can be reenacted and revised. Their actions, and the graphic geometric actions to which they connect, can become objects of mathematization and reflection (Clements & Sarama, 1997; Kieren, 1992; Noss & Hoyles, 1992a). Students can also return to test ideas and thus return to action-oriented ways of thinking as they extend higher-order ideas (Kieren, 1992).

- Software programs can provide feedback. The feedback of some programs, such as Logo, DGS, manipulative, and other construction programs, is distinct from "right or wrong" feedback associated with CAI materials. Instead, the feedback is an instantiation of students' expression of geometric ideas.

- Software environments can require and so facilitate precision and exactness in mathematical thinking (Gallou-Dumiel, 1989; Johnson-Gentile et al., 1994).

- Software programs can provide a window to students' mathematical thinking (Noss & Hoyles, 1992b; Weir, 1987). Such environments provide a fruitful setting in which teachers can work with and listen to students.

- Because students may test ideas for themselves on the computer, software environments can aid students in moving from naive to empirical to logical thinking (Clements et al., 2001). These environments also encourage students to make and test conjectures. Thus, such environments can facilitate students' development of autonomy in learning (as opposed to seeking authoritative opinion) and of positive beliefs about the creation of mathematical ideas.

- Even if used for less exploratory purposes, software environments allow students to search for relationships that seem beyond their grasp at that moment; they can try a range of possibilities on the computer (Battista & Clements, 1986; Hoyles & Noss, 1989), whereas on paper they often have little recourse but to ask for help or quit.

Unfortunately, research also identifies a reality that is far different from this vision. Few teachers from early childhood through middle school use computers for anything other than drill and practice (Clements & Nastasi, 1992; Manoucherhi, 1999). This leads to three critical, related caveats. First, most of the software that is produced and used for learning mathematics in the elementary and middle grades is not research-based in any sense—it was not developed with formative research nor is there adequate summative research substantiating its effectiveness. This situation should be changed (Clements, 2002). Second, most of the advantages
identified for geometric computer environments that have been studied have been observed in small, usually well-supported educational contexts. We need to know more about the scaling up of those environments and their effectiveness in more varied contexts. Third, broad-based improvement in elementary and middle school geometry education attributable to the use of computers probably cannot be expected without dramatic change in the type of software and instructional strategies used in most classrooms.

Several additional findings across studies using different computer environments are notable. First, extended experience with such software environments may be important to realizing their educational potential (Clements et al., 2001). Second, embedding computer activities within the geometry curriculum may also be important to realizing their full benefits. Third, in a similar vein, curriculum and pedagogy need to be consistent with the software used; in the case of innovative geometry environments, investigative and inquiry approaches need to be substantially integrated. Fourth, researchers and teachers consistently report that in such contexts students cannot “hide” what they do not understand. That is, difficulties and misconceptions that are easily masked by traditional approaches emerge and must be dealt with, leading to some frustration on the part of both teachers and students, but also to greater development of mathematics abilities (Clements & Battista, 1989). This leads to the next point. Fifth, evaluation of learning in such environments must be reconsidered (see Galindo, 1998), as traditional approaches did not assess the full spectrum of what was learned; in some cases, these approaches made little sense.

Finally, it is evident that we need to ascertain whether we are focusing on appropriate questions in a context that seems to be under the rule of an economic rationalist philosophy. Policymakers seem to be primarily concerned with value for their dollar. This is a feature that is mirrored in treatment-control studies, often for short periods of time, which attempt to find cause-and-effect features to support their arguments for or against the use of computers. With such marginal use of computers for learning it is of little surprise they have minimal effects in CAI contexts, since their use is not integrated into the whole program. It would seem that governments and policymakers need to justify that for each dollar spent on computers there should be a demonstration of commensurate increases in test scores. On the other hand, curriculum developers (in general) seem to continue their work as though computers serve only a marginal purpose in subject areas and need to mimic popular culture. Their design and approaches reflect content and suggest pedagogies more suited to traditional curricula. In this way the power of potentially revolutionary technologies is often ignored and we miss opportunities to create new
sequenced topics that incorporate the embedded use of new information technologies. At present, most schools are continuing the practice of fitting new technologies into old curricula or within the context of traditional didactic teaching methods. What we need to see are ways in which curricula can be reconceptualized to illustrate the ways in which they can be modified to reap the benefits of new technological environments for learning and teaching geometry.

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NOTES

1. Note that our own interpretation of theories of geometric thinking is that visual/spatial knowledge remains essential at all levels of thinking (see Clements, Battista, & Sarama, 2001). That is, geometric knowledge continues to include nonverbal, imagistic components; every mental geometric object includes one or more image schemes, that is, a recurrent, dynamic pattern of kinesthetic and visual actions (M. Johnson, 1987). Thus, imagistic knowledge is not left untransformed and merely “pushed into the background” by higher levels of thinking. Imagery has a number of psychological layers, from more primitive to more sophisticated (each of which connect to a different level of geometric thinking) that play different (but always critical) roles in thinking, depending on which layer is activated. Thus, even at the highest levels, geometric relationships are intertwined with images, though these may be abstract images.

2. Here, we use “conceptualizations” to refer to mental schemes—mental networks of relationships connecting concepts and processes in specific patterns.
3. Visual here refers to the use of only visual data and the neglect of analysis.
4. For further information about the Shape Makers program, see Battista (2008).

REFERENCES


Learning and Teaching Geometry With Computers


