Background Research on Early Mathematics

Douglas H. Clements, Arthur J. Baroody, and Julie Sarama

Background Research for the
National Governor's Association (NGA) Center Project on Early Mathematics

Supervisors: Amanda Szekely and Albert Wat

Table of Contents

Research Brief #1: The Rationale for Focusing on Early Childhood Mathematics

Questions addressed by this brief:
1. *Why is a focus on improving early childhood mathematics education important?*
2. *What individual and group differences do children show at the beginning of school?*
3. *What is the quality of early mathematics education in the U.S. and what impact does it have on different groups of children?*

1. Why is a focus on improving early childhood mathematics education important?

Early mathematics is surprisingly important. There are several reasons.

Our increasingly technology- and information-based society, mathematical proficiency has become as important a gatekeeper as literacy and, thus, critically important for all members of society to achieve (*1-5*). Unfortunately, U.S. students are not achieving this proficiency. The 2011 National Assessment of Educational Progress indicated that 18% of a representative sample of U.S. pupils scored below a basic level of mathematics achievement and another 35% scored below a proficient level.[1] International comparisons indicate that U.S. students have relatively low levels of mathematical achievement, as early as first grade and kindergarten (*6*), and often as early as three to five years of age (*7, 8*).

This knowledge gap is most pronounced in the performance of U.S. children living in economically deprived urban and rural communities (*9-14*). There are two primary reasons for the achievement gap between disadvantaged students and their peers. One is the relatively low quality school (formal) instruction that disadvantaged students receive. A second reason is that such students have not had experiences in the home, community, and early childhood classrooms that would prepare them for more formal instruction in the primary grades and above. Because informal mathematical knowledge, such as making fair shares of a group of treats, provides a key basis for understanding and learning formal mathematics (*15*), learning difficulties arise when children have not had opportunities to develop such knowledge in meaningful everyday situations (*16-18*). Serious individual differences in this foundational knowledge appear as early as three years of age (*19-25*). Disadvantaged children who have not developed the requisite informal mathematics knowledge struggle with formal mathematics from the start of school and fall further and further behind their peers (*26-29*). Children who begin with the lowest achievement levels show the lowest growth in mathematics from kindergarten to the third grade (*28*).

This is especially alarming because children's early knowledge of math strongly predicts their later success in math (*30*), even into high school (*5, 25, 31*). Persistent problems with mathematics is the best predictor of failing to graduate from high school or enter college (*32*). More surprising is that it also predicts later reading achievement, even better than early reading skills (*33-35*). In fact, research shows that doing more mathematics in preschool increases oral language abilities when measured during the following school year. These include vocabulary,

---

[1]See http://nationsreportcard.gov/math_2011/summary.aspx.

inference, independence, and grammatical complexity (*36*). In short, mathematical thinking is cognitively foundational (*37, 38*) and, given the importance of mathematics to academic success in all subjects (*39*), all children need a robust knowledge of mathematics in their earliest years. For all these reasons, early and effective intervention is needed to level the playing field for disadvantaged children.

Research shows that research-based experiences in early mathematics can promote learning and close the gaps in informal and formal knowledge (*1, 40, 41*). Many approaches and programs not, however, (e.g., *42, 43-45*); therefore, policy should ensure the implementation of high-quality, empirically-proven programs. In brief, given appropriate opportunities, all young children can develop substantive mathematical ideas (*1, 15, 46-48*).

2. What individual and group differences do children show at the beginning of school?

Children from low-income families Children who live in poverty and who are members of linguistic and ethnic minority groups demonstrate significantly lower levels of achievement on mathematics (*30, 49-59*), and such disparities in student outcomes have been increasing for decades (*60-62*). The achievement gaps have origins in the earliest years, with low-income children possessing less extensive math knowledge than middle-income children of pre-K and kindergarten age (*9, 13, 17, 24, 30, 63, 64*). For example, according to one study, about 80% of entering kindergartners whose mothers had a bachelor's degree passed a difficult mathematics assessment, compared to only 32% of those whose mothers had less than a high school degree. Large differences were also found between ethnic groups (*65*).

Children from low-income families show specific difficulties. They do not understand the relative magnitudes of numbers and how they relate to the counting sequence (*11, 66, 67*). They have more difficulty solving addition and subtraction problems (*66-69*) and are less able to adapt their strategies to new situations (*14*).

If high-quality mathematics education does not start in preschool and continue through the early years, children can be trapped in a trajectory of failure (*9, 26, 28, 56, 70*). These are not individual deficits (e.g., lower IQ): Poverty and lack of opportunities to learn are the strongest predictors (*37*). Even small reductions in poverty lead to increases in positive school behavior and better academic performance (*71*).

Parents' beliefs and behaviors related to math also contribute to individual and group differences among children. Studies show that low-income parents, compared to middle-income parents, believe that math education is the responsibility of the school and that children cannot learn many aspects of math that research indicates they can learn (*7*).

Children who are members of linguistic minority groups also deserve special attention (*72*). Too many people mistakenly believe that language is less of a concern in math, compared with other subjects, because math is based on "numbers" or "symbols." Children learn math mostly from oral language, rather than textbooks or mathematical symbolism (*73*). Teaching English language learners (ELLs) specific vocabulary terms ahead of time, emphasizing words that have similar roots in English and the second language (e.g., Spanish cognates), is both useful and necessary, but not sufficient. Teachers need to help these students see multiple meanings of terms in both languages (and conflicts between the two languages), and address the language of mathematics, not just the "terms" of mathematics. That is, mathematical language involves relationships and reasoning, not just particular vocabulary words. Building on the resources that bilingual children bring to mathematics is also essential. For example, all cultures have "funds of knowledge"—the mathematics used in building, farming, cooking, and so forth— that can be used to develop mathematical contexts and understandings (*74*). Thus, "talking math" is far more than just using math vocabulary. Ideally, these children should have an opportunity to learn in their first language (*75, 76*). The long-term goal should be to help children maintain and build the first language while adding fluency and literacy skills in English, not replacing the child's home language with English (*77*). At least, teachers of ELLs need to understand the linguistic characteristics of classroom language and also master ways to connect everyday language with the language of math (*73*).

Children with mathematical learning difficulties and disabilities. Similar to those at-risk for other reasons, children with special needs often do not fare well in the typical early childhood classroom. Many children show specific learning difficulties in mathematics at young ages. Unfortunately, they are often not identified, or categorized broadly with other children as "developmentally delayed." This is unfortunate because focused mathematical interventions at early ages can be effective with children who have special needs (*78-80*).

Two categories are often used (*78*). Children with *mathematical difficulties (MD)* are those who are struggling to learn math for any reason. Sometimes defined as all those below the 35th percentile, estimates can be as high as 40 to 48% of the population. Those with a specific *mathematics learning disability (MLD)* have some form of memory or cognitive deficit that interferes with their ability to learn concepts and/or procedures in one or more domains of math (*81*). They are, therefore, a small subset of all those with MD, about 6% to 7% of the population (*78, 82*). Studies have found that such classifications are not stable for many children in the early and primary grades; only 63% of those classified as MLD in kindergarten were still so classified in third grade (*82*).

Children with MD or MLD differ in important ways, and policy and programs will be more effective if they recognize and plan for these differences. For example, some children have an underlying deficit in "number sense." Others lack spatial capacities, or have language or other specific cognitive deficits.

Children who are gifted and talented in mathematics. Although often perceived by educators as "doing just fine," children with special needs due to their exceptional potential or abilities also do not do well in early childhood (and later) programs (*25*). They actually decline in some arithmetic skills relative to others, especially in the preschool and primary grades (*83*). This may be a result of mathematics curricula that are most suited to the least advanced children. Advanced children learned little or nothing (*84*) and may become bored and frustrated (*85*). Even if identified, they usually are exposed to concepts traditionally found in early childhood programs (*86*), rather than more advanced knowledge of measurement, time, and fractions. Children who are economically disadvantaged are not only at high education risk; they are also at risk of not being identified as gifted (*87*).

3. What is the quality of early mathematics education in the U.S. and what impact does it have on different groups of children?

High-quality instruction has meaningful effects on children’s mathematics knowledge (*36, 40, 88-92*). However, the quality of mathematics education for young children in the U.S. varies, and is generally disappointing, especially in the earliest years. For example, 60% of 3-year-olds had no mathematical experience of any kind across multiple observations of their classrooms (*43*).

Studies find that, even if an early childhood education program adapts an ostensibly "complete" curriculum, mathematics is often inadequate, with the most commonly used curriculum engendering no more math instruction than a “business-as-usual” comparison group (*93, 94*). It is little surprise, then, that evaluations show little or no learning of mathematics in these schools (*67, 95*). As an example, observations of *Opening the World of Learning (OWL)*, which includes mathematics in its curriculum, found that out of a 360-minute school day, only *58 seconds* were devoted to mathematics. Most children made no gains in math skills and some lost mathematics competence over the school year (*96*).

Kindergarten classrooms include more mathematics, about 11% of the day. However, again there are many missed opportunities, with average kindergarten students not engaged in any instructional activity in 39% of the observed intervals (*5*), instead performing routines like hand-washing or “lining up.”

Preschool and kindergarten teachers often believe that they are “doing mathematics” when they provide puzzles, blocks, and songs. Even when they teach mathematics, that content is usually not the main focus, but is “embedded” in a fine-motor or reading activity (*5, 37*). Unfortunately, evidence suggests such an approach is ineffective (*5*).

In summary, typical early childhood classrooms underestimate children’s ability to learn mathematics and are ill suited to help them learn. *C*hildren may *regress* on some math skills during pre-K (*96*) and kindergarten (*70*). Children who do not build a robust understanding of mathematics in the early years too often come to believe math is a guessing game and system of rules without reason (*97*). Without high-quality instruction, they develop destructive habits of mind. For all children, we need more structured, sophisticated, and better-developed and sequenced mathematics in early childhood education.

The low quality of early mathematics education is most damaging to those groups and individuals who are at risk. Children who begin kindergarten with the least mathematics knowledge have the most to gain (or to lose) from their engagement with learning. It is essential to find ways to keep these children engaged in learning tasks and to increase their initial knowledge (*28*).

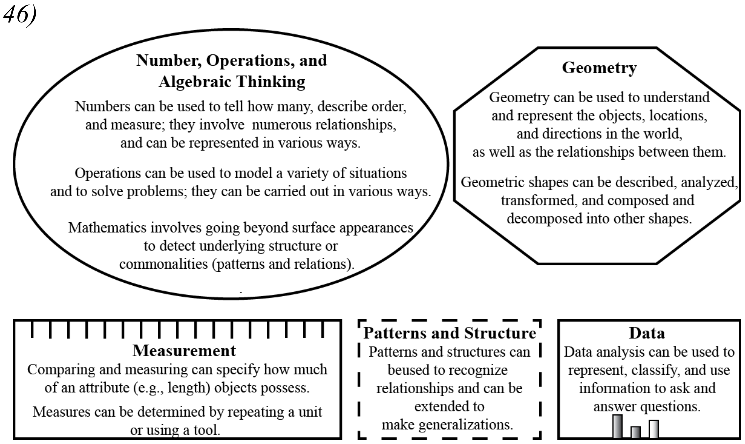
Brief #2: Key Math Concepts and Skills for Pre-K to Grade 3 and their Typical Trajectory of Development

1. What are the key goals for early childhood mathematics education?
2. How do specific goal competencies develop and how should they be taught?

1. What are the key goals for early childhood mathematics education?

Instruction that focuses on mathematical proficiency involves (a) content goals (namely conceptual understanding and procedural fluency), (b) process goals (expertise with the techniques of mathematical inquiry including strategic competence and adaptive reasoning), and (c) affective goals (a predictive disposition (see Appendix A2-1). The most important *content* goals for early mathematics are the *big ideas of mathematics education*— ideas that connect concepts and skills within a domain and across domains (*98-100*). An instructional focus on big ideas promotes coherent, meaningful, efficient, and adaptive/self-generated learning. For example, a big ideas such as equal partitioning and its informal analogy of fair sharing can provide the conceptual basis for understanding division, fractions, measurement and the rationale for related skills, such as using a ruler to measure length. Big ideas are consistent with goals and starts from the National Council of Teachers of Mathematics (*101*), the National Mathematics Advisory Panel (*25*), and the Common Core State Standards (*102*). A representation of the most general big ideas, from a conference on early standards that contributed to each of these documents, is shown in Figure 1.

Figure 1. The Main Content Areas and Big Ideas of Early Mathematics (adapted from the work of Clements, Baroody, and Sarama, 46)



However, mathematics goals includes more than ideas, facts, and skills. *Processes* and *attitudes* are also essential outcomes for all children.
Processes include general *mathematical practices*, such as problem solving, reasoning, modeling, and the use of structure and patterns (see the eight practices of the Common Core, *102*). There are also specific mathematical processes (*5*). For example, the process of *composition*—putting together and taking apart—is fundamental to both number and arithmetic (e.g., adding and subtracting) and geometry (shape composition).

Finally, other general educational goals must never be neglected. The "habits of mind" include curiosity, imagination, inventiveness, risk-taking, creativity, and persistence. These are some of the components of the essential goal of *productive disposition.* Children need to view mathematics as sensible, useful, and worthwhile and view themselves as capable of thinking mathematically. Children should also come to appreciate the beauty and creativity that is at the heart of mathematics.

All these should be involved in a high-quality early childhood mathematics program. This brief discusses the content goals and their learning trajectories. However, this brief and all others (especially "Best Practices" and "Professional Development") attend to children's development of processes and positive attitudes in mathematics.

2. How do specific goal competencies develop and how should they be taught?

a. Learning Trajectories for Number and Operations

Basic number and arithmetic concepts and skills are an essential foundation for understanding and learning higher-levels and other aspects of school mathematics. Such knowledge is also critical for everyday functioning in our information- and technology-based society. A comprehensive discussion of the teaching and learning of number and arithmetic concepts and skills from birth to grade 3 is not possible in a short brief (see Appendix A2-2 for an overview). Instead, the brief will serve to summarize (a) a developmental trajectory of early (preschool to grade K) development involving number, counting, numerical relations, and basic arithmetic and (b) how a big idea can serve as the basis for primary-grade (grade K to 3) learning across the domains of number and operations.

Early Learning Trajectory Involving Number and Counting, Numerical Relations, and Operations on Number.

Language, including number words and quantitative terms such as "more," is critical for the development of a verbal-based number concepts and skills. Summarized below (and discussed in more detail in Appendix A2-2) is a learning trajectory that includes children's earliest concepts and skills involving number, counting, numerical relations, and operations on number. With each step in the learning trajectory, the focus initially should be on working with small collections of objects (one to three items) and then gradually moving to progressively larger collections of objects. Indeed, children may start a new step with small numbers before moving to larger numbers with the previous step.

Step 1. Verbal subitizing: Immediately recognizing the total (cardinal value) of a collection without counting and labeling that total with an appropriate number word. The ability to verbally subitize collections up to about three is a key foundation for other verbal-based number and arithmetic concepts and skills.

Step 2. Meaningful object counting: Counting a collection using one and only one number word per object to determine the total or cardinal value of a collection. Children begin to learn the counting sequence ("one, two, three, four…") early even but counting by rote many hold little meaning. Children must learn to coordinate number words with pointing to each item in a collection *and* that the last counting word also represents the total of all the items counted (the cardinality principal).

Step 3. Verbal-based magnitude comparisons: Using verbal subitizing or one-for-one counting to determine the numerical relation (e.g., "same number" or "more") between collections first and *then using a mental representation of the counting sequence to specify the numerical relation between number words*. Even if they have achieved the first two steps, children many not understand the numerical relations among number words. They need to learn concepts and vocabulary for "more" and "fewer" and then use subitizing, matching, and counting to determine which of two collections has more or fewer items.

Step 4. Adding and subtracting: Once children can quantify collections meaningfully and understands verbal-based number relations, they are ready to solve basic addition and subtraction word problems. Once children have completed steps 1 to 3, they are ready to apply the basic ideas they learned to solving word problems, starting with small collections.

The Big Idea of Equal Partitioning and Its Relation to Various Aspects of Number and Operations in the Primary Grade

Children beginning school are typically well acquainted or are quickly becoming acquainted with the issue of sharing a number of items fairly between two people or perhaps among three or four people. Fair sharing is an informal analogy for the big idea of equal partitioning, and can serve as the conceptual foundation for formal (school) instruction on a variety number and arithmetic concepts and skills and, as illustrated in Section 3, the domain of measurement as well. In this section, special attention is paid to how fair sharing can help children understand key aspects of fractions, because this topic is frequently not taught well by teachers or understood by pupils.

Division. Whole-number division can be informally viewed as fairly sharing a collection of items among a given number of people. (Discussion and examples of this and the following equal partitioning ideas are in Appendix A2-3.)

Even and odd numbers. A useful informal analogy for the concept of even number is a collection of items in which all the items can be shared fairly by exactly two people. An odd number can be informally thought of as collection that cannot be shared fairly in this manner.

Fractions. Relating fractions to the familiar situation of fair sharing can help children understand otherwise mysterious concepts and skills and empower them in solving problems involving fractions. Appendix A2-3 describes the quotient or division meaning of fractions, the part-of-a-whole meaning of fractions, as well as equivalent fractions and comparing fractions.

b. Learning Trajectories for Geometry and Measurement

Is there time for spatial topics such as geometry and measurement when there is so much pressure to ensure children know number and arithmetic? Yes, for several reasons. First, the Common Core and other standards clearly indicate that geometry and measurement are essential mathematical topics. Second, research is clear that engaging children in these spatial topics does not hurt other topics (*103*), but actually supports the learning of number and arithmetic (*38*). For example, some research suggests that the ability to represent *magnitude* is dependent on visuospatial systems in regions of the parietal cortex of the brain (*104-107*).

Spatial Reasoning

Spatial reasoning is important because it is an essential human ability that contributes to mathematical ability. Some spatial competencies are present from birth or develop quickly, but the important early competencies develop with specific experiences. For example, children move along a trajectory from early spatial abilities based on their own position in space, to using landmarks to orient themselves, to understanding paths they follow that include several landmarks, to building maps, both "mental maps" and representations of space that relate all places and distances, finally resulting in the use of coordinate systems. (For more on this development, and how to support it, see Appendix A2-4.)

Geometry

Although it may seem obvious that we learn about shapes by (statically) seeing them and naming them, research shows this is not the whole story. Children also learn about shape by actively manipulating shapes in their environment and even by actively moving their eyes over its shape (*108*). Further, even if children can name a square, their knowledge might be limited. For example, if they cannot feel a hidden square and name it after exploring it with their hands, they may not have a full understanding of the concept "square."

The learning trajectory for geometry (*109, 110*) (see Appendix A2-4) shows that children first recognize and label shapes based on overall appearance rather than in terms of attributes (e.g., "The door is a rectangle because it looks like a rectangle"), then learn about the parts or attributes of shapes (e.g., a rectangle has two long parallel sides and two short parallel side, and a square has 4 equal sides and angles) and finally learn about the relationship of shapes and attributes, which requires a focus on critical attributes—the attributes that all examples of a concept share (A rectangle is an enclosed 4-sided figure with parallel opposite sides, and a square is a special rectangle because it has all the critical attributes of a rectangle and a square angle).

Measurement

Measurement is an important real-world area of math. Further, it can help develop other areas of mathematics, including reasoning and logic. Also, by its very nature it connects the two most critical domains of early mathematics, geometry and number. Unfortunately, typical measurement instruction in the U.S. does not adequately accomplish any of these goals. In international comparisons, U.S. students' performance in measurement is very low. Many young children measure in a rote fashion at best, not understanding the concepts of measurement. By understanding and using learning trajectories, we can do better for children. Children need to learn to distinguish measureable, or continuous, quantities from discrete (countable) quantities as well as to distinguish between measurable quantities (e.g., area vs. volume). They need to understand that to measure, we iterate a selected unit of that quantity, which partitions the quantity. And so forth. Appendix A2-4 presents a learning trajectory for length as an example and provides references for area and volume.

Other Topics and Processes

Space does not allow a discussion of other topics in the depth they deserve (but see *38, 111*). Here we briefly describe the topics.

Patterns and Structure (including algebraic thinking). The big idea here is seeing the world through mathematical lenses, or mathematizing our experiences— going beyond appearances and uncovering underlying commonalities (relations) or regularities (patterns). The breadth of ways the term "patterns" is used illustrates a main strength and weakness of the notion as a goal in mathematics. (Appendix A2-5 provides examples.) Children's competencies with pattern and structure *writ large* have been shown to predict and be an important component of their mathematics learning (*112*). So, the concept of "pattern" goes far beyond sequential repeated patterns. Patterning is the search for mathematical regularities and structures. Identifying and applying patterns helps bring order, cohesion, and predictability to seemingly unorganized situations and allows you to make generalizations beyond the information in front of you. Although it can be viewed as a "content area," patterning is more than a content area—it is a process, a domain of study, and a habit of mind.

Data Analysis. The foundations for data analysis, especially for the early years, lie in other areas, such as counting and classification. At all ages, children classify intuitively. For example, by 2 weeks of age, infants distinguish between objects they suck and those they do not. By 2 years, toddlers form sets with objects that are similar on some properties, although not necessarily identical. Not until age 3 can most children follow verbal rules for sorting. In the preschool ages, many children learn to sort objects according to a given attribute, forming categories, although they may switch attributes during the sorting. Not until age 5 or 6 years do children usually sort consistently by a single attribute and re-classify by different attributes.

Problem Solving. Children make progress when they solve many problems over the course of years. Children as young as preschoolers and kindergartners, and perhaps younger, benefit from planned instruction (but not prescribed strategies), from a teacher who believes problem solving is important. They benefit from modeling a wide variety of situations (geometric, and, in arithmetic, varied problem types, including addition, subtraction, and, at least from kindergarten on, multiplication, and division with concrete objects, and also from drawing a representation to show their thinking, from explaining and discussing their solutions. Solving more complex word problems remains a challenge for primary grade students. Their conceptions must move from the many messy details of a real-world situation to more abstracted (mathematized) quantitative conceptions. For example, children might read, "Mary bought 8 candies at the store, but she ate 3 on the way home. How many did she still have when she got home?" The children have to see that the store plays little part, but that it's important that there is a group of candies and some got eaten. They might then think, she had 8 but ate 3. Then, I have to find 8 take away 3. Then they might think to model this with fingers, finally putting up 8 fingers and lowering the 3 on one hand.

Final Words: *It is better to think of patterns, structure, classification, and problem-solving not as "topics" but rather as processes that should be woven through all mathematical experiences for young children.*

Brief #3: Early Math Intervention

Questions addressed in this brief:

1. What approaches to preschool math instruction are most effective?

2. How can effective primary grade math instruction build on student gains in preschool?

3. How can families help support their children's math learning?

4. When is a response-to-intervention approach appropriate?

1. What approaches to preschool math instruction are most effective?

Evidence indicates that the most powerful preschool avenue for boosting later mathematics achievement is improving the basic competencies of children prior to kindergarten entry (*113*). This is supported by international studies such as TIMSS 2011 that should substantially higher mathematics achievement in fourth grade if children engaged in numeracy activities before primary school—at home or preschool (*114*).

The quality of mathematics education varies across early childhood setting, but is generally disappointing, especially in the earliest years. Research shows that programs designed to prevent later learning difficulties in mathematics are needed for *most* young children in the U.S. (often called "Tier 1") (*115*). Although additional assistance should be provided to children making weak progress (Tier 2) and for children with special needs (Tier 3, involving intensive assistance such as tutoring—see the subsequent section), most U.S. children are at risk due to a culture that devalues mathematics, inhospitable schools, unsatisfactory teaching, and textbooks that make little sense (*5, 116*). Thus, the main thrust in early mathematics education is arguably to provide high-quality mathematics education for all children, from the earliest years (*117*).

Several research-based programs have been proven effective (*2, 118*) and two are notable in their shared characteristics. The *Number Worlds* (*11, 119*) and *Building Blocks* (*120, 121*) have successfully improved preschoolers' mathematics knowledge (*67, 122*). *Both use research to include a comprehensive set of cognitive concepts and processes (Number Worlds* only in the domain of numbers). Both use a mix of instructional methods (including explicit, but not "direct" or didactic, instruction, which has negative outcomes for the youngest children, *123*). Perhaps most important, both are based on developmentally sequenced activities, and help teachers become aware of, assess, and remediate based on those sequences.

Building Blocks' basic approach is finding the mathematics in, and developing mathematics from, children's activity. The curriculum helps children extend and mathematize their everyday activities, from building blocks to art and stories to puzzles and games. The sequenced activities are research-based *learning trajectories.* These learning trajectories may be responsible for the success of these and several other projects (*67, 92, 122, 124-126*). In the study in which *Building Blocks children outperformed those who experienced a* different, extensive mathematics curriculum (all teachers received the same amount of professional development and resources) (*122*), only the *Building Blocks* curriculum was based on learning trajectories. From the power of learning trajectories emerges several implications for policy.

Learning trajectories: Directions for successful learning and teaching. A learning trajectory has three parts: a goal, a developmental progression, and instructional activities. To attain a certain mathematical competence in a given topic (the goal), students learn each successive level of thinking (the developmental progression), aided by tasks (instructional activities) designed to build the mental actions-on-objects that enable thinking at each higher level (*127*). (We provide examples in Brief #2: Key Math Concepts and Skills for Pre-K to Grade 3 and their Typical Trajectory of Development.)

Learning trajectories are also a core component of successful *scale ups*. In one study across multiple states, most children were taught with learning trajectories learned substantially more than the comparison children (*128*). Most subgroups—girls vs. boys, different income or ethnic groups—learned similarly. However, a notable exception was that children identifying themselves as African-American learned less than their peers in the comparison group and more than their peers in the learning trajectories group (see TRIADScaleUp.org). The learning trajectories-based *Building Blocks* curriculum and TRIAD scale-up model may be particularly effective in ameliorating the negative effects of low expectations for African American children's learning of mathematics (see *25*). Learning trajectories may have helped teachers see what children *could do* and how they progress to higher levels of mathematical thinking.

In addition, learning trajectories may be beneficial in several ways: (a) learning trajectories' levels of thinking integrate the essential aspects of concepts, skills, and problem solving (*25, 129*); (b) developmental progressions provide benchmarks for assessments; (c) research-based instructional tasks support effective teaching, (d) learning trajectories can form a foundation for curriculum development and can be tested and refined to improve mathematics education; and (d) learning trajectories support professional development. *Learning trajectories provide coherent to standards, assessments, curricula, professional development, and teaching.*

2. How can effective primary grade math instruction build on student gains in preschool?

Studies such as these indicate that mathematics programs for very young children make a meaningful, positive, difference. However, in many longitudinal studies, the effects appear to "fade" over time (*55, 94, 130-132*). Skeptics have made strong policy recommendations: Do not fund preschool if effects fade. We believe this negative point of view misinterprets and ignores the existing evidence. First, some studies do show lasting effects. Second, programs that are continued into elementary school and that offer substantial exposure to early programs have the most sustained long-term effects (*133*). Third, without such follow-through, it is simply not realistic to expect short-term early programs to last indefinitely. This is especially so because most children at risk attend the lowest-quality schools. It would be surprising if these children did not gain less than their more advantaged peers year by year (*133*). Fourth, in the previously-discussed study, only the TRIAD schools that included such follow through showed persistent effects. That is, with follow through, the effects from the pre-K program persisted; without follow through, they did not (*89, 91*).

Without the follow-through program, the mathematics-experienced preschoolers who go to kindergarten are given tasks that do not challenge—or teach—them. Children's development is stalled because no new mathematics is offered. Thus, curricula designed for the typical student often assume low levels of mathematical knowledge and focus on lower-level skills. A culture of low expectations for certain groups supports the use of such curricula. Teachers may be required to follow such curricula strictly and may have few means to recognize that students have already mastered or surpassed the content they are about to "teach" them (*5, 38, 111, 134, 135*). Even if they do so recognize students' competencies, pressure to increase the number of students passing minimal competency assessments may lead teachers to work mainly with (and/or mainly at the level of) the lowest performing students. Left without continual, progressive support, children's nascent learning trajectories revert to their original, limited course.

Such findings have quite different implications for policy. If such effects "fade" in traditional settings but do not in the context of follow-through programs, then attention to and funding for programs for both pre-K and the primary grades should arguably *increase*.

3. How can families help support children's math learning?

Parenting matters. Families that are stressed often do not provide high-quality learning experiences, such as problem solving with emotional and cognitive support. Such stress also provokes harsh, punitive interactions, which are strongly related to lower IQ scores in children (*50*). Both general parent attitudes, such as expectations for academic success, as well as specific parent behaviors, such as monitoring and scaffolding, predict children's development (*53, 136-139*).

Policy interventions can promote positive changes. Optimistic parent appraisals of achievement may serve as a protective factor for these at-risk children. They may serve as a model of motivation and persistence (*53*). These families need access to educational resources; for example, the number of books in the home is a predictor of later reading and school success (*53*).

Research describes several avenues for families to promote positive mathematics learning, including the following (for detail, see *111*): Interacting with, discussing, and support infants' play (*140*); talking about numbers 0-10 consistently, from the time children are toddlers (*141, 142*); discussing mathematical ideas when reading storybooks (*143*); playing math games (*144*) and doing home number activities of all types (*145*); talking about geometry and spatial relations (*146*) and working puzzles (*147*); keeping fathers involved (*148, 149*); and participating actively in the school's math program (*41, 150, 151*).

4. When is a response-to-intervention approach appropriate?

Also important are Tier 2 and Tier 3 interventions for young children with Mathematical Difficulties (MD) and Mathematics Learning Disabilities MLD. But first a caution: Because a chasm exists between most children's potential to learn mathematics and their knowledge of mathematics before they enter the primary grades, educators must label any young children learning disabled only with caution. Such caution is also necessary even if they have experienced "conventional instruction," because that is often flawed. As many as 80% of children labeled as learning disabled are labeled in error (*116*). We need to determine whether children so labeled benefit from good instruction. For example, some children defined as learned disabled improved after remedial education to the point where they were no longer in remedial education (*152*). Better educational experiences is indicated for such children. Other children who do not respond to such interventions may be MLD children and in need of specialized instruction.

Children who do have MD or MLD often have low skills and concepts in subitizing, counting, fact retrieval, and other aspects of computation. They appear do not use reasoning-based strategies and seem rigid in their use of immature problem solving, counting, and arithmetic strategies. Children with special needs require the earliest and most consistent interventions (*153*). Fortunately, focused mathematical interventions at early ages can be effective (*25, 80, 154-159*). Understanding specific deficits can help design programs for individual children (see *37, and the references in this brief for specific examples*).

The *most important implication for early childhood may be to prevent most learning difficulties by providing high-quality early childhood mathematics education to all children* (*49*). Equity must be complete equity, devoid of labeling, prejudice, and unequal access to opportunities to learn (see *160, for a more complete discussion*). Further, it is essential to follow through on these early interventions, as MD and MLD may be more persistent even than reading disabilities (*161*).

Brief #4—Best Practices

What is known about best practices in early math instruction, curriculum, and assessment?

Although research in early childhood cannot provide precise prescriptions ("best" practices) for teaching (*162*), the existing evidence does provide useful guidelines. This brief includes guidelines in three categories: instruction, curriculum, and assessment.

Instruction

The discussion of best instructional practices is often marred by misconceptions of young children's mathematical learning or by overheated debate about false dichotomies. This section describes empirically-supported instructional practices that are based on a balanced theoretical perspective. Some common instructional practices that might be avoided are in Appendix A4-1.

1. Instruction should focus on the meaningful learning of both skills and concepts, not memorization of facts, definitions, and procedures by rote.

As late as the 1980s, the theoretical debate focused on whether mathematical skills or concepts should be taught first. One argument for the skills-first position was that young children were not capable of understanding mathematics, learning abstract mathematical concepts, or logical reasoning and, thus, needed to be told or shown simple skills and practice these skills until they were memorized by rote. Proponents of the concepts-first position contended that young children were capable of understanding and constructing abstract mathematical ideas and engaging in various forms of reasoning and could use these capabilities to reinvent procedures mathematical procedures. Recent research shows that this skills-first versus concepts-first debate is based on a false dichotomy.

Research suggests that skills and concepts develop together (*89, 129, 163-169*). There is now general agreement that (early) mathematics instruction should emphasize both procedural and conceptual knowledge (*4*). For example, the National Mathematics Advisory Panel concluded, "[Instruction] must simultaneously develop conceptual understanding, computational fluency, and problem-solving skills" (*25, p. xix*). For children of all ability levels, focusing on both skills and concepts enables children to learn skills in a meaningful fashion, which enables them to retain and flexibly transfer this knowledge. The meaningful learning of skills, in turn, enables children to discover new mathematical ideas and solve mathematical problems.

Guidelines for fostering the meaningful learning of both skills and concepts include the following.

a. *Build on what children know—that is, connect novel procedures and ideas to familiar (e.g., everyday) experiences* (*2, 162, 170*). New information is understandable and learned meaningfully only if learners can relate it to what they already know, such as familiar experiences. Instruction that fails to build on children's existing knowledge will be ignored, mislearned, or—at best—memorized by rote (*16, 165*). In terms of a learning trajectory, instruction needs to build on a child's present level of thinking to prompt the meaningful achievement of the next level. Children's informal knowledge is a critically important basis for comprehending formal ("school-based") mathematics and learning it in a meaningful manner (*171*). In brief, the *fundamental principle of developmentally appropriate instruction is:* new formal procedures or ideas need to be connected to everyday experiences, familiar analogies, or formal ideas previously learned (*172*).

b. *Help children connect a skill to its rationale (purpose and conceptual basis)*. Too often children are taught *how to* do a mathematical procedure without helping them understand *why* we use the procedure and *why* it works. For example, children are typically taught measurement skills without helping them understand the purpose of a ruler via the big idea of equal partitioning (dividing a length into equal size units that we can count).

Help children to connect different mathematical topics (*172*). Too often children learn skills and concepts separately or mathematical topics, such as whole numbers and fractions, in isolation. Helping children learn the similarities as well as the differences between whole numbers and fractions can deepen their understanding of both topics.

2. Early childhood mathematics instruction should focus on fostering *the capacity for mathematical inquiry* and a *positive disposition* toward mathematics.

Early childhood mathematics instruction needs to go beyond fostering the meaningful learning of content (concepts and skills discussed in Point 1) to promote mathematical proficiency. A central goal of instruction should be fostering the processes of mathematical inquiry: problem solving, reasoning, communicating, and justifying strategies or solutions. Learning to think mathematically is as important as learning content. Another central goal should be fostering a positive disposition (e.g., the belief that mathematics is an important set of tools everyone needs and can learn, the desire and confidence to tackle challenging problems). Piaget noted that if cognition was the front of a coin, affect was its reverse side. By this he meant that cognitive development and affect went hand-in-hand, and the former could not be separated from motivational and other affective factors.

When instruction is meaningful (Point 1) and makes sense to children, two things happen: One is that it enables their mathematical thinking (e.g., children are far more likely to engage in effective problem solving and logical reasoning when they understand a problem). A second is that it empowers children's affect (e.g., makes it more likely they will attend to mathematical activities and be motivated to learn and practice mathematics). In addition to ensuring that instruction is meaningful, inquiry skills and/or a positive disposition can be promoted by the following practices:

a. *Instruction and practice should be **engaging** or **purposeful** to pupils*. Ideally, children should see an intrinsic reason for learning and practicing mathematics. That is, it should be interesting or challenging to pupils. Carefully selected math games, for instance, can provide the motivation to learn new concepts and skills and practice their application. Carefully selected projects, such as creating a calendar as a holiday gift for parents can provide the basis for learning how to write numerals to 9 and practice writing numerals to 31, as well learning about the months and season.

b. *Instruction should often involve genuine **mathematical inquiry***. Learning mathematical problem-solving, reasoning, justification, and communication skills effectively can be learned without engaging in and practicing these processes on a regular basis. For example, only through becoming a more effective problem solver can children develop the confidence and persistence to tackle challenging mathematical problems.

c. *Instruction should engage children in a* ***various forms of mathematical reasoning***: intuitive reasoning (making an educated guess), inductive reasoning (discovering patterns), deductive (logical) reasoning, and informal mathematical induction (using a discovered a pattern about consecutive counting numbers to logically arrive at a new idea). Instruction of the basic addition and subtraction, for instance, should be focus on discovering patterns and relations and using these mathematical regularities to devise reasoning strategies for logically determining unknown sums and differences. Naming a pattern for the child who discovered it can serve to promote a positive disposition by making the child proud and prompting other pupils to look for new patterns.

d. *Instruction should encourage children to do as much for themselves as possible*. Instead of simply telling children answers or procedures, teachers should prompt pupils to find their own solutions or strategies. This encourages mathematical thinking and autonomy/self-reliance.

3. Early childhood instruction should involve a combination of informal and child-centered activities and formal teacher-centered instruction.

This debate about whether early childhood mathematics instruction should be child- ***or*** teacher-centered is frequently heated, with both sides claiming that evidence is on their side. One difficulty is that "child-centered" has been used to label everything from a laissez-faire classroom where teachers do not teach much, to well planned teacher-child interactions. Likewise, "teacher-directed" has meant everything from appropriately scaffolded activities to a rigid routine of teacher lecture and student imitation (*173*).

Again, child- *or* adult-centered instruction appears to be a false dichotomy. When planned and implemented carefully, child-centered activities can support the development of underlying cognitive and social emotional skills necessary for school readiness and performance on academic tasks. Done well, direct, or the more general term, explicit, instruction, can be effective, especially for low achievers (*25, p. xix*). High-quality early math programs combine an explicit focus on content with equally explicit focus on promoting play and self-regulatory behaviors. Curricula designed to improve self-regulation skills and enhance early academic abilities are most effective in helping children succeed in school (e.g., *174*). Further, research has shown that children in classrooms with intentional focus on mathematics do better in mathematics and are more likely to engage in high-level free play (*93*).

Results are similar for the related dichotomy of discovery vs. direct instruction. Although unassisted-discovery is often ineffective, it can be effective under some circumstances. Although guided instruction is generally more effective than unguided instruction, this advantage recedes "when learners have sufficiently high prior knowledge to provide 'internal' guidance" (*175, p.*

75). More generally, though, enhanced or guided discovery was generally better than direct teaching, providing explanations, or unassisted discovery learning. So, children learn best when construct their own of explanations and participate in guided discovery (*15, 176, 177*).

4. Instruction should help children move from the concrete to the abstract

The use of a "concrete to abstract" learning progression and "concrete" manipulatives are commonly recommended instructional approaches that stem partly from Piaget's theory. Although this conventional wisdom has some validity, care needs to be used in applying these recommendations. For example, it does not mean that young children should never be encouraged to think about abstract ideas or that simply giving children manipulatives or demonstrating a strategy with manipulative will help them to understand mathematical ideas (*178-180*). For example, a 2-year-old's construction of a concept of "two" or kindergartner's insight that "there is no last number "(i.e., the counting sequence is infinite) is abstract in a real sense.

In many ways, instruction should begin "concretely." We have *Sensory-Concrete* knowledge when we need to *use* sensory input to construct ideas, such as number or addition (*179, 180*). By "concrete" Piaget actually meant "familiar," and familiar experiences are typically specific. So as noted in Point 1, it is important to build on familiar and specific knowledge, but it is also important to help children work toward general (abstract) knowledge. Some guidelines include:

a. *Use a wide variety of examples to illustrate a concept (prompt generalization) and a variety of non-examples to help define the concept's boundaries.* Using a wide variety of examples can prompt a generalization (*181*). The use of non-examples can help prevent overgeneralizations. Simultaneous and contrasting instruction has been found to be an effective strategy (*182, 183*).

b. *Encourage children to use informal terms and methods to represent mathematical ideas, processes, and solutions.*

c. *Help children link formal vocabulary, symbols, and procedures to their informal knowledge or experiences*. This includes encouraging children to recognize mathematical regularities or applications in everyday situations.

d. Encourage children to look beyond appearances and find an underlying commonality. For example, by labeling red-blue-red-blue-red-blue, up-down-up-down-up-down, boy-girl-boy-girl-boy-girl, 121212, circle-square-circle-square-circle-square as ABABAB, children may recognize physical appearances are not always relevant.

e. Use a variety of media. For example, computers can provide representations that are just as personally meaningful to students as physical objects, and computer manipulatives may be more manageable, "clean" and flexible than their physical counterparts (*179, 180*)

.

Curricula

Play and mathematics

When children 'play,' they are often doing much more than that. Preschoolers can learn to invent solutions to solve simple arithmetic problems as they engage in dramatic play in a "store" or play dice games, for example, and almost all of them engage in substantial amounts of pre-mathematical activity in their free play (*184, 185*). Importantly, this is shown to be true regardless of the children's income level or gender (*185*). However, this is not sufficient. Simply "letting children play" does not provide high-quality, or even barely adequate, mathematics education (*186, see also the next section on "teachable moments"*). "Free play" classrooms have the lowest gains in several domains (*187*). Traditional approaches to early childhood, such as "developmentally appropriate practice" (DAP) have not been shown to increase children's learning of mathematics (*45*). Children, especially those at risk, need intentional and sequenced instruction (*94, 188*). They must learn to mathematize their experiences—talk about them and think about them in mathematical language. High-quality education can help children do so (*1, 41*).

Teachers support mathematics in play by providing a fertile environment and intervening appropriately. For toddlers, play is enhanced with realistic objects. All children should also play with materials such as Legos and blocks, because such play is linked with mathematical activity and learning about patterns and shapes.   U.S. preschools have many toys, but some of these toys do not encourage mathematical activity. In symbolic play, teachers need to structure settings, observe play for its potential, provide materials based on their observations (e.g., if children are comparing sizes, teachers might introduce objects with which to measure), highlight and discuss mathematics as it emerges within play, and ask questions such as "How do you know?" and "Are you sure?" (about your answer or solution) (*189, 190*).

These examples bring us another type, mathematical play, or play with mathematics itself (cf. *191*). For example, Abby was playing with three of the five identical toy train engines her father had brought home. Abby said,"I have 1, 2, 3. So [pointing in the air] foooour, fiiiive…two are missing, four and five. [pause] No! I want these to be [pointing at the three engines] one, three, and five. So, two and four are missing. Still two missing, but they're numbers two and four." Abby transformed her symbolic play into playing with the idea that counting words themselves could be counted.

Learning through play and intentional teaching are *not* in conflict. Early childhood programs that have more mathematics have more high-level free play, all of which promotes self-regulation and executive function (*93*). Through higher-level play, children explore patterns, shapes, and spatial relations; compare magnitudes; and count objects. They support each other.

Teachable moments

If play has so much potential to elicit mathematical thinking, should educators simply use "teaching moments"? Using teachable moments is an important pedagogical strategy. The teacher carefully observes children and identifies elements in the spontaneously emerging situations that can be used to promote learning of mathematics (*192*). However, there are serious problems with *depending* on this approach. For example, most teachers spend little time in careful observation necessary to find such moments (*192, 193*). They spend little time with children during their free play (*185*). As we have seen, many teachers also have a difficult time engaging children in tasks at their mathematical level (*134*). Most teachers do not have the mathematics language and concepts at their command. For example, they do not tend to think about relational terms in mathematics. According to researchers, their language in general may influence their ability to see opportunities for teaching mathematics throughout the curriculum (*192, 194*). Finally, it is unrealistic for them to see opportunities for multiple children to build multiple concepts (*192*). Therefore, educators should capitalize on teachable moments but also recognize that these will constitute only a small portion of the mathematics activities children, especially those at risk, will need.

Project approach

Mathematics should be gleaned from myriad everyday situations, including play, but go beyond it as well. For example, a group of young children investigated many measurement ideas as they attempted to draw plans for a carpenter, so that he could build them a new table (*195*). However, studies have found no differences in children's development of mathematics in a project approach, compared to control classrooms (*94*). Projects appear useful, but careful, intentional teaching may be necessary as well.

Assessment

Educational assessments serve a variety of purposes. Some “assessments” are equated with “high-stakes.” Alternatively, the term can suggest an identification function (e.g., identifying children with special needs). Within the classroom, “assessment” can serve to guide instruction and learning. The purposes of assessment should determine the content, methods, and use of the assessment. Misuse of tests often stems from confusion of purpose.

Also inappropriate are several common practices. In early childhood, group-administered, multiple-choice tests often are not adequate assessment tools (*196, 197*). A different, but also potentially harmful, practice is the use of timed tests as a method of promoting memorization of basic fact/ (*198*).

Instead, individual assessment, observations, documentation of children’s talk, interviews, samples of student work, and performance assessments that illuminate children’s thinking constitute a positive approach to assessing children’s strengths and needs (*197*). This is especially useful when used as an instructional strategy, *formative assessment* (*41, 199, 200*). Formative assessment is the ongoing monitoring of student learning *to inform instruction.* Teachers should observe not just answers, but *strategies.* Using learning trajectories (see Brief #2: Key Math Concepts and Skills for Pre-K to Grade 3 and their Typical Trajectory of Development), the key questions are: What is my mathematical goal? What level of thinking does the child display? What are the instructional activities that will help the child develop the *next* level of thinking? When children err, teachers ask questions such as the following (*200*). What is the key error? What is the probably reason the child made this error? How can I guide the child to avoid this error in the future?

Research Brief #5: Relationships between Early Mathematics and Other Learning and Developmental Domains

Questions addressed by this brief:

1. *How do learning of early mathematics and learning in other domains, particularly, early language and literacy support one another?*
2. *What are the implications for teaching early mathematics to English Language Learners (ELLs)?*
3. *What is the relationship between early mathematics and other cognitive processes?*

Mathematics is a language based on structure and logic. In this brief, we examine the relationships between math competencies and learning and that of language, literacy, and other cognitive domains. Do they connect, and, if so, do they potentially support each other?

1. How do learning of early mathematics and learning in other domains, particularly, early literacy and language support one another?

Early mathematics competence predicts later mathematics achievement (*201, 202*) and does so above and beyond other cognitive competencies such as verbal, memory, or spatial skills (*203, 204*). Surprisingly, though, early mathematics knowledge is a better predictor of children's later reading or science achievement than early literacy (*32, 33, 113, 205*). In this section we look at the relationships between these domains of early learning.

In addition, the relation between learning early mathematics and other domains is a two-way street. Some children have high levels of mathematical intuition, but if they cannot "talk mathematics," they often cannot participate fully in mathematics instruction in school (*37, 206*). So mathematical language, and language competencies in general, are important to learning mathematics. Developing language and literacy competencies supports mathematics learning (*207, 208*). For example, two- and three-year-olds whose language, such as English or Russian, involved plural markers were more successful in understanding small number words such as "one," "two," and "three" than those whose language, such as Chinese or Japanese, did not have singular and plural forms for nouns (*209, 210*). Children, then, may first use "two" and "three" to indicate "many" instead of a specific quantity. Both vocabulary and knowledge of print predict later numeracy scores (*211*). The more teachers engage in "math talk" the more mathematics their students learn (*212*). The more students talk about mathematics, the more engaged and competent they become (*213, 214*). Engaging in math conversation, either with a teacher or peers, can improve achievement (*215*). Interventions that include a focus on specific math vocabulary, math stories, songs, or questions have a positive effect on numeracy or math achievement (*40, 122, 128, 216*). The ability to retrieve verbal or visual–verbal associations from long-term memory predicts learning of basic facts as well as reading (*217*). In another study, early numerical skills and phonological processing influenced growth in mathematics from kindergarten to third grade (*218*).

Care must take care in interpreting the results of these correlational studies. The results of one training experiment, though, speak to *causation.* The *Building Blocks* curriculum and TRIAD scale-up model have shown to effectively teach young children mathematics, even in large-scale studies (see Brief #3: Early Math Prevention and Intervention and *38, 89, 92, 111, 122, 128,* *219*). But with increasing pressure on educators to achieve benchmarks across multiple areas of development and learning, it is important to know what, if any, impacts these early mathematics. Some teachers in earlier studies said they liked the math, but worried that children's learning of language and literacy might suffer. So, the researchers also assessed children's express oral language (story retell) and letter recognition (*36*). Children who were taught mathematics using the curriculum performed the same as controls children on letter recognition, and on two of the oral language subtests, sentence length and inferential reasoning (emotive content). However, children in the *Building Blocks* group outperformed children in the control group on four oral language subtests: ability to recall key vocabulary words, use of grammatically complex utterances, ability and willingness to reproduce narratives independently (autonomy), and inferential reasoning (practical content). In classrooms where children are explicitly provided the opportunity to explain and discuss their mathematical thinking, children become more confident and competent in their thinking and verbal expressions. Answering, "How do you know?" in mathematics contexts challenges children to "dig deep," reflecting on and explaining their thinking, and thus generalizes to a wide range of language skills (*36*).

2. What are the implications for teaching early mathematics to English Language Learners (ELLs)

Teaching English language learners (ELLs) specific vocabulary terms ahead of time, emphasizing words that have similar roots in English and the second language (e.g., Spanish cognates), is both useful and necessary. But it is not sufficient. An overview of issues for ELLs and global recommendations can be found in Research Brief #1: The Rationale for Focusing on Early Childhood Mathematics. Here we emphasize complementary research-based recommendations for policy and practice. In general, children who are members of linguistic and ethnic minority groups need more math and better math programs (*56*). They need programs that emphasize the higher-order concepts and skills at each level, as well as basic knowledge and skills (*220, 221*). Other guidelines include the following elements that should be found in programs and classrooms (much of the following is from *77*).

- Bilingual instructional support (including paraprofessionals, instructional assistant, parent volunteers, and older and more competent students).
- Instruction in children's home language (*222*) and use of cognates and other means of explaining math concepts with familiar language (*73*).
- Simple print material in the children's home language in learning centers and labeled objects.
- Age-appropriate books and stories in the child's home language (school and loan to the home). This might include E-Books (*223*).
- Discussion between the children and the teacher and between children, explaining solutions and working toward more formal mathematical language and ideas.
- Word problems that are created from students' personal narratives, helping children "mathematize" situations (*73*).
- Activities involving creating mathematical problems through story telling, which help ELL kindergartners learn problem solving (*224*), as does giving additional time to problem

solving, posing a broad range of problems involving multiplication, division, and multiple steps, and providing consistent access to children's first language (*225*).

- Programs and interventions in preschool through the primary grades, preferably with bilingual components (*89, 91, 92, 226*).
- Encouragement to parents and other family members to use the home language during family activities and early literacy and mathematics development in the primary language, as well as to visit school and share where mathematics is used in the home and community (*74, 227*).
- Encouragement to families of children with specific language impairments to talk about mathematics, numbers, and arithmetic with their children, as they tend to do so less than other families (*228*).

Of course, parents and families are important to all children, so we turn to that topic next.

3. What is the relationship between early mathematics and other cognitive processes?

As children think and learn, they build mental representations, act on them with cognitive processes, and control these acts with executive control ("metacognitive") processes. Policies and practices need to support the development of critical processes, such as higher-order processes and motivational factors, along with mathematics competencies. They also should support the use of these processes in diagnosing children who seem to have mathematical difficulties.

Higher-order processes: Executive function (self-regulation)

Thinking and problem solving involve taking in and interpreting information, operating on it, and responding to it. At the beginning of this process is attention—a focusing process that cannot be taken for granted.

A broader competence that includes focusing attention is self-regulation or executive function—the process of intentionally controlling ones impulses, attention, and behavior. It may involve avoiding distractions, and maintaining a focus on setting goals, planning, and monitoring one's attention, actions, and thoughts. Self-regulation has emerged as a significant influence on certain components of mathematics learning (*174*). Further, the lack of social-emotional self-regulation can stand in the way of a child's ability to have positive teacher child interactions in kindergarten, which, in turn, predicts later poor academic performance and behavior problems (*229*). Self-regulation and cognitive competencies appear to be related, but develop somewhat independently (*230*).

Executive function processes control other cognitive processes. For example, they select steps to put together to form a strategy for solving a problem or monitor the overall problem-solving process. Executive function competence predicts math achievement (*174, 231-237*), especially on complex and unfamiliar tasks (*238*). Mathematics competence correlates to some measures of executive function more than literacy and language do (*237*). Particular difficulties for children of lower mathematical ability are lack of inhibition and poor working memory, resulting in their having difficulty switching "mind sets" and evaluating new strategies for dealing with a particular task (*237, 239*). Persistence was significantly predictive of math achievement for both 3- and 4-year-olds (*240*). Finally, the executive function of updating (keeping items in memory and adding to that list) predicts patterning and number skills (*241*). Others agree with factors such as working memory and processing speed are important, but remind us that domain-specific competencies such as numerical competence also are critical to subsequent math achievement (*217, 242*).

Educators need to improve both executive function skills as well as enhance early academic abilities to help children succeed in school. Most students need substantial work in learning these processes, for example, to monitor their reasoning and problem solving. Helping children *understand* mathematical ideas, and engaging them in conversations about mathematics and how they solved mathematical problems promotes the development of executive processes. Research has also identified certain environments and teaching practices that can help children pay attention, and grow in their ability to do so, as well as to develop general self-regulation competencies. Carefully guiding children to attend to specific mathematical features, such as the number in a collection or the corners of a polygon, is likely to improve their learning. The predisposition to spontaneously recognizing number, for example, is a skill but also a habit of mind, including the ability to direct attention to number (*243*). These habits of mind generate further developmental of specific mathematical knowledge and the ability to direct attention to mathematics in situations in which it is relevant; that is, the generalize and transfer knowledge to new situations.

Engagement and motivation

A common, critical, component of these studies may be engagement mathematical thinking and learning. One large study confirmed the importance of engagement, or "approaches to learning," which was the single behavior predictor of learning as far out as fifth grade (*244*). Such engagement in learning, including persistence at tasks, eagerness to learn, attentiveness, learning independence, flexibility, and organization, was especially important for girls and minority students.

Returning to the fundamental importance of math, early math skills also predict *classroom engagement* (*236*). We believe developing self-regulation, learning-to-learn, and early math competencies all go hand in hand, each supporting the development of the other.

The role of other competencies

Other cognitive measures are also predictive of mathematics achievement, and assessment of these may be for children with mathematical difficulties or disabilities. For example, these children may have weaknesses in working memory (e.g., reverse digit span) (*117, 245, 246*), general intelligence, or processing speed (*238*). The lack of these cognitive processes may interfere with learning the critical skills of counting, use of strategies, and magnitude comparison (*117, 247*). We examine some of these processes in more detail. The first, working memory, is often closely related to executive function.

Working memory. When children pay attention to something, information can be encoded into their working memories—the amount of mental "space" they have to think about mathematics and solving mathematical problems (indeed, another useful metaphor is that working memory is children's capacity to attend to multiple items in memory). This allows children to consciously think about the task or problem. Working memory affects children's ability to solve problems, to learn, and to remember (*248*). Processes that are slower and more complex put additional demands on working memory. Unsurprisingly, then, limits on working memory may be one cause of learning difficulties or disabilities (*249*) and a particular large working memory one cause of superior competence in mathematics.

Children develop greater working memory capacity as they age, probably due to greater self-regulation and executive control and the ability to represent content more efficiently (*250*). At all ages, one way people's minds deal with limits on working memory is to make certain processes *automatic*—fast and easy. Such automatic processes do not take much working memory (*251*). Some automatic processes are "bootstrap" abilities, such as the ability to recognize faces. In mathematics, most must be learned and experienced many times. A familiar example knows arithmetic combinations so well that one "just knows" and does not have to figure them out while performing a more complicated task. Such automaticity requires much practice. Such practice could be "drill," but a broader definition is *repeated experiencing,* which might include drill, but also includes use of the skill or knowledge in multiple difference situations, which promotes both automaticity and transfer to new situations.

Long-term memory and mental representations. Long-term memory is how people store information. Concepts ("understandings") take effort and time to be built in long-term memory. People have difficulty transferring their knowledge to new situations (different from those in, or about, which they were taught), but without conceptual knowledge, this would be even more difficult.

Helping children build rich representations of concepts (*180*) and see how something they know can be used to solve new problems helps them remember and transfer what they have learned. Varied situations are not necessarily radically different. In one study, 6- and 7-year-old children practiced using flashcards or workshops. They had similar performance if tests in the same format, but if the format was switched, their performance was significantly lower (*252*). Although material that is easy to understand can promote fast initial learning, it does not help store knowledge in long-term memory. Challenging materials and activities lead to better longer-term memory, because children have to process it and understand it more thoroughly. Their extra effort translates into more active processing, and thus more likely storage, of information. This helps children remember information longer and retrieve ("remember") it more easily. Thus, they can retrieve the information better and are more likely to transfer its use to new situations.

Competencies interact. Early competencies such as those discussed here interact, some compensating for others. As previously stated, cognitive processes, self-regulation, and social skills may develop somewhat independently of each other (*230*). Moreover, skills in one area may help some children compensate for a lack of skills in another. For example, those of average or low cognitive ability had higher grades in first grade if they had good social skills. In comparison, those with high cognitive ability but mild externalizing problems did not suffer from the latter, outperforming all the other groups on achievement.

Final Words

To fully support the academic success of all children policies need to encourage and support the development of a wide range of mathematical concepts, skills, and higher-order thinking processes as well as general cognitive and social-emotional competencies. Policies and practices need to improve both executive function skills as well as enhance early academic abilities to help children succeed in school. Finally, mathematics should be considered at least as important foundation for school learning as reading.

BRIEF #6: Professional Development

Questions addressed in this brief:

1. *What do early childhood (preschool through the primary grades) teachers need to know in order to teach mathematics effectively?*
2. *Why does the professional development of early childhood educators need to be improved?*
3. *What are the consequences of low-quality professional development in early mathematics?*
4. *How can we remedy these problems and promote high-quality professional development for early childhood educators?*

1. What do early childhood (preschool through the primary grades) teachers need to know in order to teach mathematics effectively?

High quality professional development can have a positive impact on the development of pupils' mathematical proficiency (*37, 38*). John Dewey (*253*) argued that an providing an educative experience—one that leads to meaningful learning—required ensuring that *external* factors, such as the level of instructional content and instructional methods, mesh with *internal* factors, such as the students' needs or developmental level. To teach mathematics effectively, then, early childhood teachers need to understand (a) the mathematics they are teaching (external factor), (b) the nature of children's mathematical thinking/knowledge and how it develops (internal factor), and best practices for ensuring that mathematics instruction meshes with children's developmental needs and level (external factor) (*38, 254-256*). (Note that this corresponds precisely with the three components of learning trajectories: the mathematical goal, the developmental progression, and the correlated instructional activities. See Brief #3: Early Math Intervention.)

2. Why does the professional development of early childhood educators need to be improved?

Structural barriers. Some researchers argue that we do not know enough about the teaching and learning of mathematics to help teachers effectively prepare for the task of teaching mathematics (cf. *257*). Certainly, there is much more we need to learn. However, research has already revealed a wealth of information about what constitutes effective professional development, how children's mathematical thinking and knowledge develops, and effective teaching strategies or tools for fostering children's mathematical learning and thinking. Applying this wealth of knowledge could greatly improve early childhood mathematics education. The key challenge in translating current research and theory into effective practice is disseminating this knowledge to those responsible for teacher professional development and pre-service and in-service teachers.

Structural barriers, such as inadequate preparation or support for continuing professional growth, are a primary reason for the challenge(*23*). One fundamental structural barrier in the recent past was there were minimal or no requirements for mathematics, psychology of mathematical development, or mathematics methods courses for educators of preschoolers, and little for primary grade teachers. Indeed, because they are the most math phobic of any college major (*258*), many education majors gravitated to early childhood or special education because there were no or at least minimal requirement for mathematics and mathematics methods course, and little perceived demand of teaching this subject (*46, 258*). A result of the 2000 Conference on Standards for Preschool and Kindergarten Mathematics Education (sponsored by the National Science Foundation and Exxon Mobil Foundation), many states adopted mathematics content and methods courses for early childhood education certification (*46*). Unfortunately, the number of required content, developmental, and methods courses is still minimal at best. For instance, most college programs still offer one or fewer "mathematics for teachers" courses (*23*).

Nature of pre-service courses. To make matters worse, the nature of existing pre-service training commonly hinders or prevents effective content, developmental, and pedagogical training. Frequently, faculty members from different departments teach mathematical content, educational/developmental psychology, and instructional methods independently as separate, uncoordinated courses. Moreover, there are fundamental problems with the content and instructional approach of most current pre-service courses, such as an overreliance on direct instruction (the lecture method).

- *Mathematics content courses,* even if labeled "mathematics for elementary teachers," often may be taught by a faculty member or teaching assistant from the mathematics department who has little or no experience teaching children and little or no interests in teaching, especially that involving non-mathematics majors. Moreover, these courses typically focus on memorizing formal and abstract (relatively inaccessible) facts/definitions/procedures, such as the algorithm for converting a number in one base system to a number in another base system, by rote. Little or no effort focuses on helping pre-service teachers *understand* the concepts and procedures early childhood teachers need to teach (e.g., the grouping and place-value concepts children need to construct to understand our Hindu-Arabic base-ten, place-value system of written numbers) or developing their mathematical problem solving, reasoning, and communicating abilities. Unfortunately, mathematics content courses can do more damage than good, including inculcating misinformation, reinforcing ineffective methods for teaching mathematics and a negative disposition towards mathematics and teaching it. This may help explain why research often does not find an association between the number of college mathematics courses taken and teaching effectiveness have at best yielded mixed results (*25, 259*).

- *Educational/developmental psychology courses* are typically taught in departments of educational psychology or psychology and focus on general developmental issues and theories. As with mathematics content courses, instructors of such courses may have little or no training or experience regarding children's mathematical development. As a result, there is seldom an emphasis on the psychology of mathematical learning and development.

- *Mathematics methods courses*, as the above, too often do not model best practices. All too frequently, the instructional approach largely involves the lecture method and sometimes fails to focus on the rationale for best practices. Two limitations of the direct instruction approach are "the do as I say, not as I do" approach does not provide a convincing case for best practices and, in the pressure cooker of teaching, new teachers tend to adopt the traditional methods by which they were taught mathematics and other content throughout their lives as students (e.g., K-16). Furthermore, mathematics methods courses are often taught in combination with science methods or STEM methods, and the instructors of such courses do not necessarily have mathematics training or teaching experience.

Limitations of in-service professional development. In-service professional development has similar limitations. Much such training has long been criticized for being too unfocused, too superficial, too brief, too sporadic, and without adequate support or follow through (*5*). For example, professional development frequently does not focus on mathematical content, development, or pedagogy (*260, 261*). When it does, the focus is often on a bag of tricks—uncoordinated activities without a clear rationale—a focus on how but not why (*262*). Moreover, the training and quality of the providers in-service professional development is uneven and unregulated (*263*).

3. What are the consequences of low-quality professional development in early mathematics

The consequences of inadequate professional development is that few pre-service and in-service teachers have themselves achieved mathematical proficiency with elementary-level mathematics and are consequently ill equipped to foster the mathematical proficiency of young children (*5*).

- *Lack of concern for teaching mathematics.* Many early childhood teachers take a "careless attitude towards mathematics" (*264*) and do not appreciate its role in children's development (see Brief #1: The Rationale for Focusing on Early Childhood Mathematics).

- *Negative disposition toward mathematics and teaching mathematics.* Many early childhood educators lack confidence in their own mathematical ability and ability to teach mathematics, beliefs that lead to undervaluing the teaching of mathematics, avoiding or minimizing the mathematics instruction, or interfere with effective mathematics teaching (*38, 258, 265-267*).

- *Lack of content knowledge and mathematical thinking ability.* Most early childhood pre-service and in-service teachers do not explicitly, fully, and clearly understand the mathematical concepts and procedures they teach. Specifically, they lack *mathematical knowledge for teaching*—the mathematical knowledge needed to carry out the work of teaching mathematics. (e.g., how mathematical knowledge is interconnected and connected to the real world, including everyday analogies that would make school mathematics comprehensible to children). (See, for instance, *5, 259, 268, 269-273*). The problem solving, reasoning, and communicating ability of pre- and in-service teachers has been studied less, but the results are not positive.

- *Lack of knowledge of children's developmental of mathematical concepts and skills*. The vast majority of early childhood teachers lack a sufficiently detailed knowledge of mathematical learning trajectories. A common result is that most teachers do not possess a viable process for diagnosing and remedying the difficulties facing students with mathematical learning problems (e.g., carefully considering whether a child even has the developmental prerequisites/readiness for the problematic content). Indeed, elementary-level children in special education are too often taught material two or more years above their developmental level instead of the missing developmental foundations for the instruction dictated by their individualized education programs).

- *Lack of pedagogical knowledge.* Many early childhood educators lack a coherent philosophy of teaching and learning mathematics and knowledge of best practices (*274-276*). For example, as implied in the previous paragraph, many are unaware of how a learning progression can provide guidance in screening children, progress monitoring, diagnostic testing, and remedying gaps or difficulties in learning.

As a result of their low-quality professonal development, many early childhood educators focus on extremely limited objectives—fostering the memorization of the counting sequence, basic addition facts, and shape names by rote and, as a result, have minimal impact on children's mathematical proficiency (*5*).

4. How can we remedy these problems and promote high-quality professional development for early childhood educators?

There is a critical need for high quality professional development opportunities for teachers of young children in the area of mathematics (*38*) in both initial preparation programs and ongoing professional development offerings.

Goals of reform efforts. The structural barriers to effective professional development for early childhood educators need to be addressed so as to prevent the negative consequences of inadequate pre- and in-service professional development previously discussed (*277-279*). Policy initiatives should have the following effects (for detailed suggestions, see *38*).

- significantly increasing early childhood teachers' knowledge of, and ability to use, learning trajectories (this would involve each of the following);
- significantly upgrading early childhood teachers content (mathematics) knowledge and—more importantly—their mathematics knowledge for teaching (*259, 272*)[2] and their proficiency with the processes of mathematical inquiry (mathematical problem solving, reasoning, and communicating skills);
- significantly enhancing early childhood teachers' knowledge of the psychology of mathematical teaching and learning (*27, 280-282*).
- significantly improving early childhood teachers' pedagogical knowledge (*38, 274-276*); and
- thoughout, addressing affective issues of early childhood teachers, such as unproductive

---

[2] Hill, Rowan, and Ball 259. H. C. Hill, B. Rowan, D. L. Ball, Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal* **42**, 371 (2005). concluded that professional development that focuses on mathematics for teaching is more effective in promoting pupil achievement than that which focuses on particular teaching behaviors or materials.

beliefs or math anxiety (*38, 280, 283, 284*)[3];

A case for deep and integrated instruction. Simply increasing the number of mathematics, educational/developmental psychology, and mathematics methods courses required for undergraduate and graduate degrees in early childhood education may help but—by itself—will be insufficient to achieve the goals outlined in the previous subsection. A focus on reform is nature of such professional development:

- Mathematical content courses need to focus on fostering a deep understanding of mathematics knowledge for teaching (e.g., the connections among mathematical concepts and procedures and between school mathematics and everyday applications and analogies such as "division can be viewed as a fair-sharing problem") and fostering inquiry by engaging pre- and in-service teachers in guided inquiry-based learning.

- Educational/developmental psychology course for teachers need to include mathematical learning trajectories that embody common learning goals such as those laid out by the Common Core (and, especially for pre-K, the Curriculum Focal Points, *101*).

- Mathematical methods courses need to *model* best practices, such as guided inquiry-based learning.

- The instruction across these three domains should be coordinated and include instructors who have competence in all three areas and experience teaching young children mathematics.

- Policies must be adopted to reach the wide variety of settings and teachers who teach preschool children, and who often receive information about professional development from mailings, bulletin boards, and supervisors(*38*).

- Policies and professional development practices should also encourage sharing, risk taking, and learning from and with peers. Including coaching in classrooms, this approach situates work in the classroom, formatively evaluates teachers' fidelity of implementation and provides feedback and support in real time. (*285-296*).

- *The best way to achieve the first four points is offer courses and coaching that <u>integrate</u> mathematics knowledge for teaching, the psychology of mathematical development, and mathematical pedagogy (best practices).* For example, by using and thus modeling the best practice of "building on pupils' existing (informal or everyday) mathematical knowledge" to promote pre- and in-service teachers own mathematical proficiency, they will have a

---

[3]A teacher's beliefs can significantly impact on teaching practices 283. D. I. Cassidy, M. J. Buell, S. Pugh-Hoese, S. Russell, The effect of education on child care teachers' beliefs and classroom quality: Year one evaluation of the TEACH early childhood associate degree scholarship program. *Early Childhood Research Quarterly* **10**, 171 (1995); 284. D. J. Stipek, P. Byler, Early childhood teachers: Do they practice what they preach? *Early Childhood Research Quarterly* **12**, 305 (1997).. Beliefs about teaching and learning in general and their own mathematical ability can shape what they teach and how they teach it, as well as their confidence in their ability to teach mathematics.

better appreciation and understanding of this practice and be more likely to use it with their own.

- Successful scale-up efforts can serve as models for systemic improvement of in-service PD. (see *38, and TRIADScaleUp.org, 122*).

By tracking their own progress through learning trajectories as they learn mathematical content, teachers will develop a deeper appreciation of the challenges that confront children and the value of learning trajectories. As pre- and in-service teachers' mathematical proficiency and their positive experience with best practices grow, so will their disposition toward teaching mathematics.

Appendices

Appendix A2-1a: Major Goals for Mathematics Education from Adding it Up.

The overriding premise of our work is that throughout the grades from pre-K through 8 all students can and should be mathematically proficient. [p. 10].

Mathematical proficiency…has five strands:

• conceptual understanding—comprehension of mathematical concepts, operations, and relations

• procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately

• strategic competence—ability to formulate, represent, and solving mathematical problems

• adaptive reasoning—capacity for logical thought, reflection, explanation, and justification

• productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a believe in diligence and one's own efficacy (*4, 5*).

Appendix A2-1b: Major Goals for Mathematics Education from Adding it Up.

Levels of Learning	Development, including How Previous Levels in the Developmental Progression Serve as a Basis	Relation of Level to the Common Core State Standards (CCSS) Math Content Standards	Example of Instructional Activities for Promoting a Level
Verbal subitizing of collections of 1 and 2 first, then 3, and—in time—4 to 6: *Concept: Recognition that number words represent a specific total (cardinal value)* Skill: Reliably and immediately identify small collections with an appropriate number word.	By seeing different examples of a number labeled with a unique number word (e.g., "*two* eyes," "*two* hands," "*two* socks," "*two* shoes," "*two* cars") and non-examples labeled with other number words ("take one cookie, not two"), children construct precise, verbal-based cardinal concepts one and two and then progressively larger numbers up to about six (Palmer &	***Verbal subitizing*** is NOT a grade K CCSS goal. It *should* be because (a) many at-risk kindergartners have not mastered verbal subitizing up to 3 (and such a deficiency is a major handicap), and (b) many kindergartners have not yet mastered conceptual subitizing (e.g.,	***The Number—Not the Number Game.*** *P*layers take turns pointing out an example and a non-example of a number. The game can be made more challenging by putting a time limit on the pointing out process. **Games involving a die**

	Baroody, 2011). Children recognize that number words in general represent a specific number of items about 4 years of age (Sarnecka & Carey, 2008; Sarnecka & Gelman, 2004).	see a set of six as "3 and 3 is 6").	**or dice.**
<u>Meaningful object counting</u> *Concepts: Understanding (a) 1-to-1 object counting is another way of determining the total number of items and (b) the principles of how to execute accurate counting.* <u>Skill: Accurately use 1-word-to-1-object counting to discern the cardinal value of sets</u>.	***Verbal subitizing*** enables children to understand the principles underlying ***meaningful object counting:*** *stable order, one-to-one,* and *count-cardinality principles*. For example, by watching an adult count a small collection a child can recognize as "three," s/he can understand why the last number word in the count is emphasized or repeated—it represents the total or how many (the cardinal value of the collection).	**KCC.B.4** (Understand the relationship between numbers and quantities; connect counting to cardinality); and **K.CC.B.5** (Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects).	***Hidden Stars*** (Baroody, 1987). The hider [the teacher] shows the player(s) some stars pasted on 5 x 8 card. The player counts the stars. Then the hider covers them up and says, "How many stars am I hiding?" The player tries to tell how many stars the hider is hiding.
<u>Producing collections</u> *Concepts:* ***Cardinal-count principle*** *=a cardinal number specifies when to stop taking or counting a collection of a specified number of objects).* <u>Skill: Can accurately put pout or count out a specified number of objects up to 5, then 10, and</u>	***Verbal subitizing*** enables children to put out a specified number of objects. Building on verbal subitizing and the count-cardinality principle can help children understand the cardinal-count principle. For instance, seeing another respond to a request for "three" items by counting out three items, a child can see that the counting-out process stopped when the	***Producing collections*** is NOT a grade K CCSS goal but *should* be as a quarter to one-half of kindergartners cannot count out collections up to 20.	***Animal Spots*** (Wynroth, 1986). On their turn, 1 to 4 players throw a die with 0 to 5 (or 10) dots to determine how many pegs ("spots") they can take for their leopard or giraffe (an animal figure cut out of wood with holes drilled for pegs. After a child counts the number of dots on a die/card,

eventually 20.	requested amount ("three") was reached.		the child takes or counts out the specified number of pegs.
Concrete ordinal knowledge of number and counting *Concept:* ***Increasing magnitude principle*** = recognizing that the counting sequence represent increasingly larger quantities. Skill: Can use counting to determine the larger/largest of several collections (the collection requiring the longest count).	***Verbal subitizing*** enables children to see that "2 is *more* than 1" item and that "3 is more than 2" items (understand the term "more" and that numbers have an ***ordinal meaning***, as well as cardinal meaning. ***Verbal subitizing*** and ***ordinal number concept*** permit discovery of the **increasing magnitude principle**. This enables them to use the ***meaningful object counting*** to determine the larger of two collections (e.g., 7 items is more than 6 items because you have to count further to get to *seven* than you do for *six*).	**K.CC.C.6** (Identify whether the number of objects in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies).	***Cards More Than*** (Wynroth, 1986). Played like the card game *War*, except that cards have homogeneous and heterogeneous collections of shapes. Children count the shapes on the card they drew to determine who has more (counts further).
Number-after knowledge **of the counting sequence** *Concept: Counting is a breakable chain.* Skills: (a) Specify the number after another without counting from "one" and (b) counting-on from a specified point in the counting sequence.	Familiarity with the counting sequence enables a child to enter the mental representation of the sequence at any point and ***specify the next number*** instead of always counting from one.	**K.CC.A.2** (Count forward beginning from a given number within the known sequence (instead of having to begin at 1).	***Dominoes Number After*** (Wynroth, 1986). Played like Dominoes, except that a child must find a domino with the number after an end domino.
Abstract & precise ordinal knowledge of number and counting. Skill: Mental	The use of the ***increasing magnitude principle*** and ***number-after***	This level is the basis for **K.CC.C.7** (Compare two numbers	***Race Game*** (Baroody, 1987). Player is asked which of two number

comparisons of close/neighboring number (number after = more)	***knowledge*** enables children to determine efficiently and ***mentally compare even close numbers*** such as the larger of two neighboring numbers (e.g., "Which is more 7or 8?").	between 1 and 10 presented as written numerals.)	neighbors is more and moves token on a race track the chosen number of times.
Re-representing the counting sequence as the positive integers (n, n+1, [n+1]+1, …) *Concept:* ***Successor principle*** *= each successive number in the count sequence is exactly one more than the previous number* Skills: (a) Can specify that, e.g., it takes 1 more make 4 into 5 & (b) "4+1 is 5."	**Verbal subitizing** enables children to see that "two" is exactly one more than "one" items and that "three" is exactly one more than "two" items, and this can help them understand the ***successor principle***.	**K.CC.B.4c** (Understand that each successive number name refers to a quantity that is one larger.)	***Monkey Successor Game****.* *Monkey sees a banana on,* e.g., the fifth tree of a series of trees labeled in order from 1 to 10. Monkey swings to get banana but comes up 1 (or 2) tree short. Child asked how many more trees the monkey must go to get from Tree 4 to Tree 5.

Appendix A2-2: Learning Trajectories for Number and Operations: The Steps

Step 1. Verbal subitizing: *Immediately recognizing the total (cardinal value) of a collection without counting and labeling that total with an appropriate number word*. Initially number words may have little or no meaning. For example, 18-month Arianne bounces down the stairs while saying "Two, two, two, two." Soon though they have a sense that number words have something to do with quantity but this understanding is inexact. For instance, responding to a question about "how many fingers?" and seeing two, three, four, five, and ten fingers held up in turn, Arianne responds "two" to all. In time, children may use "one" and "two" with reliable accuracy, but treat other number words such as "three" as meaning "many" (as label for 3, 4, or more things). Gradually, other number words take on an exact meaning. As the discussion of succeeding steps illustrates, the ability to verbally subitize collections up to about three is a key foundation for other verbal-based number and arithmetic concepts and skills. The development of verbalizing appears to be dependent on experiences with identifying collections with number words. Two key guidelines for fostering this foundational knowledge—suggestions that include number-targeted teacher talk—follow.

- *Experiences that involve labeling a wide variety of examples of homogeneous collections and then heterogeneous collections may help children develop verbal subitizing more* *quickly.* Labeling various examples of single instances "one," different examples of pairs "two," and diverse cases of triplets "three" may help children abstract a concept of one, two, and three. This can help them understand that a wide variety of physical characteristics are irrelevant to number concepts and prompt their search for a common attribute (number). Consider, for instance, the *Can You Find?* game in which a parent or teacher might put out a large and a small blue, red, and yellow block and point to the two red blocks and announce "two red blocks." Follow up questions for a child or children might include "Can you find and give me two blue blocks?" "Can you find the two blocks on this end [point]," "Can you find and give me two big blocks?" This illustrates that the color, location, or size of an object can define what constitutes a particular collection but that they do not apply to examples of a number and thus are not critical (defining) attributes of a number. Once children can reliably recognize homogeneous collections of a particular number, questions about heterogeneous collections (e.g., "How many toys are in your toy box here?") may help deepen or broaden their understanding of a cardinal number.

- *Explicitly pointing out non-examples of a number may more readily define the boundaries of a number concept.* (a) One way of comparing and contrasting number words is to introduce them in pairs. Adults might help children construct cardinal concepts by first focusing on one and two, next on two and three, and then on three and four. For instance, reliable identification of one and two can serve as the lower boundary for "three" and examples of "four" can help define its upper boundary. (b) A second method is systematically labeling collections as "*n*" and "not *n*." For example, after labeling two fingers "two," a parent or teacher could hold up one, three, four, and five fingers, in turn, and label each as "not two." Contrasting examples and non-examples can be done in the context of a game such as *Number—Not the Number***.** For example, a child can point to all the collections of two s/he can see and then to all the non-examples of two s/he sees ("Point to something that is *not* two"). For small groups of children, players can take turns pointing out an example and a non-example of a number. In either case, the game can be made more challenging by putting a time limit on the pointing out process or done in small groups, where children take turns identifying an example of number and a non-example. (c) Children's errors can serve as an opportunity to point out non-examples and provide precise feedback. For instance, if a child misidentifies a picture of three bears as "two," a parent or preschool teacher could say, "That's not two bears; it's *three* bears." (d) Perhaps less discomfiting and often highly enjoyable, adults can use error-detection games, in which children indicate when an adult or puppet makes a number identification error.

Step 2. Meaningful object counting: *Counting a collection using one and only one number word per object to determine the total or cardinal value of a collection.* Children begin to learn the counting sequence ("one, two, three, four…") relatively early but even verbally counting up to ten may done with little l quantitative understanding or meaning. Even after children have learned to use number words in a one-to-one fashion with pointing to each item in a collection, many exhibit behavior that indicates they do not really understand the purpose of counting (to determine the number in the collection). Specifically, they may not understand that such a process is another way of determining the total or cardinality of a collection and that the last counting word used in this process has special significance because this number word not only serves to indicate mark the last item as counted but also represents the total of all the items counted. This understanding is called the cardinality principle (of counting). For instance, they often recount a collection when asked "how many?" A key reason for this is that young children are often taught to do one-for-one object counting by rote. Key guidelines for fostering this foundational knowledge follow.

- *Introduce one-for-one object counting in a meaningful manner by modeling the counting process with collections a child can already verbally subitize.* When adults model the one-for-one counting process, they often either emphasize the last number word or repeat it to indicate that it has special significance (indicates the total or cardinal value of the collection). For example, when illustrating counting with a collection of three items, they will often either count "One, two, ***t-h-r-e-e***" or say, "One, two, three—see *three*." If a child can verbally subitize three and already knows there are three ("Yah, there's three there"), then there is a decent chance the child will understand the emphasis put on the last number word or why its repeated. That is, the child has a reasonable chance of recognizing that one-for-one object is another way of determining the total and learning the cardinality principle.
- *As with all content areas, instruction and practice counting one-for-one should be done in a purposeful manner—that is, within the context of a real or interesting situation in which one wants to know "how many."* There are numerous everyday opportunities to use and discuss counting collections (e.g., counting the number of children at a table so that the correct number of crackers, milk containers, project items, or instructional materials can be distributed to the group). Dice games, card games and numerous other games involve counting collections either to play the game or keep score. For example, the *Hidden Stars* game (*27*) entails showing a child a card with a collections of stars (or other object, shapes, or symbols), asking the child to count the collection, then turning the card over to hide the collection, and finally asking: "How many stars am I hiding?" This game creates a real reason in the child's mind to learn or apply the cardinality principle.

Step 3. Verbal-based magnitude comparisons: *Using verbal subitizing or one-for-one counting to determine the numerical relation (e.g., "same number" or "more") between collections first and then using a mental representation of the counting sequence to specify the numerical relation between number words*. Although young children may use verbal subitizing and one-for-one counting to determine the total or cardinality of a collection, it does guarantee they understand the numerical relations among number words. For example, although they may accurately count one-for-one a collection of six and a collection of seven and determine their cardinal value ("six" and "seven," respectively), they do not necessarily understand that the collection of "seven" is *more than* the collection of "six." The sub-steps in helping children understand the order-of-magnitude (ordinal relations) of number words follow.

- *First, ensure that children understand relational terms as "more" or "fewer."* This can be done with collections that are obviously differ (collections involving one to three items or with any two collections in which the larger is more twice as large as the smaller. Such experiences can be done in the context of everyday situations, playing math games, or teaching other content, such as reading a children's story.
- *Next, have children name the larger of two collections they can verbally subitize.* By literally seeing that "two" is more than "one" and "three" is more than "two," verbalizing these relations, and relating these relations to the order these number words in

the counting sequence (“one, two, three”), children can construct the *increasing-magnitude principle*—that is, realize that the latter a number word comes in the counting sequence, the larger collection it represents.

- *Once children understand the increasing-magnitude principle, encourage them to apply to one-for-one counting with larger collections.* For example, if Jacob has a score of five (represented by five blocks) and Derye has six (represented by six blocks), the children can each count their collection of blocks to see who counted the furthest. If necessary, a teacher can provide scaffolding by counting Jacob blocks and then counting Derye’s blocks and emphasize that Jacob’s count has been surpassed: “Jacob has five, and Derye has ‘one, two, three, four, f-i-v-e, SIX. If further, explicit scaffolding is needed, the teacher can add: “Six is more than five, because six comes after five when we count.”
- *After children have mastered making concrete comparisons using one-to-one object counting and number-after relations, teachers can help them make abstract comparisons of neighboring number words*. First ensure that a child is fluent with number-after relations (e.g., knows that “when we count, after seven comes … eight). In this way, children can then mentally apply the increasing-magnitude principle to any two number neighbors for which they know the number-after relation. This can be practice with a math game, such as *Car Race*, in which a player draws a card and must decide which of two number neighbors shown on the card is larger in order to move his/her racecar the most spaces on a racetrack.

Step 4. Adding and subtracting: Once children can quantify collections meaningfully and understands verbal-based number relations, they are ready to solve basic addition and subtraction word problems. Children construct a basic informal understanding of addition and subtraction by operating on small collections. By seeing one block added to two blocks, for example they can formulate the idea that adding more to a collection makes it larger. Similarly, by seeing one block taken from two, they can construct the idea that taking away items from a collection makes it small. Once children have completed Steps 1 to 3, they are ready to apply their basic ideas to solving word problems. Solving meaningful word problems informally can provide an important basis for learning and solving symbolic expressions such as 3 + 2 or 3 – 2 or symbolic equations such as 3 + 2 = 5 or 3 – 2 = 1.

- *First encourage children to solve nonverbal addition and subtraction problems with small collections they can verbally subitize*. For instance, the *Super Hiding* game involves showing a child an initial collection on a mat, covering the collection, placing additional item(s) next to the covered collection for a moment, and then pushing the additional item(s) under the cover also. With subtraction, item(s) are removed from covered collection, shown for a few moments, and then removed from sight.
- *Next encourage children to solve word problems using their own self-invented strategy and sharing their informal strategy with other children.* Young children frequently model the meaning of simple addition and subtraction. This usually takes the form of counting objects or verbal counting

Appendix A2-3: The Big Idea of Equal Partitioning and Its Relation to Various Aspects of Number and Operations in the Primary Grade

Division. A primary-level teacher can lay the conceptual foundation for the formal instruction on division in the intermediate grades by taking advantage of every day situations that involve sharing (e.g., "Here's a plate of 8 crackers, how many crackers will each of the four of you get if the crackers are shared fairly?") or by posing fair-sharing problems (e.g., While reading the Door Bell Rang by Pat Hutchins, a teacher can ask: If there are now four children and there are 12 cookies, how many cookies will each child get if the cookies are shared fairly?"). By building on the familiar experience of fair sharing, Children as young as kindergarten can use their familiarity with fair sharing to invent strategies for solving such real or imaginary problems. Children commonly use a "dealing out" strategy—give each person one item until all nothing else can be shared. Later, formal notation for division, such as 7 ÷ 2 = 3 r1, can be introduced meaningfully by relating it to fair sharing: seven cookies shared fairly between two children results in each getting three cookies with one cookie leftover (remaining).

Even and odd numbers. A useful informal analogy for the concept of even number is a collection of items in which all the items can be shared fairly by exactly two people. An odd number can be informally thought of as collection that cannot be shared fairly in this manner. For example, 6 cookies, but not 7, cookies can be shared fairly between two children and, thus, an even number. With 7 cookies either one child gets an extra cookie or, after six of the cookies are shared fairly, one is left over. Using this fair analogy, children easily determine which numbers in the counting sequence are even and which are odd and conclude that every other number starting with 1 is odd and every number starting with 2 is even. Labeling the odd and even numbers a number list or number line that includes 0 can lead a class to debate whether 0 is odd or even. It is even if you follow the "every other number" pattern (and because no cookies shared by two children yields the same fair, if disappointing, share of "none"). Such fair-sharing experiences can provide a basis for understanding the more formal definition of an even number as "an integer evenly divisible by two and for discovering the rules for adding even (e) and odd (o) numbers: e + e = e, e + o = o, and o + o = e. For example, it can help students understand what otherwise may seem like a paradox or strange rule—that *adding two odd numbers results in an even number*. However, if a plate of 3 cookies and a plate of 5 cookies are shared fairly between two children, then each gets 1 cookie from the first plate and 2 cookies from the second plate, and the remaining 2 cookies can then be divvied up fairly to give each a total of 4 cookies—an even number of cookies.

Fractions. Relating fractions to the familiar situation of fair sharing can help children understand otherwise mysterious concepts and skills and empower them in solving problems involving fractions.

- *Quotient* or division meaning of fractions. Fractions can represent various meanings—one of which is division (e.g., 3/4 = 3 divided by 4). A division meaning of fractions can be viewed informally as fairly sharing a (continuous) quantity such as a length of string, (the area of) a pizza or rectangular cake among a given number of people. For example, 3/4 can be related to a fair-sharing problem, such as: If three small pizzas (the numerator

3) are shared fairly among (the fraction bar) four people (the denominator), namely Priscilla, Queen, Ramella, and Shifra, what is the size of each person's share?

- *Part-of-a-whole meaning of fractions*. A part-of-a-whole interpretation of the fraction ¾, for instance, indicates 3 parts of a whole divided into four equal parts. Note that the solution to a fair-sharing problem, such as the one in the previous bullet (i.e., determining each person's share), requires thinking in terms of a part-of-a-whole meaning (i.e., What part of a whole pizza does each of the four girls get as a fair share?; again see Figure 2.1). Relating fractions to such fair-sharing problems has two important advantages: (a) Such problems underscores what many children (and teachers) do not fully appreciate—that a part-of-the-whole meaning of fractions involves the special case where all the parts are equal in size. A common error in identifying fractions is to count the part(s) of interest and the total number of parts and use these to write a fraction—even though the parts are in equal in size. In Figure 2.2, for instance, the child wrote 1/3 because one of three (unequal) parts was shaded. (b) Children's familiarity with fair-sharing situation enables them to devise or invent their own strategies for solving problems involving fraction. Figure 2.3 illustrates three different strategies a second-grade class devised on their own to solve a problem involving sharing 8 pizzas among 5 people.

- *Equivalent fractions.* A fair-analogy can help underscore equivalent fractions. For example, sharing 1 pizza between 2 people results in the same size share as sharing twice as many pizzas (2 pizzas) among twice as many people (4 people) or three as many pizzas (3 pizzas) among three times as many people (6 people). This is important, because it underscores that equivalent fractions are related by multiplication, not addition (i.e., involve multiplicative, not additive, reasoning. A common error in solving an equivalence problem such as 1/2 = 2/□ is think: "Well one was added to the top number (sic) 1 of the first fraction to make the top number (sic) of the second fraction 2, so I'll add 1 to it to make the bottom number (sic) of the second fraction—so the answer is 2/3. Note in Figure 2.3 that the three solutions underscore that 1 + 1/2 + 1/10 (Solution A) = 1-3/5 (Solution B) = 8/5 (Solution C).

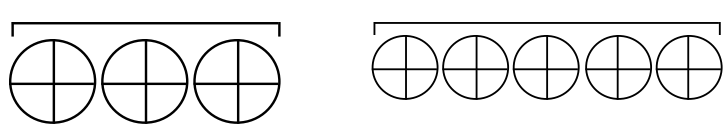
- Comparing fractions. Children are often confused about comparing fractions, such as 1/3 and 1/4, and determining which is larger, because they tend to think in terms of whole numbers (i.e., 4 is larger than 3, so 1/4 must be larger than 1/3). However, a fair-sharing analogy provides children with a powerful tool for reasoning about relations between fractions. For instance, makes clear that 1/3 is larger than 1/4. Which results in a larger share of pizza for each person: 1 pizza shared fairly among 3 people or 1 pizza shared fairly among 4 people?

Figure 2. How Fair-Sharing Problems Can Illustrate Both a Division and a Part-of-a-Whole Meaning of Fractions and Serve as a Bridge Between Them (280)

A. Divide each pizza into four parts (pieces).

3 pizzas shared by 4

5 pizzas shared by 4



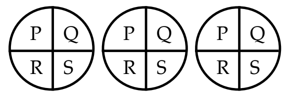
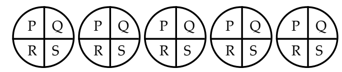
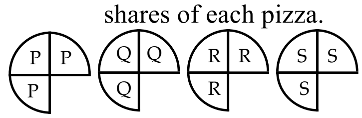
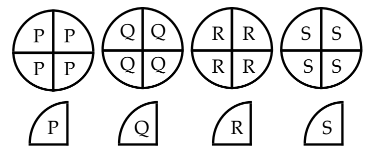
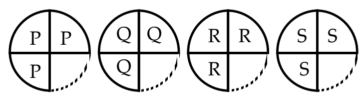
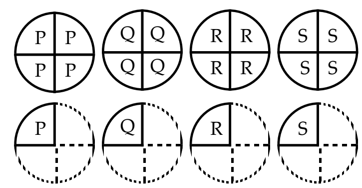
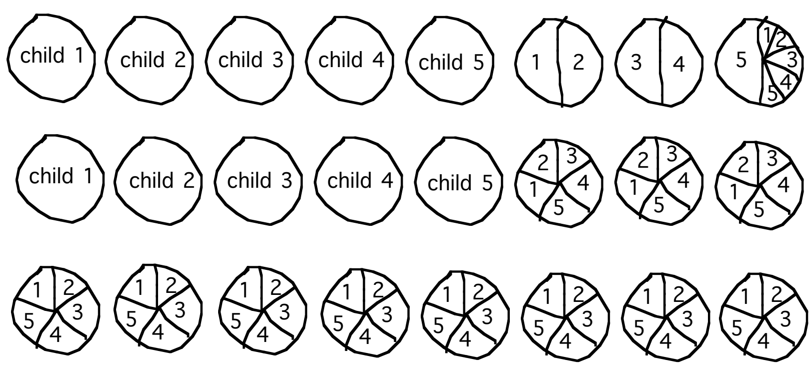
B. Divvy up the pieces among the four girls: Priscilla (P), Queen (Q), Ramella (R), & Shifra (S).	 Each girl gets one of four equal shares of each pizza.	 Each girl gets one of four equal shares of each pizza.
C. The results of divvying up the pieces.		
D. Naming the size of each girl's share relative to a whole pizza. (Note that this involves a part-of-the whole meaning.)		
E. Symbolic representation of a share.	$\frac{3}{4}$	$\frac{5}{4}$

Figure 3. Three Informal Strategies Devised by Second Graders for Solving Problems That Involve Dividing 8 Pizzas Among 5 People



Appendix A2-4: Spatial Reasoning, Geometry, and Measurement

Spatial Reasoning

Spatial reasoning is an essential human ability that contributes to mathematical ability. Some competencies, including the ability to actively and selectively seek out pertinent information and certain interpretations of ambiguous information, are present from birth (*297*). This does not mean that these competencies do not develop (they do, cf. *298*), only that initial "bootstrap" abilities4 guide development, such as infants early learning of the ability to focus their eyes on objects and then follow moving objects. As another example, children as young as oddlers use geometric information about the overall shape of their environment to solve location tasks. And, as with number, we share some of these abilities with animals. For example, baby chicks can use geometric information from their surrounds to reorient themselves in space (*299, 300*).

As stated, children move along a trajectory from early spatial abilities based on their own position in space, to using landmarks to orient themselves, to understanding paths they follow that include several landmarks, to building maps, both "mental maps" and representations of space that relate all places and distances, finally resulting in the use of coordinate systems.

Teaching isolated spatial skills especially to children with special needs, has a long history, most of which has been unsuccessful. For example, school experiences with map skills are typically limited and fail to connect such skills to other curriculum areas, including number and operations, measurement, and spatial reasoning. A more fruitful approach includes—

- Create school environments that include interesting layouts inside and outside classrooms and that invite incidental learning such as using landmarks and routes.
- Use regularly planned experiences with landmarks and routes, with frequent discussion about spatial relations, including distinguishing parts of children's bodies and spatial movements (forward, back), finding a missing object ("under the table that's next to the door"), putting objects away, and finding the way back home from an excursion.
- Provide specific instruction about models and maps. Teachers need to raise four mathematical questions: Direction—which way?, distance—how far?, location—where?, and identification—what objects? To answer these questions, children need to develop a variety of skills. Children must learn to deal with mapping processes of abstraction, generalization, and symbolization.

To truly understand space as organized into grids or coordinate systems, children must learn spatial structuring. Spatial structuring is the mental operation of constructing an organization or form for an object or set of objects in space. Children may first view a grid as a collection of squares, rather than as sets of perpendicular lines. They only gradually come to see them as organized into rows and columns, learning the order and distance relationships within the grid. For coordinates, labels must be related to grid lines and, in the form of ordered pairs of coordinates, to points on the grid. Eventually these, too, must be integrated with the grid's order and distance relationships to be understood as a mathematical system. Done well, students learn to understand and eventually quantify what grid labels on a coordinate grid represent. They connect their counting acts to those quantities and to the labels. They learn to mentally structure grids as two-dimensional spaces. That is, they understand coordinates as a way to organize 2D space by coordinating two perpendicular number lines—every location is the place where measures along each of these two number lines meet.

Geometry

A husband-and-wife team of researchers, Pierre and Dina van Hiele, agreed that children build their geometric ideas. They also describe levels of thinking through which children do so:

- Children recognize and label shapes based on overall appearance rather than in terms of attributes. (e.g., “The door is a rectangle because it looks like a rectangle”).
- Then they learning about the parts or attributes of shapes (e.g., a rectangle has two long parallel sides and two short parallel side, and a square has 4 equal sides and angles).
- Finally learn about the relationship of shapes and attributes, which requires a focus on critical attributes—the attributes that all examples of a concept share (A rectangle is an enclosed 4-sided figure with parallel opposite sides, and a square is a special rectangle because it has all the critical attributes of a rectangle and a square angle).

Experiences and instruction play a large role in shaping children’s knowledge of geometry. If children lack experience of shapes, and if the examples and nonexamples of shapes they do experience are rigid, not including a variety of variants of that shape class, children’s mental images and ideas about that shape will also be rigid and limited. For example, many children learn to accept as triangles only isosceles triangles with a horizontal base. This is important. *Children’s ideas stabilize as early as the primary grades. It is therefore critical to provide better, richer opportunities to learn about geometric figures to all children between 3 and 8 years of age.*

The ability to describe, use, and visualize the effects of composing and decomposing geometric regions is important in and of itself. It also provides a foundation for understanding other areas of mathematics, especially number and arithmetic, such as part–whole relationships, fractions, and so forth. Young children move through levels in the composition and decomposition of both two- and three-dimensional figures. From lack of competence in composing geometric shapes, they gain abilities to combine shapes into pictures, then synthesize combinations of shapes into new shapes (composite shapes), eventually operating on and iterating those composite shapes.

Research suggests the following guidelines for teaching early geometry (for more details, see *38, 111*).

- Follow the learning trajectory of developing rich visual experiences with whole shapes, then directing attention to the parts of shapes, then the attributes (relationships such as side and angle sizes) of shapes.
- Provide varied examples and nonexamples to help children understand attributes of shapes that are mathematically relevant as well as those (orientation, size) that are not. This will prevent children forming narrow ideas about any class of shapes. Use of prototypes may bootstrap initial learning, but examples should become more diverse as soon as possible.
- Include more shapes than the typical circle, square, triangle, and rectangle. Include hexagons, rhombuses, and trapezoids, for example.
- Discuss categories of shapes and what attributes each has. Encourage children’s descriptions while encouraging the development of mathematical language. Build on

informal experiences and terms (e.g., relate the formal term “angle” to the “amount of turn” and children’s everyday experience of turning.

- Include challenging distractors of shapes. Show nonexamples and compare them to similar examples helps focus children’s attention on the critical attributes of shapes and prompts discussion. This is especially important for classes that have more diverse examples, such as triangles. In Figure 1, each triangle on the left can be paired with and compared to the non-triangle on the right.
- Follow the learning trajectory for composing and decomposing geometry shapes (see Chapter 9 in both *38, 111*).
- Finally, challenge children with a broad array of interesting tasks. Activities that promote reflection and discussion include building models of shapes from components. Matching, identifying, exploring, and even making shapes with computers is particularly motivating (*301*). Work with Logo’s “turtle graphics” is accessible even to kindergartners (*302*), with results indicating significant benefits for that age group (e.g., more than older children, they benefitted in learning about squares and rectangles).

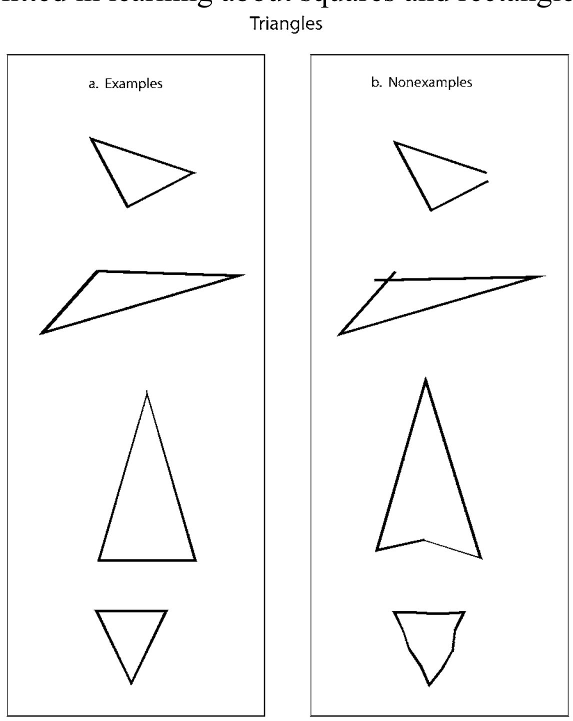


Figure 1 Examples and matched nonexamples of triangles

Measurement

Subitizing and much of counting involve quantifying discrete collections (i.e., “telling how many” separate, individual items in a group). We also need to quantify *continuous* quantities, such as length, area, and volume (i.e., “telling how much” of, or the magnitude, of, a given attribute an object has). We quantify continuous quantities by *measuring* them. We divide them up into equal parts (units) and count the units.

Thus, measurement is an important real-world area of math. Many children measure in a rote fashion. By understanding learning trajectories, we can do better for children. In this brief, we use length as an example but for important information on the learning trajectories for area and volume, see (*38, Chapter 11 in each, 111*).

Length is a characteristic of an object found by quantifying how far it is between the endpoints of the object. "Distance" is often used similarly to quantify how far it is between any two points in space. Measuring length or distance consists of two aspects, identifying a unit of measure and subdividing (mentally and physically) the object by that unit, placing that unit end to end (iterating) alongside the object. Subdividing, or equal partitioning, and unit iteration are complex mental accomplishments that are too often ignored in traditional measurement curriculum materials and instruction. Children learn to recognize length as a separate attribute (not confusing it with volume, weight, or simply "bigness"), then they learn to directly compare lengths (aligning endpoints), then to *measure* by laying done many copies of a unit end-to-end, then to *iterate* or repeat one copy of a unit, and finally to use rulers with understanding.

Research includes guidelines for teaching length (*38, 111*).

- Teach measurement as more than a simple skill—measurement is a complex combination of concepts—especially the idea of equal partitioning—and skills that develops over years. Understand the foundational concepts of measurement so that you will be better able to interpret children's understanding and ask questions that will lead them to construct these ideas. For example, when children count as they measure, focus children's conversations on what they are counting—not "points" but equal-sized units of length. That is, if a child iterates a unit five times, the "five" represents five units of length. For some students "five" signifies the hash mark next to the numeral five instead of the amount of space covered by five units.
- Use initial informal activities to establish the attribute of length and develop concepts such as "longer," "shorter," and "equal in length" and strategies such as direct comparison.
- Encourage children to solve real measurement problems, and, in so doing, to build and iterate units, as well as units of units
- Help children closely connect the use of manipulative units and rulers. When conducted in this way, measurement tools and procedures become tools for mathematics and tools for thinking about mathematics.

Appendix A2-5: Other Topics and Processes

Patterns and Structure (including algebraic thinking). As stated, the breadth of ways the term "patterns" is used illustrates a main strength and weakness of the notion as a goal in mathematics. Consider some examples from other topics.

- Perceptual patterns, such as subitized domino patterns, finger patterns, or auditory patterns (e.g., three beats).
- Patterns in the number words of counting.
- The "one-more" pattern of counting, which also connects counting with arithmetic
- Numerical patterns, such as a mental representation of 3 as a triangle; or a similar pattern of 5 that can be broken into 2 and 3 and then put them back together to make 5 again.
- Arithmetic patterns that are especially powerful and easy for children to see: doubles (3 + 3, 7 + 7), which allow access to combinations such as 7 + 8, and fives (6 made as 5 + 1, 7 as 5 + 2, etc.), which allow for decomposition into fives
- Spatial patterns, such as the composition of shapes.

None of these examples of patterns in early mathematics illustrates the most typical practice of "doing patterns" in early childhood classrooms. Typical practice involves activities such as making paper chains that are "red, blue, red, blue . . ." and so forth. Such sequential repeated patterns may be useful, but educators should be aware of the role of patterns in mathematics and mathematics education and of how sequential repeated patterns such as the paper chains fit into (but certainly do not, alone, constitute) the large role of patterning and structure that these examples convey. From this broad perspective, children begin this development from the first year of life

Appendix A4-1: Instructional Practices to Avoid

Inappropriate drill

Children need substantial practice in many mathematical competencies. However, the *nature* and timing of this practice can mean the difference between appropriate and inappropriate instruction. For example, even preschools can play games such as "Snapshots" in which the teacher holds up 2 fingers on one hand and 3 on other (held apart) for only two seconds and children "think, pair, and share"—whispering "five" to each other. This visually-based "number composition" builds a firm foundation for arithmetic. However, "bare bones" drill on worksheets filled with "facts" such as 2 + 3 = __ do not build the same foundation and are neither motivating nor appropriate for the way preschoolers' learn.

Fluency in single-digit addition and subtraction combinations is an appropriate goal for the primary grades, but again, there are appropriate and inappropriate ways to practice. For example, "Snapshots" can be played with all combinations. Children should also use *thinking strategies* to figure out basic combinations. Inappropriate practice may be useless or even potentially harmful. For example, in one large study, the more first grade teachers used timed tests of additional and subtract facts in first grades, the *fewer* facts children knew (*198*). Also negatively related to children's mastery of facts was the use of textbooks with a specific goal of memorization. Not necessarily harmful, but not helpful either, was the use of flash cards and extensive work on small sums to the exclusion of larger sums.

Fluency with arithmetic combinations is important. Fluency "frees the mind" to solve more difficult problems (*251*). To achieve true fluency—automaticity *and* able to think flexibly, children need to learn to solve simple addition and subtraction strategies several different ways (even if they are slow, such as "counting all"), then learning thinking strategies (such as "counting on" or "break-apart-to-make tens"— 9 + 6 = 9 + (1 + 5) = 10 + 5 = 15, all done mentally and quickly), then internalize all these until they are automatic. Once they react that last stage, distributed, spaced practice—often game-like and involving a variety of contexts—is appropriate (*303*).

Inappropriate use of calendar activities

Our National Research Council report stated, "Generalized teaching strategies and activities are defined as those that can apply to a variety of the NCTM mathematics standards. The most prominent generalized strategy was calendar-related activities, which occurred on a daily basis in over 90 percent of the classrooms surveyed, this despite the fact that mathematics educators do not consider most calendar activities to be useful early childhood mathematics instruction and have serious questions about the efficacy of "doing the calendar" every day (*5, p. 241*). The report goes on to say that despite the calendar's potential usefulness in teaching simple time concepts such as "yesterday" and "today," these are not core mathematical concepts. The calendar groups days into rows of 7 rather than 10, the basis of our number system. "Time spent on the calendar would be better used on more effective mathematics teaching and learning experiences."

Worse, many teachers spend long times on the calendar, and engage only one child (or one child at a time) in “doing” the calendar. Because most children are not involved, this is one example of the type of activity that harms, rather than builds, self-regulation.

Math mistakes

Although it seems obvious that errors in mathematics must be avoided, many teachers and curricula propagate just such mistakes. For example, it is not true that “multiplication always makes numbers bigger.”

Teaching practices that harm the development of executive function (self-regulation) competencies

Also important is eliminating the dull routines and overly authoritarian environments that do not develop and can harm children's developmental of executive function competencies (*304-306*). For example, too many mathematics classes include unfortunate features such as calling on one student at a time while the rest passively listen (and often do not attend) or having long “dead times” with no instructional or other activities (e.g., waiting for children to line up or pass out materials).

Instead, children learn more as they “think, pair, and share,” talking to their neighbor to give an answer or discuss a solution strategy, or as they answer chorally. Movement games might be used in which children follow more complex rules or to switch from one set of rules to another. For example, in “Shape Step,” a variety of shapes are outline on the floor (with painters’ tape) and the teacher or a child challenges children to step on “only the shapes with four sides” and then “all the shapes with at least one right angle” (and so on).

References

1. B. Doig, B. McCrae, K. Rowe, "A good start to numeracy: Effective numeracy strategies from research and practice in early childhood" (Australian Council for Educational Research, Canberra ACT, Australia, 2003).
2. D. Frye *et al.*, "Teaching math to young children: A practice guide" (National Center for Education Evaluation and Regional Assistance (NCEE), Institute of Education Sciences, U.S. Department of Education, Washington, DC, 2013).
3. N. C. Jordan, L. B. Hanich, H. Z. Uberti, in *The development of arithmetic concepts and skills: Constructing adaptive expertise,* A. J. Baroody, A. Dowker, Eds. (Erlbaum, Mahwah, NJ, 2003), pp. 359-383.
4. J. Kilpatrick, J. Swafford, B. Findell, *Adding it up: Helping children learn mathematics*. (National Academy Press, Washington, DC, 2001).
5. National Research Council, *Mathematics in early childhood: Learning paths toward excellence and equity*. C. T. Cross, T. A. Woods, H. Schweingruber, Eds., (National Academy Press, Washington, DC, 2009).
6. J. W. Stigler, S.-Y. Lee, H. W. Stevenson, *Mathematical knowledge of Japanese, Chinese, and American elementary school children*. (National Council of Teaching of Mathematics, Reston, VA, 1990).
7. P. Starkey *et al.*, in *Society for Research in Child Development*. (Albuquerque, NM, 1999).
8. M. Yuzawa, W. M. Bart, L. J. Kinne, S. Sukemune, M. Kataoka, The effects of "origami" practice on size comparison strategy among young Japanese and American children. *Journal of Research in Childhood Education* **13**, 133 (1999).
9. D. H. Arnold, G. L. Doctoroff, The early education of socioeconomically disadvantaged children. *Annual Review of Psychology* **54**, 517 (2003).
10. D. C. Geary, C. C. Bow-Thomas, L. Fan, R. S. Siegler, Even before formal instruction, Chinese children outperform American children in mental addition. *Cognitive Development* **8**, 517 (1993).
11. S. Griffin, R. Case, R. S. Siegler, in *Classroom lessons: Integrating cognitive theory and classroom practice,* K. McGilly, Ed. (MIT Press, Cambridge, MA, 1994), pp. 25-49.
12. V. E. Lee, D. T. Burkam, *Inequality at the starting gate*. (Economic Policy Institute, Washington, DC, 2002).
13. G. B. Saxe, S. R. Guberman, M. Gearhart, Social processes in early number development. *Monographs of the Society for Research in Child Development* **52**, (1987).
14. R. S. Siegler, in *Contributions of psychology to science and mathematics education,* L. A. Penner, G. M. Batsche, H. M. Knoff, D. L. Nelson, Eds. (American Psychological Association, Washington, DC, 1993), pp. 341-366.
15. A. J. Baroody, M.-l. Lai, K. S. Mix, in *Handbook of research on the education of young children,* B. Spodek, O. N. Saracho, Eds. (Erlbaum, Mahwah, NJ, 2006), pp. 187-221.
16. H. P. Ginsburg, *Children's arithmetic*. (Pro-ed, Austin, TX, 1977).
17. H. P. Ginsburg, R. L. Russell, Social class and racial influences on early mathematical thinking. *Monographs of the Society for Research in Child Development* **46**, (1981).

18. J. C. Hiebert, *Conceptual and procedural knowledge: The case of mathematics*. (Lawrence Erlbaum, Hillsdale, NJ, 1986).
19. J. L. Booth, R. S. Siegler, Developmental and individual differences in pure numerical estimation. *Developmental Psychology* **41**, 189 (2006).
20. B. A. Clarke, D. M. Clarke, J. Cheeseman, The mathematical knowledge and understanding young children bring to school. *Mathematics Education Research Journal* **18**, 81 (2006).
21. A. Dowker, *Individual differences in arithmetic: Implications for psychology, neuroscience, and education*. (Psychology Press, New York, NY, 2005).
22. R. Gelman, Preschool thought. *American Psychologist* **34**, 900 (1979).
23. H. P. Ginsburg *et al.*, in *Critical issues in early childhood professional cevelopment,* M. Zaslow, I. Martinez-Beck, Eds. (Brookes, Baltimore, MD, 2006), pp. 171–202.
24. N. C. Jordan, J. Huttenlocher, S. C. Levine, Differential calculation abilities in young children from middle- and low-income families. *Developmental Psychology* **28**, 644 (1992).
25. National Mathematics Advisory Panel, *Foundations for success: The final report of the National Mathematics Advisory Panel*. (U.S. Department of Education, Office of Planning, Evaluation and Policy Development, Washington D.C., 2008).
26. K. Aunola, E. Leskinen, M.-K. Lerkkanen, J.-E. Nurmi, Developmental dynamics of math performance from pre-school to grade 2. *J. Ed. Psych.* **96**, 699 (2004).
27. A. J. Baroody, *Children's mathematical thinking*. (Teachers College, New York, NY, 1987).
28. K. Bodovski, G. Farkas, Mathematics growth in early elementary school: The roles of beginning knowledge, student engagement, and instruction. *The Elementary School Journal* **108**, 115 (2007).
29. N. C. Jordan, D. Kaplan, M. N. Locuniak, C. Ramineni, Predicting first-grade math achievement from developmental number sense trajectories. *Learning Disabilities Research and Practice* **22**, 36 (2006).
30. K. Denton, J. West, "Children's reading and mathematics achievement in kindergarten and first grade" (U.S. Department of Education, National Center for Education Statistics, Washington, DC, 2002).
31. H. W. Stevenson, R. S. Newman, Long-term prediction of achievement and attitudes in mathematics and reading. *Child Development* **57**, 646 (1986).
32. G. J. Duncan, K. Magnuson, in *Whither opportunity? Rising inequality and the uncertain life chances of low-income children,* G. J. Duncan, R. Murnane, Eds. (Russell Sage Press, New York, NY, 2011), pp. 47-70.
33. G. J. Duncan *et al.*, School readiness and later achievement. *Developmental Psychology* **43**, 1428 (2007).
34. D. C. Farran, C. Aydogan, S. J. Kang, M. Lipsey, in *Handbook of early literacy research,* D. Dickinson, S. Neuman, Eds. (Guilford, New York, NY, 2005), pp. 257-268.
35. M.-K. Lerkkanen, H. Rasku-Puttonen, K. Aunola, J.-E. Nurmi, Mathematical performance predicts progress in reading comprehension among 7-year-olds. *European Journal of Psychology of Education* **20**, 121 (2005).
36. J. Sarama, A. Lange, D. H. Clements, C. B. Wolfe, The impacts of an early mathematics curriculum on emerging literacy and language. *Early Childhood Research Quarterly* **27**, 489 (2012).

37. D. H. Clements, J. Sarama, *Learning and teaching early math: The learning trajectories approach*. (Routledge, New York, NY, 2009).
38. J. Sarama, D. H. Clements, *Early childhood mathematics education research: Learning trajectories for young children*. (Routledge, New York, NY, 2009).
39. P. M. Sadler, R. H. Tai, The two high-school pillars supporting college science. *Science* **317**, 457 (2007).
40. D. H. Clements, J. Sarama, Early childhood mathematics intervention. *Science* **333**, 968 (Aug 19, 2011).
41. S. Thomson, K. Rowe, C. Underwood, R. Peck, "Numeracy in the early years: Project Good Start" (Australian Council for Educational Research, Camberwell, Victoria, Australia, 2005).
42. A. M. B. Austin, B. Blevins-Knabe, S. L. K. Lindauer, Informal and formal mathematics concepts: Of children in center and family child care. *Unpublished manuscript*, (2008).
43. J. R. H. Tudge, F. Doucet, Early mathematical experiences: Observing young Black and White children's everyday activities. *Early Childhood Research Quarterly* **19**, 21 (2004).
44. T. A. Graham, C. Nash, K. Paul, Young children's exposure to mathematics: The child care context. *Early Childhood Education Journal* **25**, 31 (1997).
45. M. L. Van Horn, E. O. Karlin, S. L. Ramey, J. Aldridge, S. W. Snyder, Effects of developmentally appropriate practices on children's development: A review of research and discussion of methodological and analytic issues. *Elementary School Journal* **105**, 325 (2005).
46. D. H. Clements, J. Sarama, A.-M. DiBiase, *Engaging young children in mathematics: Standards for early childhood mathematics education*. (Erlbaum, Mahwah, NJ, 2004).
47. R. Gelman, C. R. Gallistel, *The child's understanding of number*. (Harvard University Press, Cambridge, MA, 1978).
48. B. Perry, S. Dockett, in *Handbook of International Research in Mathematics Education,* L. D. English, Ed. (Erlbaum, Mahwah, NJ, 2002), pp. 81-111.
49. National Research Council, *Eager to learn: Educating our preschoolers*. B. T. Bowman, M. S. Donovan, M. S. Burns, Eds., (National Academy Press, Washington, DC, 2001).
50. J. Brooks-Gunn, G. J. Duncan, P. R. Britto, in *Developmental health and the wealth of nations,* D. P. Keating, C. Hertzman, Eds. (Guilford, New York, NY, 1999), pp. 94-124.
51. P. F. Campbell, E. A. Silver, *Teaching and learning mathematics in poor communities*. (National Council of Teachers of Mathematics, Reston, VA, 1999).
52. D. R. Entwisle, K. L. Alexander, Beginning school math competence: Minority and majority comparisons. *Child Development* **61**, 454 (1990).
53. T. G. Halle, B. Kurtz-Costes, J. L. Mahoney, Family influences on school achievement in low-income, African American children. *J. Ed. Psych.* **89**, 527 (1997).
54. I. V. S. Mullis *et al.*, *TIMSS 1999 international mathematics report*. (The International Study Center, Boston College, Lynch School of Education, Boston, 2000).
55. G. Natriello, E. L. McDill, A. M. Pallas, *Schooling disadvantaged children: Racing against catastrophe*. (Teachers College Press, New York, NY, 1990).
56. C. Rouse, J. Brooks-Gunn, S. McLanahan, Introducing the issue. *The Future of Children* **15**, 5 (2005).
57. W. G. Secada, in *Handbook of research on mathematics teaching and learning,* D. A. Grouws, Ed. (Macmillan, New York, NY, 1992), pp. 623-660.

58. K. Sylva, E. Melhuish, P. Sammons, I. Siraj-Blatchford, B. Taggart, "The effective provision of pre-school education [EPPE] project: A longitudinal study funded by the DfEE (1997-2003)" (EPPE Project, Institute of Education, University of London, London, 2005).
59. G. Thomas, A. Tagg, *An evaluation of the Early Numeracy Project 2003*. (Ministry of Education, Wellington, NZ, 2004).
60. D. C. Geary, in *Handbook of child psychology: Volume 2: Cognition, perception, and language (6th ed.),* D. Kuhn, R. S. Siegler, W. Damon, R. M. Lerner, Eds. (Wiley, Hoboken, NJ, 2006), pp. 777-810.
61. J. Lee, Racial and ethnic achievement gap trends: Reversing the progress toward equity? *Educational Researcher* **31**, 3 (2002).
62. S. F. Reardon, in *Whither Opportunity? Rising Inequality, Schools, and Children's Life Chances,* G. J. Duncan, R. Murnane, Eds. (Russel Sage Foundation, New York, NY, 2011), pp. 91-116.
63. S. Griffin, R. Case, A. Capodilupo, in *Teaching for transfer: Fostering generalization in learning,* A. McKeough, J. Lupart, A. Marini, Eds. (Erlbaum, Mahwah, NJ, 1995), pp. 121-151.
64. J. T. Sowder, in *Analysis of arithmetic for mathematics teaching,* G. Leinhardt, R. Putman, R. A. Hattrup, Eds. (Erlbaum, Mahwah, NJ, 1992).
65. NCES, *America's kindergartners (NCES 2000070)*. (National Center for Education Statistics, U.S. Government Printing Office, Washington, DC, 2000).
66. D. H. Clements, J. Sarama, *Building Blocks* for early childhood mathematics. *Early Childhood Research Quarterly* **19**, 181 (2004).
67. D. H. Clements, J. Sarama, Effects of a preschool mathematics curriculum: Summative research on the *Building Blocks* project. *JRME* **38**, 136 (2007).
68. N. C. Jordan, D. Kaplan, L. N. Oláh, M. N. Locuniak, Number sense growth in kindergarten: A longitudinal investigation of children at risk for mathematics difficulties. *Child Development* **77**, 153 (2006).
69. M. Hughes, Can preschool children add and subtract? *Educational Psychology* **1**, 207 (1981).
70. R. J. Wright, A study of the numerical development of 5-year-olds and 6-year-olds. *Educational Studies in Mathematics* **26**, 25 (1994).
71. D. C. Berliner, Our impoverished view of educational research. *Teachers College Record* **108**, 949 (2006).
72. N. i. S. Nasir, P. Cobb, *Improving access to mathematics: Diversity and equity in the classroom*. (Teachers College Press, New York, NY, 2007).
73. J. Janzen, Teaching English language learners. *Review of Educational Research* **78**, 1010 (2008).
74. L. C. Moll, C. Amanti, D. Neff, N. Gonzalez, Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. *Theory into Practice* **31**, 132 (1992).
75. S. Celedón-Pattichis, S. I. Musanti, M. E. Marshall, in *Mathematics teaching & learning in K-12: Equity and professional development,* M. Q. Foote, Ed. (Palgrave Macmillan, New York, NY, 2010), pp. 7-24.
76. J. P. Espada, The native language in teaching kindergarten mathematics. *Journal of International Education Research* **8**, 359 (2012).

77. L. M. Espinosa, Curriculum and assessment considerations for young children from culturally, linguistically, and economically diverse backgrounds. *Psychology in the Schools* **42**, 837 (2005).
78. D. B. Berch, M. M. M. Mazzocco, Eds., *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities*, (Paul H. Brooks, Baltimore, MD, 2007).
79. A. Dowker, *What works for children with mathematical difficulties? (Research Report No. 554)*. (University of Oxford/DfES, Nottingham, UK, 2004).
80. J. Lerner, *Learning disabilities*. (Houghton Mifflin Company, Boston, 1997).
81. D. C. Geary, Mathematics and learning disabilities. *Journal of Learning Disabilities* **37**, 4 (2004).
82. M. M. M. Mazzocco, G. F. Myers, Complexities in identifying and defining mathematics learning disability in the primary school-age years. *Annuals of Dyslexia* **53**, 218 (2003).
83. T. Mooij, G. Driessen, Differential ability and attainment in language and arithmetic of Dutch primary school pupils. *British Journal of Educational Psychology* **78**, 491 (Sep, 2008).
84. B. Wright, What number knowledge is possessed by children beginning the kindergarten year of school? *Mathematics Education Research Journal* **3**, 1 (1991).
85. C. Harrison, Giftedness in early childhood: The search for complexity and connection. *Roeper Review* **26**, 78 (2004).
86. E. Wadlington, J. M. Burns, Instructional practices within preschool/kindergarten gifted programs. *Journal for the Education of the Gifted* **17**, 41 (1993).
87. V. J. Molfese *et al.*, Examining associations between classroom environment and processes and early mathematics performance from pre-kindergarten to kindergarten. *Gifted Children* **5**, (2012).
88. J. Sarama, D. H. Clements, in *Applying implementation science in early childhood programs and systems,* T. G. Halle, A. J. Metz, I. Martinez-Beck, Eds. (Brookes, Baltimore, MD, 2013), pp. 173-191.
89. D. H. Clements, J. Sarama, C. B. Wolfe, M. E. Spitler, Longitudinal evaluation of a scale-up model for teaching mathematics with trajectories and technologies: Persistence of effects in the third year. *American Educational Research Journal* **50**, 812 (2013).
90. D. H. Clements, J. Sarama, in *Reconceptualizing early mathematics learning,* L. D. English, J. T. Mulligan, Eds. (Springer, Dordrecht, Germany, 2013), pp. 121-147.
91. J. Sarama, D. H. Clements, C. B. Wolfe, M. E. Spitler, Longitudinal evaluation of a scale-up model for teaching mathematics with trajectories and technologies. *Journal of Research on Educational Effectiveness* **5**, 105 (2012).
92. D. H. Clements, J. Sarama, M. E. Spitler, A. A. Lange, C. B. Wolfe, Mathematics learned by young children in an intervention based on learning trajectories: A large-scale cluster randomized trial. *JRME* **42**, 127 (2011).
93. C. Aydogan *et al.*, paper presented at the NAEYC, Washington, DC, April 2005.
94. Preschool Curriculum Evaluation Research Consortium, "Effects of preschool curriculum programs on school readiness (NCER 2008-2009)" (Government Printing Office, Washington, DC, 2008).
95. DHHS, *Head Start impact study: First year findings*. (U.S. Department of Health and Human Services; Administration for Children and Families, Washington, DC., 2005).

96. D. C. Farran, M. W. Lipsey, B. Watson, S. Hurley, paper presented at the American Educational Research Association, Chicago, IL, April 2007.
97. P. Munn, Mathematics in early childhood—The early years math curriculum in the UK and children's numerical development. *International Journal of Early Childhood* **38**, 99 (2006).
98. A. J. Baroody, M. Cibulskis, M.-l. Lai, X. Li, Comments on the use of learning trajectories in curriculum development and research. *Mathematical Thinking and Learning* **6**, 227 (2004).
99. A. J. Baroody, Y. Feil, A. R. Johnson, An alternative reconceptualization of procedural and conceptual knowledge. *JRME* **38**, 115 (2007).
100. D. H. Clements, Conference Working Group, in *Engaging young children in mathematics: Standards for early childhood mathematics education,* D. H. Clements, J. Sarama, A.-M. DiBiase, Eds. (Erlbaum, Mahwah, NJ, 2004), pp. 1-72.
101. National Council of Teachers of Mathematics, *Curriculum focal points for prekindergarten through grade 8 mathematics: A quest for coherence*. (Author, Reston, VA, 2006).
102. CCSSO/NGA, "Common core state standards for mathematics" (Council of Chief State School Officers and the National Governors Association Center for Best Practices, Washington, DC, 2010).
103. M. K. Gavin, T. M. Casa, J. L. Adelson, J. M. Firmender, The impact of challenging geometry and measurement units on the achievement of grade 2 students. *JRME* **44**, 478 (2013).
104. S. Dehaene, E. S. Spelke, P. Pinel, R. Stanescu, S. Tsivkin, Sources of mathematic thinking: Behavioral and brain-imaging evidence. *Science* **284**, 970 (1999).
105. D. C. Geary, An evolutionary perspective on learning disability in mathematics. *Developmental Neuropsychology* **32**, 471 (2007).
106. P. Pinel, D. Piazza, D. Le Bihan, S. Dehaene, Distributed and overlapping cerebral representations of number, size, and luminance during comparative judgments. *Neuron* **41**, 983 (2004).
107. M. Zorzi, K. Priftis, C. Umiltà, Brain damage – Neglect disrupts the mental number line. *Nature* **417**, 138 (2002).
108. J. Piaget, B. Inhelder, *The child's conception of space*. (W. W. Norton, New York, NY, 1967).
109. D. van Hiele-Geldof, in *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. van Hiele,* D. Fuys, D. Geddes, R. Tischler, Eds. (Brooklyn College, School of Education. (ERIC Document Reproduction Service No. 289 697), Brooklyn, NY, 1984), pp. 1-214.
110. P. M. van Hiele, *Structure and insight: A theory of mathematics education*. (Academic Press, Orlando, FL, 1986).
111. D. H. Clements, J. Sarama, *Learning and teaching early math: The learning trajectories approach*. (Routledge, New York, NY, ed. 2nd, 2014).
112. M. M. Lüken, Young children's structure sense. *J Math Didakt* **33**, 263 (2012/10/01, 2012).
113. A. Claessens, G. J. Duncan, M. Engel, "Kindergarten skills and fifth grade achievement: Evidence from the ECLS-K" (Northwestern University, Evanston, IL, 2007).

114. I. V. S. Mullis, M. O. Martin, P. Foy, A. Arora, "TIMSS 2011 International Results in Mathematics" (TIMSS & PIRLS International Study Center, Boston College, Chestnut Hill, MA, 2012).
115. R. C. Pianta, W. S. Barnett, M. R. Burchinal, K. R. Thornburg, The effects of preschool education: What we know, how public policy is or is not aligned with the evidence base, and what we need to know. *Psychological Science in the Public Interest* **10**, 49 (August 1, 2009, 2009).
116. H. P. Ginsburg, Mathematics learning disabilities: A view from developmental psychology. *Journal of Learning Disabilities* **30**, 20 (1997).
117. R. Gersten, N. C. Jordan, J. R. Flojo, Early identification and interventions for students with mathematical difficulties. *Journal of Learning Disabilities* **38**, 293 (2005).
118. A. J. Baroody, M. Eiland, Y. Su, B. Thompson, paper presented at the American Educational Research Association, Chicago, IL, April 2007.
119. R. Case, S. Griffin, W. M. Kelly, in *Developmental health and the wealth of nations,* D. P. Keating, C. Hertzman, Eds. (Guilford, New York, NY, 1999), pp. 125-149.
120. D. H. Clements, Curriculum research: Toward a framework for 'research-based curricula'. *JRME* **38**, 35 (2007).
121. D. H. Clements, J. Sarama, *Building Blocks—SRA Real Math Teacher's Edition, Grade PreK*. (SRA/McGraw-Hill, Columbus, OH, 2007).
122. D. H. Clements, J. Sarama, Experimental evaluation of the effects of a research-based preschool mathematics curriculum. *American Educational Research Journal* **45**, 443 (2008).
123. D. Stipek, R. Feiler, D. Daniels, S. Milburn, Effects of different instructional approaches on young children's achievement and motivation. *Child Development* **66**, 209 (1995).
124. B. A. Clarke, D. M. Clarke, M. Horne, in *Proceedings of the 30th Conference of the International Group for the Psychology in Mathematics Education,* J. Novotná, H. Moraová, M. Krátká, N. a. Stehlíková, Eds. (Charles University, Prague, Czecho, 2006), vol. 2, pp. 329-336.
125. G. Thomas, J. Ward, *An evaluation of the Count Me In Too pilot project*. (Ministry of Education, Wellington, NZ, 2001).
126. R. J. Wright, J. Martland, A. K. Stafford, G. Stanger, *Teaching number: Advancing children's skills and strategies*. (Paul Chapman/Sage, London, 2002).
127. D. H. Clements, J. Sarama, Learning trajectories in mathematics education. *Mathematical Thinking and Learning* **6**, 81 (2004).
128. J. Sarama, D. H. Clements, P. Starkey, A. Klein, A. Wakeley, Scaling up the implementation of a pre-kindergarten mathematics curriculum: Teaching for understanding with trajectories and technologies. *Journal of Research on Educational Effectiveness* **1**, 89 (2008).
129. J. C. Hiebert, D. A. Grouws, in *Second handbook of research on mathematics teaching and learning,* F. K. Lester, Jr., Ed. (Information Age Publishing, New York, NY, 2007), vol. 1, pp. 371-404.
130. R. Fish, Masters Thesis, University of Buffalo, State University of New York (2003).
131. R. C. Turner, G. W. Ritter, in *American Educational Research Association*. (San Diego, CA, 2004).

132. U.S. Dept. of Health and Human Services — Administration for Children and Families, "Head Start impact study. Final report" (Office of Planning, Research and Evaluation, U.S. Department of Health and Human Services, Washington, D.C., 2010).
133. J. Brooks-Gunn, Do you believe in magic? What we can expect from early childhood intervention programs. *Social Policy Report* **17**, 1; 3 (2003).
134. N. Bennett, C. Desforges, A. Cockburn, B. Wilkinson, *The quality of pupil learning experiences*. (Erlbaum, Hillsdale, NJ, 1984).
135. B. Thomas, "An abstract of kindergarten teachers' elicitation and utilization of children's prior knowledge in the teaching of shape concepts" (Unpublished manuscript, School of Education, Health, Nursing, and Arts Professions, New York University, 1982).
136. D. R. Entwisle, K. L. Alexander, Summer setback: Race, poverty, school composition, and mathematics achievement in the first two years of school. *American Sociological Review* **57**, 72 (1992).
137. C. S. Huntsinger, P. J. Jose, Chinese American and Caucasian American family interaction patterns in spatial rotation puzzle solutions. *Merrill-Palmer Quarterly* **41**, 471 (1995).
138. S. Jimerson, B. Egeland, A. Teo, A longitudinal study of achievement trajectories: Factors associated with change. *J. Ed. Psych.* **91**, 116 (1999).
139. M. W. Pratt, D. Green, J. MacVicar, M. Bountrogianni, The mathematical parent: Parental scaffolding, parent style, and learning outcomes in long-division mathematics homework. *Journal of Applied Developmental Psychology* **13**, 17 (1992).
140. G. A. Cook, L. A. Roggman, L. K. Boyce, Fathers' and mothers' cognitive stimulation in early play with toddlers: Predictors of 5th grade reading and math. *Family Science* **2**, 131 (2012).
141. E. A. Gunderson, S. C. Levine, Some types of parent number talk count more than others: Relation between parents' input and children's number knowledge. *Developmental Science* **14**, 1021 (2011).
142. S. C. Levine, L. W. Suriyakham, M. L. Rowe, J. Huttenlocher, E. A. Gunderson, What counts in the development of young children's number knowledge? *Developmental Psychology* **46**, 1309 (2010).
143. A. Anderson, J. Anderson, J. Shapiro, Mathematical discourse in shared storybook reading. *JRME* **35**, 5 (2004).
144. J. P. Benigno, S. Ellis, Two is greater than three: Effects of older siblings on parental support of preschoolers' counting in middle-income families. *Early Childhood Research Quarterly* **19**, 4 (2004).
145. J.-A. LeFevre, E. Polyzoi, S.-L. Skwarchuk, L. Fast, C. Sowinskia, Do home numeracy and literacy practices of Greek and Canadian parents predict the numeracy skills of kindergarten children? *International Journal of Early Years Education* **18**, 55 (2010).
146. S. M. Pruden, S. C. Levine, J. Huttenlocher, Children's spatial thinking: Does talk about the spatial world matter? . *Developmental Science* **14**, 1417 (2011).
147. S. C. Levine, K. R. Ratliff, J. Huttenlocher, J. Cannon, Early puzzle play: A predictor of preschoolers' spatial transformation skill. *Developmental Psychology* **48**, 530 (2012).
148. K. E. McFadden, C. S. Tamis-LeMonda, N. J. Cabrera, Quality matters: Low-income fathers' engagement in learning activities in early childhood predict children's academic performance in fifth grade. *Family Science* **2**, 120 (2011).

149. L. M. McKelvey *et al.*, Father teaching interactions with toddlers at risk: Associations with later child academic outcomes. *Family Science* **2**, 146 (2011).
150. R. Crosnoe, C. E. Cooper, Economically disadvantaged children's transitions into elementary school: Linking family processes, school contexts, and educational policy. *American Educational Research Journal* **47**, 258 (2010).
151. M. Vandermaas-Peeler, E. Boomgarden, L. Finn, C. Pittard, Parental support of numeracy during a cooking activity with four-year-olds. *International Journal of Early Years Education* **20**, 78 (2012).
152. D. C. Geary, A componential analysis of an early learning deficit in mathematics. *Journal of Experimental Child Psychology* **49**, 363 (1990).
153. A. Gervasoni, T. Hadden, K. Turkenburg, in *Mathematics: Essential research, essential practice (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia),* J. Watson, K. Beswick, Eds. (MERGA, Hobart, Australia, 2007), vol. 3, pp. 305-314.
154. S. Baker, R. Gersten, D.-S. Lee, A synthesis of empirical research on teaching mathematics to low-achieving students. *Elementary School Journal* **103**, 51 (2002).
155. D. P. Bryant, B. R. Bryant, R. Gersten, N. Scammacca, M. M. Chavez, Mathematics intervention for first- and second-grade students with mathematics difficulties: The effects of Tier 2 intervention delivered as booster lessons. *Remedial & Special Education* **29**, 20 (2008).
156. A. Dowker, *What works for children with mathematical difficulties? The effectiveness of intervention schemes*. (Nottingham, UK, DCSF Publications, 2009).
157. A. Dowker, G. Sigley, Targeted interventions for children with arithmetical difficulties. *British Journal of Educational Psychology Monographs* **II**, 65 (2010).
158. L. S. Fuchs, D. Fuchs, Principles for the prevention and intervention of mathematics difficulties. *Learning Disabilities Research & Practice* **16**, 85 (2001).
159. R. Gersten *et al.*, "Teaching mathematics to students with learning disabilities: A meta-analysis of the intervention research" (RMC Research Corporation, Center on Instruction, Portsmouth, NH, 2008).
160. A. J. Bishop, H. J. Forgasz, in *Second handbook of research on mathematics teaching and learning,* F. K. Lester, Jr., Ed. (Information Age Publishing, New York, NY, 2007), vol. 2, pp. 1145-1167.
161. S. R. Powell, L. S. Fuchs, D. Fuchs, Reaching the mountaintop: Addressing the common core standards in mathematics for students with mathematical disabilities. *Learning Disabilities Research & Practice* **28**, 38 (2013).
162. W. James, *Talks to teachers on psychology: And to students on some of life's ideas*. (Norton, New York, NY, 1892/1958).
163. A. J. Baroody, in *Pathways to number: Developing numerical abilities,* J. Bideaud, C. Meljac, J.-P. Fischer, Eds. (Erlbaum, Mahwah, NJ, 1992), pp. 99-126.
164. A. J. Baroody, H. P. Ginsburg, in *Conceptual and procedural knowledge: The case of mathematics,* J. Hiebert, Ed. (Erlbaum, Hillsdale, NJ, 1986), pp. 75–112.
165. A. J. Baroody, in *The development of arithmetic concepts and skills: Constructing adaptive expertise,* A. J. Baroody, A. Dowker, Eds. (Erlbaum, Mahwah, NJ, 2003), pp. 1-33.
166. A. W. Blöte, E. van der Burg, A. S. Klein, Students' flexibility in solving two-digit addition and subtraction problems: Instruction effects. *J. Ed. Psych.* **93**, 627 (2001).

167. K. C. Fuson, D. J. Briars, Using a base-ten blocks learning/teaching approach for first- and second-grade place-value and multidigit addition and subtraction. *JRME* **21**, 180 (1990).
168. B. Rittle-Johnson, M. W. Alibali, Conceptual and procedural knowledge of mathematics: Does one lead to the other? *J. Ed. Psych.* **91**, 175 (1999).
169. B. Rittle-Johnson, R. S. Siegler, M. W. Alibali, Developing conceptual understanding and procedural skill in mathematics: An iterative process. *J. Ed. Psych.* **93**, 346 (2001).
170. J. Piaget, *The science of education and the psychology of the child*. (Orion Press, New York, NY, 1970).
171. H. P. Ginsburg, A. Klein, P. Starkey, in *Handbook of child psychology. Volume 4: Child psychology in practice,* W. Damon, I. E. Sigel, K. A. Renninger, Eds. (John Wiley & Sons, New York, NY, 1998), pp. 401-476.
172. A. J. Baroody, A. Dowker, *The development of arithmetic concepts and skills: Constructing adaptive expertise*. (Erlbaum, Mahwah, NJ, 2003).
173. M.-K. Lerkkanen *et al.*, The role of teaching practices in the development of children's interest in reading and mathematics in kindergarten. *Contemporary Educational Psychology* **37**, 266 (2012).
174. C. Blair, R. P. Razza, Relating effortful control, executive function, and false belief understanding to emerging math and literacy ability in kindergarten. *Child Development* **78**, 647 (2007).
175. P. A. Kirschner, J. Sweller, R. E. Clark, Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist* **41**, 75 (2006).
176. L. Alfieri, P. J. Brooks, N. J. Aldrich, H. R. Tenenbaum, Does discovery-based instruction enhance learning? *J. Ed. Psych.* **103**, 1 (2010).
177. A. Levesque, (Master's thesis), Concordia University (2010).
178. A. J. Baroody, Manipulatives don't come with guarantees. *Arithmetic Teacher* **37**, 4 (1989).
179. D. H. Clements, S. McMillen, Rethinking "concrete" manipulatives. *Teaching Children Mathematics* **2**, 270 (1996).
180. J. Sarama, D. H. Clements, ''Concrete'' computer manipulatives in mathematics education. *Child Development Perspectives* **3**, 145 (2009).
181. A. Palmer, A. J. Baroody, Blake's development of the number words "one," "two," and "three". *Cognition and Instruction* **29**, 265 (2011).
182. S. Hattikudur, M. Alibali, Learning about the equal sign: Does comparing with inequality symbols help? *Journal of Experimental Child Psychology* **107**, 15 (2010).
183. B. Rittle-Johnson, J. R. Star, Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *J. Ed. Psych.* **99**, 561 (2007).
184. E. Reikerås, I. K. Løge, A.-M. Knivsberg, The mathematical competencies of toddlers expressed in their play and daily life activities in Norwegian kindergartens. *International Journal of Early Childhood*, (2012).
185. K.-H. Seo, H. P. Ginsburg, in *Engaging young children in mathematics: Standards for early childhood mathematics education,* D. H. Clements, J. Sarama, A.-M. DiBiase, Eds. (Erlbaum, Mahwah, NJ, 2004), pp. 91-104.

186. J. Sarama, D. H. Clements, in *Contemporary debates in childhood education and development,* S. Suggate, E. Reese, Eds. (Routledge, New York, NY, 2012), pp. 71-80.
187. N. C. Chien *et al.*, Children's classroom engagement and school readiness gains in prekindergarten. *Child Development* **81**, 1534 (2010).
188. A. Klein, P. Starkey, D. H. Clements, J. Sarama, R. Iyer, Effects of a pre-kindergarten mathematics intervention: A randomized experiment. *Journal of Research on Educational Effectiveness* **1**, 155 (2008).
189. B. van Oers, Are you sure? Stimulating mathematical thinking during young children's play. *European Early Childhood Education Research Journal* **4**, 71 (1996).
190. B. van Oers, M. Poland, in *Developmental Education for Young Children* B. van Oers, Ed. (Springer Netherlands, 2012), vol. 7, pp. 121-136.
191. L. P. Steffe, H. G. Wiegel, Cognitive play and mathematical learning in computer microworlds. *Journal of Research in Childhood Education* **8**, 117 (1994).
192. H. P. Ginsburg, J. S. Lee, J. Stevenson-Boyd, Mathematics education for young children: What it is and how to promote it. *Social Policy Report* **22**, 1 (2008).
193. J. Lee, Correlations between kindergarten teachers' attitudes toward mathematics and teaching practice. *Journal of Early Childhood Teacher Education* **25**, 173 (2004).
194. B. Moseley, Pre-service early childhood educators' perceptions of math-mediated language. *Early Education & Development* **16**, 385 (2005).
195. L. Malaguzzi, *Shoe and meter*. (Reggio Children, Reggio Emilia, Italy, 1997).
196. K. C. Fuson, in *Engaging young children in mathematics: Standards for early childhood mathematics education,* D. H. Clements, J. Sarama, A.-M. DiBiase, Eds. (Erlbaum, Mahwah, NJ, 2004), pp. 105-148.
197. M. M. Lindquist, J. N. Joyner, in *Engaging young children in mathematics: Standards for early childhood mathematics education,* D. H. Clements, J. Sarama, A.-M. DiBiase, Eds. (Erlbaum, Mahwah, NJ, 2004), pp. 449-455.
198. V. J. Henry, R. S. Brown, First-grade basic facts: An investigation into teaching and learning of an accelerated, high-demand memorization standard. *JRME* **39**, 153 (March, 2008).
199. P. Black, D. Wiliam, Assessment and classroom leanring. *Assessment in education: Principles, Policy & Practice* **5**, 7 (1998).
200. L. A. Shepard, in *Preparing teachers for a changing world,* L. Darling-Hammond, J. Bransford, Eds. (Jossey-Bass, San. Francisco, 2005), pp. 275-326.
201. C. Aubrey, S. Dahl, R. Godfrey, Early mathematics development and later achievement: Further evidence. *Education Research Journal* **18**, 27 (2006).
202. N. C. Jordan, D. Kaplan, C. Ramineni, M. N. Locuniak, Early math matters: Kindergarten number competence and later mathematics outcomes. *Developmental Psychology* **45**, 850 (2009).
203. M. N. Locuniak, N. C. Jordan, Using kindergarten number sense to predict calculation fluency in second grade. *Journal of Learning Disabilities* **41** 451 (2008).
204. M. M. M. Mazzocco, R. E. Thompson, Kindergarten predictors of math learning disability. *Learning Disability Quarterly Research and Practice* **20**, 142 (2005).
205. A. Claessens, M. Engel, How important is where you start? Early mathematics knowledge and later school success. *Teachers College Record* **115**, 1 (April, 2011).
206. C. Sophian, Learning about what fits: Preschool children's reasoning about effects of object size. *JRME* **33**, 290 (2002).

207. P. Aunio, C. Aubrey, R. Godfrey, Y. Pan, Y. Liu, Children's early numeracy in England, Finland and People's Republic of China. *International Journal of Early Years Education* **16**, 203 (2008).
208. I. T. Miura, Y. Okamoto, in *The development of arithmetic concepts and skills: Constructing adaptive expertise,* A. J. Baroody, A. Dowker, Eds. (Erlbaum, Mahwah, NJ, 2003), pp. 229-242.
209. X. Li, Y. Sun, A. J. Baroody, D. J. Purpura, The effect of language on Chinese and American 2- and 3-Year olds' small number identification. *European Journal of Psychology of Education*, (in press).
210. B. W. Sarnecka, V. G. Kamenskaya, Y. Yamana, T. Ogura, Y. B. Yudovina, From grammatical number to exact numbers: early meanings of "one", "two", and "three" in English, Russian, and Japanese. *Cognitive Psychology* **55**, 136 (2007).
211. D. J. Purpura, L. E. Hume, D. M. Sims, C. J. Lonigan, Early literacy and early numeracy: The value of including early literacy skills in the prediction of numeracy development. *Journal of Experimental Child Psychology* **110**, 647 (2011).
212. R. S. Klibanoff, S. C. Levine, J. Huttenlocher, M. Vasilyeva, L. V. Hedges, Preschool children's mathematical knowledge: The effect of teacher "math talk". *Developmental Psychology* **42**, 59 (2006).
213. K. C. Fuson, A. Murata, Integrating NRC principles and the NCTM process standards to form a class learning path model that individualizes within whole-class activities. *NCSM Journal of Mathematics Education Leadership* **10**, 72 (Spring, 2007).
214. R. Gersten *et al.*, in *Foundations for Success: The Final Report of the National Mathematics Advisory Panel,* National Mathematics Advisory Panel, Ed. (U.S. Department of Education, Office of Planning, Evaluation and Policy Development, Washington D.C., 2008).
215. L. S. Fuchs, D. Fuchs, K. Karns, Enhancing kindergartners' mathematical development: Effects of peer-assisted learning strategies. *Elementary School Journal* **101**, 495 (2001).
216. J. W. Fantuzzo, V. L. Gadsden, P. A. McDermott, An integrated curriculum to improve mathematics, language, and literacy for Head Start Children. *American Educational Research Journal* **48**, 763 (2011).
217. T. Koponen, K. Aunola, T. Ahonen, J.-E. Nurmi, Cognitive predictors of single-digit and procedural calculation and their covariation with reading skill. *Journal of Experimental Child Psychology* **97**, 220 (2007).
218. R. K. Vukovic, Mathematics difficulty with and without reading difficulty: Findings and implications from a four-year longitudinal study. *Exceptional Children* **78**, 280 (2012).
219. D. H. Clements, J. Sarama, in *Instructional strategies for improving student learning: Focus on early mathematics and reading,* J. S. Carlson, J. R. Levine, Eds. (Information Age Publishing, Charlotte, NC, 2012), vol. 3 of Psychological perspectives on contemporary educational issues, pp. 107-162.
220. J. Fryer, Roland G. , S. D. Levitt, Understanding the black-white test score gap in the first two years of school. *The Review of Economics and Statistics* **86**, 447 (2004).
221. J. Sarama, D. H. Clements, in *Contemporary perspectives on mathematics in early childhood education* O. N. Saracho, B. Spodek, Eds. (Information Age Publishing, Charlotte, NC, 2008), pp. 67-94.

222. M. R. Burchinal, S. Field, M. L. López, C. Howes, R. Pianta, Instruction in Spanish in pre-kindergarten classrooms and child outcomes for English language learners. *Early Childhood Research Quarterly* **27**, 188 (2012).
223. A. Shamir, I. Lifshitz, E-Books for supporting the emergent literacy and emergent math of children at risk for learning disabilities: can metacognitive guidance make a difference? *European Journal of Special Needs Education* **28**, 33 (2013/02/01, 2012).
224. E. E. Turner, S. Celedón-Pattichis, M. E. Marshall, in *Promoting high participation and success in mathematics by Hispanic students: Examining opportunities and probing promising practices,* R. S. Kitchen, E. A. Silver, Eds. (2008), vol. 1, pp. 19-42.
225. E. E. Turner, S. Celedón-Pattichis, Problem solving and mathematical discourse among Latino/a kindergarten students: An analysis of opportunities to learn. Journal of Latinos and Education,. *Promoting high participation and success in mathematics by Hispanic students: Examining opportunities and probing promising practices* **10**, 146 (2011).
226. L. S. Fuchs *et al.*, Effects of first-grade number knowledge tutoring with contrasting forms of practice. *J. Ed. Psych.* **105**, 58 (2013).
227. N. González, R. Andrade, M. Civil, L. Moll, Bridging funds of distributed knowledge: Creating zones of practices in mathematics. *Journal of Education for Students Placed at Risk* **6**, 115 (2001).
228. T. Kleemans, E. Segers, L. Verhoeven, Relations between home numeracy experiences and basic calculation skills of children with and without specific language impairment. *Early Childhood Research Quarterly* **28**, 415 (2013).
229. B. K. Hamre, R. C. Pianta, Early teacher-child relationships and the trajectory of children's school outcomes through eighth grade. *Child Development* **72**, 625 (2001).
230. T. R. Konold, R. C. Pianta, Empirically-derived, person-oriented patterns of school readiness in typically-developing children: Description and prediction to first-grade achievement. *Applied Developmental Science* **9**, 174 (2005).
231. J. R. Best, P. H. Miller, J. A. Naglieri, Relations between executive function and academic achievement from ages 5 to 17 in a large, representative national sample. *Learning and Individual Differences* **21**, 327 (2011).
232. C. Blair, School readiness: Integrating cognition and emotion in a neurobiological conceptualization of children's functioning at school entry. *American Psychologist* **57**, 111 (2002).
233. R. Bull, G. Scerif, Executive functioning as a predictor of children's mathematics ability: Inhibition, switching, and working memory. *Developmental Neuropsychology* **19**, 273 (2001).
234. C. A. Clark, V. E. Pritchard, L. J. Woodward, Preschool executive functioning abilities predict early mathematics achievement. *Dev Psychol* **46**, 1176 (Sep, 2010).
235. J.-A. LeFevre *et al.*, The role of executive attention in the acquisition of mathematical skills for children in grades 2 through 4. *Journal of Experimental Child Psychology* **114**, 243 (2013).
236. L. S. Pagani, C. Fitzpatrick, S. Parent, Relating kindergarten attention to subsequent developmental pathways of classroom engagement in elementary school. *J Abnorm Child Psychol*, (Jan 11, 2012).
237. C. C. Ponitz, M. M. McClelland, J. S. Matthews, F. J. Morrison, A structured observation of behavioral self-regulation and Its contribution to kindergarten outcomes. *Developmental Psychology* **45**, 605 (2009).

238. D. C. Geary, Cognitive addition: A short longitudinal study of strategy choice and speed-of-processing differences in normal and mathematically disabled children. *Developmental Psychology* **47**, 1539 (2011).
239. X. Lan, C. H. Legare, C. C. Ponitz, S. Li, F. J. Morrison, Investigating the links between the subcomponents of executive function and academic achievement: A cross-cultural analysis of Chinese and American preschoolers. *Journal of Experimental Child Psychology* **108**, 677 (2011).
240. M. F. Maier, D. B. Greenfield, paper presented at the Institute of Education Science 2007 Research Conference, Washinton, DC, June 2008.
241. K. Lee *et al.*, The cognitive underpinnings of emerging mathematical skills: Executive functioning, patterns, numeracy, and arithmetic. *British Journal of Educational Psychology* **82**, 82 (2012).
242. M. C. Passolunghi, S. Lanfranchi, Domain-specific and domain-general precursors of mathematical achievement: A longitudinal study from kindergarten to first grade. *British Journal of Educational Psychology* **82**, 42 (2012).
243. E. Lehtinen, M. M. Hannula, in *Instructional psychology: Past, present and future trends. Fifteen essays in honour of Erik De Corte,* L. Verschaffel, F. Dochy, M. Boekaerts, S. Vosniadou, Eds. (Elsevier, Amsterdam, 2006), vol. 49, pp. 39-55.
244. K. Bodovski, M.-J. Youn, The long term effects of early acquired skills and behaviors on young children's achievement in literacy and mathematics. *Journal of Early Childhood Research* **9**, 4 (February 1, 2011, 2011).
245. D. C. Geary, in *Handbook of learning disabilities,* H. L. Swanson, K. Harris, S. Graham, Eds. (Guilford, New York, NY, 2003).
246. S. W. M. Toll, S. Van der Ven, E. Kroesbergen, J. E. H. Van Luit, Executive functions as predictors of math learning disabilities. *Journal of Learning Disabilities* **20**, 1 (December, 2010).
247. N. C. Jordan, L. B. Hanich, D. Kaplan, A longitudinal study of mathematical competencies in children with specific mathematics difficulties versus children with comorbid mathematics and reading difficulties. *Child Development* **74**, 834 (2003).
248. M. H. Ashcraft, in *The Neural Basis of Mathematical Development*. (Vanderbilt University, Nashville, TN, 2006).
249. D. C. Geary, M. K. Hoard, C. O. Hamson, Numerical and arithmetical cognition: Patterns of functions and deficits in children at risk for a mathematical disability. *Journal of Experimental Child Psychology* **74**, 213 (1999).
250. N. Cowan, J. S. Saults, E. M. Elliott, The search for what is fundamental in the development of working memory. *Advances in Child Development and Behavior* **29**, 1 (2002).
251. R. M. Shiffrin, W. Schneider, Controlled and automatic human information processing: II. Perceptual learning, automatic attending, and a general theory. *Psychological Review* **84**, 127 (1984).
252. T. K. Nishida, A. S. Lillard, in *Society for Research in Child Development*. (Boston, MA, 2007).
253. J. Dewey, *Experience and education*. (Simon & Schuster, New York, NY, 1938/1997).
254. T. P. Carpenter, E. H. Fennema, P. L. Peterson, D. A. Carey, Teacher's pedagogical content knowledge of students' problem solving in elementary arithmetic. *JRME* **19**, 385 (1988).

255. M. Kennedy, *Form and substance of inservice teacher education (Research Monograph No. 13).* (National Institute for Science Education, University of Wisconsin–Madison, Madison, WI, 1998).
256. P. L. Peterson, T. P. Carpenter, E. H. Fennema, Teachers' knowledge of students' knowledge in mathematics problem solving: Correlational and case analyses. *J. Ed. Psych.* **81**, 558 (1989).
257. K. S. Yoon, T. Duncan, S. W.-Y. Lee, B. Scarloss, K. L. Shapley, "Reviewing the evidence on how teacher professional development affects student achievement (Issues & Answers Report, REL 2007–No. 033)." (U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance, Regional Educational Laboratory Southwest, Washington, DC, 2007).
258. M. H. Ashcraft, E. P. Kirk, D. Hopko, in *The development of mathematical skills,* **C. Donlan**, Ed. (Psychology Press, Hove, U.K., 1998).
259. H. C. Hill, B. Rowan, D. L. Ball, Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal* **42**, 371 (2005).
260. S. M. Wilson, S. T. Lubienski, S. L. Mattson, paper presented at the American Educational Research Association, New York, NY, 1996.
261. P. S. Wilson, Wilson, P. (1999). Thoughts about professional development from a mathematics education perspective.

Pat Wilson. Retrieved from: <http://conferences.illinoisstate.edu/NSA/papers/Wilson.pdf%3E, 1999.
262. H. C. Hill, S. G. Schilling, D. L. Ball, Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal* 11 (2004).
263. S. LeMoine, *Workforce designs: A policy blueprint for state early childhood professional development systems*. (National Association for the Education of Young Children, Washington, DC, 2008).
264. J. S. Lee, H. P. Ginsburg, What is appropriate mathematics education for four-year-olds? *Journal of Early Childhood Research* **5**, 2 (2007).
265. R. Balfanz, in *Mathematics in the early years,* J. V. Copley, Ed. (National Council of Teachers of Mathematics, Reston, VA, 1999), pp. 3-10.
266. J. S. Lee, Preschool teachers' shared beliefs about appropriate pedagogy for 4-year-olds. *Early Childhood Education Journal* **33**, 433 (2006).
267. J. S. Lee, H. P. Ginsburg, Preschool teachers' beliefs about appropriate early literacy and mathematics education for low- and middle-socioeconomic status children. *Early Education & Development* **18**, 111 (2007).
268. D. L. Ball, in *Advances in research on teaching,* J. E. Brophy, Ed. (JAI Press, Greenwich, CT, 1991), vol. 2, pp. 1-48.
269. D. L. Ball, H. C. Hill, H. Bass, in *American Educator*. (2005), pp. 14-22; 43-46.
270. M. Lampert, When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal* **27**, 29 (1990).
271. G. Leinhardt, D. E. Smith, Expertise in mathematics instruction: Subject matter knowledge. *J. Ed. Psych.* **77**, 247 (1985).

272. L. Ma, *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. (Erlbaum, Mahwah, NJ, 1999).
273. M. A. Simon, Prospective elementary teachers' knowledge of division. *JRME* **24**, 233 (1993).
274. D. L. Ball, Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education* **51**, 241 (2000).
275. L. S. Shulman, Those who understand: Knowledge growth in teaching. *Educational Researcher* **15**, 4 (1986).
276. L. S. Shulman, Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review* **57**, 1 (1987).
277. L. R. Davenport, C. M. Grant, M. A. Carter, J. Gorman, J. Mark, paper presented at the National Council of Teachers of Mathematics, St. Louis, MO, 2006.
278. D. B. Erchick *et al.*, paper presented at the National Council of Teachers of Mathematics, Atlanta, GA, 2007.
279. L. West, F. C. Staub, *Content-focused coaching: Transforming mathematics lessons*. (Heinneman, Portsmouth, NH:, 2003).
280. A. J. Baroody, R. T. Coslick, *Fostering children's mathematical power: An investigative approach to K-8 mathematics instruction*. (Erlbaum, Mahwah, NJ, 1998).
281. A. J. Baroody, in *Engaging young children in mathematics: Standards for early childhood mathematics education,* D. H. Clements, J. Sarama, A.-M. DiBiase, Eds. (Erlbaum, Mahwah, NJ, 2004), pp. 149-172.
282. R. A. Philipp *et al.*, Effects of early field experiences on the mathematical content knowledge and beliefs of prospective elementary school teachers: An experimental study. *JRME* **38**, 438 (2007).
283. D. I. Cassidy, M. J. Buell, S. Pugh-Hoese, S. Russell, The effect of education on child care teachers' beliefs and classroom quality: Year one evaluation of the TEACH early childhood associate degree scholarship program. *Early Childhood Research Quarterly* **10**, 171 (1995).
284. D. J. Stipek, P. Byler, Early childhood teachers: Do they practice what they preach? *Early Childhood Research Quarterly* **12**, 305 (1997).
285. S. J. Bodilly, *Lessons from New American Schools' scale-up phase*. (RAND Education, Santa Monica, CA, 1998).
286. G. D. Borman, G. M. Hewes, L. T. Overman, S. Brown, Comprehensive school reform and achievement: A meta-analysis. *Review of Educational Research* **73**, 125 (2003).
287. D. Clarke, in *Professional development for teachers of mathematics,* D. B. Aichele, A. F. Coxford, Eds. (National Council of Teachers of Mathematics, Reston, VA, 1994), pp. 37-48.
288. D. K. Cohen, in *Rewards and reforms: Creating educational incentives that work,* S. H. Fuhrman, J. A. O'Day, Eds. (Jossey Bass, San Francisco, CA, 1996), pp. 61-112.
289. R. F. Elmore, Getting to scale with good educational practices. *Harvard Educational Review* **66**, 1 (1996).
290. M. S. Garet, A. C. Porter, L. Desimone, B. F. Birman, K. S. Yoon, What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal* **38**, 915 (2001).

291. T. R. Guskey, Ed., *Evaluating professional development*, (Corwin Press, Thousand Oaks, CA, 2000).
292. G. E. Hall, S. M. Hord, *Implementing change: Patterns, principles, and potholes*. (Allyn and Bacon, Boston, MA, 2001).
293. J. S. Kaser, P. S. Bourexis, S. Loucks-Horsley, S. A. Raizen, *Enhancing program quality in science and mathematics*. (Corwin Press, Thousand Oaks, CA, 1999).
294. J. K. Klingner, S. Ahwee, P. Pilonieta, R. Menendez, Barriers and facilitators in scaling up research-based practices. *Exceptional Children* **69**, 411 (2003).
295. H. L. Schoen, K. J. Cebulla, K. F. Finn, C. Fi, Teacher variables that relate to student achievement when using a standards-based curriculum. *JRME* **34**, 228 (2003).
296. B. Showers, B. Joyce, B. Bennett, Synthesis of research on staff development: A framework for future study and a state-of-the-art analysis. *Educational Leadership* **45**, 77 (1987).
297. R. Gelman, E. M. Williams, in *Cognition, perception, and language. Volume 2: Handbook of Child Psychology,* D. Kuhn, R. Siegler, Eds. (John Wiley & Sons, New York, NY, 1997), pp. 575-630.
298. E. J. Gibson, *Principles of perceptual learning and development*. (Appleton-Century-Crofts, Meredith Corporation, New York, NY, 1969).
299. S. A. Lee, E. S. Spelke, G. Vallortigara, Chicks, like children, spontaneously reorient by three-dimensional environmental geometry, not by image matching. *Biology Letters* **8**, 492 (Aug 23, 2012).
300. G. Vallortigara, V. A. Sovrano, C. Chiandetti, Doing Socrates experiment right: controlled rearing studies of geometrical knowledge in animals. *Current Opinion in Neurobiology* **19**, 20 (Feb, 2009).
301. J. Sarama, D. H. Clements, S. Swaminathan, S. McMillen, R. M. González Gómez, Development of mathematical concepts of two-dimensional space in grid environments: An exploratory study. *Cognition and Instruction* **21**, 285 (2003).
302. D. H. Clements, M. T. Battista, J. Sarama, Logo and geometry. *Journal for Research in Mathematics Education Monograph Series* **10**, (2001).
303. N. J. Cepeda, H. Pashler, E. Vul, J. T. Wixted, D. Rohrer, Distributed practice in verbal recall tasks: A review and quantitative synthesis. *Psychological Bulletin* **132**, 354 (2006).
304. W. S. Barnett, D. J. Yarosz, J. Thomas, A. Hornbeck, "Educational effectiveness of a Vygotskian approach to preschool education: A randomized trial" (National Institute of Early Education Research, 2006).
305. E. Bodrova, D. J. Leong, J. Norford, D. Paynter, It only looks like child's play. *Journal of Staff Development* **24**, 47 (2003).
306. A. Diamond, W. S. Barnett, J. Thomas, S. Munro, Preschool program improves cognitive control. *Science* **318**, 1387 (2007).