"Concrete" Computer Manipulatives in Mathematics Education

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ABSTRACT—*The use of "concrete manipulatives" in mathematics education is supported by research and often accepted as a sine qua non of "reform" approaches. This article reviews the research on the use of manipulatives and critiques common notions regarding concrete manipulatives. It presents a reformulation of the definition of concrete as used in educational psychology and educational research and provides a rationale of how, based on that reformulation, computer manipulatives may be pedagogically efficacious. The article presents 7 hypothesized, interrelated affordances of manipulatives and briefly reviews evidence for their empirical validity.*

KEYWORDS—*mathematic education; child development; manipulatives; cognition; abstract; technology; computers; concrete models*

The notion of "concrete," from concrete manipulatives to pedagogical sequences such as "concrete to abstract," is embedded in educational theories, research, and practice, especially in mathematics education (see Kaminski, Sloutsky, & Heckler, 2009; Martin, Lukong, & Reaves, 2007; McNeil & Uttal, 2009). Like many widely accepted notions that have a good deal of truth behind them, this one has become nearly immune from critical reflection. In this article, we consider research on the use of mathematics manipulatives—for example, "colored counters, miscellaneous 'junk' items, patterning material, blocks of various colors, shapes and sizes, linking cubes, and base-ten blocks" (English, 2004, pp. 205–206)—in early and elementary education and offer a critique of common notions concerning concrete manipulatives and concrete ideas. From a reformulation of these notions, we reconsider the role computer manipulatives may play in helping students learn mathematics.

EARLY RESEARCH ON MANIPULATIVES

Early research on mathematics learning with manipulatives supported the notion that students who use manipulatives in their mathematics classes usually outperform those who do not (Driscoll, 1983; Greabell, 1978; Johnson, 2000; Lamon & Huber, 1971; Raphael & Wahlstrom, 1989; Sowell, 1989; Suydam, 1986). The studies also showed an increase in scores on retention and problem-solving tests.

Not all the early research was confirmative, however. Fennema (1972), for example, found that on a test of transfer, students who did not use Cuisenaire (colored) rods to learn multiplication as repeated addition outperformed students randomly assigned to an instructional intervention that used manipulatives. Other researchers reported a lack of connection between students' representations; for example, students did not connect the arithmetic they performed with manipulatives to their paper-and-pencil computations (Resnick & Omanson, 1987; Thompson & Thompson, 1990). Taken together, this early research, supported by subsequent studies (e.g., Gagatsis, 2003; Martin et al., 2007; Uttal, O'Doherty, Newland, Hand, & DeLoache, 2009), suggests that instruction should begin "concretely" but also that

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manipulatives are not sufficient to guarantee meaningful learning. It is important to define what various people mean by the term *concrete*.

THE MEANING OF ''CONCRETE'' IN EDUCATION

By "concrete," most practitioners mean physical objects that students can grasp with their hands. This sensory nature ostensibly makes manipulatives "real," connected with one's intuitively meaningful personal self, and therefore helpful. There are, however, problems with this view (cf. Metz, 1995).

First, it cannot be assumed that concepts can be "read off" manipulatives. That is, the physical objects may be manipulated without illuminating the concepts. Second, even if children begin to make connections between manipulatives and nascent ideas, physical actions with certain manipulatives may suggest different mental actions than those students are to learn. For example, such a mismatch occurs when students use number lines to perform addition. When adding 6 + 3, students locate 6, count "one, two, three," and read the answer, "9." This usually does not help them solve the problem mentally, for to do so they have to count "seven, eight, nine" and at the same time *count the counts*—7 is one, 8 is two, and so on. These actions are quite different (Gravemeijer, 1991; see also Sarama & Clements, 2009).

Therefore, although manipulatives have an important place in learning, their physicality does not carry—and may not even be essential in supporting—the meaning of the mathematical idea. Students may require physically concrete materials to build meaning initially, but they must *reflect* on their *actions* with manipulatives to do so. When educators speak of concrete understanding, they are not always referring to physicality. Teachers of intermediate and later grades expect students to have a kind of "concrete" understanding that goes beyond manipulatives. Quantities themselves are mentally manipulated as if they were concrete objects. For example, a child who is mentally adding 43 and 38 could operate on mental representations of physical objects—"I took 4 tens blocks and 3 tens blocks and put them together and then took the 11 ones blocks and made that 1 more ten and 1 left . . . so 81." A children with a higher level "concrete" understanding of the quantities may "break apart" the number itself, as in "I took 2 off the 43 and put it on the 38 and that made 40 . . . so 80, but one left, so 81" (Cobb, Perlwitz, & Underwood, 1996; Sarama & Clements, 2009). Such "concrete" understandings are not direct analogs. They emerge from complex semantic webs that connect numbers and number relations to meaningful experiences with physical and social contexts. Research indicates that such concrete connections support arithmetic and algebraic thinking (Schliemann, Carraher, & Brizuela, 2007).

We posit that there are two different types of concrete knowledge. Students with *sensory-concrete* knowledge need sensory material to make sense of a concept or procedure. For example, most children do not solve larger number problems without the support of concrete objects until 5.5 years of age (Levine, Jordan, & Huttenlocher, 1992). Young children can solve complex equivalence problems with manipulatives when presented in a nonsymbolic context (Sherman & Bisanz, 2009), and may not be able to solve even the simplest of problems without such physical, concrete support (Baroody, Eiland, Su, & Thompson, 2007). The physical material supports students' action schemes (Correa, Nunes, & Bryant, 1998; see also, Martin, 2009). *Abstract* knowledge is generalized, often symbolic, knowledge. Some theorists have argued that the abstract has primacy over the concrete, for example, that to recognize an object as an instance of an abstraction, the person must already possess that abstraction (Lehtinen & Hannula, 2006). Whether or not one adopts that position, most people accept that mathematics is an abstract subject matter area, with generalizations (e.g., the meaning of "three") at its core.

Integrated-concrete knowledge is knowledge that is *connected* in special ways (Clements & McMillen, 1996). What gives sidewalk concrete its strength is the combination of separate particles in an interconnected mass (the etymology of *concrete* is *to grow together*). What gives integrated-concrete thinking its strength is the combination of separate ideas in an interconnected structure of knowledge. For students with this type of interconnected knowledge, knowledge of physical objects, actions performed on them, and symbolic representations are all interrelated in a strong mental structure. This is illustrated by the child who "took 2 off the 43 and put it on the 38"—using a direct instantiation of physical, concrete objects and actions applied to mental, symbolic, representations. This is integrated-concrete knowledge.

Therefore, an idea is not simply concrete or not concrete. Depending on what kind of *relationship* you have with the knowledge (Wilensky, 1991), it might be sensory-concrete, abstract (only), or integrated-concrete (Clements, 1999). What ultimately makes mathematical ideas integrated-concrete is not their physical characteristics but how "meaning-full"—connected to other ideas and situations—they are. Good manipulatives are those that aid students in building, strengthening, and connecting various representations of mathematical ideas (cf. Uttal et al., 2009).

Comparing the two levels of concrete knowledge reveals a shift in what the adjective *concrete* describes. In "sensory-concrete," it refers to the support of concrete objects and their manipulation. In "integrated-concrete," it refers to knowledge that is "concrete" at a higher level because it is connected to other knowledge, both physical knowledge that has been abstracted and thus distanced from concrete objects and abstract knowledge of a variety of types. Such knowledge consists of units that "are primarily *concrete*, embodied, incorporated, lived" (Varela, 1999, p. 7). Ultimately, these are descriptions of changes in the configuration of knowledge as children develop. Consistent with other theoreticians (Anderson, 1993), we do not believe that there are fundamentally different types of knowledge, such as "concrete" versus "abstract" or "concrete" versus "symbolic" (Sarama & Clements, 2009).

COMPUTER MANIPULATIVES

Even if readers agree that "concrete" cannot simply be equated with physical manipulatives, they might have difficulty accepting objects on the computer screen as valid manipulatives. However, computers may provide representations that are just as personally meaningful to students as physical objects (Yerushalmy, 2005). Further, research indicates that, compared with their physical counterparts, computer representations may be more manageable, flexible, extensible, and "clean" (i.e., free of potentially distracting features; see Brown, McNeil, & Glenberg, 2009; Kaminski et al., 2009; Uttal et al., 2009). For example, compared with the use of physical "bean stick" materials, a computer environment offered students greater control and flexibility, allowing them to, among other things, duplicate and modify the computer bean sticks (Char, 1989). In another study, a single-group pre–post design revealed that third graders working with computer manipulatives made statistically significant gains learning fractional concepts (Reimer & Moyer, 2004). Qualitative evidence indicated that the computer manipulatives helped the students because they were easier and faster to use than physical manipulatives and provided immediate and specific feedback. Finally, students who used both physical and software manipulatives demonstrated a greater sophistication in classification and logical thinking than did a control group that used physical manipulatives alone (Olson, 1988).

We believe that an overarching but underemphasized reason for the positive effects of computer manipulatives in such studies is that computer manipulatives provide unique affordances for the development of integrated-concrete knowledge. Perhaps the most powerful is embodying the processes children are to develop and internalize as mental actions (as opposed to uses of physical manipulatives that can direct attention to the completely different processes or perspectives; see Uttal et al., 2009). Following are seven hypothesized, interrelated affordances, along with evidence (admittedly limited in some cases) for their empirical validity.

Bringing Mathematical Ideas and Processes to Conscious Awareness

Most students can use physical manipulatives to perform motions such as slides, flips, and turns; however, they make intuitive movements and corrections without being aware of these geometric motions. For example, even young children can move puzzle pieces into place without conscious awareness of the geometric motions that can describe these physical movements. Using computer tools to manipulate shapes brings those geometric motions to an explicit level of awareness (Clements & Sarama, 2007a). In one study, Pre-K children were unable to explain the motions needed to make the pieces fit in a physical puzzle. However, within one class session, these children adapted to using computer tools and were able to explain their actions to peers. A caveat is that when they moved to a "free explore" environment, the children did not use the same tools to manipulate shapes, so their potential benefit was not realized without specific task or guidance.

Encouraging and Facilitating Complete, Precise Explanations

Compared with students using paper and pencil, students using computers work with more precision and exactness (Clements, Battista, & Sarama, 2001; Gallou-Dumiel, 1989; Johnson-Gentile, Clements, & Battista, 1994). One study included two treatments to teach geometric transformations, symmetry, and congruence. One of the treatments used specially designed Logo computer environments to provide computer actions (geometric motions) on computer manipulatives (geometric figures). The other treatment group used physical manipulatives and paper and pencil. Otherwise, the curriculum and tasks were identical. Pretreatment and posttreatment interviews revealed that both treatment groups, especially the Logo group, performed at a higher level of geometric thinking than did a nontreatment control group. Although the two treatment groups did not significantly differ on the immediate posttest, the Logo group outperformed the non-Logo group on a second posttest that was administered 1 month after the end of the treatments. The Logo-based version enhanced the construction of higher level conceptualizations of motion geometry, aiding retention (Johnson-Gentile et al., 1994).

Supporting Mental ''Actions on Objects''

The flexibility of computer manipulatives allows them to mirror mental "actions on objects" better than physical manipulatives do. For example, physical base-ten blocks can be so clumsy, and the manipulations so disconnected from one another, that students see only the trees—manipulations of many pieces—and miss the forest—place-value ideas. In addition, students can break computer base-ten blocks into ones, or "glue" ones together, to form tens. Such actions are more in line with the *mental actions* that students are to learn (cf. Thompson, 1992).

Geometric computer manipulatives can encourage mental composition and decomposition of shapes (Clements & Sarama, 2007b; Sarama, Clements, & Vukelic, 1996). In an observational study of young children's use of physical and computer manipulatives, kindergartner Mitchell started making a hexagon out of triangles on the computer (Sarama et al., 1996). After placing two, he counted with his finger on the screen around the center of the incomplete hexagon, imaging the other triangles. Whereas off-computer, Mitchell had to check each placement with a physical hexagon, the intentional and deliberate actions on the computer led him to form mental images (decomposing the hexagon imagistically) and predict each succeeding placement.

Actions on computer manipulatives can include precise decompositions that are not easily duplicated with physical manipulatives—for example, cutting a shape (e.g., a regular hexagon) into other shapes (e.g., not only into two trapezoids but also into two pentagons and a variety of other combinations). Computer manipulatives have supported dramatic gains in this competency (Clements & Sarama, 2007b; Sarama et al., 1996; Spitler, 2009).

Changing the Very Nature of the Manipulative

In a similar vein, the flexibility of computer manipulatives allows children to explore geometric figures in ways that they cannot with physical shape sets. For example, children can change the size of the computer shapes, altering all shapes or only some. One study (Moyer, Niezgoda, & Stanley, 2005) compared how linguistically and economically diverse populations of kindergartners worked and learned with physical versus computer manipulatives. The researchers stated that the flexibility of the computer manipulatives had several positive effects on kindergartners' patterning. Specifically, the children made a greater number of patterns, and used more elements in their patterns, when working with computer manipulatives than when working with physical manipulatives or drawing. Finally, only when working with computer manipulatives did they create new shapes (by partial occlusion).

Symbolizing Mathematical Concepts

Computer manipulatives can also serve as symbols for mathematical ideas, often better than physical manipulatives can. For example, the computer manipulative can have just the mathematical features that developers wish it to have and just the actions on it that they wish to promote—and not additional properties that may be distracting. An example is a computer game to teach motion geometry. In research with this game, three modes were compared (Sedig, 2008; Sedighian & Sedighian, 1996): direct manipulation (DM), in which a student might drag a shape to turn it; direct concept manipulation (DCM), in which the student manipulated a representation of turning and angle measure, not the shape directly; and reflective DCM (RDCM), which included faded scaffolding (teaching help that is gradually withdrawn). Students using RDCM performed significantly and substantially better on assessments than did those using DCM versions, who, in turn, performed significantly better than did students using versions with conventional DM.

Linking the Concrete and the Symbolic With Feedback

The computer can help link manipulatives to symbols—the notion of multiple linked representations. For example, the number represented by the base-ten blocks is dynamically linked to the students' actions on the blocks, so that when the student changes the blocks, the number displayed is automatically changed as well. This helps students make sense of their activity and the numbers.

Computer manipulatives, more so than physical manipulatives, can also *connect* objects that students make, move, and change to other representations. For example, when students draw rectangles by hand, they may never think further about them in a mathematical way. In the Logo environments, however, students must analyze the (visual, concrete) figure to construct a sequence of (symbolic) commands, such as "forward 75" and (turn) "right 90" to direct the Logo "turtle" to draw a rectangle. So, they have to apply numbers to the measures of the sides and angles (turns). This helps them become explicitly aware of such characteristics as "opposite sides equal in length." The link between the symbols, the actions of the turtle, and the figure are direct and immediate (Clements et al., 2001).

Is it too restrictive or too hard to have to operate on symbols rather than directly on the manipulatives? Ironically, less "freedom" might be *more* helpful. In a study of place value, one group of students worked with a computer base-ten manipulative. The students could not move the computer blocks directly. Instead, they had to operate on symbols (Thompson, 1992; Thompson & Thompson, 1990). Another group of students used physical base-ten blocks. Although teachers frequently guided students to see the connection between what they did with the blocks and what they wrote on paper, the group using physical blocks did not feel constrained to write something that represented what they did with blocks. Instead, they appeared to look at the two as separate activities. In comparison, the computer group used symbols more meaningfully, tending to connect them to the base-ten blocks. As in the Logo example, it appears that this was caused by the "natural consequences" feedback—that is, when students manipulated the computer manipulatives, the connected symbols provided immediate feedback on their actions.

Because the computer offers immediate direct feedback, in environments such as computer base-tens blocks or computer programming, students may not be able to overlook the consequences of their actions. Thus, computer manipulatives can help students build on their physical experiences, tying them tightly to symbolic representations. In this way, computers help students link sensory-concrete and abstract knowledge, enabling them to build integrated-concrete knowledge.

Recording and Replaying Students' Actions

Once they finish a series of actions with physical manipulatives, it is often difficult for students to reflect on them. But computers allow students to store more than static configurations: They enable students to record *sequences* of their actions on manipulatives, and later replay, change, and reflect on them at will.

FINAL WORDS: CONCRETE MANIPULATIVES AND INTEGRATED-CONCRETE IDEAS

Manipulatives are meaningful for learning only *with respect to learners' activities and thinking*. Physical and computer manipulatives can be useful, but they will be more so when used in comprehensive, well-planned, instructional settings. Their physicality is not important—their *manipulability* and *meaningfulness* make them educationally effective (cf. Martin, 2009, who also reports that it is the manipulations that help children develop new concepts). In addition, some studies suggest that computer manipulatives can encourage students to make their knowledge explicit, which helps them build integrated-concrete knowledge, but rigorous causal studies have not been conducted to our knowledge. Such research, using randomized control trials, must be conducted to investigate the specific contributions of physical and computer manipulatives to particular aspects of mathematics teaching and learning.

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