Developing young children's mathematical thinking and understanding

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DEVELOPING YOUNG CHILDREN’S MATHEMATICAL THINKING AND UNDERSTANDING

Douglas H. Clements and Julie Sarama

Introduction

In contrast to the view that mathematics for very young children is developmentally inappropriate and that only simple number tasks are appropriate for the primary grades (see Balfanz, 1999; Hughes, 1986; Sun Lee and Ginsburg, 2009), research has shown that young children can think and understand mathematics content that is surprisingly broad and deep. Recent research and developmental work has suggested that learning trajectories can help early childhood educators respect children’s developmental processes and constraints, and their potential for thinking about and understanding mathematical ideas (Bobis et al., 2005; Clarke, 2008; Clements and Sarama, 2009; Sarama and Clements, 2009b; Wright, 2003). In this chapter we briefly discuss young children’s natural mathematical thinking. Then, we give two concrete examples of learning trajectories, illustrating how they can be used to enhance teaching and learning.

Young children’s natural mathematical thinking

It seems probable that little is gained by using any of the child’s time for arithmetic before grade 2, though there are many arithmetic facts that he [sic] can learn in grade 1.

(Thorndike, 1922, p. 198)

Children have their own preschool arithmetic, which only myopic psychologists could ignore.

(Vygotsky, 1935/1978, p. 84)
For more than 100 years, views of young children’s mathematics have differed widely, as these contradictory quotes from two psychologists show. Across that time, many have reported observations of children enjoying pre-mathematical activities. However, others have expressed fears of the inappropriateness of mathematics for young children, although these opinions are based on broad social theories or trends, not observation (Balfanz, 1999). The institutionalization of early childhood education often extinguished promising mathematical movements.

Consider Edward Thorndike, quoted above. He wished to emphasize that building blocks, rather than learning mathematics. But the original inventor, Caroline Pratt (1948), created today’s unit blocks to teach mathematics. She speaks of children making enough room for a horse to fit inside a stable. The teacher told Diana that she could have the horse when she had made a stable for it. Diana and Elizabeth began to build a small construction, but the horse did not fit. Diana had made a large stable with a low roof. After several unsuccessful attempts to get the horse in, she removed the roof, added blocks to the walls to make the roof higher, and replaced the roof. She then tried to put into words what she had done. ‘Roof too small.’ The teacher gave her new words, ‘high’ and ‘low’, and she gave a new explanation to the other children. Just like Pratt, we believe that ‘doing mathematics’ is natural and appropriate for children of all ages — if engendered and supported well. First, to be educative (Dewey, 1938/1997), mathematical experiences should involve mathematical processes or practices such as problem-solving, reasoning, communicating, modeling, and connecting (discusses these in depth is beyond the scope of this chapter, but the references contain many elaborations and examples). That is, every educative experience should involve helping children mathematize (De Lange, 1987; Kaartinen and Kumpulainen, 2012) their world: representing and elaborating their world mathematically — creating models of everyday situations with mathematical objects, such as numbers and shapes, with mathematical actions, such as counting or transforming shapes; and with structural relationships, such as ‘one more’ or ‘equal length’ — and using those models to solve problems. Second, to avoid being miseducative (Dewey, 1938/1997), experiences should not include inappropriate and harmful routines, such as flash cards and timed tests to promote ‘memorization’ of basic facts (especially before thinking strategies are well established) (Henry and Brown, 2008) or dull calendar exercises in which one child performs routine actions while others passively wait for it to be over (National Research Council, 2009, note this as miseducative for self-regulation as well as mathematics competencies). Third, to be educative, experiences should be challenging but achievable, generative of future learning, and consistent with young children’s ‘natural’ ways of thinking and learning (Clements et al., 2004; Trundle, 2008). We believe research-based learning trajectories are useful for ensuring that experiences are maximally educative.
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Learning trajectories: paths for successful learning

Why learning trajectories?

Children generally follow certain developmental paths in learning mathematics. When teachers understand the progression of levels of thinking along these paths, and sequence and individualize activities based on them, they can build effective mathematics learning environments. Research has suggested learning trajectories are effective in this way (Clements and Sarama, in press; Sarama and Clements, 2009b). Similarly, several recent efforts have based their reports on learning trajectories (e.g. Horne and Rowley, 2001; Nis, 2009). The National Research Council report on early mathematics (2009) is subtitled, 'Learning paths toward excellence and equity'. The Early Numeracy Research Project (ENRP) in Victoria, Australia was built around using 'growth points' to inform planning and teaching (e.g. Clarke, 2008; Horne and Rowley, 2001; Perry et al., 2008). The authors of the Common Core (CCSSO/NGA, 2010) started by writing learning trajectories for each major topic. These were used to determine what the sequence would be and were 'cut' into grade-level specific standards. Similar approaches are used in the New Zealand Numeracy Development Project, the Victorian Early Numeracy Research Project and the Count Me In Too program in New South Wales, Australia (Bobis et al., 2005) as well as Mathematics Recovery (Wright, 2003).

Each learning trajectory as we define it has three parts: a goal, a developmental progression, and instructional activities (Sarama and Clements, 2009b). To develop a certain mathematical competence (the goal), children construct each level of thinking in turn (the developmental progression), aided by tasks and teaching (instructional activities) designed to build the mental actions-on-objects that enable thinking at each higher level (Clements and Sarama, in press; Sarama and Clements, 2009b).

As an initial example, take the goal of measuring length, a common goal for mathematics (e.g. MacDonald et al., 2012), but one that challenges children (e.g. in the iteration of standard units, Nunes et al., 2009). A typical goal is for children to learn, by the end of second grade (ages 7–8 years), to measure the length of objects using appropriate tools, relate the size of the unit to the number of units, determine how much longer one object is than another, and so forth. That is the long-range goal.

Children develop through a series of levels of thinking as they achieve that goal; that is, as they learn the ideas and skills that constitute accurate and meaningful measurement of length. At each level, children can solve a new type of problem. These levels form a developmental progression (cf. MacDonald and Lowrie, 2011; McDonough and Sullivan, 2011). The second column in Figure 28.1 describes several levels of thinking in the counting learning trajectory. This includes the name of each level, a description, and a brief concrete example of a behavior indicative of that level of thinking (the first, or leftmost, column is the approximate age at which children achieve each level of thinking. These are present-day averages and not the goal – with good education; children often develop these levels earlier).

The right-most column in Figure 28.1 provides examples of instructional tasks that are designed to develop each of the levels of thinking. That is, educators can use
<table>
<thead>
<tr>
<th>Age</th>
<th>Developmental progression</th>
<th>Instructional tasks</th>
</tr>
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<tbody>
<tr>
<td></td>
<td><strong>Length quantity recognizer</strong></td>
<td>Teachers and other caregivers listen for and extend conversations about things that are &quot;long,&quot; &quot;tall,&quot; &quot;high,&quot; and so forth.</td>
</tr>
<tr>
<td>3</td>
<td>Identifies length/distance as attribute. May understand length as an absolute descriptor (e.g., all adults are tall), but not as a comparative (e.g., one person is taller than another). &quot;I'm tall, see?&quot;</td>
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<td></td>
<td><strong>Length direct comparator</strong></td>
<td>In &quot;Compare Lengths,&quot; teachers encourage children to compare lengths throughout the day, such as the lengths of blocks, ropes, or heights of furniture, and so forth.</td>
</tr>
<tr>
<td>4</td>
<td>Physically aligns two objects to determine which is longer or if they are the same length. Stands two sticks up next to each other on a table and says, &quot;This one's bigger.&quot;</td>
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<tr>
<td></td>
<td><strong>Indirect length comparator</strong></td>
<td>In &quot;Comparisons,&quot; children simply click on the object that is longer (or wider, etc.) In &quot;Line Up By Height,&quot; children order themselves with teacher's assistance by height in groups of five during transitions.</td>
</tr>
<tr>
<td>5</td>
<td>Compares the length of two objects by representing them with a third object. Compares length of two objects with a piece of string. May be able to measure with a ruler, but often lacks understanding or skill (e.g., ignores starting point). Measures two objects with a ruler to check if they are the same length, but does not accurately set the &quot;zero point&quot; for one of the items.</td>
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<td></td>
<td>Children solve everyday tasks that require indirect comparison, such as whether a doorway is wide enough for a table to go through. Children often cover the objects to be compared, so that indirect comparison is actually not possible. Give them a task with objects such as felt strips so that, if they cover them with the third object such as a (wider) strip of paper (and therefore have to visually guess), they can be encouraged to then directly compare them. If they are not correct, ask them how they could have used the paper to better compare. Model laying it next to the objects if necessary.</td>
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<td></td>
<td><strong>End-to-end length measurer</strong></td>
<td>In &quot;Deep Sea Compare,&quot; children move the ruler to compare the lengths of two fish, then click on the longer fish.</td>
</tr>
<tr>
<td>6</td>
<td>Lays units end-to-end. May not recognize the need for equal-length units. The ability to apply resulting measures to comparison situations develops later in this level. Lays 9-inch cubes in a line beside a hose to measure how long it is.</td>
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<tr>
<td></td>
<td>&quot;Length Riddles&quot; asks questions such as, &quot;You write with me and I am 7 cubes long. What am I?&quot;</td>
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<tr>
<td></td>
<td>&quot;Workin' on the Railroad&quot; In this computer activity, children lay units end-to-end to repair a railroad bridge.</td>
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</table>
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<table>
<thead>
<tr>
<th>Age</th>
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</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>Length unit ruler and repeater</td>
<td>Measure with physical or drawn units. Focus on long, thin units such as toothpicks cut to 1 inch sections. Explicit emphasis should be given to the linear nature of the unit. That is, children should learn that, when measuring with, say, centimeter cubes, it is the length of one edge that is the linear unit — not the area of a face or volume of the cube.</td>
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<td></td>
<td></td>
<td>Repeat 'Length Riddles' (see above) but provide fewer cases (e.g. only the length) and only one unit per child so they have to iterate (repeatedly 'lay down') a single unit to measure.</td>
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<td></td>
<td></td>
<td>'Mr. Mixup’s Measuring Menu' can be used at several levels, adapted for the levels before and after this one. For example, have the puppet leave gaps between units used to measure an object (for the End-to-End Length Measure level, gaps are between multiple units, for this level, gaps would be between iterations of one unit). Other errors include overlapping units and not aligning at the starting point.</td>
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<tr>
<td></td>
<td></td>
<td>'Draw a line.' Use line-drawing activities to emphasize how you start at the 0 (zero point) and discuss how, to measure objects, you have to align the object to that point. Similarly, explicitly discuss what the intervals and the numbers represent, connecting these to end-to-end length measuring with physical units.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>'Different Units.' Confront children with measurements using different units and discuss how many of each unit will fill a linear space. Help children make an explicit statement that the longer the unit the fewer are needed.</td>
</tr>
<tr>
<td>7-8</td>
<td>Length measure</td>
<td>Children should be able to use a physical unit and a ruler to measure line segments and objects that require both an iteration and subdivision of the unit. In learning to subdivide units, children may fold a unit in half, mark the fold as a half, and then continue to do so, to build fourths and eighths.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Children create units of units, such as a ‘footstrip’ consisting traces of their feet glued to a roll of adding-machine tape. They measure in different-sized units (e.g. 15 paces or 3 footstrips each of which has five paces) and accurately relate these units. They also discuss how to deal with leftover space, to count it as a whole unit or as part of a unit.</td>
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</tbody>
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Figure 28.1 Learning trajectory for length measurements (adapted from Clements and Sarama, in press)

these types of tasks to promote children’s growth from the previous level. More complete learning trajectories provide multiple illustrations of tasks for each level (e.g. see Clements and Sarama, in press); however, keep in mind that these illustrations are simply examples — many approaches are possible and children’s cultures and individual characteristics also need consideration.

Note the consistency between the standards for the grades preceding grade 2. The kindergarten standards include comparing the lengths of two objects directly.
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(by comparing them with each other). This is Figure 28.1's 'Length direct compare' level. The first grade Common Core standards include comparing the lengths of two objects indirectly by using a third object — Figure 28.1's 'Indirect length compare' level and the ability to 'Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps', precisely, the 'End-to-end length measurer' level of Figure 28.1. Finally, by the end of second grade, we reach the 'Length unit relater and repeater' and 'Length measurer' levels that are consistent with the Grade 2 Common Core goals previously described.

Teaching using learning trajectories

Learning trajectories' instructional tasks might offer a 'sketch' of a curriculum — a sequence of activities. However, research suggests that they can and should offer more. They should support teachers' use of formative assessment — the ongoing monitoring of student learning to inform and guide instruction. Research indicates that formative assessment is an effective teaching strategy (Clarke, 2008; Clarke et al., 2002; National Mathematics Advisory Panel, 2008; Shepard, 2005). However, the strategy is useless for teachers unless they can accurately assess 'where students are' in learning a mathematical topic and know how to support them in learning the following level of thinking. The goal of learning trajectories help define the mathematical content that teachers have to teach and so have to understand well themselves. The developmental progresses give teachers a tool to understand the level of thinking at which their students are operating, along with the next level of thinking that each student should learn. Then, matched instructional tasks provide guidance as to the type of educational activity to support that learning and help explain why those activities would be particularly effective. Such knowledge helps teachers be more effective professionals. Next, we look at ways to put this all into practice.

The key to the use of formative assessment is knowing what standards or goals one is trying to reach, where the students are starting, and how to help them move from there to the goal. Notice that these three formative assessment questions align with the three components of learning trajectories, as shown in Figure 28.2.

Our second example is from the learning trajectory of early counting-based addition and subtraction. A study of textbooks in California showed the importance of teaching core concepts and meaningful strategies for arithmetic, not simply 'facts' (Henry and Brown, 2008). The learning trajectory, therefore, should include core knowledge, strategies, and skills.

Note that the following is one of two approaches to addition and subtraction involving counting-based strategies. A critical complement to these is that of conceptual subitizing and related visually and structurally based part-whole approaches (see examples in the patterns and structure curriculum, Mulligan et al., 2006, and chapter 6 in both; Sarama and Clements, 2009b). Both begin in the earliest years, not represented here (see references).
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<table>
<thead>
<tr>
<th>Formative assessment questions</th>
<th>Learning trajectories’ components</th>
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<tbody>
<tr>
<td>1. Where are you trying to go?</td>
<td>The goal – Describes the mathematical concepts, structures, and skills</td>
</tr>
<tr>
<td>2. Where are you now?</td>
<td>The developmental progression – Helps determine how the children are thinking now and on the ‘next step’</td>
</tr>
<tr>
<td>3. How can you get there?</td>
<td>The instructional activities – Provide tasks linked to each level of the developmental progression that are designed to engender the kind of thinking that will form the next level. Suggests feedback for specific errors.</td>
</tr>
</tbody>
</table>

Figure 28.2 Relationships between the major questions of formative assessment and the components of a learning trajectory

1 The goal. Mathematically, whole-number addition can be viewed as an extension of counting (National Research Council, 2009). The sum \( 7 + 5 \) is the whole number that results from counting up 5 numbers starting at 7; that is, 7, 8, 9, 10, 11, 12. As tedious as it would be to solve this way, the sum 194 + 746 is the number resulting from counting up 746 numbers starting at 194.

As they move through the learning trajectory, students learn to solve increasingly difficult problems. Some problems are more difficult simply because they involve larger numbers, of course. Unfortunately, such difficulties are often greater than they should be because too many curricula and teachers provide far more practice on problems with smaller digits and neglect the larger single-digit numbers (Hamann and Ashcraft, 1986). Teachers should ensure students receive more balanced experiences.

Beyond the size of the number, however, it is the type, or structure of the word problem that determines its difficulty. Type depends on the situation and the unknown. The situation can be a ‘Join’ problem (have 2 apples, got 4 more) or ‘Separate’ problem (had 2 apples, ate 1); a ‘Part-part-whole’ problem (3 are girls and 4 are boys – no action is suggested); or a ‘Compare’ problem (John has 4, Emily has 6). For each of these categories, there are three quantities that play different roles in the problem, any one of which could be the unknown. In some cases, such as the unknown parts of ‘Part-part-whole’ problems, there is no real difference between the roles, so this does not affect the difficulty of the problem. In others, such as the ‘result unknown’, ‘change unknown’ (had 2, got some more, now has 6), or ‘start unknown’ (had some, got 4 more, now has 6) of ‘Join’ problems, the differences in difficulty are large. ‘Result unknown’ problems are easy, ‘change unknown’ problems are moderately difficult, and ‘start unknown’ problems are the most difficult. This is due in large part to the increasing difficulty children have in modeling, or ‘act outing’, each type. In summary, a main goal of this addition and subtraction learning trajectory is that children learn to solve arithmetic problems of different types using counting strategies.

2 The developmental progression. A few selected levels for this component of learning trajectories are shown in the second column in Figure 28.3. Children

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### Find Result
Finds sums by joining (e.g., had 3 apples and get 3 more, how many do you have in all?) and part-part-whole (there are 5 girls and 5 boys on the playground, how many children were there in all?) problems by direct modeling, counting-adding whole with objects.

- *Questions:* Have 5 red balls and 5 blue balls, how many in all? Count out 2 red balls then away 2, how many are left?
- *Solves problems by separating with objects.*

### Counting strategies
Finds sums for joining (you had 8 apples and get 3 more) and part-part-whole (5 girls and 5 boys) problems with finger patterns and/or by counting on.

- *Questions:* How much is 4 and 5 more? 10 more, 10 more, seven (i.e., 7-finger or Finger pattern to keep track). Seven!
- *Counting strategies:* May solve thinking aloud (9 + 7 = 16; or compare problem with counting up: e.g., count on 4, 5, 6, 7 while putting up fingers, and then count or manipulate the 4 fingers raised.

### Derive
Uses flexible strategies and derived combinations (e.g., 7 + 3 = 10, so 7 + 4 is 11) to solve all types of problems. Can simultaneously think of 3 numbers within a sum, and can move part of a number to another, amount of the increase in one and the decrease in another.

- *Questions:* Who’s got 15? How’s 15? 1 + 14 = 15 17 + 3 = 20 = 15

### Instructional tasks

#### Word problems. Children solving all the above problem types using manipulatives or their fingers to represent objects.

- **For Separate, result sentence (tally-expect):** Have 5 balls and give 2 to friend. How many do you have left? Children might count out 5 balls, then take away 2, and then count the remaining 3.
- **For Part-part-whole, whole sentence problems, they might solve:** Have 2 red balls and 5 blue balls. How many are in all?

#### Dinosaur Shop 3.
Context or the shop asks students to combine the two orders and add the contents of two boxes of toy dinosaurs (number known) and click a target numeral that represents the sum.

#### Off the Tree. Students add two amounts of data to identify their total number value, and then move forward a corresponding number of spaces on a game board, which is now marked with numerals.

#### How Many Now? Have the children count objects as you place them in a box. Ask, "How many are in the box now?" Add one, repeating the question, then ask the children’s responses by counting all the objects. Repeat, checking occasionally. When children are ready, sometimes add two, and eventually more, objects.

- *Variations:* Place more in a coffee cup. Decline that a given number of objects are in the cup. Then have the children add more copies and count on by focusing on additional objects already dropped in.

#### Find Result Unknown and Part-Part-Whole, Whole Unknown
- *How many is 4 and 5 more?*

#### Bridge Idea. Students are given a numeral and a frame with dots. They count on from this numeral to identify the total amount, and then move forward a corresponding number of spaces.

#### Easy as Pie. Students add two numerals to find a total number, and then move forward a corresponding number of spaces on a game board.

#### All types of single-digit problems.

- **RACE.** See text.

#### Tic-Tac-Toe.** Draw a tic-tac-toe board and write the numbers 1 to 10. Players take turn crossing out one of the numbers and writing it in the board. Whoever makes 15 first wins (Kamii, 1985).

#### 21. Play cards, where Ace is worth either 1 or 11 and 2 to 10 are their values.

1. **Deal:** Each player receives 5 cards, including himself.

2. **Ongoing game:** Each player, if sum is less than 21, can request another card, or "hit." If any total card makes the sum more than 21, the player is out.

3. **Continue until everyone is ‘out.’" The player whose sum is closest to 21 wins.

### Figure 28.3 Partial learning trajectory for addition and subtraction (emphasizing counting strategies) (adapted from Clements and Sarama, in press)
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develop increasingly sophisticated counting strategies to solve increasingly difficult problem types. For example, most initially use a counting-all procedure. At the 'Find result' level, given a problem of 7 + 2, such children count out objects to form a set of 7 items, then count out 2 more items, and finally count all those and say 'nine.' Children use such counting methods to solve story situations if they understand the language in the story.

After children develop such methods, they eventually curtail them. Often independently, children as young as 4 or 5 years invent 'counting on', solving the previous problem by counting, 'Seven... 8, 9, 9!' The elongated pronunciation may be substituting for counting the initial set one-by-one. It is as if they counted a set of 7 items.

Children then move to the counting-from-larger strategy, which is preferred by most children once they invent it. Problems such as 4 + 25, where the most 'counting on' work is saved by reversing the problem, often prompt children to start counting with the 25. Counting-on when increasing collections and the corresponding counting-back-from when decreasing collections are important numerical strategies for students to learn. However, they are only beginning strategies. In the case where the amount of increase is unknown, children use counting-up-to to find the unknown amount. If 6 items are increased so that there are now 9 items, children may find the amount of increase by counting and keeping track of the number of counts, as in 'Six... 7, 8, 9, 3!' And if 9 items are decreased so that 6 remain, children may count from 9 down to 6 to find the unknown decrease, as follows: 'Nine... 8, 7, 6, 3!' However, counting backwards, especially more than two or three counts, is difficult for most children unless they have consistent instruction. Instead, children might learn counting-up-to the total to solve a subtraction situation. For example, 'I took away 6 from those 9, so 7, 8, 9 (raising a finger with each count) — that's 3 more left in the 9.' Students then learn to incorporate place value and other ideas.

3 The instructional tasks. As stated, instructional tasks are not the only way to guide children to achieve the levels of thinking embedded within the learning trajectories. However, those in the right-most column of Figure 28.3 are (simply) examples of the type of instructional activity that helps promote thinking at the subsequent level. Thus, teachers implement, adapt, or use them as a template to gauge the appropriateness and expected effectiveness of other lessons, including those in published curricula. Further, these tasks are often useful as problems for children to solve (via guided discovery), but teachers must make critical decisions concerning pedagogical strategies (Anthony and Walshaw, 2009), such as whether these problems might be posed in a play context (e.g. Harsh-Pasek and Golinkoff, 2008; Lee, 2010; Sarama and Clements, 2009a; van Oers, 2003, 2010), presented as small group activities (Clarke et al., 2002; Clements and Sarama, 2007, 2013; Griffin, 2004; Griffin et al., 2007), or other approaches. Some principles identified by researchers include (a) that social and cultural contexts that make sense to the children (Anthony and Walshaw, 2009; Perry et al., 2008), (b) that they involve thoughtful and sensitive discourse (Anderson et al., 2004; Anthony and Walshaw, 2009), (c) that a careful synthesis of child-centered and teacher-guided (intentional, sequential) experiences are included,
and that connections between home and school, and educators from birth through the primary years, be strong and continuous (Anthony and Walshaw, 2009; National Research Council, 2009).

In some cases, there is evidence that certain aspects of the instructional tasks are especially effective. For example, if students need extra help in learning counting on skills, there is theory and empirical work that provides specific instructional strategies. After setting up the problem situation with objects (say, 5 + 3), the teacher guides children to connect the numeral signifying the first addend to the objects in the first set (Carruthers and Worthington, 2006, describe the development of written symbols and drawings beyond the scope of this chapter). Students then learn to recognize that the last object of that set is assigned the counting word ('five'). Next, the teacher helps the children understand that the first object in the second set will always be assigned the next counting number ('six'). Students learn that they can start with the 'five' immediately and count on. These understandings and skills are reinforced with additional problems and a variety of specific, focused questions.

Besides carefully addressing necessary ideas and subskills, this instructional activity is successful because it promotes psychological containment (Clements and Burs, 2000; Kluetskeii, 1976), an encapsulation process in which one mental activity gradually stands in for another mental activity. Children must learn that it is not necessary to enumerate each element of the first set. The teacher explains this, then demonstrates by naming the number of that set with an elongated number word and a sweeping gesture of the hand before passing on to the second addend. Elkonin and Davydov (1975) claim that such abbreviated actions are not eliminated but are transferred to the position of actions which are considered as if they were carried out and are thus 'implicit'. A sweeping movement gives rise to a 'mental plan' by which addition is performed, because only in this movement does the child begin to view the group as a unit. The child becomes aware of addition as distinct from counting. This construction of counting on must be based on physically present objects. Then, through introspection (considering the basis of one's own ways of acting), the object set is transformed into a symbol.

A second example of instructional activities supported by specific research evidence, found in the next level in Figure 28.3, Derive +/-, is the Japanese approach to developing the Break-Apart-to-Make-Ten (Murata, 2004; note this is known as 'bridging through 10' in the UK and other countries, e.g. Heidreich, 2005; BAMT, see Murata and Fuson, 2006). The BAMT strategy actually consists of a series of instructional activities involving several interrelated learning trajectories (see Murata and Fuson, 2006; Sarama and Clements, 2009b, for descriptions). Before lessons on BAMT, children work on several related learning trajectories. They develop solid knowledge of numerals and counting (i.e. move along the counting learning trajectory). This includes the number structure for teen numbers as 10 + another number, which is more straightforward in Asian languages and Hindi than in English, Spanish, and other languages ('13' is '10 and 3' — teachers in the latter languages must be particularly attentive to this competence). They learn to solve addition and subtraction of numbers with totals less than 10, often chunking numbers into 5 (e.g. 8 as 5-plus-3) and using visual models. With these levels of thinking established, children develop several levels of thinking within the composition/decomposition developmental
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9 + 4
1

The line slants between the numbers, indicating that we need to find a partner for 9 to make 10.

9 + 4
1 3

Four is separated into two partners, 1 and 3.

9 + 4
1 3

The ring shows how the numbers combine to make 10.

9 + 4 = 13

Ten and three are shown to add to 13.

Figure 28.4 Teaching BAMT

progression. For example, they work on 'break-apart partners' of numbers less than or equal to 10. They solve addition and subtraction problems involving teen numbers using the 10's structure (10 + 5 = 15), and addition and subtraction with three addends using 10's (e.g. 6 + 4 + 7 = 10 + 7 = 17).

Teachers then introduce problems such as 9 + 4. They first elicit, value, and discuss child-invented strategies and encourage children to use these strategies to solve a variety of problems. Only then do they proceed to the use of BAMT. They provide supports to connect visual and symbolic representations of quantities. In the example 9 + 4, they show 9 counters (or fingers) and 4 counters, then move 1 counter from the group of 4 to make a group of 10. Next, they highlight the 3 left in the group. Then children are reminded that the 9 and 1 made 10. Next, children see 10 counters and 3 counters and think 10-3. Last, representational drawings serve this role in a sequence such as shown in Figure 28.4.

Conclusion

Young students can learn more mathematics than many current programs provide. Learning trajectories can help teachers support their students' learning of more profound ideas in mathematics (space constraints have not allowed explorations of subitizing, early number relationships, multiplication reasoning and so forth, see Nunes et al., 2009). Current research in learning trajectories points the way toward more effective and efficient, yet also more creative and enjoyable, mathematics.

References


Developing young children's mathematical thinking and understanding


Douglas H. Clements and Julie Sarama


