Ph.D. Preliminary Examination in Analysis

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- The duration of this exam is four hours.
- Each exercise is worth ten points.
- Please submit no more than eight problems. You are free to choose any eight out of the eleven offered in this exam.
- A score of sixty out of eighty guarantees a pass for this exam.
- Your work will be assessed for its quality and rigor.
- No document, computer, calculator, or cell phone is allowed to be used during this exam.

GOOD LUCK!

1 Analysis

1. State and prove Rolle's Theorem.
2. Prove of disprove: $f(x) = x \ln(x)$ is uniformly continuous on
   (a) $(0, 1]$
   (b) $[1, \infty)$.
3. Assume that $(a_n)_{n\in\mathbb{N}}$ satisfies $\sum_{n\geq 1} |a_n|$ converges. Prove that
   $$f(x) = \sum_{n\geq 1} a_n \cos(nx)$$
   converges in $\mathbb{R}$ and that the function f is continuous on $\mathbb{R}$.
4. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and let
   $$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$
   for all $n \in \mathbb{N}, n > 0$. Prove that if $a_n \to a$ then $b_n \to a$.

5. Let f be a real-valued continuous function defined for all $0 \leq x \leq 1$, such that $f(0) = 1$, $f(1/2) = 2$ and $f(1) = 3$. Show that

$$\lim_{n \to \infty} \int_0^1 f(x^n)\, dx$$

exists and compute this limit. Justify your answer.

2 Topology

1. Prove that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed when $X \times X$ is given the product topology.

2. Let (E, τ_E) and (F, τ_E) be topological spaces. Let $f : E \to F$ be given. Show: f is continuous if and only if for all $A \subset E$, $f\left(\overline{A}\right) \subset \overline{f(A)}$.

3. Let X be a compact topological space, Y a Hausdorff space, and $f : X \to Y$ a continuous bijection.

   (a) Show that f is a homeomorphism.

   (b) Give an example where Y is not Hausdorff, while the other assumptions in part (a) are satisfied, and the conclusion fails.

3 Metric Spaces

1. Let X be a complete metric space and let $(A_n)_{n \in \mathbb{N}}$ be a sequence of closed non-empty sets such that

$$A_{n+1} \subset A_n \text{ and } \operatorname{diam}(A_n) \to 0.$$

   Recall that $\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}$.

   (a) Prove that $\cap_{n \geq 1} A_n \neq \emptyset$.

   (b) Is the result true if we don't assume that X is complete? Justify your answer.

2. Using only the definition of compactness (every open cover has a finite subcover), prove that a non-empty compact set in a metric space has a countable dense set.

3. Let (X, d) be a complete metric space and assume that $T : X \to X$ is a contraction (i.e., there exists $\alpha \in (0, 1)$ such that $d(T(x), T(y)) \leq \alpha d(x, y)$ for all $x, y \in X$).

   (a) Prove that T has a unique fixed point (i.e., there exists a unique $x \in X$ such that $T(x) = x$).

   (b) Suppose that T^2 is a contraction but T is not. Must T have a fixed point? If so, must this fixed point be unique?