Ph.D. Preliminary Examination in Analysis
Department of Mathematics

University of Denver
Fall 2012

- The duration of this exam is four hours.
- Each exercise is worth ten points.
- Please submit no more than eight problems. You are free to choose any eight out of the eleven offered in this exam.
- A score of sixty out of eighty guarantees a pass for this exam.
- Your work will be assessed for its quality and rigor.
- No document, computer, calculator, or cell phone is allowed to be used during this exam.

GOOD LUCK!

1 Real Analysis

1. Let $f:[0,1]\to\mathbb{R}$ and $g:[0,1]\to\mathbb{R}$ be continuous functions satisfying $f\left(\frac{1}{3}\right)<g\left(\frac{1}{3}\right)$. Prove that there exists $\delta>0$ such that $f(x)<g(x)$ for all $x\in[0,1]$ satisfying $\frac{1}{3}-\delta<x<\frac{1}{3}+\delta$.

2. Suppose that for every $n\in\mathbb{N}$, there exists $M_n>0$ and $f_n:[0,1]\to\mathbb{R}$ such that every x in $[0,1]$ satisfies $|f_n(x)|\leq M_n$. Prove that if $\sum_{n=1}^{\infty}M_n<\infty$, then $\sum_{n=0}^{\infty}f_n$ converges uniformly on $[0,1]$.

3. Let $f:[0,2]\to\mathbb{R}$ be a Riemann integrable function and define $F:[0,2]\to\mathbb{R}$ by
$$F(x)=\int_0^x f(t)\,dt.$$
   (a) Use the definition of derivative to prove that if f is continuous at x_0, then F is differentiable at $x_0\in(1,2)$.
   (b) Is the converse true? That is, if F is differentiable at $x_0\in(1,2)$, does it follow that f is continuous at x_0? If true give a proof, and if false give a counter example.

4. Let f be a real continuous function defined for all $0 \leq x \leq 1$, such that $f(0) = 1$, $f(1/2) = 2$ and $f(1) = 3$. Prove that

$$\lim_{n \to \infty} \int_0^1 f(x^n)\, dx$$

exists and compute the limit. Justify your answer.

5. Suppose that for every $n \in \mathbb{N}$, $f_n : \mathbb{R} \to \mathbb{R}$ is a differentiable function satisfying $|f_n'(x)| \leq 1$ for every $x \in \mathbb{R}$, and that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that (f_n) converges pointwise to g.

   (a) Prove that $g : \mathbb{R} \to \mathbb{R}$ is continuous.

   (b) Does it follow that (f_n) converges uniformly to g? If the answer is yes, give a proof and if the answer is no give a counter-example.

2 Metric Spaces

1. Let (M, d) be a metric space satisfying the following property:

   If $(A_n)_{n\in\mathbb{N}}$ is a sequence of non-empty closed subsets of M that satisfy $A_{n+1} \subset A_n$ for every $n \in \mathbb{N}$ and diam $(A_n) \to 0$, then $\cap_{n=1}^{\infty} A_n \neq \emptyset$.

   Prove that (M, d) is a complete metric space.

2. Let $M = C[0,1]$ be the space of continuous functions on $[0,1]$ and consider the following metrics on $C[0,1]$

$$\begin{aligned} \rho_1(f,g) &= \int_0^1 |f(x) - g(x)|\, dx \\ \rho_2(f,g) &= \max_{0 \leq x \leq 1} |f(x) - g(x)|\,. \end{aligned}$$

   Prove that the metrics (M, ρ_1) and (M, ρ_2) are not equivalent. (Recall that (M, d_1) and (M, d_2) are equivalent if there exist constants $c_1 > 0$ and $c_2 > 0$ such that for every $x, y \in M$,

$$c_1 d_1(x,y) \leq d_2(x,y) \leq c_2 d_1(x,y)\,.)$$

3. (a) State and prove Baire's Category Theorem.

   (b) Prove that that a non-empty complete metric space with no isolated points is uncontable.

3 Topology

1. Let (X, τ) be a topological space. Prove that the diagonal $\Delta = \{(x,x) : x \in X\} \subset X \times X$ is closed in the product topology of $(X \times X, \tau \times \tau)$ if and only if X is Hausdorff.

2. Let X and Y be topological spaces and let $C(X,Y)$ be the set of continuous functions from X to Y. If $A \subset X$ and $B \subset Y$, define the set

$$W(A,B) = \{f \in C(X,Y) : f(A) \subset B\}.$$

The compact-open topology of $C(X,Y)$ is defined by the subbasis

$$\{W(A,B) : A \text{ is compact and } B \text{ is open}\}$$

(Recall that the base for the topology of $C(X,Y)$ will consist of finite intersections of these sets).

Prove that if Y is Hausdorff, then $C(X,Y)$ with the compact open topology is also Hausdorff.

3. Let $\mathbb{N} = \{1,2,3,\ldots\}$ and endow $\mathbb{N}$ with the topology with basic elements given by $n\mathbb{N} = \{n, 2n, 3n, \ldots\}$ for $n \in \mathbb{N}$.

   (a) Prove that $\{n\mathbb{N} : n \in \mathbb{N}\}$ is a base for a topology in $\mathbb{N}$.

   (b) Find the closure of the set $\{6\}$.

   (c) Consider $P = \{2,3,5,7,\ldots\}$, the set of prime numbers in $\mathbb{N}$. Is $P \cup \{1\}$ open, closed, or neither? justify your answer.