Analysis Preliminary Exam

Name: ________________________________

Instructions

- The duration of this exam is four hours.
- Each exercise is worth ten points.
- Please submit no more than eight problems. You are free to choose any eight out of the twelve problems offered in this exam. There are six problems are from the Real Analysis curriculum, three from the Metric Spaces curriculum and three from the Topology curriculum.
- A score of sixty out of eighty guarantees a pass for this exam.
- Your work will be assessed for its quality and rigor. Unless otherwise specified, proper justification is expected.
- No documents, computers, calculators, or cell phones are allowed to be used during this exam.

GOOD LUCK!

Real Analysis

1. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences in $\mathbb{R}$. Prove: $\frac{1}{N}\sum_{i=1}^{N} a_i b_{N-i}$ converges.

2. (a) Show that $\int_0^{2\pi} |\sin x|^n dx$ converges to 0 as $n \to \infty$.

   (b) Show that $\int_0^{2\pi n} |\sin x|^n dx$ does not converge to 0 as $n \to \infty$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ a function. Prove that the following definitions of lower-semi continuous functions are equivalent.

   (a) We say that f is lower semi-continuous if for all z, $f(z) \leq \liminf_{x \to z} f(x)$.

   (b) We say that f is lower semi-continuous if for all $c \in \mathbb{R}$, $f^{-1}(c, \infty)$ is open.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Show $f(x + \frac{1}{n})$ converges uniformly to $f(x)$.

5. Let f be a continuous function in $[-1, 2]$. Given $x \in [0, 1]$, define $f_n(x) = \frac{n}{2}\int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)dt$.

   (a) Prove: For $n > 1$, f_n is differentiable on $[0, 1]$.

   (b) Prove: $\{f_n\}$ converges uniformly to f in $[0, 1]$.

6. Let $f : [a, b] \to \mathbb{R}$ be an increasing function. Prove: f is Riemann integrable.

Metric Spaces

1. Suppose that S and T are closed subsets of the metric space (X, d).

   (a) Prove: If S and T are compact then there are points $s_0 \in S$ and $t_0 \in T$ such that

   $$d(s_0, t_0) = \inf \{d(s, t) : s \in S, t \in T\}$$

   (b) Must the conclusion of part (a) remain true if S and T are only assumed to be closed (not necessarily compact) subsets of (X, d)? Either prove it or give a counterexample.

2. Let (X, d) and (Y, d') be metric spaces.

   (a) Prove that if X is compact and $f : X \to Y$ is continuous, then f is uniformly continuous.

   (b) Give an example of a non-compact metric space (X, d) and a continuous function $f : X \to \mathbb{R}$ which is not uniformly continuous.

3. Prove the Lebesgue Number Theorem: If (X, d) is a compact metric space and $\{U_i\}$ is an open cover of X then there is a $\delta > 0$ such that for any $x \in X$, $B(x, \delta) \subset U_i$ for some i.

Topology

1. Let X be an uncountable space. Consider the collection τ of sets $\tau = \{U \subset X : X \setminus U \text{ is countable.}\} \cup \{\emptyset\}$.

   (a) Prove that τ forms a topology on X.

   (b) Prove that (X, τ) is not compact, but is Lindelöf (every open cover has a countable subcover).

   (c) Prove that there is an infinite subset of (X, τ) with no limit points.

2. Let S^1 denote the unit circle in $\mathbb{R}^2$. Define $f : S^1 \to \mathbb{R}$ be a continuous function. Prove the following.

   (a) f is not surjective.

   (b) There is a $z \in S^1$ such that $f(z) = f(-z)$.

3. Let X and Y be connected spaces. Prove: $X \times Y$ is connected.