PH.D. PRELIMINARY EXAMINATION IN ANALYSIS

ANALYSIS FACULTY GROUP

Number

- This exam lasts four hours.
- No document, computer, calculator, cell phone and any other aid is allowed.
- Each problem is worth ten points.
- Please indicate which problems you wish to be graded.
  - Only marked problems will be graded,
  - Do not mark more than eight problems.
  - You are free to choose any eight problems you wish.
- A score of 60 would ensure a pass to this exam.
- Your work will be assessed on its *quality and rigor*.

GOOD LUCK!

1. REAL ANALYSIS

(1) Let $a < b \in \mathbb{R}$. Let $f : [a,b] \to [0,\infty)$ be a continuous nonnegative function. Show that:

$$\lim_{n\to\infty} \sqrt[n]{\int_a^b (f(x))^n \, dx} = \sup\{f(x) : x \in [a,b]\}.$$

(2) Let $f : [0,1] \to \mathbb{R}$ be a Riemann integrable function which is also continuous at 0, and with $f(0) = 0$. Show that:

$$\lim_{x\to 0^+} x \int_x^1 \frac{f(t)}{t^2} \, dt = 0.$$

Date: October 6, 2015.

(3) Does the series:

$$\left(\sum \sqrt{n}\left(1-\cos\left(\frac{1}{n}\right)\right)\right)_{n\in\mathbb{N}}$$

converge? Prove whatever series convergence or divergence tests you use.

(4) Let $f:\mathbb{R}\to\mathbb{R}$ be continuous. For all $n\in\mathbb{N}, n>0$ and $x\in\mathbb{R}$, we let:

$$f_n(x)=\frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right).$$

Prove that on any compact interval $[a,b]$ (with $a<b\in\mathbb{R}$), the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly, and provide its limit in terms of f.

(5) Prove that an increasing function $f:[0,1]\to\mathbb{R}$ is the pointwise limit of a sequence of continuous functions over $[0,1]$. *Hint: Prove that for any interval I of $[0,1]$, the function:*

$$x\in[0,1]\mapsto\begin{cases}1 \text{ if } x\in I\\ 0 \text{ otherwise}\end{cases}$$

is a pointwise limit of continuous functions on $[0,1]$.

2. Metric Spaces

(1) Let (E,d) be a compact metric space and let $f:E\to\mathbb{R}$ be a continuous function. Prove that f is uniformly continuous on E.

(2) Let (E,d) be a metric space. For any $A\subseteq E$, let:

$$\operatorname{diam}(A)=\sup\{d(x,y):x,y\in A\}.$$

Show that (E,d) is complete if, and only if, the following property holds: if $(H_n)_{n\in\mathbb{N}}$ is a sequence of nonempty closed subsets of (E,d) such that:

$$\lim_{n\to\infty}\operatorname{diam}(H_n)=0,$$

and

$$H_{n+1}\subseteq H_n \text{ for all } n\in\mathbb{N},$$

then there exists $x\in E$ such that $\bigcap_{n\in\mathbb{N}} H_n=\{x\}$.

(3) Prove that $\mathbb{Q}$ is not the intersection of a countable collection of open subsets of $\mathbb{R}$.

3. Topology

(1) Let (E,τ) be a topological space. Let $A \subseteq E$. Prove that:

$$\{x \in E : \forall V \in \tau \quad x \in V \implies V \cap A \neq \emptyset\}.$$

is the closure of A, i.e. the smallest closed set in τ containing A. Let A' be the set:

$$A' = \{x \in E : \forall V \in \tau \quad x \in V \implies V \cap (A \setminus \{x\}) \neq \emptyset\}$$

i.e. A' is the set of limit points of A. Prove that if τ is T1, then:

$$A' = \{x \in E : \forall V \in \tau \quad x \in V \implies V \cap A \text{ is infinite}\}.$$

(2) Let (E,τ_E) and (F,τ_F) be two compact Hausdorff spaces. Let $f : E \to F$. Prove the following assertions are equivalent:

(a) f is continuous,

(b) the graph:

$$\text{Graph}(f) = \{(x, f(x)) : x \in E\}$$

is closed in $(E \times F, \tau_E \otimes \tau_F)$.

(3) Let E be the set of all functions from $[0,1)$ to $\{0,1\}$, endowed with the topology τ of pointwise convergence, where $\{0,1\}$ is endowed with the discrete topology. Is there a metric on E which induces the topology τ?

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