Ph.D. Preliminary Examination 1 for Analysis

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Duration: 4 hours

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MATH PRELIM(Analysis) – Winter 2013

Examination Information: This examination lasts for 4 hours and consists of 16 pages. Please be very precise with your answers. Proper justification is *always* expected. The general examination rules detailed in the syllabus and the exam rules document apply. Failure to comply with these rules will result in disciplinary procedures and at best a failing grade for this examination. In particular, for the whole duration of this examination, no document or electronic device is allowed, and no communication is permitted with anyone except with the proctor. Follow the directions of the proctor. You may only start when the proctor gives you permission, and you *must* end when the proctor asks you to. *Good luck!*

Your name (first,last) and sid:

- The duration of this exam is four hours.
- Each exercise is worth ten points.
- Please submit no more than eight problems. You are free to choose any eight out of the twelve problems offered in this exam.
- A score of sixty out of eighty guarantees a pass for this exam.
- Your work will be assessed for its *quality and rigor*.
- No document, computer, calculator, or cell phone is allowed to be used during this exam.

Good Luck!

1 Real Analysis

Exercise 1. Let $a_0 \geqslant b_0 \in [0, \infty)$. We define two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ by induction as follows:

$$\forall n \in \mathbb{N} \quad a_{n+1} = \frac{a_n + b_n}{2} \text{ and } b_{n+1} = \sqrt{a_n b_n}.$$

Show that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converge to the same limit.

Your answer

Score: 10

Exercise 2. Prove the following theorem:

Theorem 1. *Let I be an open interval of $\mathbb{R}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions on I converging uniformly on I to some function $f : I \to \mathbb{R}$. Then f is continuous on I.*

Your answer

Score: 10

Exercise 3. Let $f : [a, b] \to [0, \infty)$ be a Riemann integrable function such that $\int_a^b f = 0$. Show that:

$$\{x \in [a, b] : f(x) = 0\}$$

is dense in $[a, b]$.

Your answer

Score: 10

Exercise 4. Let I be a nonempty interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be given.

1. Show that if f is monotone, then f is continuous on I if and only if $f(I)$ is an interval.
2. Show that the following are equivalent:
   (a) f is strictly monotone on I,
   (b) f is injective on I,
   (c) f is an homeomorphism from I onto $f(I)$.

Your answer

Score:	10

Exercise 5. Let $a < b \in \mathbb{R}$ and I an open interval such that $[a, b] \subseteq I$.

1. Assume that $f : I \to \mathbb{R}$ is differentiable on I and that f' is continuous on I. Show that:

$$\lim_{n\to\infty} \int_a^b f(t) \sin(nt)\, dt = 0.$$

2. Assume now that $f : I \to \mathbb{R}$ is merely Riemann integrable on $[a, b]$. Show again that:

$$\lim_{n\to\infty} \int_a^b f(t) \sin(nt)\, dt = 0.$$

Hint: Part 1 of this problem is useful in the proof of this more general result.

Your answer

Score: | 10

Exercise 6. Let $a < b \in \mathbb{R}$. For each $n \in \mathbb{N}$, we are given a decreasing continuous function $f_n : [a, b] \to \mathbb{R}$, such that the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to a continuous function $f : [a, b] \to \mathbb{R}$. Show that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on $[a, b]$

Your answer

Score: 10

2 Metric Spaces

Exercise 7. Let (E, d) and (F, δ) be two metric spaces, $f : E \to F$ a uniformly continuous function, and $(x_n)_{n\in\mathbb{N}}$ a Cauchy sequence in E. Show that $(f(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in F.

Your answer

Score: 10

Exercise 8. Let $C([0,1])$ be the space of continuous functions on $[0,1]$ with the metric defined, for all $f, g \in C([0.1])$, by:

$$\delta(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}.$$

Show that the set

$$\left\{ f \in C([0,1]) : \int_0^1 t f(t)\, dt = 0 \right\}$$

is closed.

Your answer

Score: 10

Exercise 9. Let (E, d) and (F, δ) be two metric spaces and $f : E \to F$ be given. Let:

$$C = \{x \in E : f \text{ is continuous at } x\}.$$

Show that there exists a sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of E such that $C = \bigcap_{n\in\mathbb{N}} U_n$ and such that $U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$.

Your answer

Score:	10

3 Topology

Exercise 10. Let (E, τ_E) be a compact Hausdorff space and A, B be two disjoint closed subsets of (E, τ_E). Show that there exist two disjoint open sets U, V of (E, τ_E) such that $A \subseteq U$ and $B \subseteq V$.

Your answer

Score: 10

Exercise 11. Let (E, τ) be a compact Hausdorff space and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed connected subsets of (E, d) such that for all $n \in \mathbb{N}$ we have $A_{n+1} \subseteq A_n$. Show that $\bigcap_{n\in\mathbb{N}} A_n$ is connected. *Hint: you may use for this problem, without proof, the fact that given two disjoint closed subsets A, B of E, there exist two disjoint open subsets U, V of E such that $A \subseteq U$ and $B \subseteq V$.*

Your answer

Score:	10

Exercise 12. Let (E, τ_E) be a topological metric space and (F, τ_F) be a Hausdorff topological space. Let $f : E \to F$ be continuous. Show that:

$$\{(x, f(x)) : x \in E\}$$

is closed in $E \times F$, where $E \times F$ has the product topology.

Your answer

Score: 10

Score

- Score for Exercise 1:
- Score for Exercise 2:
- Score for Exercise 3:
- Score for Exercise 4:
- Score for Exercise 5:
- Score for Exercise 6:
- Score for Exercise 7:
- Score for Exercise 8:
- Score for Exercise 9:
- Score for Exercise 10:
- Score for Exercise 11:
- Score for Exercise 12:
- **Total Score out of 120:**
- **Grade:**

Scratch Page 1

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