Ph.D. Preliminary Examination
in Real Analysis
Fall 2004

May 4, 2005

Instructions. Answer all of the following questions.

1. Let

\[ f_n(x) = \begin{cases} 
  n^2 x & \text{if } 0 \leq x \leq \frac{1}{n} \\
  -n^2 x + 2n & \text{if } \frac{1}{n} < x \leq \frac{2}{n} \\
  0 & \text{if } \frac{2}{n} < x \leq 1 
\end{cases} \]

(a) Compute \( \lim \int_0^1 f_n(x) \, dx \) and \( \int_0^1 \lim f_n(x) \, dx \).

(b) If the answers in (a) agree give an explanation. And if they disagree give conditions such that if \( g_n \) converges pointwise to \( g \), then \( \int_0^1 g_n(x) \, dx \) converges to \( \int_0^1 g(x) \, dx \).

2.

(a) Define: The series \( \sum_{n=1}^{\infty} a_n = a \).

(b) Using the definition you gave in part (a), prove that if \(-1 < x < 1\), then

\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}. \]

(c) Prove this version of the ratio test: Suppose that \( a_n > 0 \) for \( n = 1, 2, \ldots \) and

\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1. \]

Then the series \( \sum_{n=0}^{\infty} a_n \) converges.

(d) Prove this version of the root test: Suppose that \( a_n > 0 \) for \( n = 1, 2, \ldots \) and

\[ \lim_{n \to \infty} \sqrt[n]{a_n} < 1. \]

Then the series \( \sum_{n=0}^{\infty} a_n \) converges.
3. Let $X$ be a compact metric space.

(a) Suppose that $f : X \to \mathbb{R}$ is continuous and that for every $x \in X$ there exists $x' \in X$ such that $|f(x')| \leq \frac{1}{2} |f(x)|$. Prove that $f$ has a zero. (That is, show that there is an $x \in X$ so that $f(x) = 0$.)

(b) Show that if $G_1, G_2, \ldots$ are dense open subsets of $X$, then $\cap_{i=1}^{\infty} G_i$ is dense in $X$.

(c) Show that if $X = \cup_{i=1}^{\infty} F_i$ and each $F_i$ is closed, then one of the $F_i$’s has non empty interior.

4. Definition: Let $X$ be a metric space and $f : X \to \mathbb{R}$ a function. We say that $f$ is lower semicontinuous on $X$ if, for each $r \in \mathbb{R}$,

$$\{x \in X : f(x) > r\}$$

is open in $X$.

(a) Prove that the following are equivalent for $f : X \to \mathbb{R}$.

i. $f$ is lower semicontinuous on $X$.

ii. For every sequence $(x_n)_{n=1}^{\infty}$ which converges to some $x \in X$,

$$f(x) \leq \liminf_{n \to \infty} f(x_n).$$

(b) For $f, g : X \to \mathbb{R}$, define $f \vee g : X \to \mathbb{R}$ by

$$(f \vee g)(x) = \max\{f(x), g(x)\}.$$ 

Prove that if $f, g : X \to \mathbb{R}$ are lower semicontinuous on $X$, so is $f \vee g$.

(c) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is lower semicontinuous on $\mathbb{R}$ but not continuous on $\mathbb{R}$.

5.

(a) Prove that the dual space of $\ell_1$ is $\ell_\infty$.

(b) Let $j : \ell_1 \to (\ell_\infty)^*$ be the natural embedding, i.e.

$$j(x)(f) = f(x)$$

for $x \in \ell_1$ and $f \in \ell_\infty$. Prove that $j$ is not onto.