The Varieties of Loops of Bol-Moufang Type

J. D. Phillips and Petr Vojtěchovský

Abstract. A loop identity is of Bol-Moufang type if two of its three variables occur once on each side, the third variable occurs twice on each side, and the order in which the variables appear on both sides is the same, viz. $((xy)x)z = x(y(xz))$. Loop varieties defined by one identity of Bol-Moufang type include groups, Bol loops, Moufang loops and C-loops. We show that there are exactly 14 such varieties, and determine all inclusions between them, providing all necessary counterexamples, too. This extends and completes the programme of Fenyves [5].

1. Introduction

An identity $\varphi = \psi$ is said to be of *Bol-Moufang type* if: (i) the only operation appearing in $\varphi = \psi$ is a binary operation, (ii) the number of distinct variables appearing in φ (and thus in ψ) is 3, (iii) the number of variables appearing in φ (and thus in ψ) is 4, (iv) the order in which the variables appear in φ coincides with the order in which they appear in ψ.

The *variety of loops* consists of universal algebras $(L, \cdot, \backslash, /, e)$ whose binary operations $\cdot$, $/$, $\backslash$ satisfy

$$a \cdot (a \backslash b) = b, \quad (b/a) \cdot a = b, \quad a \backslash (a \cdot b) = b, \quad (b \cdot a)/a = b,$$

and whose nullary operation e satisfies

$$e \cdot a = a \cdot e = a.$$

Recall that any loop satisfies the identities $(x/y) \backslash x = y$, $x/(y \backslash x) = y$.

For the rest of the paper, all identities of Bol-Moufang type $\mathcal{B}$ are loop identities with $\cdot$ as the binary operation.

We say that all identities in a subset of $\mathcal{B}$ are *equivalent* if each of them defines the same variety of loops. In this sense, most of the varieties of Bol-Moufang type can be defined in several equivalent ways. It is then a nontrivial task of practical importance to describe all maximal subsets of equivalent identities in $\mathcal{B}$. In fact, this work was partially motivated by the authors' frustration with the inconsistencies in the literature concerning definitions of loop varieties.

This paper presents the classification of all varieties of *loops of Bol-Moufang type*, i.e., varieties of loops defined by one identity from $\mathcal{B}$. We determine the variety defined by each identity $\mathcal{B}$ and conclude that there are 14 such varieties, including groups, Bol loops, Moufang loops and C-loops. We then describe the inclusions among all these varieties and provide all necessary distinguishing examples.

Many of the results below were known already to Fenyves [4], [5]. See the Acknowledgement and historical remarks for more information.

1991 *Mathematics Subject Classification*: 20N05.
Key words and phrases: loops of Bol-Moufang type, Moufang loops, Bol loops, C-loops, alternative loops.

2. Systematic notation

Let x, y, z be all the variables appearing in the identities of $\mathcal{B}$. Without loss of generality, we can assume that they appear in the terms in alphabetical order. Then there are exactly 6 ways in which the 3 variables can form a word of length 4, and there are exactly 5 ways in which a word of length 4 can be bracketed, namely:

A	$xxyz$
B	$xyxz$
C	$xyyz$
D	$xyzx$
E	$xyzy$
F	$xyzz$

1	$o(o(oo))$
2	$o((oo)o)$
3	$(oo)(oo)$
4	$(o(oo))o$
5	$((oo)o)o$

Let Xij with $X \in \{A, \dots, F\}$, $1 \leq i < j \leq 5$ be the identity from $\mathcal{B}$ whose variables are ordered according to X, whose left-hand side is bracketed according to i, and whose right-hand side is bracketed according to j. For instance, $C25$ is the identity $x((yy)z) = ((xy)y)z$.

It is now clear that any identity in $\mathcal{B}$ can be transformed into some identity Xij by renaming the variables and interchanging the left-hand side with the right-hand side. There are therefore $6 \cdot (4 + 3 + 2 + 1) = 60$ “different” identities in $\mathcal{B}$.

The *dual* of an identity I is the identity obtained from I by reading it backwards, i.e., from right to left. For instance, the dual of $(xy)(xz) = ((xy)x)z$ is the identity $z(x(yx)) = (zx)(yx)$. With the above conventions in mind, we can rewrite the latter identity as $x(y(zy)) = (xy)(zy)$. One can therefore identify the dual of any identity Xij with some identity $X'j'i'$. The name $X'j'i'$ of the dual of Xij is easily calculated with the help of the following rules:

$$A' = F, \quad B' = E, \quad C' = C, \quad D' = D, \quad 1' = 5, \quad 2' = 4, \quad 3' = 3.$$

Finally, we will use the following notational conventions: we usually omit $\cdot$ while multiplying two elements (eg $x \cdot y = xy$), we declare $\backslash$ and $/$ to be less binding than the omitted multiplication (eg $x/yz = x/(yz)$), and if $\cdot$ is used, we consider it to be less binding than any other operation (eg $x \cdot yz \backslash y = x((yz) \backslash y)$).

3. Canonical definitions of some varieties of loops

Table 1 defines 15 varieties of loops. With the exception of the 3-power associative loops, all these varieties can be defined by some identity Xij. Namely, GR is equivalent to $A12$ (cancel x on the left), LA to $A45$ (substitute e for z), RA to $F12$ (duality), and FL to $B45$ (substitute e for z).

We have carefully chosen the defining identities in such a way that they are either self-dual (GR, EL, CL, FL, MN, 3PA) or coupled into dual pairs (LB′=RB, LC′=RC, LA′=RA, LN′=RN). The only exception to this rule is the Moufang identity $D34$. We will often appeal to this duality in our proofs.

Only four of the above varieties were not previously named in the literature, namely the *left*, *middle* and *right nuclear square loops*, and the 3-*power associative loops*. Since we have no desire to swamp the field with new definitions, we opted for these longer, descriptive names. The reader should note that 3-power associative loops are not necessarily *power associative*, i.e., the subloop generated by x does not have to be a group in a 3-power associative loop (viz. Example 3.1). We have included 3PA as a technical variety that will allow us to make several arguments faster.

TABLE 1. Definitions of varieties of loops of Bol-Moufang type.

variety	abbrev.	defining identity	its name	ref.
groups	GR	$x(yz) = (xy)z$		folklore
extra loops	EL	$x(y(zx)) = ((xy)z)x$	$D15$	[4]
Moufang loops	ML	$(xy)(zx) = (x(yz))x$	$D34$	[8], [1, p. 58], [10]
left Bol loops	LB	$x(y(xz)) = (x(yx))z$	$B14$	[11]
right Bol loops	RB	$x((yz)y) = ((xy)z)y$	$E25$	[1, p. 116], [11]
C-loops	CL	$x(y(yz)) = ((xy)y)z$	$C15$	[5]
LC-loops	LC	$(xx)(yz) = (x(xy))z$	$A34$	[5]
RC-loops	RC	$x((yz)z) = (xy)(zz)$	$F23$	[5]
left alternative loops	LA	$x(xy) = (xx)y$		folklore
right alternative loops	RA	$x(yy) = (xy)y$		folklore
flexible loops	FL	$x(yx) = (xy)x$		[10, p. 89]
left nuclear square loops	LN	$(xx)(yz) = ((xx)y)z$	$A35$	
middle nuclear square loops	MN	$x((yy)z) = (x(yy))z$	$C24$	
right nuclear square loops	RN	$x(y(zz)) = (xy)(zz)$	$F13$	
3-power associative loops	3PA	$x(xx) = (xx)x$		

Example 3.1. This is a loop that is 3-power associative but not power associative (since $(1 \cdot 1)(1 \cdot 1) \neq 1(1(1 \cdot 1))$):

0	1	2	3	4	5
1	2	0	4	5	3
2	0	3	5	1	4
3	4	5	0	2	1
4	5	1	2	3	0
5	3	4	1	0	2

A loop L is said to have the *left inverse property* if $e/x \cdot xy = y$ for every $y \in L$. Dually, L has the *right inverse property* if $yx \cdot x \setminus e = y$ for every $y \in L$. If L has both the left and right inverse property, it is called an *inverse property loop*.

When L has the left inverse property, it also has two-sided inverses since $e/x = e/x \cdot x(x \setminus e) = x \setminus e$. The same conclusion holds when L has the right inverse property. The two-sided inverse of x will be denoted by x^{-1}. However, when a loop has two-sided inverses, it does not have to have any inverse properties, as the following example shows:

Example 3.2. This is a loop that has two-sided inverses but is neither a left inverse property loop (since $1^{-1}(1 \cdot 2) \neq 2$), nor a right inverse property loop (since $(2 \cdot 1)1^{-1} \neq 2$):

0	1	2	3	4
1	0	3	4	2
2	4	0	1	3
3	2	4	0	1
4	3	1	2	0

4. Equivalences

We now begin the exhaustive search for equivalent Bol-Moufang identities.

Proposition 4.1. *The following Bol-Moufang identities are equivalent to the defining group identity $x(yz) = (xy)z$: $A12$, $A23$, $A24$, $A25$, $B12$, $B13$, $B24$, $B25$, $B34$, $B35$, $C13$, $C23$, $C34$, $C35$, $D12$, $D13$, $D14$, $D25$, $D35$, $D45$, $E13$, $E14$, $E23$, $E24$, $E35$, $E45$, $F14$, $F24$, $F34$, $F45$.*

Proof. We have already noted that $A12$ defines groups. We now briefly describe how each of the remaining identities listed in this Proposition can be seen to be equivalent to groups.

For $A23$: let $z = e$, deduce LA, then use LA to rewrite $A23$ into $A12$. For $A24$: note that given x, any u can be written as xy for some y; then use $u = xy$ in $A24$. For $A25$: let $z = e$, deduce LA, then use LA to rewrite $A25$ into $A24$. For $B12$: cancel x on the left. For $B13$: let $u = xz$. For $B24$: let $u = yx$. For $B25$: let $z = e$, deduce FL, then use FL to rewrite $B25$ as $B24$. For $B34$: let $z = e$, deduce FL, then use FL to write $B34$ as $B35$, let $u = xy$ in $B35$. For $B35$: see $B34$. For $C13$: let $u = yz$. For $C23$: let $x = e$, deduce LA, then use LA to write $C23$ as $C13$. For $D12$: cancel x on the left. For $D13$: let $u = zx$. For $D14$: let $z = e$, deduce FL, then use FL to write $D14$ as $D12$.

The remaining 15 identities are duals of the already investigated 15 identities. Since the defining group identity is self-dual, we are done. □

Proposition 4.2. *The following Bol-Moufang identities are equivalent to the defining extra identity $D15$: $B23$, $D15$, $E34$.*

Proof. See [4, Thm. 1]. □

Proposition 4.3. *The following Bol-Moufang identities are equivalent to the defining Moufang loop identity $D34$: $B15$, $D23$, $D34$, $E15$.*

Proof. In [10, p.88–89], Pflugfelder defines identities (MI), (M_5), (M_6) and (M_7), and shows in [10, Thm. IV.1.4] that these identities are equivalent. They correspond to our identities $D34$, $D23$, $B15$ and $E15$, respectively. □

Remark 4.4. We would like to point out that [10, Thm. IV.1.4] also says that (MI) is equivalent to (M_4)=B14. This is true only if flexibility holds in the loop in question, which is what Pflugfelder tacitly assumes.

Lemma 4.5. *Let L be an LC-loop. Then L is left alternative, has the left inverse property, is a middle nuclear square loop, and satisfies $C14$.*

Proof. The left alternative law follows from $A34$ with $z = e$. Hence $A34$ implies $A14$. By $A14$, we have $x(x \cdot (x \setminus e)z) = x(x \cdot x \setminus e) \cdot z = xz$, and thus $x \cdot (x \setminus e)z = z$. With $x = e/y$, we obtain $z = e/y \cdot ((e/y) \setminus e)z = e/y \cdot yz$, and L has the left inverse property.

By $A14$ and the left inverse property, $x(x \cdot (x^{-1}y)z) = x(x \cdot x^{-1}y) \cdot z = xy \cdot z$. With $(x^{-1}y)^{-1}z$ instead of z, we get $xy \cdot (x^{-1}y)^{-1}z = x(xz) = (xx)z$. Therefore $(x^{-1}y)^{-1}z = (xy)^{-1} \cdot (xx)z$, which reduces to $(x^{-1}y)^{-1} = (xy)^{-1}(xx)$ with $z = e$. But then $(xy)^{-1} \cdot (xx)z = (x^{-1}y)^{-1}z = (xy)^{-1}(xx) \cdot z$, and thus L is a middle nuclear square loop. The identity $C14$ follows by LA. □

Lemma 4.6. *Assume that L is a loop satisfying $C14$. Then L is an LC-loop.*

Proof. The left alternative law follows from $C14$ with $x = e$. By $C14$, we have $e = x \cdot x \setminus e = (x/xx \cdot xx)(x \setminus e) = (x/xx)(x \cdot x(x \setminus e)) = x/xx \cdot x$, and hence $e/x = x/xx$, or $e/x \cdot xx = x$. Then $C14$ yields $e/x \cdot xy = e/x \cdot x(x \cdot x \setminus y) = (e/x \cdot xx)(x \setminus y) = x(x \setminus y) = y$, and L has the left inverse property.

By $C14$ and the left inverse property, $x(yy) \cdot y^{-1}(y^{-1}x^{-1}) = x \cdot y(y \cdot y^{-1}(y^{-1}x^{-1})) = e$, and thus $(x \cdot yy)^{-1} = y^{-1} \cdot y^{-1}x^{-1}$. Applying the left inverse property to $(x \cdot yy)z = x(y \cdot yz)$ yields $y^{-1}(y^{-1}(x^{-1} \cdot (x \cdot yy)z)) = z$. With $u = (x \cdot yy)z$, the last identity becomes $y^{-1}(y^{-1}(x^{-1}u)) = z = (x \cdot yy)^{-1}u = (y^{-1} \cdot y^{-1}x^{-1})u$, and $A34$ follows by LA. □

Proposition 4.7. *The following Bol-Moufang identities are equivalent to the defining LC-loop identity $A34$: $A14$, $A15$, $A34$, $C14$.*

Proof. Note that any of the three identities $A14$, $A15$, $A34$ yield LA (let $y = e$ in $A14$, $z = e$ in $A15$, use Lemma 4.5(i) for $A34$). With LA, the three identities are immediately seen to be equivalent. Lemmas 4.5 and 4.6 show that C14 is equivalent to A34. □

By the duality, we obtain:

Proposition 4.8. *The following Bol-Moufang identities are equivalent to the defining RC-loop identity $F23$: $C25$, $F15$, $F23$, $F25$.*

Proposition 4.9. *The following Bol-Moufang identities are equivalent to the defining left alternative identity $x(xy) = (xx)y$: $A13$, $A45$, $C12$.*

Proof. Both $A13$ and $A45$ yield LA with $z = e$, while $C12$ yields LA with $x = e$. On the other hand, LA obviously implies each of the three identities. They are therefore equivalent. □

By the duality, we obtain:

Proposition 4.10. *The following Bol-Moufang identities are equivalent to the defining right-alternative identity $x(yy) = (xy)y$: $C45$, $F12$, $F35$.*

Proposition 4.11. *The following Bol-Moufang identities are equivalent to the defining flexible identity $x(yx) = (xy)x$: $B45$, $D24$, $E12$.*

Proof. As the defining identity FL is self-dual and $(B45)' = E12$, it suffices to show that the identities $B45$ and $D24$ are equivalent to FL. It is obviously true for $B45$ (cancel z on the right). With $z = e$, $D24$ reduces to FL. Using FL with yz instead of y yields $D24$. □

The only identities not covered by Propositions 4.1–4.11 are $A35$, $B14$, $C15$, $C24$, $E25$ and $F13$. These are the defining identities of LN, LB, CL, MN, RB and RN, respectively.

5. Implications

We now show how the 14 varieties of loops of Bol-Moufang type are related to each other.

Lemma 5.1. *The following inclusions hold among the varieties of loops of Bol-Moufang type and* 3PA*:* GR⊆EL*,* EL⊆ML*,* EL⊆CL*,* ML⊆LB*,* ML⊆RB*,* CL⊆LC*,* CL⊆RC*,* ML⊆FL*,* LB⊆LA*,* RB⊆RA*,* LC⊆LA*,* RC⊆RA*,* LC⊆LN*,* LC⊆MN*,* RC⊆MN*,* RC⊆RN*,* FL⊆3PA*,* LA⊆3PA*,* RA⊆3PA*. The situation is depicted in Figure* 1*.*

Proof. GR is contained in any variety of loops listed in Table 1. It is shown in [5, Thm. 2] and [3, Corollary 2] that extra loops are precisely Moufang loops where every square belongs to the nucleus. In [5], Fenyves shows that extra loops are C-loops [5], and that C-loops are both LC-loops and RC-loops [5, Thm. 4]. It is well-known (cf [11, Thm. 2.7]) that Moufang loops are both left Bol and right Bol. Moufang loops are flexible, as one can see upon letting $z = e$ in $B15$. Robinson [11, Thm. 2.1] makes the simple observation that right Bol loops are right alternative. The dual of this statement then holds, too. LC-loops are left alternative by Lemma 4.5, and, dually, RC-loops are right alternative. With the left alternative law at our disposal, we see immediately that LC-loops are left nuclear square. The dual of this statement then holds, too. Clearly, any of FL, LA or RA implies 3-power associativity. It remains to show that both LC-loops and RC-loops are middle nuclear square. This follows from Lemma 4.5(iv) and its dual. □

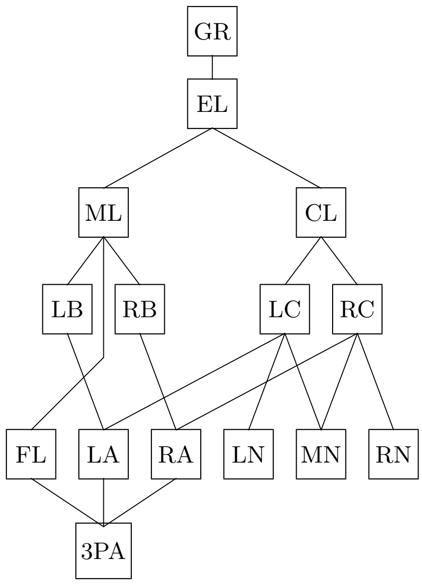

FIGURE 1. Varieties of loops of Bol-Moufang type and 3-power associative loops. If A, B are varieties such that $A \subseteq B$ then A is depicted above B.

6. Distinguishing examples

We proceed to show that all the 14 varieties of loops of Bol-Moufang type are indeed distinct, and that no edges (inclusions) are missing in Figure 1. Our intention is to come up with as few examples as possible to accomplish this. It turns out that 8 examples and their duals suffice.

For the convenience of the reader, we provide Table 2 that points to examples distinguishing any two given varieties of loops of Bol-Moufang type. If the cell in row A and column B of Table 2 is empty then A is a subvariety of B. If the cell contains the integer n, then the loop of Example 6.n belongs to $A \setminus B$. A primed number n' indicates that one should use the dual of the respective example.

All multiplication tables below have 0 as a neutral element. We believe that all examples below are as small as possible (when all properties are to be satisfied at the same time).

Example 6.1 (Extra loop that is not a group)**.** The Moufang loop that Goodaire et al. [6] call 16/1 is a nonassociative extra loop. Instead of giving its multiplication table, we recall a general construction due to Chein [2] that produces the loop 16/1.

For a group G, let $M(G, 2) = G \times \{0, 1\}$, where $(g, 0)(h, 0) = (gh, 0)$, $(g, 0)(h, 1) = (hg, 1)$, $(g, 1)(h, 0) = (gh^{-1}, 1)$, and $(g, 1)(h, 1) = (h^{-1}g, 0)$. Then $M(G, 2)$ is a nonassociative Moufang loop if and only if G is nonabelian.

Then 16/1 is the loop $M(D_4, 2)$, where D_4 is the dihedral group of order 8.

Example 6.2 (Moufang loop that is neither left nuclear square nor middle nuclear square)**.** Take the loop $M(S_3, 2)$, where S_3 is the symmetric group on 3 points.

TABLE 2. Distinguishing varieties of loops of Bol-Moufang type.

	GR	EL	ML	CL	LB	RB	LC	RC	LA	FL	RA	LN	MN	RN
GR														
EL	1													
ML	2	2		2			2	2				2	2	$2'$
CL	3	3	3		3	$3'$				3				
LB	2	2	4	2		4	2	2		4	4	2	2	$2'$
RB	2	2	$4'$	2	$4'$		2	2	$4'$	$4'$		2	2	$2'$
LC	3	3	3	5	3	$3'$		5		3	5			5
RC	3	3	3	$5'$	3	$3'$	$5'$		$5'$	3		$5'$		
LA	2	2	3	2	3	$3'$	2	2		4	5	2	2	$2'$
FL	2	2	6	2	6	$6'$	2	2	6		$6'$	2	2	$2'$
RA	2	2	3	2	3	$3'$	2	2	$5'$	$4'$		2	2	$2'$
LN	3	3	3	7	3	$3'$	7	7	7	3	7		7	5
MN	3	3	3	8	3	$3'$	8	8	8	3	8	$5'$		5
RN	3	3	3	$7'$	3	$3'$	$7'$	$7'$	$7'$	3	$7'$	$5'$	$7'$	

Example 6.3 (C-loop that is neither flexible, nor left Bol)**.** This example first appeared in [7].

0	1	2	3	4	5	6	7	8	9	10	11
1	2	0	4	5	3	7	8	6	10	11	9
2	0	1	5	3	4	8	6	7	11	9	10
3	4	5	0	1	2	10	11	9	8	6	7
4	5	3	1	2	0	11	9	10	6	7	8
5	3	4	2	0	1	9	10	11	7	8	6
6	7	8	11	9	10	0	1	2	4	5	3
7	8	6	9	10	11	1	2	0	5	3	4
8	6	7	10	11	9	2	0	1	3	4	5
9	10	11	7	8	6	5	3	4	0	1	2
10	11	9	8	6	7	3	4	5	1	2	0
11	9	10	6	7	8	4	5	3	2	0	1

The loop is not flexible since $8(9 \cdot 8) \neq (8 \cdot 9)8$, and it is not left Bol since $5(8(5 \cdot 5)) \neq (5(8 \cdot 5))5$.

Example 6.4 (Left Bol loop that is neither flexible, nor right alternative)**.**

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	5	1	7	0	6	2	4
4	2	6	0	7	1	5	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

The loop is not flexible since $1(2 \cdot 1) \neq (1 \cdot 2)1$, and it is not right alternative since $6(4 \cdot 4) \neq (6 \cdot 4)4$.

Example 6.5 (LC-loop that is neither right nuclear square, nor right alternative).

0	1	2	3	4	5	6	7	8	9	10	11
1	0	3	2	5	4	7	6	9	8	11	10
2	3	0	1	6	7	4	5	10	11	8	9
3	2	6	7	0	1	10	11	4	5	9	8
4	5	1	0	8	9	2	3	11	10	6	7
5	4	8	9	1	0	11	10	2	3	7	6
6	7	11	10	2	3	8	9	1	0	4	5
7	6	10	11	3	2	9	8	0	1	5	4
8	9	5	4	11	10	1	0	7	6	2	3
9	8	4	5	10	11	0	1	6	7	3	2
10	11	7	6	9	8	3	2	5	4	0	1
11	10	9	8	7	6	5	4	3	2	1	0

The loop is not right nuclear square since $1(2(3\cdot 3)) \neq (1\cdot 2)(3\cdot 3)$, and it is not right alternative since $1(2\cdot 2) \neq (1\cdot 2)2$.

Example 6.6 (Flexible loop that is not left alternative).

0	1	2	3	4
1	0	3	4	2
2	4	0	1	3
3	2	4	0	1
4	3	1	2	0

The loop is not left alternative since $1(1\cdot 2) \neq (1\cdot 1)2$.

Example 6.7 (Left nuclear square loop that is neither middle nuclear square, nor 3-power associative).

0	1	2	3	4	5
1	5	0	4	3	2
2	0	4	5	1	3
3	4	5	0	2	1
4	2	3	1	5	0
5	3	1	2	0	4

The loop is not middle nuclear square since $1((2\cdot 2)3) \neq (1(2\cdot 2))3$, and it is not 3-power associative since $1(1\cdot 1) \neq (1\cdot 1)1$.

Example 6.8 (Middle nuclear square loop that is not 3-power associative).

0	1	2	3	4	5
1	2	3	0	5	4
2	4	5	1	3	0
3	5	4	2	0	1
4	0	1	5	2	3
5	3	0	4	1	2

The loop is not 3-power associative since $1(1\cdot 1) \neq (1\cdot 1)1$.

7. Summary

There are 14 varieties of loops of Bol-Moufang type. Their definitions can be found in Table 1. They are related according to Figure 1. One can look up examples distinguishing any two varieties in Table 2. Since we believe this paper will be used as a quick reference, we also include

TABLE 3. Loop varieties determined by identities of Bol-Moufang type.

$B \setminus A$	1	2	3	4	5
1		GR	LA	LC	LC
2	GR		GR	GR	GR
3	GR	EL		LC	LN
4	LB	GR	GR		LA
5	ML	GR	GR	FL	

$D \setminus C$	1	2	3	4	5
1		LA	GR	LC	CL
2	GR		GR	MN	RC
3	GR	ML		GR	GR
4	GR	FL	ML		RA
5	EL	GR	GR	GR	

$F \setminus E$	1	2	3	4	5
1		FL	GR	GR	ML
2	RA		GR	GR	RB
3	RN	RC		EL	GR
4	GR	GR	GR		GR
5	RC	RC	RA	GR	

Table 3, that determines the variety defined by any of the equations Xij in $\mathcal{B}$, although the same information is given in Section 4. To save space, we list identities of type A, C, E as Xij with $i < j$, and identities of type B, D, F as Xij with $i > j$.

8. Acknowledgement and historical remarks

The classification of varieties of loops of Bol-Moufang type was initiated by Fenyves [4], [5]. He was aware of all results of Section 4 with the exception of the fact that C14 was equivalent to the LC-identity A34, of the dual statement, and of some parts of Lemma 4.6. He mentions all inclusions of Figure 1 with the exception of LC⊆MN and, dually, RC⊆MN. He only provides a few distinguishing examples.

In the introduction of [5], Fenyves claims: *"Our results make possible to decide of any two identities of Bol-Moufang type whether one of them imply the other or not."* This statement has now been justified.

The systematic notation is ours, and makes the discussion more transparent, in our opinion.

Our investigations were aided by the equational reasoning tool `Otter` and by the finite model builder `Mace4`. Both of these tools were developed by McCune [9]. Nevertheless, all proofs needed for the classification (including those we only refer to) are now presented in full, without any usage of computers.

We would like to thank the referee for several useful comments that allowed us to substantially shorten the proofs of Lemmas 4.5 and 4.6.

References

[1] R. Hubert Bruck, A survey of binary systems, third printing, corrected, *Ergebnisse der Mathematik und Ihrer Grenzgebiete*, New series **20**, Springer-Verlag(1971).

[2] Orin Chein, *Moufang loops of small order*, Mem. Amer. Math. Soc. **13**(1978), no. 197.

[3] Orin Chein and D. A. Robinson, *An "extra" law for characterizing Moufang loops*, Proc. Amer. Math. Soc. **33**(1972), 29–32.

[4] Ferenc Fenyves, *Extra loops I.*, Publ. Math. Debrecen **15**(1968), 235–238.

[5] Ferenc Fenyves, *Extra loops II. On loops with identities of Bol-Moufang type*, Publ. Math. Debrecen **16**(1969), 187–192.

[6] Edgar G. Goodaire, Sean May, Maitreyi Raman, The Moufang loops of order less than 64, Nova Science Publishers, Inc., Commack, NY (1999).

[7] M. K. Kinyon, K. Kunen, and J. D. Phillips, *A generalization of Moufang and Steiner loops*, Algebra Universalis **48**(2002), 416–430.

[8] Ruth Moufang, *Zur Struktur der Alternativkoerpern*, Math. Ann. **110**(1935).

[9] W. McCune, `Otter`, equational reasoning tool, and `Mace4`, finite model builder. Argonne National Laboratory, 2003. Available at http://www-unix.mcs.anl.gov/AR.

[10] Hala O. Pflugfelder, Quasigroups and Loops: Introduction, *Sigma Series in Pure Mathematics* **7**, Heldermann Verlag Berlin(1990).

[11] D. A. Robinson, *Bol loops*, Trans. Amer. Math. Soc. **123**(1966), 341–354.

Department of Mathematics & Computer Science, Wabash College, Crawfordsville, Indiana 47933, U.S.A.

E-mail address: phillipj@wabash.edu

Department of Mathematics, University of Denver, 2360 S Gaylord St, Denver, Colorado 80208, U.S.A.

E-mail address: petr@math.du.edu