A NEW CHARACTERIZATION OF THE CONTINUOUS FUNCTIONS ON A LOCALE

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ABSTRACT. Within the category **W** of archimedean lattice-ordered groups with weak order unit, we show that the objects of the form $C(L)$, the set of continuous real-valued functions on a locale L, are precisely those which are divisible and complete with respect to a variant of uniform convergence, here termed indicated uniform convergence. We construct the corresponding completion of a **W**-object A purely algebraically in terms of Cauchy sequences. This completion can be variously described as c^3A, the "closed under countable composition hull of A," as $C(Y_\ell A)$, where $Y_\ell A$ is the Yosida locale of A, and as the largest essential reflection of A.

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1. Introduction

We introduce a natural convergence on **W**-objects called indicated uniform convergence. Though not topological, it is closely related to ordinary uniform convergence. We then develop the corresponding completion, called the indicated uniform completion, purely algebraically, i.e., by means of Cauchy sequences without reference to any representation of the **W**-objects. In fact, this completion is of the nicest sort, namely an essential epireflection (Theorem 3.2.10), and since such reflections have been thoroughly investigated (in [12] and [6], among other places), this fact raises the question of exactly what the complete **W**-objects are. We settle this question by showing that the **W**-objects which are divisible and complete are precisely those closed under countable composition. This famous class is of central importance in **W**, and coincides with the **W**-objects of the form $C(L)$ for a locale L, a theorem of Isbell [16]; see also [18]. Thus our construction provides a purely algebraic characterization of such objects.

Furthermore, indicated uniform completion bears exactly the same relationship to the localic Yosida representation of **W**-objects as ordinary uniform completion bears to the classical Yosida representation of **S**-objects, where **S** is the category of archimedean lattice-ordered groups with strong order unit, i.e., the full subcategory of bounded **W**-objects. In each case the embedding of the object in its completion is the unit of the corresponding Yosida adjunction. We elaborate. If one starts with an arbitrary **S**-object A, passes to its compact Hausdorff Yosida space YA, and then passes back to $C(YA)$, one obtains a complete **S**-object in which A naturally embeds so as to be dense, where both completeness and density in this context are understood to be with respect to ordinary uniform convergence. Thus $A \leq C(YA)$ is the uniform completion of A. On the other hand, if one starts with an arbitrary **W**-object A, passes to its regular Lindelöf Yosida locale $Y_\ell A$ (notation to be introduced later), and then passes back to $C(Y_\ell A)$, one obtains a complete **W**-object in which A naturally embeds so as to be dense, where both completeness and density in this context are understood to be with respect to indicated uniform convergence. Thus $A \leq C(Y_\ell A)$ is the indicated uniform completion of A.

1.1. Notation and ground-clearing.

A *lattice ordered group*, or *ℓ-group* for short, is a structure of the form $(A, +, -, 0, \vee, \wedge)$, where $(A, +, -, 0)$ is a group, $(A, \vee, \wedge)$ is a lattice, and the group and lattice operations are compatible in the sense that

$$a + (b \vee c) = (a + b) \vee (a + c)$$

for all $a, b, c \in A$. We employ the abbreviations $a_+ \equiv a \vee 1$, $a_- \equiv (-a) \vee 1$, $|a| \equiv a_+ \vee a_- = a \vee (-a)$. Likewise $A_+ \equiv \{a \in A : a \geq 0\}$. An *$\ell$-homomorphism* is a mapping $f : A \to B$ between ℓ-groups which is simultaneously a group and lattice homomorphism. A subset $B \subseteq A$ is an *ℓ-subgroup* if it is a subgroup and sublattice, and it is a convex ℓ-subgroup if it also satisfies

$$b_1 \leq a \leq b_2 \Longrightarrow a \in B$$

for all $b_1, b_2 \in B$ and $a \in A$. The kernels of ℓ-homomorphisms are precisely the convex normal ℓ-subgroups. Fine references on the general theory of lattice ordered groups are [1], [9], and [10].

In this article we will be interested only in ℓ-groups which are abelian; we use **alGp** to designate the category of such ℓ-groups with ℓ-homomorphisms. Abelian ℓ-groups have the virtue of unique extraction of roots, i.e., if $na_1 = na_2$ for $a_i \in A$ and $n \in N$ then $a_1 = a_2$.

This justifies use of the notation $\frac{p}{q}a$, $p \in Z$ and $q \in N$, designating that element $b \in A$ satisfying $qb = pa$ if such exists. We say that A is divisible if $\frac{p}{q}a$ exists in A for all $a \in A$, $p \in Z$, and $q \in N$.

An ℓ-group A is said to be *archimedean* if for all $0 \leq a, b \in B$ such that $na \leq b$ for all $n \in N$ we have that $a = 0$. Archimedean ℓ-groups are necessarily abelian. An element u of an ℓ-group A is said to be a *strong order unit* if for all $a \in A$ there is some $n \in N$ for which $|a| \leq nu$. The category **S** has objects of the form $(A, 1_A)$, where A is an archimedean ℓ-group and 1_A is the designated strong order unit if A. The morphisms of **S** are required to preserve the strong unit, i.e., an **S**-morphism $f : A \to B$ satisfies $f(1_A) = 1_B$. A prototypical **S**-object is $(C(X), 1)$, where $C(X)$ is the ℓ-group of continuous real-valued functions on a compact Hausdorff space X, and 1 is the constantly 1 function.

An element u is said to be a *weak order unit* if $u \geq 0$ and for all $a \in A$, $|a| \wedge u = 0$ implies $a = 0$. The category **W** has objects of the form $(A, 1_A)$, where A is an archimedean ℓ-group and 1_A is the designated weak order unit of A. The morphisms of **W** are required to preserve the weak unit, i.e., a **W**-morphism $f : A \to B$ satisfies $f(1_A) = 1_B$. A prototypical **W**-object is $(C(X), 1)$, where $C(X)$ is the ℓ-group of continuous real-valued functions on a not-necessarily-compact Tychonoff space X, and 1 is the constantly 1 function. Of course, these objects are much-studied [11].

2. Indicated uniform convergence and completion

Our objective in this section is the development in **W** of indicated uniform convergence, together with the associated notions of completion, density, and closure. The development is abstract inasmuch as it uses only constructs built into **W**, and avoids reference to any representation of **W** objects. The first step in this development is to analyze uniform convergence with a designated indicator, and that requires introducing the auxiliary category **Wi**.

2.1. **Uniform convergence and completion in Wi.** We enrich each **W**-object by designating an indicator.

Definition 2.1.1. *The category* **Wi** *has objects of the form* (A, r)*, where* $1 \leq r \in A \in \mathbf{W}$*. We refer to the element* r *as the* indicator *of* A*, and sometimes write it as* r_A*. We also sometimes simply say that* A *is a* **Wi***-object, the unit and indicator being understood. A* **Wi***-morphism is a map* $f : A \to B$ *such that* $f \in \mathbf{W}$*, i.e.,* $f(1_A) = 1_B$*, and such that* $f(r_A) \leq r_B$*.*

Wi provides the context for this subsection; the letters A, B, and C stand for **Wi**-objects unless otherwise explicitly stipulated. We say that A *is a* **Wi***-subobject of* B, and write $A \leq B$, if A is a subset of B such that the insertion map is a **Wi**-morphism. That is to say that A is an ℓ-subgoup of B, that $1_A = 1_B$, and that $r_A \leq r_B$. Notice that $A \leq B$ does not imply that $r_A = r_B$.

We use the indicator to define a convergence which will play a central role in our investigation.

Definition 2.1.2. *Let* $\{a_i\}$ *be a sequence and* a *an element in* A*. We say that* $\{a_i\}$ converges uniformly to a*, and write* $a_i \to a$*, provided that for all* $p, k \in N$ *there is some* $m \in N$ *such that for all* $i \geq m$*,*

$$(*) \qquad k[(p - r) \wedge |a_i - a|] \leq 1.$$

We use the phrase *ordinary uniform convergence* to refer to uniform convergence with indicator 1, and we assume the reader to be familiar with the properties of this convergence and its completion by Cauchy sequences ([14], [13]). In the context of **Wi**, the unqualified phrase *uniform convergence*, or sometimes for emphasis *indicated uniform convergence*, refers to the more general convergence of Definition 2.1.2. Indicated uniform convergence is closely related to ordinary uniform convergence, and in fact the two coincide whenever the indicator r is bounded. But even when r is not bounded, $a_i \to a$ in A if and only if, in any representation of A as a **Wi**-subobject of $D(X)$ for some space X (see Subsection 4.1), the functions a_i approach a uniformly on sets where r is bounded.

We get the working form of Definition 2.1.2 by dividing $(*)$ by k and replacing $1/k$ by ε.

$$\forall p \, \forall \varepsilon > 0 \, \exists m \, \forall i \geq m \, \left((p - r) \wedge |a_i - a| \leq \varepsilon \right).$$

We henceforth adopt this more evocative terminology, and assure the squeamish reader that every reference to real numbers like ε can be replaced by the formalism of $(*)$ so as to cast the entire discussion strictly in terms which make sense in **Wi**. To be sure, some of the results do require the hypothesis of divisibility in order that A be closed under rational multiplication. In fact, the reader who wishes to replace this hypothesis by the stronger hypothesis that A is a vector lattice, i.e., that A is closed under real multiplication, may do so without damage to the ideas and with little loss of generality. That is because the indicated uniform completion is uniformly complete in the classical sense, and a **W**-object which is both divisible and uniformly complete is automatically a vector lattice.

The elementary properties of indicated uniform convergence follow directly from the definition. Here and in what follows, we use $\{\dot{a}\}$ to designate the constant sequence whose every term is a.

Proposition 2.1.3. *Let $\{a_i\}$ be a sequence and a an element in A, and let $g : A \to B$ be a* **Wi**-*morphism.*

1. *g is uniformly continuous, i.e., if $a_i \to a$ in A then $g(a_i) \to g(a)$ in B. In particular, if $A \leq B$ and $a_i \to a$ holds in A then $a_i \to a$ holds also in B.*
2. *Suppose g is injective and $g(r_A) = r_B$. Then $a_i \to a$ holds in A if and only if $g(a_i) \to g(a)$ holds in B. In particular, if $A \leq B$ and $r_A = r_B$ then $a_i \to a$ holds in A if and only if $a_i \to a$ holds in B.*
3. *If $a_i \to a$ then $a_{i_j} \to a$ for any subsequence $\{a_{i_j}\}$ of $\{a_i\}$.*
4. *If $a_i = a$ for all $i \in N$ then $a_i \to a$. That is, $\{\dot{a}\}$ converges uniformly to a.*

It is important to point out that the hypothesis that $g(r_A) = r_B$ in Proposition 2.1.3(2) is necessary. Here is an example of a **Wi**-subobject $A \leq B$ with sequence $\{a_i\}$ and element a in A such that $a_i \to a$ holds in B but not in A.

Example 2.1.4. *Let $X \equiv [-1,1] \setminus \{0\} = [-1,0) \cup (0,1]$, and let*

$$A \equiv \left\{ a \in C(X) : \lim_{x \to 0^-} a(x) \text{ and } \lim_{x \to 0^+} a(x) \text{ exist} \right\},$$

$$B \equiv \left\{ b \in C(X) : \exists a \in A \, \exists s \in R \, \forall x \in X \, \left(b(x) = a(x) + \frac{s}{|x|} \right) \right\}.$$

If we agree that both A and B have unit 1, that A has indicator 1, and that B has indicator $\frac{1}{|x|}$, then clearly $A \leq B$ in **Wi**. *Consider now the sequence $\{a_i\}$ and element a in A defined* *by*

$$a(x) \equiv \begin{cases} -1 & \text{if } -1 \leq x < 0 \\ 1 & \text{if } 0 < x \leq 1 \end{cases},$$

$$a_i(x) \equiv \begin{cases} -1 & \text{if } -1 \leq x \leq -\frac{1}{i} \\ ix & \text{if } -\frac{1}{i} < x < \frac{1}{i},\ x \neq 0 \\ 1 & \text{if } \frac{1}{i} \leq x \leq 1 \end{cases}, \quad i \in N.$$

Then a little reflection leads to the conclusion that $a_i \to a$ in B but not in A.

Uniform convergence on A is equivalent to ordinary uniform convergence on certain quotients of A. This is the content of Proposition 2.1.9, which is crucial for our development. In order to formulate a convenient notation for the quotients involved, we begin by reminding the reader of the standard notion of the polar complementary to an element $a \in A$. This polar, designated and defined by

$$a^{\perp} \equiv \{b \in A : |b| \wedge |a| = 0\},$$

is a convex ℓ-subgroup of A which gives rise to an archimedean quotient. The polars of interest to us here are of the form $(n-r)_+^{\perp}$, where r is the indicator for A and $n \in N$. Note that

$$(n-r)_+^{\perp} = \{f \in A : \forall x \in X\ (r(x) < n \Longrightarrow f(x) = 0)\}$$

in any representation of A as a **Wi**-subobject of $D(X)$ for some space X (see Subsection 4.1). For each $n \in N$ and $a \in A$ we abbreviate the coset $(n-r)_+^{\perp} + a$ to ${}_n a$, and we write the quotient as

$${}_n A \equiv A/(n-r)_+^{\perp} = \{{}_n a : a \in A\}.$$

We let ${}_n\pi^A : A \to {}_n A$ stand for the natural projection defined by the rule

$${}_n\pi^A(a) \equiv {}_n a, \quad a \in A.$$

We make ${}_n A$ into a **Wi**-object by taking the unit to be ${}_n 1$ and the regulator to be ${}_n r$, and this also makes ${}_n\pi^A$ into a **Wi**-surjection.

Proposition 2.1.5. *Uniform convergence in ${}_n A$ is of the ordinary type because the indicator is bounded. In fact,*

$${}_n r \leq n({}_n 1) = {}_n n.$$

Proof. The chain of equivalent conditions

$$\begin{aligned} {}_n r \leq {}_n n &\Longleftrightarrow ({}_n r - {}_n n) \vee {}_n 0 = {}_n 0 \Longleftrightarrow {}_n\left((r-n)_+\right) = {}_n 0 \Longleftrightarrow (r-n)_+ \in (n-r)_+^{\perp} \\ &\Longleftrightarrow (r-n)_+ \wedge (n-r)_+ = 0 \end{aligned}$$

ends with a condition known to hold. □

For $n \geq p$ in N, the inclusion

$$(n-r)_+^{\perp} \subseteq (p-r)_+^{\perp}$$

induces the natural projection ${}_p\pi^n : {}_n A \to {}_p A$ defined by the rule

$${}_p\pi^n({}_n a) \equiv {}_p a, \quad a \in A.$$

We note the important properties of this **Wi**-surjection in Proposition 2.1.7.

Lemma 2.1.6. *For any $r \in A_+$ we have $\bigcap_N (n-r)_+^{\perp} = 0$.*

Proof. Consider $0 \le c \in \bigcap_N (n-r)_+^{\perp}$. By replacing c by $c \wedge 1$ if necessary, we may assume that $c \le 1$. Then for any $n \in N$, the fact that $(nc-r)_+ \le (n-r)_+$ implies that

$$(nc-r)_+ \in (n-r)_+^{\perp\perp} \cap (n-r)_+^{\perp} = 0,$$

i.e., $nc \le r$. The archimedean property of A yields $c = 0$. □

Proposition 2.1.7. *Let a and b lie in A and let $n \ge p \ge q$ lie in N.*

1. *$a \le b$ if and only if ${}_na \le {}_nb$ for all $n \in N$. In particular, $a = 0$ if and only if ${}_na = 0$ for all $n \in N$.*
2. *${}_p\pi^n {}_n\pi^A = {}_p\pi^A$.*
3. *${}_q\pi^p {}_p\pi^n = {}_q\pi^n$.*
4. *For any* **Wi***-morphism $f : A \to B$ and $n \in N$ there is a unique* **Wi***-morphism ${}_nf : {}_nA \to {}_nB$ such that ${}_nf {}_n\pi^A = {}_n\pi^B f$. This morphism is given by the rule*

$$({}_nf)({}_na) \equiv {}_n(f(a)), \quad a \in A.$$

Proof. (1) follows from Lemma 2.1.6, while (2) and (3) follow from the containment relations on the kernels of the maps. The equation ${}_nf {}_n\pi^A = {}_n\pi^B f$ in (4) mandates the definition given there, and it is routine to verify that this definition gives a well-defined **Wi**-morphism satisfying the equation. □

Lemma 2.1.8. *If elements $a, b, c \in A$ satisfy $a \wedge b \le c$ and $b \ge c+1$ then $a \le c$.*

Proof. Subtract c from the first inequality to get

$$(a-c) \wedge 1 \le (a-c) \wedge (b-c) \le 0.$$

Then the status of 1 as a weak order unit implies that $a - c \le 0$, i.e., that $a \le c$. □

Proposition 2.1.9. *The following are equivalent for a sequence $\{a_i\}$ and element a in A.*

1. *$a_i \longrightarrow a$.*
2. *$f(a_i) \longrightarrow f(a)$ for any* **Wi***-morphism f out of A such that $f(r)$ is bounded (by a multiple of the unit).*
3. *${}_na_i \to {}_na$ for every $n \in N$.*

Proof. Suppose $a_i \longrightarrow a$, and consider a **Wi**-morphism $f : A \to B$ for which $f(r)$ is bounded, say $f(r) \le p$ for some $p \in N$. Given ε, $1 > \varepsilon > 0$, use the uniform convergence to find an index m such that for all $i \ge m$,

$$(p+2-r) \wedge |a_i - a| \le \varepsilon$$

Applying f to this inequality gives

$$(p+2-f(r)) \wedge |f(a_i) - f(a)| \le \varepsilon,$$

and since $(p+2-f(r)) \ge 2 > \varepsilon + 1$, it follows from Lemma 2.1.8 that $|f(a_i) - f(a)| \le \varepsilon$ for all $i \ge m$. That is, $f(a_i) \longrightarrow f(a)$ in B. This shows that (1) implies (2).

Since (3) is just the specialization of (2) to the morphisms ${}_n\pi^A$, it remains only to show that (3) implies (1). For that purpose suppose that $\{a_i\}$ and a satisfy (3), fix p and $\varepsilon > 0$, and put $b \equiv (p-r)_+$. Then use the fact that ${}_pa_i \to {}_pa$ to find an index m for which $|{}_pa_i - {}_pa| \le {}_p\varepsilon$ for all $i \ge m$. But

$$\begin{aligned}|{}_pa_i - {}_pa| \le {}_p\varepsilon &\iff (|{}_pa_i - {}_pa| - {}_p\varepsilon) \vee {}_p0) = {}_p0 \iff {}_p\left((|a_i - a| - \varepsilon)_+\right) = {}_p0 \\ &\iff (|a_i - a| - \varepsilon)_+ \in b^\perp \iff (|a_i - a| - \varepsilon)_+ \wedge b = 0 \\ &\iff (|a_i - a| - \varepsilon) \wedge b \le 0 \iff |a_i - a| \wedge (b+\varepsilon) \le \varepsilon \\ &\implies |a_i - a| \wedge b \le \varepsilon.\end{aligned}$$

This shows that $a_i \to a$. □

Corollary 2.1.10. *The following hold for sequences $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ and elements a, b in A.*

1. *If $a_i \to a$ and $b_i \to b$ then $(a_i + b_i) \to (a+b)$, $(a_i \vee b_i) \to (a \vee b)$, and $(-a_i) \to (-a)$.*
2. *If $a_i \le b_i \le c_i$ for all $i \in N$ and if $a_i \to a$ and $c_i \to a$ then $b_i \to a$.*
3. *Uniform convergence is Hausdorff. That is, if $a_i \to a$ and $a_i \to b$ then $a = b$.*

Proof. In light of Propositions 2.1.7 and 2.1.9, these facts follow directly from the corresponding facts for ordinary uniform convergence. □

Any well-behaved convergence gives rise to a completion by Cauchy sequences [4]. This is the case with uniform convergence.

Definition 2.1.11. *We say that a sequence $\{a_i\}$ is* Cauchy *provided that for all $\varepsilon > 0$ and for all $p \in N$ there is some $m \in N$ such that for all $i, j \ge m$*

$$(p-r) \wedge |a_i - a_j| \le \varepsilon.$$

We say that two Cauchy sequences $\{a_i\}$ and $\{b_i\}$ are equivalent, *and write $\{a_i\} \sim \{b_i\}$, provided that $(a_i - b_i) \to 0$.*

Thus if r is bounded then a sequence $\{a_i\}$ is Cauchy precisely when it is Cauchy in the ordinary sense of the term, and $\{a_i\} \sim \{b_i\}$ means that Cauchy sequences $\{a_i\}$ and $\{b_i\}$ are equivalent in the ordinary sense.

Proposition 2.1.12. *The following are equivalent for a sequence $\{a_i\}$ in A.*

1. *$\{a_i\}$ is Cauchy.*
2. *$\{f(a_i)\}$ is Cauchy for any **Wi**-morphism f out of A for which $f(r)$ is bounded (by some multiple of the unit).*
3. *$\{{}_na_i\}$ is Cauchy for all $n \in N$.*

Proof. Argue as in the proof of Proposition 2.1.9. □

Proposition 2.1.13. *Let $g : A \to B$ be a **Wi**-morphism.*

1. *If $\{a_i\}$ is a Cauchy sequence in A then $\{g(a_i)\}$ is a Cauchy sequence in B, and if two Cauchy sequences $\{a_i\}$ and $\{b_i\}$ are equivalent in A then $\{g(a_i)\}$ and $\{g(b_i)\}$ are equivalent in B. In particular, if $A \le B$ then any Cauchy sequence in A is also Cauchy in B, and two equivalent Cauchy sequences in A are also equivalent Cauchy sequences in B.*
2. *Suppose g is injective and $g(r_A) = r_B$. Then $\{a_i\}$ is a Cauchy sequence in A if and only if $\{g(a_i)\}$ is a Cauchy sequence in B, and two Cauchy sequences $\{a_i\}$ and $\{b_i\}$ are equivalent in A if and only if $\{g(a_i)\}$ and $\{g(b_i)\}$ are equivalent Cauchy sequences in B. In particular, if $A \le B$ and $r_A = r_B$ then a sequence is Cauchy in*

A if and only if it is Cauchy in B, and two Cauchy sequences are equivalent in A if and only if they are equivalent in B.

(3) *Suppose g is injective and $g(r_A) = r_B$. If $g(a_i) \to b$ for some $b \in B$ then $\{a_i\}$ is Cauchy in A. In particular, if $A \leq B$ and $r_A = r_B$, and if $a_i \to b$ for some $b \in B$, then $\{a_i\}$ is Cauchy in A.*

(4) *If $\{a_i\}$, $\{a_i'\}$, $\{b_i\}$, and $\{b_i'\}$ are Cauchy sequences in A such that $\{a_i\} \sim \{a_i'\}$ and $\{b_i\} \sim \{b_i'\}$, then $\{a_i + b_i\} \sim \{a_i' + b_i'\}$, $\{a_i \vee b_i\} \sim \{a_i' \vee b_i'\}$, and $\{-a_i\} \sim \{-a_i'\}$.*

Proof. In light of Propositions 2.1.9 and 2.1.12, (4) follows from its well-known truth for ordinary Cauchy sequences. The rest of the conditions can be routinely verified from the definitions. □

It is important to understand that the converse of Proposition 2.1.13(1) is false. The sequence $\{a_i\}$ of Example 2.1.4 is Cauchy in B but not in A.

We continue to follow the general development of the Cauchy completion from [4].

Definition 2.1.14. *We designate the equivalence class of a Cauchy sequence $\{a_i\}$ by*

$$[a_i] \equiv \{\{b_i\} : \{b_i\} \text{ is Cauchy and } \{a_i\} \sim \{b_i\}\}.$$

The uniform completion of A *is the set of equivalence classes*

$$uA \equiv \{[a_i] : \{a_i\} \text{ is a Cauchy sequence in } A\}.$$

We endow uA with the ℓ-group operations in the natural way:

$$[a_i] + [b_i] \equiv [a_i + b_i], \quad [a_i] \vee [b_i] \equiv [a_i \vee b_i], \quad -[a_i] \equiv [-a_i].$$

We define the map $u^A : A \to uA$ by the rule

$$u^A(a) \equiv [\dot{a}], \quad a \in A.$$

Observe that if the indicator is bounded (by a multiple of the unit) then u^A is just the injection of A into its ordinary uniform completion uA. However, the situation is more interesting when the indicator is unbounded. Our objective in this subsection is to reveal the algebraic structure of uA as the inverse limit in **Wi** of the uniform completions of the quotients ${}_nA$, $n \in N$. This result is Theorem 2.1.19, and it gives useful information because the uniform completions of the quotients are of the ordinary type and are therefore familiar and well understood. Curiously, the basic fact that uA lies in **Wi** will not be established until Theorem 2.1.19 is proven. We begin by observing that uA is an abelian ℓ-group.

Lemma 2.1.15. *uA is an abelian ℓ-group with zero element $[\dot{0}]$, and the map u^A is an ℓ-injection.*

Proof. The ℓ-group operations on uA are well-defined by Proposition 2.1.13(4), and it is routine to verify that they make uA an abelian ℓ-group with zero $[\dot{0}]$ and that they make u^A an ℓ-injection. □

For the sake of brevity we abbreviate the terminology of Definition 2.1.14 when it applies to the quotients ${}_nA$. For $n \in N$ let ${}_nuA$ designate the uniform completion of ${}_nA$, i.e., ${}_nuA \equiv u({}_nA)$. Because uniform convergence in ${}_nA$ has bounded indicator by Proposition 2.1.5, both the convergence and its completion are of the ordinary sort, and so ${}_nuA$ is archimedean. Let us also abbreviate the injection $u^{{}_nA}$ to u^n and $u^n{}_n\pi^A$ to ${}_nu^A$, so that

$${}_nu^A(a) \equiv [({}_na)^{\cdot}], \quad a \in A.$$

When we designate unit $[({}_n1)^{\cdot}]$ and indicator $[({}_nr)^{\cdot}]$, we make ${}_nuA$ into a **Wi**-object and the maps u^n and ${}_nu^A$ into **Wi**-morphisms. For $n \geq p$ in N define ${}_pu^n : {}_nuA \to {}_puA$ by the rule

$${}_pu^n\left([{}_na_i]\right) \equiv \left[{}_p\pi^n\left({}_na_i\right)\right] = \left[{}_pa_i\right], \quad \{{}_na_i\} \text{ a Cauchy sequence in } {}_nA.$$

We recognize ${}_pu^n$ as the ordinary uniform lifting of ${}_p\pi^n$, and we ask the reader to recall that the ordinary uniform lifting process preserves composition of maps, i.e., is functorial. We prove that u is functorial in general in Proposition 2.1.24.

Proposition 2.1.16. *Let $n \geq p \geq q$ in N.*

1. *The maps ${}_nu^A$ and ${}_pu^n$ are* **Wi***-morphisms.*
2. ${}_pu^n{}_nu^A = {}_pu^A$.
3. ${}_qu^p{}_pu^n = {}_qu^n$.
4. *For any* **Wi***-morphism $f : A \to B$ and any $n \in N$ there is a unique* **Wi***-morphism ${}_nuf : {}_nuA \to {}_nuB$ such that $\left({}_nuf\right)_n u^A = {}_nu^Bf$.*

Proof. (1), (2), and (3) follow from the preceding remarks. In (4), ${}_nuf$ is just $u\left({}_nf\right)$, the ordinary uniform lifting of the map given by Proposition 2.1.7(4). □

The next task is to establish the connection between the uniform completion of A and the uniform completions ${}_nuA$ of the quotients ${}_nA$, $n \in N$. That connection is provided by the maps ${}_n\upsilon : uA \to {}_nuA$ given by the rule

$${}_n\upsilon\left([a_i]\right) \equiv \left[{}_n\pi^A\left(a_i\right)\right] = \left[{}_na_i\right], \quad \{a_i\} \text{ a Cauchy sequence in } A.$$

Proposition 2.1.17. *For each $n \in N$, the map ${}_n\upsilon$ is a well-defined ℓ-homomorphism which satisfies ${}_n\upsilon u^A = {}_nu^A$ and ${}_pu^n{}_n\upsilon = {}_p\upsilon$ for $n \geq p$ in N, and which takes $\left[\dot{1}\right]$ to $[({}_n1)^{\cdot}]$ in ${}_nuA$.*

Proof. The map ${}_n\upsilon$ is well defined by Proposition 2.1.13(1), and it is routine to verify that it preserves the group and lattice operations, that ${}_n\upsilon u^A = {}_nu^A$ and ${}_pu^n{}_n\upsilon = {}_p\upsilon$ hold for $n \geq p$ in N, and that ${}_n\upsilon\left(\left[\dot{1}\right]\right) = [({}_n1)^{\cdot}]$. □

The algebraic structure of uA as an abelian ℓ-group is that of the inverse limit of the family $\{{}_nuA : n \in N\}$.

Proposition 2.1.18. *The uniform completion uA of A is the inverse limit in* **alGP** *of the family $\{{}_nuA : n \in N\}$. The projections are the maps ${}_n\upsilon$, $n \in N$, and the bonding maps are of the form ${}_pu^n$ for $n \geq p$ in N.*

Proof. Given a family $\{f_n : B \to {}_nuA : n \in N\}$ of **alGp** morphisms satisfying ${}_pu^nf_n = f_p$ for all $n \geq p$ in N, define $f : B \to uA$ as follows. Given $b \in B$, for each $n \in N$ write $f_n(b)$ as $[{}_na(n,i)]$ for some sequence $\{a(n,i)\}$ in A such that $\{{}_na(n,i)\}$ is Cauchy in ${}_nA$. Then find for each $n \in N$ an index m_n such that $\left|{}_na(n,i) - {}_na(n,j)\right| \leq {}_n\left(\frac{1}{n}\right)$ for all $i, j \geq m_n$. We aim to show that the sequence $\{a(n,m_n)\}$ is Cauchy in A.

We claim that for all $k, l \geq n$,

$$\left|{}_na\left(k,m_k\right) - {}_na\left(l,m_l\right)\right| \leq {}_n\left(\frac{1}{k} + \frac{1}{l}\right).$$

To verify this claim, first note that for $i \geq m_k \vee m_l$ we have

$$\begin{aligned}\left|{}_n a\left(k, m_k\right) - {}_n a\left(l, m_l\right)\right| &\leq \left|{}_n a\left(k, m_k\right) - {}_n a\left(k, i\right)\right| + \left|{}_n a\left(k, i\right) - {}_n a\left(l, i\right)\right| \\ &\qquad + \left|{}_n a\left(l, i\right) - {}_n a\left(l, m_l\right)\right| \\ &\leq {}_n\left(\frac{1}{k}\right) + \left|{}_n a\left(k, i\right) - {}_n a\left(l, i\right)\right| + {}_n\left(\frac{1}{l}\right).\end{aligned}$$

Now the middle term in the last sum can be made arbitrarily small by choosing i large enough, for the fact that

$$\begin{aligned}\left[{}_n a\left(k, i\right)\right] &= {}_n u^k\left(\left[{}_k a\left(k, i\right)\right]\right) = {}_n u^k f_k\left(b\right) = f_n\left(b\right) = {}_n u^l f_l\left(b\right) \\ &= {}_n u^l\left(\left[{}_l a\left(l, i\right)\right]\right) = \left[{}_n a\left(l, i\right)\right]\end{aligned}$$

implies that $\left({}_n a\left(k, i\right) - {}_n a\left(l, i\right)\right) \to 0$ in ${}_n A$. The claim follows.

Since the claim establishes that $\left\{{}_n a\left(i, m_i\right)\right\}$ is Cauchy in ${}_n A$ for each $n \in N$, it follows from Proposition 2.1.12 that $\left\{a\left(n, m_n\right)\right\}$ is Cauchy in A. Therefore define $f\left(b\right) \equiv \left[a\left(n, m_n\right)\right] \in uA$. It is then a routine exercise to check that f is an ℓ-homomorphism satisfying ${}_n v f = f_n$ for all n, and that f is unique with respect to this property. □

Theorem 2.1.19. *uA is an archimedean ℓ-group, and when $\left[\dot{1}\right]$ and $\left[\dot{r}\right]$ are designated as unit and indicator, respectively, uA becomes the inverse limit in* **Wi** *of the family $\left\{{}_n uA : n \in N\right\}$. And the map $u_A : A \to uA$ is a* **Wi***-injection.*

Proof. The inverse limit in **alGp** of the family $\left\{{}_n uA : n \in N\right\}$ can be realized as

$$B \equiv \left\{b \in \prod_N {}_n uA : \forall n \geq p \;\; \left({}_p u^n\left(b_n\right) = b_p\right)\right\}.$$

But B, being a subgroup of a product of archimedean ℓ-groups, is evidently archimedean and uA is ℓ-isomorphic to B by Proposition 2.1.18. One can then readily check that, with the designated unit and indicator, all the maps of Proposition 2.1.18 lie in **Wi**. □

We close this subsection by pointing out that uniform completion is functorial. This is the content of Proposition 2.1.24, which requires a lemma. We remind the reader that by Proposition 2.1.13(2), a sequence $\{a_i\}$ is Cauchy in A if and only if $\left\{u^A\left(a_i\right)\right\} = \left\{\left[\dot{a}_i\right]\right\}$ is Cauchy in uA, since u^A is injective and carries the indicator r of A to the indicator $\left[\dot{r}\right]$ of uA.

Lemma 2.1.20. *For any sequence $\{a_i\}$ in A and any element c in uA, $\left[\dot{a}_i\right] \to c$ if and only if $\{a_i\}$ is Cauchy and $c = \left[a_i\right]$.*

Proof. Suppose that $\{a_i\}$ is a Cauchy sequence in A. Given $p \in N$ and $\varepsilon > 0$, find index m such that $(p - r) \vee \left|a_i - a_j\right| \leq \varepsilon$ for all $i, j \geq m$. In particular, we have for all $i \geq m$ that

$$\varepsilon \vee \left((p - r) \vee \left|a_m - a_i\right|\right) = \varepsilon,$$

with the result that the sequence $\left\{\varepsilon \vee \left((p - r) \vee \left|a_m - a_i\right|\right) : i \in N\right\}$ is equivalent to $\{\dot{\varepsilon}\}$, i.e.,

$$\left[\dot{\varepsilon}\right] \vee \left(\left(\left[\dot{p}\right] - \left[\dot{r}\right]\right) \vee \left|\left[\dot{a}_m\right] - \left[a_i\right]\right|\right) = \left[\varepsilon \vee \left((p - r) \vee \left|a_m - a_i\right|\right)\right] = \left[\dot{\varepsilon}\right].$$

This shows that $\left[\dot{a}_i\right] \to \left[a_i\right]$.

On the other hand suppose that $\left[\dot{a}_i\right] \to c$ for some sequence $\{a_i\}$ in A and element c in uA. Then $\{a_i\}$ is Cauchy in A by Proposition 2.1.13(3). Since we have just proven that $\left[\dot{a}_i\right] \to \left[a_i\right]$, we conclude that $c = \left[a_i\right]$ by Corollary 2.1.10(3). □

Proposition 2.1.21. *Every Cauchy sequence in A converges uniformly to a unique element of uA, and every element of uA is the uniform limit of a sequence in A.*

Proof. This follows directly from Lemma 2.1.20. □

Recall that a **W**-injection $f : A \to B$ is essential if g is injective whenever gf is injective for all **W**-morphisms $g : B \to C$. This is evidently equivalent to the condition that every nontrivial **W**-kernel of B meets $f(A)$ nontrivially. A stronger condition is that $f(A)$ be a large ℓ-subgroup of B, i.e., every nontrivial convex ℓ-subgroup of B meets $f(A)$ nontrivially. The latter condition is equivalent to this: for every $0 < b \in B$ there is some $a \in A$ and $n \in N$ such that $0 < f(a) \leq nb$. An even stronger condition is that $f(A)$ be an order-dense ℓ-subgroup of B, i.e., for every $0 < b \in B$ there is some $a \in A$ such that $0 < f(a) \leq b$.

Lemma 2.1.22. *$u^A(A)$ is a large ℓ-subgroup of uA.*

Proof. Consider $0 < b \in B \equiv uA$, say $b = [a_i]$ for some Cauchy sequence $\{a_i\}$ in A. We have $a_i \xrightarrow{r} b$ by Lemma 2.1.20. Because B is archimedean there is some $k \in N$ for which $kb \not\leq 1$. By Proposition 2.1.7(1) there is some $n \in N$ for which ${}_n(kb) \not\leq {}_n1$ in ${}_nB$. Now $\{{}_na_i\}$ converges uniformly to ${}_nb$ with indicator ${}_nr$ by Proposition 2.1.9(3), and since this is ordinary uniform convergence by Proposition 2.1.5, there must be some index m for which ${}_n(2k|a_i - b|) \leq {}_n1$ in ${}_nB$ for all $i \geq m$. It follows that

$$ {}_n0 < {}_n(2(kb-1))_+ = {}_n(2kb-2)_+ \leq {}_n(2ka_m - 1)_+ \leq {}_n(2kb). $$

If we then set $a \equiv ((2ka_m - 1) \wedge (n - r))_+$, it is straightforward to verify that $0 < a \leq 2kb$. □

Proposition 2.1.23. *The map u^A is an essential epimorphism in* **Wi**.

Proof. Together with the Lemma 2.1.22, the preceding remarks establish that u^A is essential. To show that u^A is an epimorphism consider **Wi**-morphisms $f_i : uA \to B$, $i = 1, 2$, such that $f_1u^A = f_2u^A$, and consider an arbitrary element $c \in uA$. Then $c = [a_i]$ for some Cauchy sequence $\{a_i\}$ in A, and $[\dot{a}_i] \to [a_i]$ by Lemma 2.1.20. Therefore by Proposition 2.1.3(1) the sequence

$$ \{f_1([\dot{a}_i])\} = \{f_1u^A(a_i)\} = \{f_2u^A(a_i)\} = \{f_2([\dot{a}_i])\} $$

converges uniformly to both $f_1([a_i])$ and $f_2([a_i])$. These must coincide by Corollary 2.1.10(3), i.e., $f_1 = f_2$. □

Proposition 2.1.24. *For every* **Wi**-*morphism $f : A \to B$ there is a unique* **Wi**-*morphism $uf : uA \to uB$ such that $(uf)u^A = u^Bf$. This map is given by the rule*

$$ (uf)([a_i]) = [f(a_i)], \quad \{a_i\} \text{ a Cauchy sequence in } A. $$

Proof. It is straightforward to verify that the displayed rule defines a **Wi**-morphism uf with the property that $(uf)u^A = u^Bf$. But any two morphisms which satisfy this equation must coincide by Proposition 2.1.23. □

2.2. **Indicated uniform convergence and completion in W.** In this subsection we allow the indicator to vary. Thus the context here and in all that follows is **W** rather than **Wi**. In particular, the symbols A, B, and C represent **W**-objects and the notation $A \leq B$ means that A is a **W**-subobject of B, i.e., an ℓ-subgroup containing 1_B as designated unit.

Nevertheless, the development in **W** is based on the development in **Wi**. We access the **Wi** results by the simple stratagem of adjoining indicators to **W**-objects, but doing so requires that we keep track of the indicators notationally. In particular, the notation ${}^{r}A$ will be understood to mean the **Wi**-object obtained by adjoining the indicator r to the **W**-object A, and use of this notation will imply that $r \geq 1$ in A.

Once in a while it will be convenient to refer to the functor $F : \mathbf{Wi} \to \mathbf{W}$ which forgets the indicators of **Wi**-objects and fixes the morphisms. Clearly each **W**-object A has the F-universal map $(i_A, {}^1A)$ [15, 26.1], where i_A designates the identity **W**-morphism. Consequently F has as left adjoint the functor $I : \mathbf{W} \to \mathbf{Wi}$ defined by $I(A) = {}^1A$ for $A \in \mathbf{W}$.

2.2.1. *Coordinatizing the convergences and completions arising from single indicators.* Let $\{a_i\}$ be a sequence and a and r elements in A such that $r \geq 1$. We say that $\{a_i\}$ *converges uniformly to a with indicator r*, and write $a_i \xrightarrow{r} a$, provided that $a_i \to a$ in ${}^rA$. For emphasis, we cast into this terminology three simple facts from Subsection 2.1 which the reader will do well to keep in mind.

Proposition 2.2.1. *Let $\{a_i\}$ be a sequence and a, r, and s elements of A such that $s \geq r \geq 1$. Let $g : A \to B$ be a **W**-morphism and put $t \equiv g(r)$.*

1. *If $a_i \xrightarrow{r} a$ holds in A then $g(a_i) \xrightarrow{t} g(a)$ holds in B.*
2. *If g is injective then $a_i \xrightarrow{r} a$ if and only if $g(a_i) \xrightarrow{t} g(a)$. In particular, if $A \leq B$ then $a_i \xrightarrow{r} a$ holds in A if and only if it holds in B.*
3. *If $a_i \xrightarrow{r} a$ then $a_i \xrightarrow{s} a$.*

Proof. (1) and (2) follow Proposition 2.1.3 when we make g into a **Wi**-morphism by endowing A with indicator r and B with indicator t. Likewise (3) follows from the same proposition since ${}^rA \leq {}^sA$ in **Wi**. □

Let $\{a_i\}$ and $\{b_i\}$ be sequences and a and r elements of A such that $r \geq 1$. We say that $\{a_i\}$ is r-*Cauchy* provided that it is Cauchy in ${}^rA$. Two r-Cauchy sequences $\{a_i\}$ and $\{b_i\}$ are said to be r-*equivalent*, written $\{a_i\} \overset{r}{\sim} \{b_i\}$, provided that they are equivalent in ${}^rA$, i.e., that $(a_i - b_i) \xrightarrow{r} 0$. Again, we cast into this terminology the facts from Subsection 2.1 which, though simple enough, are important for the reader to keep in mind.

Proposition 2.2.2. *Let a, r, and s be elements of A such that $1 \leq r \leq s$, let $g : A \to B$ be a **W**-morphism, and put $t \equiv g(r)$.*

1. *If $\{a_i\}$ is an r-Cauchy sequence in A then $\{g(a_i)\}$ is a t-Cauchy sequence in B. And if $\{a_i\}$ and $\{b_i\}$ are r-equivalent r-Cauchy sequences in A then $\{g(a_i)\}$ and $\{g(b_i)\}$ are t-equivalent t-Cauchy sequences in B.*
2. *Suppose that g is injective. Then $\{a_i\}$ is an r-Cauchy sequence in A if and only if $\{g(a_i)\}$ is a t-Cauchy sequence in B. And $\{a_i\}$ and $\{b_i\}$ are r-equivalent r-Cauchy sequences in A if and only if $\{g(a_i)\}$ and $\{g(b_i)\}$ are t-equivalent t-Cauchy sequences in B. In particular, if $A \leq B$ then a sequence is r-Cauchy in A if and only if it is r-Cauchy in B, and two r-Cauchy sequences are r-equivalent in A if and only if they are r-equivalent in B.*
3. *Suppose that g is injective. If a sequence $\{a_i\}$ satisfies $g(a_i) \xrightarrow{t} b$ for some $b \in B$ then $\{a_i\}$ is r-Cauchy in A. In particular, if $A \leq B$ and if $a_i \xrightarrow{r} b$ for some $b \in B$ then $\{a_i\}$ is r-Cauchy in A.*
4. *If $\{a_i\}$ and $\{b_i\}$ are r-equivalent r-Cauchy sequences then they are also s-equivalent s-Cauchy sequences.*

Proof. Argue as in the proof of Proposition 2.2.1, this time basing the argument on Proposition 2.1.13. □

We need to be able to conveniently refer to the various uniform completions $u\left({}^{r}A\right)$, $1 \leq r \in A$, developed in Subsection 2.1. We designate the $\overset{r}{\sim}$-equivalence classes of the r-Cauchy sequence $\{a_i\}$ by

$$[a_i]_r \equiv \left\{ \{b_i\} : \{b_i\} \text{ is } r\text{-Cauchy and } \{a_i\} \overset{r}{\sim} \{b_i\} \right\},$$

and we designate the uniform completion of ${}^{r}A$ by

$${}^{r}uA \equiv u\left({}^{r}A\right) = \left\{ [a_i]_r : \{a_i\} \text{ is an } r\text{-Cauchy sequence in } A \right\}.$$

Let ${}^{r}u_A : A \to {}^{r}uA$ designate the canonical injection given by the rule

$${}^{r}u_A(a) = [\dot{a}]_r, \quad a \in A.$$

For $1 \leq r \leq s$ in A define ${}^{s}u_r : {}^{r}uA \to {}^{s}uA$ by the rule

$${}^{s}u_r\left([a_i]_r\right) = [a_i]_s,$$

for r-Cauchy sequences $\{a_i\}$ in A.

The proof of Proposition 2.2.3 is the only occasion on which we explicitly mention the functor $F : \mathbf{Wi} \to \mathbf{W}$ which forgets the indicators of $\mathbf{Wi}$-objects. From that point on we suppress this technicality.

Proposition 2.2.3. *Let $1 \leq r \leq s \leq t$ in A.*

1. *The maps ${}^{r}u_A$ and ${}^{s}u_r$ are well-defined $\mathbf{W}$-injections.*
2. ${}^{s}u_r{}^{r}u_A = {}^{s}u_A$.
3. ${}^{t}u_s{}^{s}u_r = {}^{t}u_r$.
4. *For any $\mathbf{W}$-morphism $f : A \to B$ and any $1 \leq r \in A$ there is a unique $\mathbf{W}$-morphism ${}^{r}uf : {}^{r}uA \to {}^{f(r)}uB$ such that $\left({}^{r}uf\right){}^{r}u_A = {}^{f(r)}u_B f$.*

Proof. ${}^{r}u_A$ is a well-defined $\mathbf{W}$-injection by Theorem 2.1.19 because it is just $F\left(u^{{}^{r}A}\right)$, and ${}^{s}u_r$ is a well-defined $\mathbf{W}$-morphism because it is $F\left(u\left({}^{s}i_r\right)\right)$, where $u\left({}^{s}i_r\right)$ is the lifting provided by Proposition 2.1.24 of the identity $\mathbf{Wi}$-morphism ${}^{s}i_r : {}^{r}A \to {}^{s}A$. The fact that ${}^{s}u_r$ is injective follows from Proposition 2.1.23. Parts (2) and (3) are direct consequences of the definitions of the maps involved, and ${}^{r}uf$ is just $F\left(u\left(f\right)\right)$, where $u\left(f\right)$ is the uniform lifting given by Proposition 2.1.24 of the $\mathbf{Wi}$-morphism $f : {}^{r}A \to {}^{f(r)}B$. □

2.2.2. *With indicators unspecified.* We arrive finally at the central convergence of this article.

Definition 2.2.4. *Let $\{a_i\}$ be a sequence and a an element in A. We say that $\{a_i\}$* $*$-converges to a, or that $\{a_i\}$ converges uniformly to a with an indicator, *and write $a_i \overset{*}{\to} a$, provided that $a_i \overset{r}{\to} a$ for some $r \geq 1$ in A. We refer to $\overset{*}{\to}$ as* indicated uniform convergence *on A.*

Proposition 2.2.5. *The following hold for sequences $\{a_i\}, \{b_i\}$ and elements a, b in A.*

1. *If $a_i \overset{*}{\to} a$ then $a_{i_j} \overset{*}{\to} a$ for any subsequence $\left\{a_{i_j}\right\}$ of $\{a_i\}$.*
2. *The sequence $\{\dot{a}\}$ converges to a with any indicator. Thus if $a_i = a$ for all $i \in N$ then $a_i \overset{*}{\to} a$.*
3. *If $a_i \overset{*}{\to} a$ and $b_i \overset{*}{\to} b$ then $(a_i + b_i) \overset{*}{\to} (a + b)$, $(a_i \vee b_i) \overset{*}{\to} (a \vee b)$, and $(-a_i) \overset{*}{\to} (-a)$.*
4. *If $a_i \leq b_i \leq c_i$ for all $i \in N$ and if $a_i \overset{*}{\to} a$ and $c_i \overset{*}{\to} a$ then $b_i \overset{*}{\to} a$.*

(5) *Indicated uniform convergence is Hausdorff. That is, if* $a_i \xrightarrow{*} a$ *and* $a_i \xrightarrow{*} b$ *then* $a = b$.
(6) *Every* **W***-morphism is continuous with respect to indicated uniform convergence. That is, if* $f : A \to B$ *is a* **W***-morphism and if* $a_i \xrightarrow{*} a$ *in* A *then* $f(a_i) \xrightarrow{*} f(a)$ *in* B.
(7) *If* $A \leq B$ *and* $a_i \xrightarrow{*} a$ *holds in* A *then* $a_i \xrightarrow{*} a$ *holds also in* B.

Proof. Each part of this proposition follows from the corresponding part of Proposition 2.1.3 or Corollary 2.1.10. □

When the indicators are suppressed, Example 2.1.4 provides a counterexample to the converse of Proposition 2.2.5(7).

Definition 2.2.6. *We say that a sequence* $\{a_i\}$ *in* A *is* $*$-Cauchy *provided that it is* r*-Cauchy for some* $r \geq 1$ *in* A*. We say that two* $*$*-Cauchy sequences* $\{a_i\}$ *and* $\{b_i\}$ *are* $*$-equivalent*, and write* $\{a_i\} \overset{*}{\sim} \{b_i\}$*, provided that there is some* $r \geq 1$ *in* A *for which* $\{a_i\} \sim \{b_i\}$ *in* A^r*, i.e., provided that* $\{a_i\}$ *and* $\{b_i\}$ *are* r*-Cauchy and* $\{a_i\} \overset{r}{\sim} \{b_i\}$.

We reiterate an important point for emphasis. To say that $\{a_i\}$ is a $*$-Cauchy sequence in A means not only that the terms a_i of the sequence lie in A but also that the sequence is r-Cauchy for some indicator r which must also lie in A.

Proposition 2.2.7. *The following hold for sequences* $\{a_i\}$ *and* $\{b_i\}$ *in* A.

(1) *If* $a_i \xrightarrow{*} a$ *for some* $a \in A$ *then* $\{a_i\}$ *is* $*$*-Cauchy.*
(2) *Suppose* $g : A \to B$ *is a* **W***-morphism. If* $\{a_i\}$ *is* $*$*-Cauchy in* A *then* $\{g(a_i)\}$ *is* $*$*-Cauchy in* B*. In particular, if* $A \leq B$ *and if* $\{a_i\}$ *is* $*$*-Cauchy in* A *then* $\{a_i\}$ *is* $*$*-Cauchy in* B.
(3) *Suppose* $A \leq B$*. If* $\{a_i\}$ *and* $\{b_i\}$ *are* $*$*-equivalent* $*$*-Cauchy sequences in* A *then they are* $*$*-equivalent* $*$*-Cauchy sequences in* B.
(4) *If* $\{a_i\}$, $\{a'_i\}$, $\{b_i\}$*, and* $\{b'_i\}$ *are* $*$*-Cauchy sequences in* A *such that* $\{a_i\} \overset{*}{\sim} \{a'_i\}$ *and* $\{b_i\} \overset{*}{\sim} \{b'_i\}$ *then* $\{a_i + b_i\} \overset{*}{\sim} \{a'_i + b'_i\}$, $\{a_i \vee b_i\} \overset{*}{\sim} \{a'_i \vee b'_i\}$*, and* $\{-a_i\} \overset{*}{\sim} \{-a'_i\}$.

Proof. This follows directly from Proposition 2.1.13. □

When the indicators are suppressed, Example 2.1.4 provides a counterexample to the converse of Proposition 2.2.7(2).

Definition 2.2.8. *We designate the equivalence class of a Cauchy sequence* $\{a_i\}$ *in* A *by*

$$[a_i]_* \equiv \left\{\{b_i\} : \{b_i\} \text{ is } *\text{-Cauchy and } \{a_i\} \overset{*}{\sim} \{b_i\}\right\}.$$

The indicated uniform completion of A *is*

$$^*uA \equiv \{[a_i]_* : \{a_i\} \text{ is a } *\text{-Cauchy sequence in } A\}.$$

We endow $^*uA$ *with the* ℓ*-group operations in the natural way:*

$$[a_i]_* + [b_i]_* \equiv [a_i + b_i]_*, \quad [a_i]_* \vee [b_i]_* \equiv [a_i \vee b_i]_*, \quad -[a_i]_* \equiv [-a_i]_*.$$

We define the map $^*u_A : A \to {}^*uA$ *by the rule*

$$^*u_A(a) \equiv [\dot{a}]_*, \quad a \in A.$$

Our objective in this subsection is to reveal the structure of ${}^*uA$ as the direct limit in **W** of the family $\{{}^ruA : 1 \leq r \in A\}$. This is the content of Theorem 2.2.12. Curiously, the basic fact that ${}^*uA$ lies in **W** will not be established until that theorem is proven. The first step is to point out that ${}^*uA$ is an abelian ℓ-group.

Lemma 2.2.9. *${}^*uA$ is an abelian ℓ-group and the map ${}^*u_A$ is an ℓ-injection.*

Proof. The ℓ-group operations on ${}^*uA$ are well-defined by Proposition 2.2.7(4), and it is routine to verify that they make ${}^*uA$ an abelian ℓ-group with zero $\left[\dot{0}\right]_*$ and that they make ${}^*u_A$ an ℓ-injection. □

The next task is to establish the connection between ${}^ruA$ and ${}^*uA$ for each $1 \leq r \in A$. That connection is provided by the map ${}^*u_r : {}^ruA \to {}^*uA$ given by the rule

$$ {}^*u_r\left([a_i]_r\right) \equiv [a_i]_*, \quad \{a_i\} \text{ an } r\text{-Cauchy sequence in } A. $$

Proposition 2.2.10. *For each $1 \leq r \in A$, the map ${}^*u_r$ is a well-defined ℓ-injection which takes $\left[\dot{1}\right]_r$ to $\left[\dot{1}\right]_*$ and which satisfies ${}^*u_r{}^ru_A = {}^*u_A$ and ${}^*u_s{}^su_r = {}^*u_r$ for all $1 \leq r \leq s$ in A.*

Proof. The map ${}^*u_r$ is well defined because two r-equivalent r-Cauchy sequences are $*$-equivalent $*$-Cauchy sequences. The rest of the verification is equally routine. □

The algebraic structure of ${}^*uA$ as an abelian ℓ-group is that of a direct limit of the family $\{{}^ruA : 1 \leq r \in A\}$.

Proposition 2.2.11. *The indicated uniform completion ${}^*uA$ of A is the direct limit in* **alGP** *of the family $\{{}^ruA : 1 \leq r \in A\}$. The injections of the factors into this limit are the maps ${}^*u_r$, $1 \leq r \in A$, and the bonding maps are of the form ${}^su_r$ for $1 \leq r \leq s$ in A.*

Proof. Given a family $\{f_r : {}^ruA \to B : 1 \leq r \in A\}$ of ℓ-homomorphisms satisfying $f_s{}^su_r = f_r$ for $1 \leq r \leq s$ in A, define $f : {}^*uA \to B$ as follows. An element $c \in {}^*uA$ has the form $c = [a_i]_*$ for some $*$-Cauchy sequence $\{a_i\}$ in A. Find $r \geq 1$ for which $\{a_i\}$ is r-Cauchy, so that $[a_i]_r \in {}^ruA$, and define $f(c) \equiv f_r\left([a_i]_r\right)$. We leave it to the reader to verify that this definition delivers a well-defined ℓ-homomorphism which satisfies $f{}^*u_r = f_r$ for $1 \leq r \in A$, and that it is unique with respect to this property. □

We are finally able to show that ${}^*uA$ is a **W**-object. The key observation is that when the bonding maps are injective **W**-morphisms, the direct limit in **alGp** of an up-directed family of **W**-objects lies in **W**, and in fact is the direct limit of the family in **W**.

Theorem 2.2.12. *${}^*uA$ is an archimedean ℓ-group, and when $\left[\dot{1}\right]$ is designated as unit, ${}^*uA$ becomes the direct limit in* **W** *of the family $\{{}^ruA : 1 \leq r \in A\}$. And the map ${}^*u_A : A \to {}^*uA$ is a* **W***-injection.*

Proof. We showed in Proposition 2.2.3 that the bonding maps are **W**-injections. One can then readily check that, with the designated unit, all the maps of Proposition 2.2.11 lie in **W**. □

We close this subsection by pointing out that indicated uniform completion is functorial.

Proposition 2.2.13. *The map ${}^*u_A$ is an essential epimorphism in* **W***.*

Proof. We pointed out in Proposition 2.1.23 that each map ${}^ru_A$ is an essential epimorphism in **Wi**, and a little additional reflection leads to the conclusion that these maps are also essential epimorphisms in **W**. It is a general fact that when the bonding maps are injective, a direct limit of essential epimorphic extensions is an essential epimorphic extension. □

Proposition 2.2.14. *For every* $\mathbf{W}$*-morphism* $f : A \to B$ *there is a unique* $\mathbf{W}$*-morphism* ${}^*uf : {}^*uA \to {}^*uB$ *such that* $({}^*uf)\,{}^*u_A = {}^*u_B f$*. This map is given by the rule*

$$({}^*uf)\left([a_i]_*\right) = \left[f\left(a_i\right)\right]_*, \quad \{a_i\} \text{ a } *\text{-Cauchy sequence in } A.$$

Proof. It is straightforward to verify that the displayed rule defines a $\mathbf{W}$-morphism ${}^*uf$ with the property that $({}^*uf)\,{}^*u_A = {}^*u_B f$. The uniqueness of this morphism follows from Proposition 2.2.13. □

3. Density and completeness

The notions of density and completeness with respect to indicated uniform convergence are complicated by the necessity of iteration. The reason is that indicated uniform convergence is not topological, and therefore its closure operator need not be idempotent. Nevertheless the iterated versions of these operators are well suited to our purposes.

At this point in the discourse it is helpful to tone down the formalism. Henceforth we identify A and its various Cauchy completions with their images under the canonical embeddings. Our point of view now is that for $1 \leq r \leq s$ in A,

$$A \leq {}^r uA \leq {}^s uA \leq {}^*uA.$$

3.1. **Density and closure.**

Definition 3.1.1. *For* $A \leq B$ *in* $\mathbf{W}$ *and* $R \subseteq B$*, define*

$$\begin{aligned}
\operatorname{cl}(A, B, R) &\equiv \left\{ b \in B : \exists \{a_i\} \subseteq A \ \exists r \in R \ \left(a_i \xrightarrow{r} b\right) \right\}, \\
\operatorname*{cl}_0 (A, B) &\equiv A, \\
\operatorname*{cl}_\alpha (A, B) &\equiv \operatorname{cl}\left(\operatorname*{cl}_\beta (A, B), B, \operatorname*{cl}_\beta (A, B) \right), \quad \alpha = \beta + 1, \\
\operatorname*{cl}_\alpha (A, B) &\equiv \bigcup_{\beta < \alpha} \operatorname*{cl}_\beta (A, B), \quad \alpha \text{ a limit ordinal,} \\
\operatorname{cl}(A, B) &\equiv \operatorname*{cl}_{\omega_1} (A, B).
\end{aligned}$$

We say that A is closed in B *whenever* $A = \operatorname{cl}(A, B)$*, and we say that* A is dense in B *whenever* $B = \operatorname{cl}(A, B)$*.*

Proposition 3.1.2. *A is dense in* $\operatorname{cl}(A, B)$*, and* $\operatorname{cl}(A, B)$ *is closed in* B*.*

Proof. Let $A \leq B$ and put $C \equiv \operatorname{cl}(A, B)$. Then an easy induction shows that $\operatorname{cl}_\alpha(A, C) = \operatorname{cl}_\alpha(A, B)$ for all α, and this shows that A is dense in $\operatorname{cl}(A, B)$. To show that C is closed in B we show that $\operatorname{cl}(C, B, C) = C$. For that purpose consider an element $b \in \operatorname{cl}(C, B, C)$ by virtue of a sequence $\{c_i\} \subseteq C$ and element $r \geq 1$ in C such that $c_i \xrightarrow{r} b$. Since the set $\{c_i\} \cup \{r\}$ is countable, it lies in $\operatorname{cl}_\beta(A, B)$ for some countable ordinal β. But then $b \in \operatorname{cl}_{\beta+1}(A, B) \leq C$. □

Proposition 3.1.3. *A dense embedding is an epimorphism in* $\mathbf{W}$*.*

Proof. Suppose A is dense in B. A simple induction shows that two maps out of B which agree on A must agree also on $\operatorname{cl}_\alpha(A, B)$. The argument uses parts (5) and (6) of Proposition 2.2.5. □

3.2. **Completeness.**

Definition 3.2.1. *A is* complete *if every $*$-Cauchy sequence in A $*$-converges, i.e., if $A = {}^{*}uA$. A* completion of *A is a complete extension $B \geq A$ in which A is dense.*

Although ${}^{*}uA$ is not generally complete, it does enjoy the following weak completeness property.

Proposition 3.2.2. *Every $*$-Cauchy sequence in A $*$-converges to a unique element of ${}^{*}uA$ with an indicator from A, and for every element $b \in {}^{*}uA$ there is a sequence $\{a_i\}$ and indicator r in A such that $a_i \xrightarrow{r} b$.*

Proof. A $*$-Cauchy sequence $\{a_i\}$ in A is r-Cauchy for some $r \geq 1$, and this sequence converges in ${}^{r}uA$ by Lemma 2.1.20. This convergence also holds in ${}^{*}uA$ by Proposition 2.2.5(7). On the other hand, any element $b \in {}^{*}uA$ has the form $[a_i]_*$ for some $*$-Cauchy sequence $\{a_i\}$ in A. But then $\{a_i\}$ is r-Cauchy for some $r \geq 1$, and $a_i \xrightarrow{r} b$ in ${}^{r}uA$ by Lemma 2.1.20. As before, this convergence takes place also in ${}^{*}uA$. □

In order to complete A we must iterate the indicated uniform completion.

Definition 3.2.3. *For ordinal numbers α we define*

$$\begin{aligned} {}^{0}uA &\equiv A, \\ {}^{\alpha}uA &\equiv {}^{*}u\left({}^{\beta}uA\right), \quad \alpha = \beta + 1, \\ {}^{\alpha}uA &\equiv \bigcup_{\beta<\alpha} \left({}^{\beta}uA\right), \quad \alpha \text{ a limit ordinal,} \\ {}^{\infty}uA &\equiv {}^{\omega_1}uA. \end{aligned}$$

Proposition 3.2.4. *${}^{\infty}uA$ is complete.*

Proof. Abbreviate ${}^{\infty}uA$ to B, and consider a Cauchy sequence $\{b_i\}$ in B. Then $\{b_i\}$ is r-Cauchy for some $r \in B$. Because $\{b_i\} \cup \{r\}$ is a countable set, there is some $\alpha < \omega_1$ for which $\{b_i\} \cup \{r\} \subseteq {}^{\alpha}uA$. Now $\{b_i\}$ is r-Cauchy in ${}^{\alpha}uA$ by Proposition 2.2.2(2), and so $b_i \xrightarrow{r} b$ for a unique element $b \in {}^{r}\left({}^{\alpha}uA\right)$ by Proposition 3.2.2. Since

$${}^{r}\left({}^{\alpha}uA\right) \leq {}^{*}\left({}^{\alpha}uA\right) = {}^{\alpha+1}uA \leq {}^{\infty}uA,$$

this convergence holds also in B by Proposition 2.2.1(2). □

Proposition 3.2.5. *For each ordinal α, $\mathrm{cl}_\alpha(A, {}^{\infty}uA) = {}^{\alpha}uA$. Consequently A is dense in ${}^{\infty}uA$.*

Proof. Abbreviate ${}^{\infty}uA$ to B. We show by induction on α that $\mathrm{cl}_\alpha(A,B) = {}^{\alpha}uA$. This reduces to showing that if $\alpha = \beta + 1$ and $\mathrm{cl}_\beta(A,B) = {}^{\beta}uA$ then $\mathrm{cl}_\alpha(A,B) = {}^{\alpha}uA$. For that purpose consider $b \in {}^{\alpha}uA = {}^{*}u\left({}^{\beta}uA\right)$. By Proposition 3.2.2 there is a sequence $\{c_i\}$ and indicator r in ${}^{\beta}uA$ for which $c_i \xrightarrow{r} b$. But since ${}^{\beta}uA = \mathrm{cl}_\beta(A,B)$, this fact puts b in $\mathrm{cl}_\alpha(A,B)$ by definition. On the other hand, consider $b \in \mathrm{cl}_\alpha(A,B)$, say $c_i \xrightarrow{r} b$ for a sequence $\{c_i\}$ and element r in $\mathrm{cl}_\beta(A,B) = {}^{\beta}uA$. But then $\{c_i\}$ is r-Cauchy by Proposition 2.2.2(2) and therefore $*$-Cauchy in ${}^{\beta}uA$, and so $c_i \xrightarrow{*} b'$ for a unique element b' of ${}^{*}u\left({}^{\beta}uA\right) = {}^{\alpha}uA \leq B$ by Proposition 3.2.2. We conclude by Proposition 2.2.5(5) that $b = b'$, and hence that $b \in {}^{\alpha}uA$. □

Proposition 3.2.6. *Suppose that A is dense in C, that B is complete, and that $f : A \to B$ is a* **W***-morphism. Then there is a unique* **W***-morphism $\theta : C \to B$ such that the restriction of θ to A is f.*

Proof. We define $\theta(a) = f(a)$ for all $a \in \mathrm{cl}_0(A, C)$. Suppose that θ has been defined so that it is a **W**-morphism from $\mathrm{cl}_\alpha(A, C)$ into B. To extend θ to $\mathrm{cl}_{\alpha+1}(A, C)$, consider an element $c \in \mathrm{cl}_{\alpha+1}(A, C)$, say $c_i \xrightarrow{r} c$ for some sequence $\{c_i\}$ and element r from $\mathrm{cl}_\alpha(A, C)$. Then $\{c_i\}$ is an r-Cauchy sequence in $\mathrm{cl}_\alpha(A, C)$ by Proposition 2.2.2(3), and so $\{\theta(c_i)\}$, which is a $\theta(r)$-Cauchy sequence in B by Proposition 2.2.2(1), converges to a unique element $b \in B$. We have no choice but to define $\theta(c) \equiv b$. We leave it to the reader to confirm that θ is a well-defined **W**-morphism from $\mathrm{cl}_{\alpha+1}(A, C)$ into B. □

If $A \leq C$ and $A \leq B$ then we say that a **W**-morphism $\theta : C \to B$ is *over* A provided that θ is the identity map on A.

Corollary 3.2.7. *If A is dense in C then there is a unique W-injection $\theta : C \to {}^\infty uA$ over A.*

Proof. We claim that, in the terminology of the proof of Proposition 3.2.6, if f is injective on $\mathrm{cl}_\alpha(A, C)$ then the extension of f is injective on $\mathrm{cl}_{\alpha+1}(A, C)$. To verify this claim suppose f is a **W**-isomorphism from $\mathrm{cl}_\alpha(C, D)$ onto $B \equiv f(\mathrm{cl}_\alpha(A, C))$, and consider elements $c, d \in \mathrm{cl}_{\alpha+1}(A, C)$ such that $f(c) = f(d)$. That means there exist in $\mathrm{cl}_\alpha(A, C)$ elements r_c and r_d and sequences $\{c_i\}$ and $\{d_i\}$, and there exists in ${}^\infty uA$ an element x, such that

$$c_i \xrightarrow{r_c} c, \quad d_i \xrightarrow{r_d} d, \quad f(c_i) \xrightarrow{f(r_c)} x, \quad f(d_i) \xrightarrow{f(r_d)} x.$$

By replacing r_c and r_d by $r \equiv r_c \vee r_d$ we may assume that $r_c = r_d$. But then $f(c_i - d_i) \xrightarrow{f(r)} 0$ in B, and if we apply the **W**-isomorphism f^{-1} to this convergence we get $(c_i - d_i) \xrightarrow{r} 0$, and this implies that $c = d$. We conclude that the extension of f is injective on $\mathrm{cl}_{\alpha+1}(A, C)$. □

The example given in [7] shows that the converse of Corollary 3.2.7 is false, that is, A need not be dense in C just because $A \leq C \leq {}^\infty uA$. Nevertheless, a partial converse to Proposition 3.2.7 is true and important.

Theorem 3.2.8. *A is dense in the complete* **W***-object C if and only if C is isomorphic to ${}^\infty uA$ over A.*

Proof. Suppose A is dense in the complete **W**-object C, and let $\theta : C \to {}^\infty uA$ be the unique **W**-injection over A. We claim that θ maps $\mathrm{cl}_\alpha(A, C)$ onto ${}^\alpha uA$ for all ordinals α. The claim clearly holds for $\alpha = 0$. Suppose that it holds for some countable ordinal α. We leave to the reader the routine verification that θ takes $\mathrm{cl}_{\alpha+1}(A, C)$ into ${}^{\alpha+1}uA$. To show that θ takes $\mathrm{cl}_{\alpha+1}(A, C)$ onto ${}^{\alpha+1}uA$ consider an element

$$b \in {}^{\alpha+1}uA = {}^*u\left({}^\alpha uA\right) = \bigcup_{r \in {}^\alpha uA} {}^r u\left({}^\alpha uA\right),$$

say $b \in {}^r u({}^\alpha uA)$ for $r \in {}^\alpha uA$. Then by Lemma 2.1.20 $b_i \xrightarrow{r} b$ for some sequence $\{b_i\} \subseteq {}^\alpha uA$. Since $\theta^{-1} : {}^\alpha uA \to \mathrm{cl}_\alpha(A, C)$ is a **W**-isomorphism, $\{\theta^{-1}(b_i)\}$ is a $\theta^{-1}(r)$-Cauchy sequence in $\mathrm{cl}_\alpha(A, C)$ which must have a limit c in C because C is complete. By definition c lies in $\mathrm{cl}_{\alpha+1}(A, C)$ and $\theta(c) = b$. □

The following result slightly strengthens the notion of completeness. We will use this stronger version without comment in what follows.

Theorem 3.2.9. *The following are equivalent for $A \in \mathbf{W}$.*

1. *A is complete, i.e., every $*$-Cauchy sequence $*$-converges.*
2. *$A = {}^{\infty}uA$.*
3. *Every r-Cauchy sequence r-converges for each $1 \leq r \in A$.*
4. *$A = {}^{r}uA$ for each $1 \leq r \in A$.*

Proof. The equivalence of (1) and (2) follows from Theorem 3.2.8, and the equivalence of (3) and (4) follows from Proposition 2.1.21. Furthermore, the implication from (3) to (1) is clear. But the fact that $A \leq {}^{r}uA \leq {}^{*}uA \leq {}^{\infty}uA$ for all $1 \leq r \in A$ makes the implication from (2) to (4) clear as well. □

Theorem 3.2.10. *The embedding $A \leq {}^{\infty}uA$ constitutes an epireflection from $\mathbf{W}$ into the class of complete $\mathbf{W}$-objects.*

Proof. Consider a map f from A into a complete $\mathbf{W}$-object B. Since A is dense in ${}^{\infty}uA$ by Proposition 3.2.5, there is by Proposition 3.2.6 a unique map $\theta : {}^{\infty}uA \to B$ such that the restriction of θ to A is f. □

4. The indicated uniform completion is the c-cubed hull

In this section we show that the indicated uniform completion ${}^{\infty}uA$ of A is isomorphic to the c-cubed hull $\mathrm{c}^3 A$ of A over A; this is Theorem 4.5.1. In contrast to the abstract approach taken in Section 2, the development here is concrete inasmuch as it takes place in the context of the classical Yosida representation, a topic briefly reviewed in Subsection 4.1. The c-cubed hull itself is outlined in Subsection 4.2.

4.1. **The classical Yosida representation.** Let Y be a compact Hausdorff space. We designate by $D(Y)$ the set of those continuous functions b from Y into the extended real numbers $\bar{R} \equiv R \cup \{\pm\infty\}$ which are almost finite in the sense that $b^{-1}(R)$ is dense in Y. $D(Y)$ is a lattice under pointwise meet and join, but the addition of its elements is not well-defined. Given $b_1, b_2 \in D(Y)$, we would like to add them by setting

$$(b_1 + b_2)(y) \equiv b_1(y) + b_2(y), \quad y \in b_1^{-1}(R) \cap b_2^{-1}(R),$$

but the resulting function in some cases cannot be continuously extended to all of Y. We say that a subset $B \subseteq D(Y)$ is a $\mathbf{W}$-object in $D(Y)$ provided that it has these properties.

1. B contains the constant 1 function.
2. B is a sublattice of $D(Y)$.
3. B is closed under addition. That is, for any $b_1, b_2 \in B$, the function $(b_1 + b_2)$ defined above can be continuously extended to all of Y.

We say that a $\mathbf{W}$-object B in $D(Y)$ *separates the points of* Y if for all $y_1 \neq y_2$ in Y there is some $b \in B$ for which $b(y_1) \neq b(y_2)$.

Theorem 4.1.1 ([20])**.** *For any $\mathbf{W}$-object A there is a compact Hausdorff space Y and a $\mathbf{W}$-isomorphism μ from A onto a $\mathbf{W}$-object in $D(Y)$ such that $\mu(A)$ separates the points of Y. The space is unique up to homeomorphism.*

The space given by Theorem 4.1.1 for the $\mathbf{W}$-object A is called *the Yosida space of* A, and we designate it YA. We refer to the isomorphism of Theorem 4.1.1 or its image as *the Yosida representation of* A, and we designate the isomorphism μ_A. The passage from A to its Yosida space is functorial in the following sense.

Theorem 4.1.2. *For any* **W**-*morphism* $f : A \to B$ *there is a unique continuous function* $\phi : YB \to YA$ *such that for all* $a \in A$ *and* $x \in YB$,

$$\mu_A(a)(\phi(x)) = \mu_B(f(a))(x).$$

f *is injective if and only if* ϕ *is surjective, and if* f *is surjective then* ϕ *is injective.*

4.2. **The c-cubed hull.** The class of **W**-objects which are closed under countable composition was introduced and characterized by Henriksen, Isbell, and Johnson in [17]. We briefly recapitulate the defining property and fundamental characterization of these objects from that pivotal paper.

Let $Y \equiv YA$, and identify A with its Yosida representation as a **W**-object in $D(Y)$. Let R^ω designate a countable product of copies of R with projections $\rho_n : R^\omega \to R$, $n \in N$. Now a countable subset $\{a_n : n \in N\} \subseteq A$ gives rise to a unique continuous map $f : S \to R^\omega$ such that $\rho_n f = a_n$ for all $n \in N$, where $S \equiv \bigcap_N a^{-1}(R)$. Then A is said to be *closed under countable composition* provided that for every countable subset $\{a_n : n \in N\}$ and for every continuous function $g : R^\omega \to R$ there is some $a \in A$ for which $a|_S = gf$.

The concrete characterization of these objects from [17] will be used as the working definition in what follows. Let $\mathcal{F}$ designate the filter of subsets of Y generated by sets of the form $\bigcap_{A_0} a^{-1}(R)$ for countable subsets $A_0 \subseteq A$. These subsets are dense by the Baire Category Theorem, since each $a^{-1}(R)$, $a \in A$, is a dense open subset of Y. For subsets $F_1, F_2 \in \mathcal{F}$ with $F_1 \subseteq F_2$, we consider $C(F_2)$ to be a **W**-subobject of $C(F_1)$ by restriction. Then we define

$$C[\mathcal{F}] \equiv \bigcup_{\mathcal{F}} C(F).$$

Theorem 4.2.1 ([17])**.** *A is closed under countable composition if and only if $A = C[\mathcal{F}]$.*

The class of **W**-objects closed under countable composition plays a central role in the investigation of archimedean ℓ-groups with order unit, and the following two results illustrate this fact. First, the class of **W**-objects closed under countable composition forms a reflective subcategory ([2], [3]), with the reflection of A given by its natural embedding in $C[\mathcal{F}]$ by virtue of the fact that each $a \in A$ lies in $C(a^{-1}(R))$. And this is the smallest essentially reflective subcategory of **W** [6]. *In what follows we use* $\mathrm{c}^3 A$ *to designate this reflection of* A. And secondly we noted in the introduction Isbell's Theorem that this class coincides with the class of **W**-objects of the form $C(L)$ for a locale L [16].

4.3. $\mathrm{c}^3 A$ **is complete.** We show in Theorem 4.5.2 that for any locale L, $C(L)$ is complete. A first step in this direction is to establish the result in the special case in which L is spatiale.

Proposition 4.3.1. *For any space S, $C(S)$ is complete.*

Proof. Abbreviate $C(S)$ to B, and consider an r-Cauchy sequence $\{b_i\} \subseteq B$ for some $1 \leq r \in B$. For each $n \in N$ put $S_n \equiv \operatorname{coz}(n-r)_+$, an open subset of S. Let $\theta_n : B \to C(S_n)$ be the restriction map defined by the rule $\theta_n(b) \equiv b\,|_{S_n}$, $b \in B$. Since θ_n and ${}_n\pi_B$ both have kernel $(n-r)_+^{\perp}$, θ_n induces a unique injection from ${}_nB$ into $C(S_n)$, and in fact this injection is just ${}_nb \longmapsto b\,|_{S_n}$. Therefore $\{\theta_n(b_i)\}$ is an ordinary Cauchy sequence in $C(S_n)$ because it corresponds under the aforementioned isomorphism to $\{{}_nb_i\}$, which is an ordinary Cauchy sequence in ${}_nB$ by Proposition 2.1.12. Since $C(S_n)$ is complete with respect to ordinary uniform convergence, there is a unique function $c_n \in C(S_n)$ such that $\theta_n(b_i) \to c_n$. Furthermore, the c_n's are consistent with one another in the sense that $c_m\,|_{S_n} = c_n$ for $m \geq n$.

Therefore the function $c : S \to R$, defined by the rule $c(s) = c_n(s)$ for $s \in S_n$, is clearly continuous. Note that $c \in B$ because $\bigcup_N S_n = S$. And since $\theta_n(b_i) \to \theta_n(c)$ for all $n \in N$, it follows that ${}_n b_i \to {}_n c$ for all $n \in N$. Therefore $b_i \xrightarrow{r} c$ by Proposition 2.1.9. □

We continue to identify A with its Yosida representation in $D(Y)$, $Y \equiv YA$, and to designate by $\mathcal{F}$ the filter of subsets of Y generated by those of the form $\bigcap_N a_n^{-1}(\mathbb{R})$, $\{a_n\} \subseteq A$.

Proposition 4.3.2. $\mathrm{c}^3 A$ *is complete.*

Proof. Let $r \in c^3 A$ and let $\{c_i\}$ be an r-Cauchy sequence in $c^3 A$. Then there is a single $S \in \mathcal{F}$ for which $r \in C(S)$ and $\{c_i\} \subseteq C(S)$. By Proposition 4.3.1, $\{c_i\}$ converges in $C(S)$ and therefore also in the larger **W**-object $c^3 A$. □

4.4. A **is dense in** $\mathrm{c}^3 A$**.** Let A be a **W**-object with Yosida space $Y \equiv YA$, and let $\{a_n : n \in N\}$ be a subset of A_+ such that $a_n \le a_{n+1}$ for all $n \in N$. Abbreviate $a_n^{-1}(R)$ to S_n for $n \in N$ and put $S \equiv \bigcap_N S_n$, a dense G_δ in Y. Let $B \equiv C(S)$ and $X \equiv YB = \beta S$, and identify B with its Yosida representation as a W-object in $D(X)$.

We impose two restrictive assumptions on A which will be in force from this point through Proposition 4.4.6: we assume that A is divisible, and we assume that $a^{-1}(R) \supseteq S$ for all $a \in A$. The second assumption allows us to regard A as embedded in B by restriction. Let $\phi : X \to Y$ be the continuous surjection which realizes this inclusion, so that for $x_i \in X$, $\phi(x_1) = \phi(x_2)$ if and only if $a(x_1) = a(x_2)$ for all $a \in A$. For any point $y \in Y$ we use $\mathcal{N}(y)$ to designate the filter of neighborhoods of Y.

We require a generalization of the classical Stone-Weierstrass Theorem, Proposition 4.4.1. (The classical theorem is achieved when ϕ is injective.) Although this result can obviously be formulated more generally, we confine our attention here to the special case needed in what follows. Let us agree to measure the distance from an element $b \in B$ to A by the sup metric:

$$d(b, A) \equiv \bigwedge_A \bigvee_X |b(x) - a(x)|.$$

Proposition 4.4.1. *For $b \in B^*$ let $\varepsilon \equiv \bigvee_{\phi(x_1)=\phi(x_2)} |b(x_1) - b(x_2)|$. Then*

$$\frac{\varepsilon}{2} \le d(b, A) \le \varepsilon.$$

Proof. First observe that if $\phi(x_1) = \phi(x_2)$ then for any $a \in A$ we have

$$\begin{aligned} |b(x_1) - b(x_2)| &\le |b(x_1) - a(x_1)| + |a(x_2) - b(x_2)| \\ &= |b(x_1) - a(x_1)| + |b(x_2) - a(x_2)|, \end{aligned}$$

from which it follows that $d(b, A) \ge \frac{\varepsilon}{2}$.

To establish that $d(b, A) \le \varepsilon$ put $u \equiv \bigvee_X |b(x)|$. Note that u is the figure which results from taking a to be 0 in the infimum which defines $d(b, A)$, so that $u \ge d(b, A) \ge \frac{\varepsilon}{2}$. Therefore if $u \le \varepsilon$ we are done. Of the two remaining cases, the first is the one in which $u > \frac{3\varepsilon}{2}$. In this case we claim that for any rational number $q \ge \frac{u}{3}$ there is some $a \in A$ for which $\bigvee_X |b(x) - a(x)| \le 2q$. In order to establish this claim let $X^+ \equiv b^{-1}[q, \infty)$ and $X^- \equiv b^{-1}(-\infty, q]$. Note that for all $x^+ \in X^+$ and $x^- \in X^-$ we have $\phi(x^+) \ne \phi(x^-)$ because $2q > \varepsilon$, from which fact follows the existence of some $a \in A$ such that $a(x_1) \ne a(x_2)$. It follows by a standard argument [19, 44.5] that there is some $a \in A$ such that $-q \le a \le q$, $a(x^+) = q$ for all $x^+ \in X^+$, and $a(x^-) = -q$ for all $x^- \in X^-$. (This argument uses the divisibility of A to produce the constant functions $\pm q$ in A.) Clearly $|b(x) - a(x)| \leq 2q$ for all $x \in X$.

In the second case we have $\varepsilon < u \leq \frac{3\varepsilon}{2}$. In this case we claim that for any rational number $q > \frac{\varepsilon}{2}$ there is some $a \in A$ for which $\bigvee_X |b(x) - a(x)| \leq 2q$. In order to establish this claim, let $X^+ \equiv b^{-1}[q, \infty)$ and $X^- \equiv b^{-1}(-\infty, q]$. Then the argument from the previous case produces some $a \in A$ such that $-q \leq a \leq q$, $a(x^+) = q$ for all $x^+ \in X^+$, and $a(x^-) = -q$ for all $x^- \in X^-$. Clearly $|b(x) - a(x)| \leq 2q$ for all $x \in X$.

The proof of the proposition proceeds by applying the first case to b to get an element $a_1 \in A$ such that $\bigvee_X |b(x) - a_1(x)|$ is close to $\frac{2}{3}u$. Then if necessary this case is applied again, this time to $b - a_1$, to get an element $a_2 \in A$ such that $\bigvee_X |b(x) - a_1(x) - a_2(x)|$ is close to $\left(\frac{2}{3}\right)^2 u$. Iterating this argument eventually produces a difference to which the second case applies. A single application of the second case then produces a final element $a \in A$ for which $\bigvee_X |b(x) - a(x)|$ is as close to ε as desired. □

Definition 4.4.2. *For $b \in B$ and $y \in Y$, the* variation in b at y *is defined to be*

$$\operatorname{var}(b, y) \equiv \bigwedge_{U \in \mathcal{N}(y)} \bigvee_{s_i \in U \cap S} |b(s_1) - b(s_2)|.$$

For a real number $\varepsilon \geq 0$, the set of points of Y at which the variation in b is at least ε is designated

$$Y(b, \varepsilon) \equiv \{y \in Y : \operatorname{var}(b, y) \geq \varepsilon\}.$$

Note that for any $b \in B$ and $\varepsilon > 0$, $Y(b, \varepsilon)$ is a closed subset of Y disjoint from S.

Definition 4.4.3. *For a closed subset $Z \subseteq Y$, define*

$$Z^0 \equiv Z,$$
$$Z^\alpha \equiv \bigcap_N \operatorname{cl}\left(S_n \cap Z^\beta\right) \quad \text{for } \alpha = \beta + 1,$$
$$Z^\alpha \equiv \bigcap_{\beta < \alpha} Z^\beta \quad \text{for } \alpha \text{ a limit ordinal,}$$
$$Z^\infty \equiv Z^\alpha \quad \text{for some } \alpha \text{ for which } Z^\alpha = Z^{\alpha+1}.$$

Note that since the Z^α's form a nested sequence of subsets of Y, there must be some ordinal α for which $Z^\alpha = Z^{\alpha+1}$. That is, Z^∞ must always exist. For the deeper meaning of Z^∞ see [8].

Lemma 4.4.4. *$Z^\infty = \emptyset$ for any closed subset $Z \subseteq Y$ disjoint from S.*

Proof. What characterizes Z^∞ is that $S_n \cap Z^\infty$ is dense in it for all $n \in N$. But if Z^∞ is nonempty then the Baire Category Theorem would furnish a point of S lying in it, contrary to hypothesis. □

The next result is the key to the findings of this section.

Proposition 4.4.5. *For all $b \in B_+$ and $\varepsilon > 0$, if $Y(b, \varepsilon)^\alpha = \emptyset$ then*

$$d\left(b, \operatorname{cl}_\alpha(A, B)\right) \leq \varepsilon.$$

Proof. We induct on α. Suppose $Y(b,\varepsilon)^0 = Y(b,\varepsilon) = \emptyset$ for some $b \in B_+$ and $\varepsilon > 0$. Then b is bounded, for otherwise $b(x) = \infty$ for some $x \in X$, in which case $\operatorname{var}(b, \phi(x)) = \infty$, contrary to hypothesis. The desired conclusion then follows from Proposition 4.4.1.

Now assume that $\alpha = \beta + 1$ and that the proposition holds at β, and consider $b \in B$ and $\varepsilon > 0$ for which $Y(b,\varepsilon)^\alpha = \emptyset$. Fix $\delta > \varepsilon$. Our objective is to produce an element $c \in \operatorname{cl}_\alpha(A,B)$ satisfying $|b - c| \le \delta$. We do this by finding an index $m \in N$ for which there is an a_m-Cauchy sequence $\{c_i\} \subseteq \operatorname{cl}_\beta(A,B)$ such that the limit of this sequence is an element $c \in \operatorname{cl}_\alpha(A,B)$ with $|b - c| \le \delta$.

The first step is to find m. Since $\bigcap_N \operatorname{cl}(S_n \cap Y(b,\varepsilon)^\beta) = \emptyset$, it follows that $S_n \cap Y(b,\varepsilon)^\beta = \emptyset$ for all but finitely many indices n; let m be any such index. For $k \in N$ set $U_k \equiv a_m^{-1}(-\infty, k) \cap S$, and put

$$d_k \equiv ((k + 1 - a_m) \wedge 1)_+, \quad b_k \equiv d_{2k} b.$$

Note that $0 \le d_k \le 1$, that $d_k(s) = 1$ for $s \in U_k$, and that $d_k(s) = 0$ for $s \notin U_{k+1}$. Therefore $0 \le b_k \le b$, $b_k(s) = b(s)$ for $s \in U_{2k}$, and $b_k(s) = 0$ for $s \notin U_{2k+1}$. Now for any $k \in N$, d_{2k} lies in A and can therefore be continuously extended from S to Y, and when so extended it clearly satisfies the formula

$$\operatorname{var}(b_k, y) = d_{2k}(y) \operatorname{var}(b, y)$$

for all $y \in Y$. It follows that $Y(b_k,\varepsilon) \subseteq Y(b,\varepsilon)$, and consequently that $Y(b_k,\varepsilon)^\beta \subseteq Y(b,\varepsilon)^\beta$. Furthermore, since b_k is zero off U_{2k+1}, it is clear that $Y(b_k,\varepsilon) \subseteq \operatorname{cl} U_{2k+1}$, hence

$$Y(b_k,\varepsilon)^\beta \subseteq Y(b_k,\varepsilon) \subseteq \operatorname{cl} U_{2k+1} \subseteq S_m.$$

Since S_m is disjoint from $Y(b,\varepsilon)^\beta$, we conclude that $Y(b_k,\varepsilon)^\beta = \emptyset$. Now for each $k \in N$ apply the inductive hypothesis to b_k to get an element $e_k \in \operatorname{cl}_\beta(A,B)$ satisfying $|b_k - e_k| \le \delta$.

Although each e_k approximates b to within δ on U_{2k}, the problem is that the e_k's may not be particularly close to one another. The solution is to sew them together to form an a_m-Cauchy sequence. For that purpose, the first step is to define

$$c_1 \equiv e_1, \quad c_{k+1} \equiv d_{2k-1} c_k + (1 - d_{2k-1}) e_{k+1}, \quad 1 < k \in N.$$

Observe the following.

(1) For $s \notin U_{2k}$, $c_{k+1}(s) = e_{k+1}(s)$ because $d_{2k-1}(s) = 0$.
(2) For $s \in U_{2k} \backslash U_{2k-1}$, $c_{k+1}(s) = d_{2k-1}(s) e_k(s) + (1 - d_{2k-1}(s)) e_{k+1}(s)$ because $c_k(s) = e_k(s)$.
(3) For $s \in U_{2k-1}$, $c_{k+1}(s) = c_k(s)$ because $d_{2k-1}(s) = 1$.
(4) For $s \in U_{2k+1} \setminus U_{2k-1}$, $|b(s) - c_{k+1}(s)| \le \delta$. This follows from (1) and (2).
(5) For $s \in U_{2k-1}$ and $k \le l$, $c_k(s) = c_l(s)$. This follows from (3).
(6) For $s \in U_{2k-1}$, $|b(s) - c_k(s)| \le \delta$. This follows from (4) and (5) and the fact that $|b(s) - e_1(s)| \le \delta$ for $s \in U_1$.

(5) makes it clear that the sequence $\{c_i\}$ is a_m-Cauchy, since for $i, j \ge k$ we have

$$((2k-1) - a_m) \wedge |c_i - c_j| \le 0.$$

Because B is complete by Proposition 4.3.1, there is a unique $c \in B$ such that $c_i \overset{a_m}{\to} c$, and the present construction in fact locates c in $\operatorname{cl}_\alpha(A,B)$. And because $c(s) = c_k(s)$ for $s \in U_{2k-1}$ and $\bigcup_N U_k = S$, (6) shows that $|b - c| \le \delta$. We conclude that the proposition holds at $\alpha = \beta + 1$. Since the least ordinal α for which $Y(b,\varepsilon)^\alpha = \emptyset$ must be a successor, the induction is complete. □

Proposition 4.4.6. *A is dense in B.*

Proof. Given $b \in B_+$ and $n \in N$, let α_n be the least ordinal such that $Y\left(b, \frac{1}{n}\right)^{\alpha_n} = \emptyset$. Find by Proposition 4.4.5 some $c_n \in \mathrm{cl}_{\alpha_n}(A, B)$ such that $|b - c_n| \leq \frac{2}{n}$. Since $c_n \to b$, it is clear that $b \in \mathrm{cl}(A, B)$. □

A close inspection of the proof of Proposition 4.4.5 leads to a surprising conclusion: the indicators used in the inductive definition of the closure of A in B may be restricted to lie in A.

Proposition 4.4.7. *Let*

$$\begin{aligned} A_0 &\equiv A \\ A_\alpha &\equiv \mathrm{cl}(A_\beta, B, A), \ \alpha = \beta + 1, \\ A_\alpha &\equiv \bigcup_{\beta < \alpha} A_\beta, \ \beta \text{ a limit ordinal.} \end{aligned}$$

Then $A_{\omega_1} = B$.

The reader should be cautioned that Proposition 4.4.7 does not imply that the indicators used in the formation of $\mathrm{cl}(A, B)$ can be restricted to lie in A in an arbitrary extension $A \leq B$. The authors wish to record their skepticism of this stronger assertion.

We now lift the assumption that $a^{-1}(R) \supseteq S$ for all $a \in A$. For the rest of this section, A will represent an arbitrary divisible **W***-object.* We continue to write Y for YA, and we write $\mathcal{F}$ for $A^{-1}(R)_\delta$, the filter of subsets of Y generated by the countable intersections of the domains of reality of the members of A, and we realize c^3A as the natural embedding of A in $C[\mathcal{F}]$ [3].

Proposition 4.4.8. *A is dense in c^3A.*

Proof. Consider arbitrary $b \in c^3A_+$, say $b \in B \equiv C(S)$ for $S \in \mathcal{F}$ of the form $\bigcap_N a_n^{-1}(R)$ for some subset $\{a_n : n \in N\} \subseteq A_+$ such that $a_n \leq a_{n+1}$ for all $n \in N$. Let

$$A_S \equiv \left\{a \in A : a^{-1}(R) \supseteq S\right\},$$

a **W**-subobject of A. A_s separates the points of Y since it contains A^*, so $YA_S = Y$. Consequently A_S is dense in B by Proposition 4.4.6. The result follows. □

4.5. $^{\infty}uA = \mathrm{c}^3 A$.

Theorem 4.5.1. *c^3A is isomorphic to $^{\infty}uA$ over A. Furthermore, this isomorphism takes $\mathrm{cl}_\alpha(A, c^3A)$ onto ^{a}uA for each ordinal α.*

Proof. The first sentence is a direct application of Theorem 3.2.8, and the second follows from Proposition 3.2.5. □

Let us summarize our results.

Theorem 4.5.2. *The following are equivalent for a divisible* **W***-object A.*

(1) *A is closed under countable composition, i.e., $A = \mathrm{c}^3 A$.*
(2) *A is of the form $C(L)$ for some locale L.*
(3) *A is complete, i.e., $A = {}^{\infty}uA$.*

Proof. The equivalence of (1) and (2) is Isbell's Theorem [16] noted in the introduction. The equivalence of (1) and (3) follows immediately from Theorem 4.5.1. □

Corollary 4.5.3. *A* **W***-object is closed under countable composition if and only if it is divisible and complete.*

We use $Y_l A$ to designate the Yosida locale of A. The localic Yosida representation is the work of Madden and Vermeer in [18]; see also [5].

Corollary 4.5.4. $^{\infty}uA$ *is isomorphic to* $C(Y_l A)$ *over* A.

Proof. The localic Yosida representation $\mu_A : A \to C(Y_l A)$ and the indicated uniform completion $A \leq {}^{\infty}uA$ are both essential epireflections, and the target objects coincide by Theorem 2.1.7. Therefore the reflections coincide, at least up to isomorphism. □

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