INJECTIVE AND PROJECTIVE T-BOOLEAN ALGEBRAS

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ABSTRACT. We introduce and analyze Boolean algebras acted on by a topological monoid T, so-called T-Boolean algebras. These are the duals of Boolean flows, and by analyzing injective T-Boolean algebras we are able to characterize projective Boolean flows. Moreover, we characterize the projective T-Boolean algebras in the case that T is a group. This characterization shows that the existence of nontrivial projective T-Boolean algebras depend on the properties of T.

1. INTRODUCTION

Boolean algebras with actions, herein termed T-Boolean algebras or T-algebras for short, is a subject of intrinsic interest and importance. Moreover, any systematic program of investigation of topological dynamics must place the topic of Boolean flows high on its list of priorities, and any investigation of Boolean flows leads directly to the subject of T-algebras via the Stone duality with actions outlined in Subsection 1.2.

In this article we take up the central issues of injective and projective T-Boolean algebras. It develops that, although the usual categorical arguments establish the existence and uniqueness of the injective hull of any T-algebra A, the structure of this hull is quite complicated even when that of A is not. Among the more significant results of our research is a characterization of injective T-algebras in terms of systems of ideals; this can be found in Subsections 4.2, 4.3, and 4.4.

It also develops that nontrivial projective T-algebras exist only for rather special topological monoids T. We characterize those monoids within a class slightly broader than topological groups in Theorem 6.2.6. A fully satisfactory characterization of which topological monoids

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admit nontrivial projective T-algebras must await a deeper understanding than is currently available to the authors.

1.1. **Actions.** We fix a topological monoid T throughout. We refer to the elements of T as *actions*, and denote them by the letters s, t, and r, often with subscripts. Let $\mathbf{C}$ be a category and C an object of $\mathbf{C}$. A *left action of T on C* (*right action of T on C*) is a monoid homomorphism (antimorphism) $\phi_C : T \to \hom_{\mathbf{C}}(C, C)$, i.e., $\phi_C(1_T) = 1_C$ and for all $t_i \in T$,

$$\phi_C(t_1)(\phi_C(t_2)) = \phi_C(t_1 t_2) \quad (\phi_C(t_1)(\phi_C(t_2)) = \phi_C(t_2 t_1)).$$

We suppress nearly all mention of ϕ_C, writing $\phi_C(t)(c)$ as $t_C c$ (ct_C) or simply tc (ct). In this simplified notation, the definition of a left (right) action is just that $1c = c$ and $(t_1 t_2)\, c = t_1 (t_2 c)$ ($c1 = c$ and $c\,(t_1 t_2) = (ct_1)\, t_2$) for all $c \in C$ and $t_i \in T$. We then have an enriched category $\mathbf{TC}$ ($\mathbf{CT}$) whose objects are the $\mathbf{C}$ objects C acted upon by T in such a way that the evaluation map $(t, c) \longmapsto tc$ ($(c, t) \longmapsto ct$) is continuous from $T \times C$ ($C \times T$) into C, where C either has a topological structure or is given the discrete topology, and $T \times C$ ($C \times T$) has the product topology. The $\mathbf{TC}$ ($\mathbf{CT}$) morphisms are the $\mathbf{C}$ morphisms $f : C \to B$ between $\mathbf{TC}$ ($\mathbf{CT}$) objects C and B which commute with the actions, i.e., those which make this diagram commute. Maps commuting with

$$\begin{array}{ccc} C & \xrightarrow{t_A} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[t_B]{} & B \end{array}$$

the actions are also called *T-equivariant* maps.

We sometimes distinguish between the simpler objects of $\mathbf{C}$ and the more complicated objects of $\mathbf{TC}$ ($\mathbf{CT}$) by referring to the former as *naked*. Of course, we could also recover what is essentially the category $\mathbf{C}$ of naked objects by specializing to the case in which $T = \{1\}$, a case which we refer to as the classical, or no-action, situation.

Thus from the category $\mathbf{Ba}$ of Boolean algebras with Boolean homomorphisms we obtain the category $\mathbf{BaT}$ of *T-Boolean algebras* and *T-Boolean homomorphisms*. We consider no other types of algebras in this article, so we simplify the terminology by dropping the word Boolean, referring to the objects of $\mathbf{Ba}$ and of $\mathbf{BaT}$ as algebras and T-algebras, respectively, and to the corresponding morphisms as morphisms and T-morphisms. We reiterate that T-algebras carry the discrete topology, and that T acts on them on the right.

Likewise from the category **Sp** of spaces with continuous functions we obtain the category **TSp** of T*-spaces*, or T*-flows*, or simply *flows*, and likewise the category **SpT** of *antiflows*. The distinction between the two is that T acts on flows on the left and on antiflows on the right. T itself has two roles to play: it is a flow under the action of left multiplication and an antiflow under the action of right multiplication. Finally, from the full subcategory **BSp** of **Sp** consisting of the Boolean spaces, i.e., compact Hausdorff spaces with a clopen base, we obtain the category **TBSp** of Boolean flows.

1.2. **Stone duality with actions.** Stone duality extends to the action categories without a hitch. In fact, our interest in injective T-algebras arose from our desire to understand their duals, projective Boolean flows. However, we emphasize T-algebras in this article, pausing only occasionally to translate the results into terms of Boolean flows.

The main purpose of mentioning duality here is to point out that the definition of T-algebra is the right one. In particular, the continuity of evaluation on a Boolean flow is equivalent to the continuity of evaluation on its clopen algebra only when the latter carries the discrete topology.

Theorem 1.2.1. *Let X be a Boolean space and B its algebra of clopen subsets. Then every left action ϕ_X on X gives rise to a right action ϕ_B on B by the rule*

$$bt = \phi_B(t)(b) \equiv \phi_X(t)^{-1}(b) = t^{-1}(b).$$

Conversely, every right action ϕ_B on B gives rise to a left action ϕ_X on X by the rule

$$tx = \phi_X(t)(x) \equiv \bigcap_{x \in \phi_B(t)(b)} b = \bigcap_{x \in bt} b.$$

These two processes are inverses of one another. Furthermore, ϕ_X renders evaluation continuous on X if and only if ϕ_B renders evaluation continuous on B. Thus are the categories **BaT** *and* **TBSp** *equivalent.*

Stone duality helps to clarify the nature of epimorphisms and monomorphisms in both **TBSp** and **BaT**.

Proposition 1.2.2. *In both* **TBSp** *and* **BaT**, *epimorphisms are surjective and monomorphisms are injective.*

Proof. Consider a **TBSp** morphism $f : X \to Y$. The product $P \equiv X \times X$ is a Boolean flow under componentwise actions, i.e., $t(x_1, x_2) \equiv (tx_1, tx_2)$ for all $t \in T$ and $(x_1, x_2) \in P$, and the projection maps

$p_i : P \to X$, $i = 1, 2$, are flow surjections. Let

$$Z \equiv \{(x_1, x_2) \in P : f(x_1) = f(x_2)\},$$

a closed subflow of P and therefore itself a **TBSp** object. If f is not injective then there are $x_1 \neq x_2$ in X such that $f(x_1) = f(x_2)$. The corresponding point $p \equiv (x_1, x_2)$ lies in Z, and $p_1(p) = x_1 \neq x_2 = p_2(p)$. Since $fp_1 = fp_2$, f is not a monomorphism. This proves that monomorphisms in **TBSp** are injective, and it follows from Stone duality that epimorphisms in **BaT** are surjective. A similar argument shows that monomorphisms in **BaT** are injective, and dualizes to show that epimorphisms in **TBSp** are surjective. □

2. Pointed antiflows

Antiflows of a particular sort arise as a means of classifying the elements of a T-algebra according to the complexity of their orbits. This is because the orbit $aT \equiv \{at : t \in T\}$ of an element a in a T-algebra A is a discrete antiflow having source a.

2.1. Pointed antiflows defined.

Definition 2.1.1. A *pointed antiflow* is an object of the form (R, s), where R is a discrete antiflow and s is a *source* for R, i.e., for all $r \in R$ there is some $t \in T$ such that $st = r$. The pointed antiflow morphisms are the antiflow morphisms which take the designated source of the domain to the designated source of the codomain. We use **pSpT** to denote the category of pointed antiflows and pointed antiflow morphisms.

We remind the reader that T acts on any antiflow on the right, and does so in such a way that evaluation is continuous. A pointed antiflow is distinguished among general antiflows by two additional features: a pointed antiflow is discrete and has a source.

Observe that a pointed antiflow is really just a discrete antiflow quotient of T, with the image of the identity as source. In fact, for every pointed antiflow (R, s) there is a *unique* antiflow surjection $\rho_R : T \to R$ such that $\rho_R(1) = s$, and it is defined by the rule $\rho_R(t) = st$ for all $t \in T$. Consequently there are, up to antiflow isomorphism, only a sets worth of pointed antiflows.

Definition 2.1.2. We use $\{R_i : i \in I\}$ to designate the set of isomorphism types of pointed antiflows of T; more precisely, every pointed antiflow is **pSpT** isomorphic to exactly one R_i. And we use $\rho_i : T \to R_i$ to designate the corresponding antiflow morphism, i.e., $\rho_i(t) = s_i t$ for all $t \in T$. Moreover, any two pointed antiflows (R_1, s_1) and (R_2, s_2) admit at most one **pSpT** morphism $\rho_2^1 : R_1 \to R_2$, given by the rule $\rho_2^1(s_1 t) = s_2 t$, and it satisfies $\rho_2^1 \rho_1 = \rho_2$ when it exists.

2.2. **suitable relations.** The reason for the uniqueness of the morphisms ρ_i and ρ^i_j, $i,j\in I$, is that pointed antiflows are in one-to-one correspondence with certain equivalence relations on T. Given $i\in I$, define

$$t'\sim_i t''\Longleftrightarrow s_it'=s_it''.$$

Then $\sim_i$ is an equivalence relation on T which is *right invariant* in the sense that for all $t,t',t''\in T$,

$$t'\sim_i t''\Longrightarrow t't\sim_i t''t.$$

And $\sim_i$ has equivalence classes which are open, and hence clopen. We refer to an equivalence relation with these two properties as a *suitable relation.*

Given a suitable relation $\sim$ on T, let R designate the discrete space $T/\sim$, i.e.,

$$R\equiv\{[t]:t\in T\},$$

where $[t]$ denotes the equivalence class of t. Let $t\in T$ act on $[t']\in R$ by the rule

$$[t']\,t=[t't]\,.$$

Then R is a pointed antiflow with source $s\equiv[1]$. Therefore there is a unique $i\in I$ for which (R,s) is antiflow isomorphic to (R_i,s_i), and $\sim$ is actually $\sim_i$.

Remark 2.2.1. The pointed antiflows are in one-to-one correspondence with the suitable relations.

(1) $(T,1)$ is a pointed antiflow if and only if T is discrete.
(2) If T is connected then its only pointed antiflow contains a single point.
(3) If T is compact then all its pointed antiflows are finite.

2.3. **The lattice of pointed antiflows.** The index set I inherits a partial order from the refinement ordering on suitable relations: we define $i\geq j$ in I if $\sim_i$ is finer than $\sim_j$, i.e., if $[t]_i\subseteq[t]_j$ for all $t\in T$, where $[t]_i$ designates the equivalence class of $t\in T$ with respect to $\sim_i$.

Proposition 2.3.1. *$i\geq j$ in I if and only if there is a* **pSpT** *morphism $\rho^i_j:R_i\to R_j$. The morphism is unique when it exists, and I is a lattice under this order.*

Proof. Consider $i,j\in I$. If $i\geq j$ then the map $[t]_i\longmapsto[t]_j$ is a **pSpT** morphism. Conversely if ρ^i_j exists then for all $t,t'\in T$ we would have

$$\begin{aligned}t\sim_i t'\Longleftrightarrow s_it=s_it'\Longrightarrow \rho^i_j\left(s_it\right)=\rho^i_j\left(s_it'\right)\Longrightarrow \rho^i_j\left(s_i\right)t=\rho^i_j\left(s_i\right)t'\\ \Longrightarrow s_jt=s_jt'\Longleftrightarrow t\sim_j t'.\end{aligned}$$

That the order is a lattice ordering depends on three observations. First, the meet or join (in the lattice of equivalence relations on T) of two right-invariant relations is right invariant. Second, two equivalence relations on T with clopen classes have a join with the same feature. And third, any equivalence relation coarser than one whose classes are clopen also has clopen classes. □

Corollary 2.3.2. *$i \geq j$ in I if and only if*

$$s_i t = s_i t' \Longrightarrow s_j t = s_j t'$$

for all $t, t' \in T$.

At the expense of a little redundancy, we offer an exterior formulation of the fact that I is an upper semilattice. We use $i \vee j$ to denote the supremum of i and j in I.

Proposition 2.3.3. *For $i, j \in I$, $R_{i \vee j}$ is* **pSpT** *isomorphic to*

$$\{(r_i, r_j) \in R_i \times R_j : s_i t = r_i \text{ and } s_j t = r_j \text{ for some } t \in T\},$$

with coordinatewise actions and source (s_i, s_j). Furthermore, the projection maps are **pSpT** *morphisms.*

We close this subsection by pointing out that when T is a topological group, I is anti-isomorphic to its lattice of open subgroups.

Definition 2.3.4. For any element r in any pointed antiflow R_i we designate *the stabilizer of r in T* by

$$\operatorname{stab} r \equiv \{t \in T : rt = r\}.$$

Note that $\operatorname{stab} r$ is a clopen submonoid of T. When r is the source s_i of R_i, we call $\operatorname{stab} r$ a *source stabilizer.*

The terminology of Definition 2.3.4 applies to any element a in a T-algebra A, since the orbit $(aT, a) \equiv (\{at : t \in T\}, a)$ is a pointed antiflow. That is, $\operatorname{stab} a = \{t \in T : at = a\}$.

Proposition 2.3.5. *Suppose that T is a topological group. Then the lattice I of isomorphism types of pointed antiflows is anti-isomorphic to the lattice of open subgroups of T via the map*

$$i \mapsto \operatorname{stab} s_i.$$

The inverse of this map is

$$U \mapsto (T/U, U),$$

where T/U denotes the pointed antiflow of right cosets of U acted upon by right multiplication. Thus I is a lattice in this case.

It follows from Proposition 2.3.5 that for topological groups T, the suitable relations are in bijective order-reversing correspondence with the source stabilizers. The broader class of topological monoids with this feature play role in Section 6.

Definition 2.3.6. Let T be a topological monoid. We say that *the suitable relations on T correspond to the source stabilizers* if for $i, j \in I$,

$$i \geq j \Longleftrightarrow \operatorname{stab} s_i \subseteq \operatorname{stab} s_j.$$

2.4. **Constructing $\hat{T}$.** For the purpose of analyzing T-algebras, the only pertinent feature of T is its actions on pointed antiflows. Thus we may exchange T for an associated topological monoid $\hat{T}$, formed by isolating this pertinent feature (Theorem 2.6.1).

We indulge in this development to point out that the action of a given topological monoid T on a T-algebra may, in effect, be other than what it first appears. For example, if T is connected then $\hat{T}$ has a single point and the action is, in fact, trivial; see Remark 2.2.1(2). On the other hand, there may be actions implicit in $\hat{T}$ which are not present in T, i.e., $\hat{T}$ may be larger than T; see Example 2.4.6. However, the reader who is only interested in injective and projective objects in **BaT**, the main content of this article, may choose to skip this development.

For each $i \in I$ we designate $\hom_{\mathbf{Sp}}(R_i, R_i) = R_i^{R_i}$ by V_i, and we make V_i into a topological monoid by using as basic neighborhoods of $v \in V_i$ sets of the form

$$\{v' \in V_i : sv' = sv \text{ for all } s \in S\},$$

for finite subset $S \subseteq R_i$. We designate the action of T on R_i by $\phi_i : T \to V_i$, we designate $\phi_i(T)$ by T_i, and for each $t \in T$ we abbreviate $\phi_i(t)$ to t_i. Let V designate the product $\prod_I V_i$, regarded as a topological monoid with componentwise multiplication and product topology. Let $\hat{\phi}_i : V \to V_i$ designate the i^{th} projection map. Finally, define $\phi : T \to V$ by the rule

$$\phi(t)(i) \equiv \phi_i(t) = t_i.$$

For $i \geq j$ in I, the **pSpT** morphism ρ_j^i naturally induces a topological monoid homomorphism $\phi_j^i : T_i \to T_j$ as follows. For any $t_i \in T_i$ we define the action of $\phi_j^i(t_i)$ on an arbitrary $r_j \in R_j$ by the rule

$$(r_j)\,\phi_j^i(t_i) \equiv \rho_j^i(r_i t_i),$$

where $r_i \in R_i$ is chosen to satisfy $\rho_j^i(r_i) = r_j$. The definition is independent of the choice of r_i because ρ_j^i commutes with the actions.

Writing r_i as $s_i t'_i$ for some $t' \in T$, so that $r_j = s_j t'_j$, gives the simpler formula

$$\left(s_j t'_j\right) \phi^i_j\left(t_i\right) = s_i t'_i t_i = s_i\left(t' t\right)_i.$$

Note that $\phi^i_j \phi_i = \phi_j$ for all $i \geq j$ in I.

The construct that emerges naturally here is the inverse limit in the category of topological monoids, held together by the bonding maps ϕ^i_j for $i \geq j$ in I. In this case we can realize $\bar{T}$ concretely as

$$\bar{T} = \lim_{\longleftarrow I} T_i = \left\{\bar{t} \in \prod_I T_i : \forall i \geq j \ \left(\phi^i_j\left(\bar{t}\left(i\right)\right) = \bar{t}\left(j\right)\right)\right\}.$$

$\hat{T}$ is defined to be the closure of $\bar{T}$ in V. Of course, $\phi\left(T\right)$ is dense in $\bar{T}$, so $\hat{T}$ is also the closure of $\phi\left(T\right)$ in V. We abbreviate $\hat{\phi}_i\left(\hat{t}\right)$ to $\hat{t}_i$ for elements $\hat{t} \in \hat{T}$.

Remark 2.4.1. The following hold for any topological monoid T.

(1) All the maps ϕ, ϕ_i, and $\hat{\phi}_i$ are continuous monoid homomorphisms, and $\hat{\phi}_i \phi = \phi_i$ for each $i \in I$.
(2) If each pointed antiflow of T is finite then $\hat{T}$ is compact.
(3) ϕ is injective if and only if the common refinement of the suitable relations is the identity relation, i.e., each pair of distinct points of T is separated by a suitable relation.
(4) $\hat{T}$ is discrete if and only if I has a greatest element.

Proposition 2.4.2. *$\hat{T}$ has the same lattice I of pointed antiflows as T does.*

Proof. For each action ϕ_i of T on one of its pointed antiflows R_i we have the corresponding action $\hat{\phi}_i$ of $\hat{T}$ on R_i, and $\phi\hat{\phi}_i = \phi_i$ by construction. Conversely any action of $\hat{T}$ on a pointed antiflow, when followed by ϕ, gives an action of T on that flow. This shows that the pointed antiflows of T are the same as those for $\hat{T}$. Furthermore, the order on I imposed by T is the same as that imposed by $\hat{T}$. For if $i \geq j$ in I by virtue of the **pSpT** morphism $\rho^i_j : R_i \to R_j$ then this ρ^i_j is also a $\mathbf{pSp\hat{T}}$ morphism, i.e., it commutes with each $\hat{t} \in \hat{T}$. The reason is that for $r_i \in R_i$ and $r_j \equiv \rho^i_j\left(r_i\right)$ there is some $t \in T$ such that $r_i \hat{t}_i = r_i t_i$ and $r_j \hat{t}_j = r_j t_j$, hence

$$\rho^i_j\left(r_i \hat{t}_i\right) = \rho^i_j\left(r_i t_i\right) = \rho^i_j\left(r_i\right) t_i = r_j t_j = r_j \hat{t}_j = \rho^i_j\left(r_i\right) \hat{t}_j.$$

Thus $i \geq j$ in the order imposed on I by $\hat{T}$. □

Corollary 2.4.3. *$\hat{\hat{T}} = \hat{T}$.*

Remark 2.4.4. Let $i \in I$.

(1) For elements $\hat{t}, \hat{t}' \in \hat{T}$, $\hat{t} \sim_i \hat{t}'$ if and only if $s_i \hat{t}_i = s_i \hat{t}'_i$.
(2) Each $\sim_i$ class of $\hat{T}$ contains an element of $\phi(T)$.

Let us have a closer look at the special case where T is a topological group. We leave the verification of Proposition 2.4.5 to the reader.

Proposition 2.4.5. *Suppose that T is a topological group. Then $\bar{T}$ is a group, but $\hat{T}$ need not be a group. The identity element of $\bar{T}$ has a neighborhood base consisting of the open subgroups of $\bar{T}$.*

Here is an example which illustrates the ideas in this subsection. It makes the point that T need not coincide with $\hat{T}$ even when $\phi : T \to \hat{T}$ is injective, and that $\hat{T}$ need not be a group even when T is.

Example 2.4.6. *Let T be the group of permutations of the natural numbers N under composition, topologized by using as neighborhoods of an element $t \in T$ sets of the form*

$$T_m(t) = \{t' : it' = it \text{ for all } i \leq m\}$$

for $m \in N$. (We choose to write the permutation to the right of its input.) T is a topological group. For each $m \in N$ define the equivalence relation $\sim_m$ by declaring

$$t \sim_m t' \Longleftrightarrow it = it' \quad \text{for all } i \leq m.$$

Then $\sim_m$ is a suitable relation, and every suitable relation is refined by one of these. Thus we may identify $[t]_m$ with the m-tuple $(1t, 2t, \ldots, mt)$, and identify $R_m \equiv T/{\sim_m}$ with the set of all m-tuples of distinct elements from N. The source of R_m is $s_m = (1, 2, \ldots, m)$, and the action of $t \in T$ on $(i_1, i_2, \ldots, i_m)$ is given by

$$(i_1, i_2, \ldots, i_m)\, t = (i_1 t, i_2 t, \ldots, i_m t).$$

The natural order on N coincides with the refinement ordering on the corresponding suitable relations, and if $m \geq n$ then the **pSpT** *morphism $\rho_n^m : R_m \to R_n$ is simply restriction, i.e.,*

$$\rho_n^m (i_1, i_2, \ldots, i_m) = (i_1, i_2, \ldots, i_n).$$

The map $\phi : T \to V = \prod_N R_n$ is injective; in fact,

$$s_n \phi(t)(n) = s_n t_n = (1, 2, \ldots, n)\, t = (1t, 2t, \ldots, nt).$$

Thus an element $\hat{t} \in \hat{T} = \mathrm{cl}_V\, \phi(T)$ is a function with consistent values, i.e., such that its value at index n is the restriction of its value at index m whenever $n \leq m$, and whose value at index m is the same as that of some permutation. However, the permutations do not all have to be the same for various indices, and so one sees that the elements of $\hat{T}$ correspond to the one-to-one functions from N into N. For example, *the (equivalent of) the function $n \longmapsto n^2$ lies in $\hat{T}$ but not in T. Note that $\hat{T}$ is not a group.*

2.5. **The type of an element of a T-algebra.** Pointed antiflows arise as a means of classifying elements of a T-algebra A according to the complexity of their orbits. Let A be an algebra on which T acts, and for each $a \in A$ let $\sim_a$ designate the relation on T defined by the rule

$$t \sim_a t' \Longleftrightarrow at_A = at'_A,$$

for $t, t' \in T$. (Here and in what follows we use t_A to abbreviate $\phi_A(t)$, where

$$\phi_A : T \to \hom A \equiv \hom_{\mathbf{Ba}}(A, A)$$

is the action of T on A.) Then $\sim_a$ is a right-invariant equivalence relation, and A is a T-algebra, i.e., evaluation is continuous, if and only if each $\sim_a$ is a suitable relation.

Definition 2.5.1. We say that *the type of an element a of a* T-algebra *A is $i \in I$*, and write $\operatorname{type} a = i$, provided that $\sim_a$ is $\sim_i$.

Remark 2.5.2. Let a be an element of a T-algebra A.

(1) To assert that a is of type at most i is to assert that for all $t, t' \in T$,
$$s_i t_i = s_i t'_i \Longrightarrow at_A = at'_A.$$
(2) If the suitable relations on T correspond to the source stabilizers, then to assert that a is of type at most i is to assert that
$$\operatorname{stab} s_i \subseteq \operatorname{stab} a.$$

Lemma 2.5.3. *Let a and b be elements of a T-algebra A. Then*

$$\operatorname{type}(a \vee b) \vee \operatorname{type}(a \wedge b) \leq \operatorname{type} a \vee \operatorname{type} b, \text{ and}$$
$$\operatorname{type} a = \operatorname{type} \bar{a},$$

where $\bar{a}$ denotes the complement of a. Therefore $\{a \in A : \operatorname{type} a \leq i\}$ is a subalgebra of A for each $i \in I$, but is generally not a T-subalgebra.

A crucial observation is that morphisms reduce type.

Proposition 2.5.4. *If $f : A \to B$ is a T-morphism then for all $a \in A$,*

$$\operatorname{type} a \geq \operatorname{type} f(a).$$

T acts on I on the right, essentially by shifting each source s_i of R_i to $s_i t$. The key observation is that if $\sim_i$ is a suitable relation on T then so is the relation $\sim_{it}$ defined by the rule

$$t' \sim_{it} t'' \Longleftrightarrow tt' \sim_i tt''$$

for $t', t'' \in T$.

Remark 2.5.5. The following hold for $i \in I$ and $t, t', t'' \in T$.

(1) $(it)\,t' = i\,(tt')$.
(2) $t' \sim_{it} t''$ if and only if $s_i t_i t'_i = s_i t_i t''_i$.
(3) A pointed antiflow corresponding to the suitable relation $\sim_{it}$ is $(s_i tT, s_i t)$, where $s_i tT \equiv \{s_i tt' : t' \in T\}$, and where the action is given by

$$(s_i tt')\,\phi_j\,(t'') = s_i \phi_i\,(tt't'')$$

for $t', t'' \in T$.

Proposition 2.5.6. *Suppose a is an element of a T-algebra A and $t \in T$. Then*

$$\operatorname{type}(at_A) = (\operatorname{type} a)\,t.$$

Proof. Let $\operatorname{type} a \equiv i$ and $\operatorname{type}(at_A) \equiv j$. Then for $t', t'' \in T$,

$$\begin{aligned} t' \sim_{it} t'' &\Longleftrightarrow tt' \sim_i tt'' \Longleftrightarrow s_i\,(tt')_i = s_i\,(tt'')_i \\ &\Longleftrightarrow a\,(tt')_A = a\,(tt'')_A \Longleftrightarrow (at_A)\,t'_A = (at_A)\,t''_A \\ &\Longleftrightarrow t' \sim_j t''. \end{aligned}$$

Since the suitable relations $\sim_{it}$ and $\sim_j$ coincide, it follows that $it = j$ in I, which is the desired conclusion. □

One might idly conjecture that, in the notation of Lemma 2.5.6, $i \geq it$ by virtue of the map $r_i \longmapsto r_i t$, $r_i \in R_i$. But this is most assuredly not the case, since this map is generally not a **pSpT** morphism.

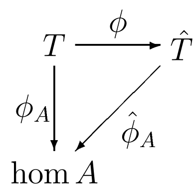
Proposition 2.5.7. *Let A be a T-algebra and $i \in I$. Then for each $i \in I$,*

$$A_i \equiv \left\{ a \in A : \operatorname{type} a \leq \bigvee_{T_0} it \text{ for some finite } T_0 \subseteq T \right\}$$

is a T-subalgebra of A, and

$$A = \varinjlim \{A_i : i \in I\}.$$

Proof. The fact that A_i is a T-subalgebra follows from Proposition 2.5.6 and Lemma 2.5.3. And since each element $a \in A$ has a type $i \in I$ and is therefore contained in A_i, it follows that A is the direct limit of the A_i's. □



2.6. **Trading T for $\hat{T}$.**

Theorem 2.6.1. *For every action ϕ_A of T on a T-algebra A there is a unique corresponding action $\hat{\phi}_A$ of $\hat{T}$ on A such that $\hat{\phi}_A \phi = \phi_A$.*

Proof. Consider $\hat{t} \in \hat{T}$ and $a \in A$ such that $\operatorname{type} a = i \in I$, and find $t \in T$ such that $\phi(t) \sim_i \hat{t}$; see Remark 2.4.4(2). If an action $\hat{\phi}_A$ is to exist satisfying this theorem, it follows from Remarks 2.4.4(1) and 2.5.2(1) that

$$a\hat{\phi}_A\left(\hat{t}\right) \equiv a\hat{t}_A = at_A \equiv a\phi_A(t).$$

Therefore take this as the definition of $\hat{\phi}_A$. First observe that $a\hat{t}_A$ is well defined, for if t' is another element of T such that $\phi(t') \sim_i \hat{t}$ then $\phi(t) \sim_i \phi t'$, hence $s_i t_i = s_i t'_i$, with the result that $at_A = at'_A$ because $\operatorname{type} a = i$. Next we claim that $\hat{t}_A$ is a Boolean morphism. For

$$\overline{a\hat{t}_A} = \overline{at_A} = \bar{a}t_A = \bar{a}\hat{t}_A$$

because the same element $t \in T$ used to define $a\hat{t}_A$ can also be used to define $\bar{a}\hat{t}_A$ since $\operatorname{type} a = \operatorname{type} \bar{a}$ by Lemma 2.5.3. And for $a, b \in A$ we may take $k = \operatorname{type} a \vee \operatorname{type} b$ and find $t \in T$ such that $\phi(t) \sim_k \hat{t}$. Then because t_A is a Boolean morphism which agrees with $\hat{t}_A$ at a, b, $a \vee b$, and $a \wedge b$, we get

$$(a \vee b)\,\hat{t}_A = a\hat{t}_A \vee b\hat{t}_A \quad \text{and} \quad (a \wedge b)\,\hat{t}_A = a\hat{t}_A \wedge b\hat{t}_A.$$

To verify that $\hat{\phi}_A$ is a monoid morphism consider $\hat{t}', \hat{t} \in \hat{T}$ and $a \in A$, let $\operatorname{type} a = i$, and find $t \in T$ such that $\phi(t) \sim_i \hat{t}$. Then $a\hat{t}_A = at_A$, and

$$\operatorname{type}\left(a\hat{t}_A\right) = \operatorname{type}(at_A) = (\operatorname{type} a)\,t = it.$$

Next find $t' \in T$ such that $\phi(t') \sim_{it} \hat{t}'$, so that

$$\left(a\hat{t}_A\right)\hat{t}'_A = (at_A)\,\hat{t}'_A = (at_A)\,t'_A = a\,(tt')_A.$$

To show that $a\,(tt')_A = a\left(\hat{t}\hat{t}'\right)_A$ we must show that $\hat{t}\hat{t}' \sim_i \phi(tt')$. But this is easy. Because $\hat{t} \sim_i \phi(t)$ we know that $s_i\hat{t}_i = s_i t_i$, hence $s_i\hat{t}_i\hat{t}'_i = s_i t_i \hat{t}'_i$, and from the fact that $\hat{t}' \sim_{it} \phi(t')$ we know from Remark 2.5.5(2) that $s_i t_i \hat{t}'_i = s_i t_i t'_i$, hence $s_i\hat{t}_i\hat{t}'_i = s_i t_i t'_i$, from which the desired conclusion follows. This completes the verification that $\hat{\phi}_A$ is a monoid morphism. Finally, evaluation is continuous by construction. $\square$

3. T-ALGEBRAS

In this section we record the basic facts concerning T-algebras which will be necessary in what follows.

3.1. T-Morphisms and T-ideals.

Definition 3.1.1. A T*-ideal* of a T-algebra A is an ideal I with the property that $at \in I$ for all $a \in I$ and all $t \in T$.

Such ideals determine the T-surjections.

Proposition 3.1.2. *For any T-morphism $f : A \to B$,*

$$I = \{a \in A : f(a) = \bot\}$$

is a T-ideal. Conversely, for a given T-ideal I there is one and only one way to have T act on the quotient A/I so as to make the actions commute with the quotient map g, namely by defining

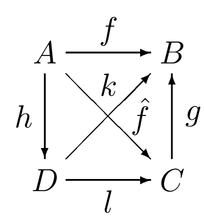
$$g(a)\,t = g(at)$$

for each $t \in T$ and $a \in A$. In this case A/I is a T-algebra and g is a T-morphism.

Lemma 3.1.3. *Any ideal I of a T-algebra A has a largest T-ideal contained in it, namely*

$$I_T \equiv \{a \in A : at \in I \text{ for all } t \in T\}.$$

Definition 3.1.4. A factorization $g\hat{f} = f$ of a T-surjection f is a **Ba**-**BaT** *factorization* if $\hat{f}$ and g are surjections such that $g \in$ **Ba** and $\hat{f} \in$ **BaT**. Such a factorization is *minimal* if it has the additional feature that any **Ba**-**BaT** factorization $kh = f$ has h as an initial factor of $\hat{f}$, i.e., $\hat{f} = kl$ for some $l \in$ **BaT**.



Proposition 3.1.5. *Every naked surjection out of a T-algebra has a minimal* **Ba-BaT** *factorization.*

Proof. If I is the kernel of f then $\hat{f}$ is the natural map from A onto A/I_T. □

Lemma 3.1.6. *Let A_0 be a subset of the T-algebra A, and let I be a T-ideal of A such that $I \cap A_0 \subseteq \{\bot\}$. Then there is a T-ideal J maximal with respect to $J \supseteq I$ and $J \cap A_0 \subseteq \{\bot\}$.*

Proof. The collection of such ideals is closed under unions of chains, and therefore has a maximal element by Zorn's Lemma. □

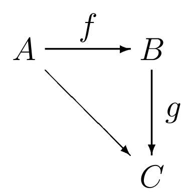
Recall that in any category **C**, a morphism $f : A \to B$ is called *essential* if it is injective, and every morphism out of B whose composition with f is injective must itself be injective.

Proposition 3.1.7. *The following are equivalent for a T-injection $f : A \to B$.*

(1) *f is essential.*
(2) *Every nontrivial T-ideal of B meets $f(A)$ nontrivially.*
(3) *For every $\bot < b \in B$ there is some $\bot < a \in A$ and finite $T' \subseteq T$ such that $f(a) \leq \bigvee_{T'} bt'$.*

Proof. By Proposition 3.1.2, the non-essentiality of f is equivalent to the existence of a T-ideal $I \subseteq B$ such that $I \cap f(A) = \{\bot\}$. An element $b > \bot$ of such an ideal would violate the condition of this proposition, and any $b \in B$ which violated this same condition would generate a proper T-ideal corresponding to a T-surjection denying essentiality. □

Proposition 3.1.8. *For every T-injection f there is a T-surjection g such that gf is essential.*



Proof. Let I be a T-ideal of B maximal with respect to $I \cap f(A) = \{\bot\}$, and let $g : B \to C \equiv B/J$ be the natural map. □

Definition 3.1.9. A T-algebra A is *simple* if it has no proper T-homomorphic images.

Proposition 3.1.10. *Every T-algebra has a simple quotient.*

Proof. Any T-algebra A has a maximal proper T-ideal by Lemma 3.1.6, with A_0 and I there taken to be $\{\top\}$ and $\{\bot\}$, respectively. And the corresponding quotient is simple by Proposition 3.1.2 □

We use **2** to denote the algebra containing only greatest element $\top$ and least element $\bot$. When regarded as a T-algebra, the action is presumed to be trivial, as indeed it must be.

Proposition 3.1.11. *The following are equivalent for a T-algebra A.*

(1) *A is simple.*
(2) *A has no proper T-ideals.*

(3) **2** *is essentially embedded in* A.
(4) *For all* $\bot < a \in A$ *there is some finite subset* $T_0 \subseteq T$ *such that* $\bigvee_{T_0} at_0 = \top$.

3.2. **The reduction to discrete T.** Let T_d denote the monoid T with discrete topology. We will construct several coreflections of a given T-algebra by the strategy of first doing so in the category $\mathbf{BaT}_d$ and then passing to the corresponding extension in **BaT** by restricting to a particular subalgebra. The first use of this strategy comes in the following subsection.

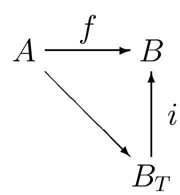
Lemma 3.2.1. *Suppose* B *is an algebra on which* T *acts, i.e., a* T_d*-algebra. Then*

$$B_T \equiv \{b \in B : \forall t \in T \; \exists T_t \in \mathcal{N}_t \; \forall t' \in T_t \;\; (bt = bt')\}$$

is the largest subalgebra of B *which forms a* T*-algebra under the relativised actions.*

Theorem 3.2.2. BaT *is a coreflective subcategory of* $\mathbf{BaT}_d$*, and the coreflection morphism for* $B \in \mathbf{BaT}_d$ *is the insertion of* B_T *in* B.

Proof. Let $f : A \to B$ be a T_d-morphism for which $A \in \mathbf{BaT}$. We

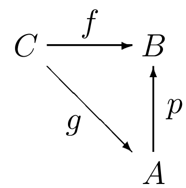


claim that $f(A) \subseteq B_T$. For if $a \in A$ then for any $t \in T$ we have a neighborhood T' of t such that $at = at'$ for all $t' \in T'$. But then

$$f(a)\,t = f(at) = f(at') = f(a)\,t'$$

for all $t' \in T'$. This proves the claim that $f(a) \in B_T$. □

3.3. **Freely adding actions to a naked algebra.** In this subsection we show that, although there are many ways to endow a naked algebra with actions, adding such actions "as freely as possible" can be done in one and only one way. More precisely, for a given naked algebra B there exist a unique T-algebra A and naked morphism p such that for any other T-algebra C and naked morphism f there is a unique **Bat** morphism g such that $pg = f$. That is, p is an F-co-universal map for B [2, VII 26.1], where $F : \mathbf{BaT} \to \mathbf{Ba}$ is the functor which forgets the actions. (We are, in effect, showing that F has a right adjoint [2, VII 27.3].) We refer to this situation by saying that **BaT** is coreflective in **Ba**.



We fix a naked algebra B, and set

$$A \equiv \prod_{t \in T} B_t,$$

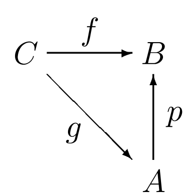
where B_t is a copy of B for each $t \in T$. We view each element of A as a map from T into B. Let $t \in T$ act on $a \in A$ according to the rule

$$(at)\,(t') \equiv a(tt')$$

for all $t' \in T$. It is easy to check that $A \in \mathbf{BaT}_d$, i.e., that T acts on A. Project A onto B by the **Ba** morphism p defined by $p\,(a) = a\,(1)$.

Proposition 3.3.1. $\mathbf{BaT}_d$ *is coreflective in* **Ba**, *and the* $\mathbf{BaT}_d$ *coreflection of a naked algebra* B *is* $p : A \to B$.

Proof. Given a **Ba** morphism f whose domain C is a T_d-algebra, a



T_d-morphism g which makes the diagram commute must satisfy

$$g\,(c)\,(t) = g\,(c)\,(t1) = (g\,(c)\,t)\,(1) = (g\,(ct))\,(1) = pg\,(ct) = f\,(ct)\,.$$

So if we take this requirement as a definition of g we clearly get a homomorphism which makes the diagram commute. To check that g commutes with the actions, observe that

$$(g\,(c)\,t)\,(t') = g\,(c)\,(tt') = f\,(c\,(tt')) = f\,((ct)\,(t')) = g\,(ct)\,(t')$$

for all $c \in C$ and $t, t' \in T$. □

Proposition 3.3.2. **BaT** *is coreflective in* **Ba**, *and the* **BaT** *coreflection of a naked algebra* B *is* $p : A_T \to B$.

Proof. Given a **Ba** morphism $f : C \to B$ whose domain is a **BaT** object, the morphism g of Proposition 3.3.1 factors through A_T by Theorem 3.2.2. □

In Proposition 2.5.7 we observed that A_T is the direct limit of the T-subalgebras $\{A_i : i \in I\}$. Therefore the following observation provides information about A_T.

Lemma 3.3.3. *An element of A has type at most i if and only if it is constant on each $\sim_i$ class. Therefore*

$$A_T = \{a \in A : \exists i \in I \ \ (a \text{ is constant on each } \sim_i \text{ class})\}.$$

Furthermore, for any $i \in I$ the elements of A_i are those $a \in A$ which are constant on $\sim_j$ classes, where $\sim_j = \bigvee_{T_0}$ it for some finite $T_0 \subseteq T$.

Proof. If type $a \leq i$ then for all $t, t' \in T$ we have

$$t \sim_i t' \iff s_i t = s_i t' \implies at = at'$$

by Remark 2.5.2(1). On the other hand, if a is constant on $\sim_i$ classes and $t \sim_i t'$ then by the right invariance of $\sim_i$ we have $tt'' \sim_i t't''$ for all $t'' \in T$. This implies that $a\,(tt'') = a\,(t't'')$, i.e., $(at)\,(t'') = (at')\,(t'')$ for all $t'' \in T$, which is to say that $at = at'$. This shows that type $a \leq i$ by Remark 2.5.2(1). □

We summarize the results of this subsection.

Theorem 3.3.4. *The* **BaT** *coreflection of B is the subalgebra of A consisting of those elements which are constant on the classes of some suitable relation on T.*

3.4. **Free T-algebras over pointed antiflows.** For $i \in I$ let F_i designate the free algebra over the generating set R_i. Each action t_i of $t \in T$ on R_i lifts uniquely to a morphism on F_i, and the actions thus defined make F_i a T-algebra.

Consider now a T-algebra A with element a of type at most i. The map $s_i t \mapsto at$ is a **pSpT** morphism from (R_i, s_i) onto the orbit (aT, a) of a. We show that this map lifts to a unique T-morphism from F_i into A. It is in this sense that F_i serves as the free T-algebra over R_i.

Theorem 3.4.1. *Let a be an element of a T-algebra A of type at most $i \in I$. Then there is a unique T-morphism $f : F_i \to A$ such that $f\,(s_i) = a$.*

Proof. In order for f to respect the actions we must have

$$f\,(s_i t) = f\,(s_i)\,t = at$$

for each $t \in T$. Since each $r_i \in R_i$ is of the form $r_i = s_i t$ for some $t \in T$, we may take this equation as the definition of the map f from R_i into A. This map is well-defined precisely because a has type at most i. The fact that F_i is the free algebra over R_i then provides a unique extension of f to a morphism from F_i into A. And f commutes with the actions on F_i because its restriction commutes with the actions on R_i. □

Since the free T-algebra F_i over R_i plays a prominent role in what follows, particularly in Section 6, we allow ourselves a closer look at its elements.

The context of this discussion is a little more general than that of Theorem 3.4.1. Suppose that F is a naked algebra with subset $R \subseteq F$. We use $\langle R\rangle_{\mathbf{Ba}}$ to designate the subalgebra generated by R; the elements of $\langle R\rangle_{\mathbf{Ba}}$ are Boolean combinations of finite subsets of R. R generates F if $\langle R\rangle_{\mathbf{Ba}} = S$, and R freely generates F as a naked algebra if every set map from S into another algebra A lifts to a unique morphism from F into A. This is equivalent to the condition that

$$\langle S_1\rangle_{\mathbf{Ba}} \cap \langle S_2\rangle_{\mathbf{Ba}} = \langle S_1 \cap S_2\rangle_{\mathbf{Ba}}$$

for finite $S_1, S_2 \subseteq R$. (See [1, V.3] for another equivalent formulation.) Therefore every $w \in F$ has a smallest subset $S \subseteq R$ for which $w \in \langle S\rangle_{\mathbf{Ba}}$; we refer to S as the *support of* w, and write $S = \operatorname{supp} w$. Note that $\operatorname{supp} w = \emptyset$ if and only if w is $\perp$ or $\top$.

We can understand the concept of support more concretely. If w is an element of $\langle S\rangle_{\mathbf{Ba}} \setminus \{\perp, \top\}$ then the laws of Boolean algebra allow w to be expressed in the form

$$w = \bigvee_{\Theta} \bigwedge_{S} s^{\theta(s)}$$

for some $\Theta \subseteq \{\pm 1\}^S$. (Here $\{\pm 1\}^S$ designates the set of all maps from S into $\{\pm 1\}$, and s^1 and s^{-1} designate s and the complement $\bar{s}$ of s, respectively.) This representation may be redundant because the laws of Boolean algebra may make it possible to omit an element s from S, restrict the functions of Θ to $S \setminus \{s\}$, and still have a representation of w. The criterion for being unable to omit s from S is exactly that there be a function $\theta \in \Theta$ such that changing its value only at s results in another function not in θ. The support of w is precisely the subset of S consisting of those elements which cannot be omitted from S in this sense. Thus

$$w = \bigvee_{\Lambda} \bigwedge_{\operatorname{supp} w} s^{\theta(s)}$$

for some $\Lambda \subseteq \{\pm 1\}^{\operatorname{supp} w}$,and this representation is a normal form, i.e., it is unique to w.

Proposition 3.4.3 will find use in Subsection 6.2.

Lemma 3.4.2. *Suppose F is a T-algebra which is freely generated as a naked algebra by a subset $R \subseteq F$. Then*

$$\operatorname{supp}(wt) \subseteq (\operatorname{supp} w)\, t$$

for any $w \in F \setminus \{\bot, \top\}$ and any $t \in T$.

Proof. Abbreviate $\operatorname{supp} w$ to S. If we write w in normal form and act on it by t, we get

$$wt = \left(\bigvee_{\Theta} \bigwedge_{S} s^{\theta(s)} \right) t = \bigvee_{\Theta} \bigwedge_{S} (st)^{\theta(s)} = \bigvee_{\Lambda} \bigwedge_{St} s^{\lambda(s)},$$

where in the rightmost expression s ranges over St and Λ is some subset of $\{\pm 1\}^{St}$. Now this expression for wt may be redundant, but in that case it can be reduced to the normal form for wt by removing extraneous elements from St. That is, $\operatorname{supp}(wt) \subseteq St = (\operatorname{supp} w)\, t$. □

Proposition 3.4.3. *Suppose F is a T-algebra which is freely generated as a naked algebra by a subset $R \subseteq F$. Then*

$$\operatorname{supp}(wt) = (\operatorname{supp} w)\, t$$

for any $w \in F \setminus \{\bot, \top\}$ and any $t \in \operatorname{stab} w$.

Proof. Abbreviate $\operatorname{supp} w$ to S. Then we have

$$S = \operatorname{supp} w = \operatorname{supp}(wt) \subseteq (\operatorname{supp} w)\, t = St.$$

But these sets are finite, and the cardinality of St does not exceed that of S. Therefore $St = S$ □

For $w \in F_i$ and $t \in T$, Theorem 3.4.1 allows for a very useful notational device. Since an element $w \in F_i \setminus \{\bot, \top\}$ has the normal form

$$w = \bigvee_{\Theta} \bigwedge_{S} s^{\theta(s)}$$

for $S \equiv \operatorname{supp} w$ and $\Theta \subseteq \{\pm 1\}^{S}$, and since each $s \in S$ is a translate of the source, say $s = s_i t_s$, we may write

$$w = \bigvee_{\Theta} \bigwedge_{S} (s_i t_s)^{\theta(s)}.$$

Note that the latter form is no longer unique to w because of the multiplicity of choices of t_s for s.

By viewing the s_i as an indeterminate, we can think of w as a Boolean word in translates of this free variable. So we often write the image of w under the mapping of Theorem 3.4.1 in the form

$$f(w) = w(a) = \bigvee_{\Theta} \bigwedge_{S} (at_s)^{\theta(s)} = \bigvee_{\Theta} \bigwedge_{S} a^{\theta(s)} t_s.$$

This notation is unambiguous precisely because a has type at most i, i.e., meaning that for $t, t' \in T$, $at = at'$ whenever $s_i t = s_i t'$. Thus all references to $w(a)$ for words $w \in F_i \setminus \{\pm 1\}$ are implicitly references to Theorem 3.4.1. In particular, the notation $w(a)$ makes no sense unless a is of type at most i.

3.5. **Free products in BaT.** We leave the routine verification of the following lemma to the reader.

Lemma 3.5.1. *Let $\{A_j : j \in J\}$ be a family of T-algebras and let $i_j : A_j \to B$ be their coproduct in* **Ba**. *Extend each action $t \in T$ to B by means of the coproduct property. Then $i_j : A_j \to B$ is also the*

$$\begin{array}{ccccc} A_1 & \xrightarrow{i_1} & B & \xleftarrow{i_2} & A_2 \\ {\scriptstyle t_1}\downarrow & & \downarrow{\scriptstyle t_B} & & \downarrow{\scriptstyle t_2} \\ A_1 & \xrightarrow[i_1]{} & B & \xleftarrow[i_2]{} & A_2 \end{array}$$

coproduct of the A_i's in **BaT**.

The coproduct of algebras, respectively T-algebras, is also called their *free product*. The following corollary easily follows from Theorem 3.4.1.

Corollary 3.5.2. *Every T-algebra is an image under an epimorphism of a coproduct of T-algebras of the form F_i, $i \in I$.*

3.6. **Free T-algebras over sets.** A T-algebra F is free over the set X provided that there is a set map $i : X \to F$ such that for every T-algebra A and every set map $f : X \to A$ there is a unique T-morphism $g : F \to A$ such that $gi = f$. We say that *the set map f lifts uniquely to the T-morphism g*. Any two T-algebras free over the same set X are isomorphic over X, and so are determined up to isomorphism solely by the cardinality of X. The trivial free T-algebra, namely the free T-algebra over $\emptyset$, always exists; it is the two-element algebra **2**. Now **BaT** is closed under free products by Lemma 3.5.1, and the free product of free T-algebras is free. Therefore free T-algebras exist over all sets if they exist over singletons.

The question is to determine when free T-algebras exist over all sets. We describe this situation succinctly by saying that nontrivial free T-algebras exist.

Theorem 3.6.1. *The following are equivalent for a topological monoid T.*

1. *Nontrivial free T-algebras exist.*
2. *The lattice I of types of T has a greatest element.*
3. *$\hat{T}$ is discrete.*

If T is a topological group, these conditions are equivalent to the following

4. *T possesses a smallest open subgoup.*
5. *$\bar{T}$ is a discrete topological group.*

Proof. We have already remarked on the equivalence of (2) and (3) in Remark 2.4.1(4). To show that (1) implies (2), suppose that F is the free algebra over a singleton set $X \equiv \{x\}$, and identify x with its image $i(x) \in F$. Let $i \equiv \operatorname{type} x$ and fix $j \in I$. Let F_j be the T-algebra of Theorem 3.4.1, and let $g : F \to F_j$ be the T-morphism which results from lifting the set map $x \mapsto s_j$. Then by Proposition 2.5.4 we get

$$i = \operatorname{type} x \geq \operatorname{type} g(x) = \operatorname{type} s_j = j.$$

This shows that i is the largest element of I.

Now suppose that I has a greatest element i. We claim that the T-algebra F_i of Theorem 3.4.1 is the free T-algebra over the singleton set $\{s_i\}$. That is because any set map f from $\{s_j\}$ into a T-algebra A takes s_j to an element of type $j \leq i$, and hence lifts to a unique T-morphism $g : F_j \to A$ by Theorem 3.4.1. Finally, if T is a topological group then the equivalence of (2) and (4) follows from Proposition 2.3.5(1). □

3.7. **Extending mappings to morphisms.** For a subset B of a T-algebra C we let

$$BT \equiv \{bt : t \in T, b \in B\}.$$

We use $\langle B \rangle_{\mathbf{Ba}}$ and $\langle B \rangle_{\mathbf{BaT}}$ to denote the subalgebra generated in the category of the subscript. Note that $\langle B \rangle_{\mathbf{BaT}} = \langle BT \rangle_{\mathbf{Ba}}$. We often shorten $\langle B \rangle_{\mathbf{BaT}}$ to $\langle B \rangle$ when doing so is unambiguous.

Lemma 3.7.1. *Suppose that A and C are T-algebras and that B is a subset of C. Then a mapping $f : B \to A$ can be extended to a T-morphism $\hat{f} : \langle B \rangle_{\mathbf{BaT}} \to A$ if and only if it satisfies the following pair of conditions.*

1. *$bt = b't' \Longrightarrow f(b)\,t = f(b')\,t'$ for all $t, t' \in T$ and $b, b' \in B$.*

(2) *For all finite subsets* $B', B'' \subseteq B$ *and* $T', T'' \subseteq T$ *we have*

$$\bigwedge_{T',B'} b't' \wedge \bigwedge_{T'',B''} \overline{b''}t'' = \bot \Longrightarrow \bigwedge_{T',B'} f(b')\,t' \wedge \bigwedge_{T'',B''} \overline{f(b'')}t'' = \bot$$

Proof. The existence of $\hat{f}$ certainly implies the conditions. Assuming them, first extend f to BT by declaring $\hat{f}(bt) = f(b)\,t$, an extension which is well-defined by the first condition. Then the second condition is a well-known criterion for the extension of $\hat{f}$ to all of $\langle BT\rangle_{\mathbf{Ba}}$; see [1, V.2]. Since $\langle BT\rangle_{\mathbf{Ba}} = \langle B\rangle_{\mathbf{BaT}}$, and since it is easy to verify that $\hat{f}$ commutes with the actions, the result follows. □

Proposition 3.7.2. *Suppose that we have* T*-algebras* C *and* A *with elements* c *and* a*, respectively, a subalgebra* $B \leq C$*, and a morphism* $f : B \to A$*. Then* f *can be extended to a morphism* $\hat{f} : \langle B, c\rangle \to A$ *such that* $\hat{f}(c) = a$ *if and only if the following conditions are satisfied.*

(1a) *The type of* a *is at most the type of* c*, i.e.,* $ct = ct' \Longrightarrow at = at'$ *for all* $t, t' \in T$*.*

(1b) *For all* $t \in T$ *and* $b \in B$*,* $ct = b \Longrightarrow at = f(b)$ *.*

(2) *For all* $b \in B$ *and all finite subsets* $T', T'' \subseteq T$ *we have*

$$b \wedge \bigwedge_{T'} ct' \wedge \bigwedge_{T''} \bar{c}t'' = \bot \Longrightarrow f(b) \wedge \bigwedge_{T'} at' \wedge \bigwedge_{T''} \bar{a}t'' = \bot$$

Proof. All these conditions are clearly necessary. To prove their sufficiency apply Lemma 3.7.1 to $B \cup \{c\}$. Conditions (1a) and (1b) above together imply condition (1) of the lemma, while condition (2) here implies its counterpart in the lemma. □

4. Injectives

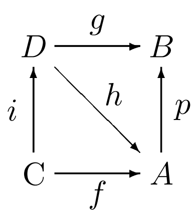
Definition 4.0.3. A T-algebra A is *injective* provided that for all morphisms $f : B \to A$ and all monomorphisms $g : B \to C$ there is a morphism $h : C \to A$ such that $hg = f$. An *injective hull* of A is an essential embedding of A into an injective object.

We first show the existence and uniqueness of injective hulls of T-algebras in Subsection 4.1. Although this follows from simple categorical principles, we outline the construction because we need to understand the structure of these hulls as concretely as possible. We then characterize injectivity in terms of systems of ideals in Subsections 4.2, 4.3, and 4.4.

4.1. **The existence and uniqueness of the injective hull.** We begin by showing that injective T-algebras exist.

Lemma 4.1.1. *Let $p : A \to B$ be the* **BaT** *coreflection of the naked complete algebra B. Then A is injective in* **BaT**.

Proof. Given the injection i and T-morphism f, let g be any morphism induced by the injectivity of B in **Ba** such that $gi = pf$. Then let h



be the T-morphism induced by the coreflective property of A such that $ph = g$. To show that $hi = f$ simply observe that for all $t \in T$,

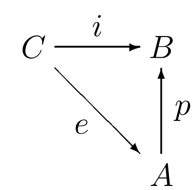
$$\begin{aligned}(hi\,(c))\,(t) &= (hi\,(c))\,(t1) = (hi\,(c)\,t)\,(1) = phi\,(ct) = gi\,(ct) = pf\,(ct) \\ &= f\,(ct)\,(1) = (f\,(c)\,t)\,(1) = f\,(c)\,(t1) = f\,(c)\,(t)\,.\end{aligned}$$

This proves the lemma. □

We continue by showing that **BaT** has enough injectives.

Lemma 4.1.2. *Every T-algebra can be embedded in an injective object.*

Proof. Let i be the insertion of C in its injective hull B in **Ba**. (B is

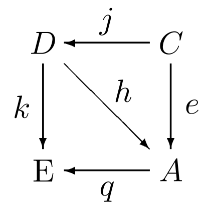


just the completion of C.) Let $p : A \to B$ be the **BaT** coreflection of B, and let e be the induced T-morphism. Then A is injective by Lemma 4.1.1, and e is injective because i is. □

For given morphisms $i_j : C \to E_j$, $j = 1, 2$, we say that a morphism $k : E_1 \to E_2$ is *over* C if $ki_1 = i_2$.

Proposition 4.1.3. *Every T-algebra C has a maximal essential extension $g : C \to E$. That is, g is essential, and every other essential extension of C embeds in E over C.*

Proof. Let e be the injection of Lemma 4.1.2, and let q be the quotient of Lemma 3.1.8. The composition $g \equiv qe$ is essential by construction. Given an essential extension j, let h be a T-morphism produced by the injectivity of A. Then $k \equiv qh$ is injective because $kj = qhj = qe$ is injective and j is essential. □



For a proof of the next result, see [1, I.20].

Proposition 4.1.4. *The following are equivalent for a T-algebra E.*

(1) *E is injective.*
(2) *E is a retract of each of its extensions.*
(3) *E has no proper essential extensions.*

We summarize the development of this subsection.

Theorem 4.1.5. *Every T-algebra has an injective hull which is unique up to isomorphism over it.*

4.2. **Systems of ideals.** Having demonstrated the existence and uniqueness of injective hulls of T-algebras in Theorem 4.1.5, we turn to the question of characterizing injective T-algebras. This is the content of Theorem 4.2.7, which requires the notion of an i-system of ideals.

Definition 4.2.1. Let A be a T-algebra and $i \in I$. Let F_i be the free T-algebra over (R_i, s_i) (see Subsection 3.4). Then an *i-system of ideals of A*, or simply an *i-system*, is a family

$$S = \{L(w) : w \in F_i\}$$

of ideals of A with the following properties.

(1) $L(\bot) = \{\bot\}$ and $L(\top) = A$.
(2) $\bigcap_K L(w_k) \subseteq L(w)$ for all finite subsets $\{w_k : k \in K\}$ and elements w of F_i such that $\bigwedge_K w_k \leq w$ in F_i.
(3) $L(w)\,t \equiv \{bt : b \in L(w)\} \subseteq L(wt)$ for all $t \in T$ and $w \in F_i$.

We sometimes suppress the i when it is clear from the context.

Proposition 4.2.2. *T-morphisms preserve i-systems. That is, if $f : A \to B$ is a T-morphism and if $\{L(w)\}$ is an i-system of ideals of A then*

$$\{\{b \in B : \exists a \in A \ \ (a \in L(w) \ \ \text{and} \ f(a) \geq b)\} : w \in F_i\}$$

is an i-system of ideals of B. And if $\{L(w)\}$ is an i-system of ideals of B and f is one-to-one then

$$\{\{a \in A : f(a) \in L(w)\} : w \in F_i\}$$

is an i-system of ideals of A.

Definition 4.2.3. We say that an element c in an extension $C \geq A$ *realizes* an i-system $S = \{L(w)\}$ if c is of type at most i and

$$L(w) \subseteq \{a \in A : a \leq w(c)\}$$

for each $w \in F_i$. We say that c *exactly realizes* S if c realizes S and the containment is an equality for each $w \in F_i$.

We characterize i-systems in Theorem 4.2.5, for which we need a simple lemma about naked algebras. In this lemma we consider the partitions of a given finite set B into two parts, B_1 and B_2. The symbol $\biguplus$ stands for disjoint union, so that we refer to the partition by writing $B_1 \biguplus B_2 = B$.

Lemma 4.2.4. *Let b be an element and B_0 a finite subset of a naked algebra B. Then b lies in the ideal generated by*

$$\left\{ \bigwedge B_2 : B_1 \biguplus B_2 = B_0,\, b \nleq \bigvee B_1 \right\}.$$

Proof. If not then by Zorn's Lemma there is a prime ideal J containing the displayed set and omitting b. Put

$$B_1 \equiv \{b_1 \in B_0 : b_1 \in J\}, \quad B_2 \equiv \{b_2 \in B_0 : b_2 \notin J\}.$$

Then the primeness of J implies that $\bigwedge B_2 \notin J$, hence $b \leq \bigvee B_1 \in J$, contrary to hypothesis. □

Theorem 4.2.5. *A collection $S = \{L(w) : w \in F_i\}$ of ideals of A is an i-system if and only if there is some element c in some extension $C \geq A$ which exactly realizes S.*

Proof. Suppose that $C \geq A$ is an extension having an element c of type at most i in C, and put

$$L(w) = \{a \in A : a \leq w(c)\}, \quad w \in F_i.$$

To verify that $\{L(w)\}$ constitutes an i-system which is exactly realized by c, it is only necessary to observe that $w(c) = f(w)$, where $f : F_i \to C$ is the map of Proposition 3.4.1 such that $f(s_i) = c$.

Conversely assume that an i-system $\{L(w)\}$ is given. Let B denote the coproduct of A with F_i, with insertion maps j_A and j_i. Let $g : B \to C$ be the surjection which results from identifying with $\perp$ all elements of B of the form $j_A(a) \wedge j_i(\bar{w})$ for $w \in F_i$ and $a \in L(w)$. Note that the ideal J generated by these elements is closed under the actions by the third defining property of a system of ideals, so that both C and g lie in **BaT** by Proposition 3.1.2. Finally set $f = gj_A$ and $c = gj_i(s_i)$. Observe that, since s_i is of type i in F_i, c is of type at most i in B by Proposition 2.5.4. And gj_i must be the function of Theorem 3.4.1 by virtue of its uniqueness, so that $gj_i(w) = w(c)$ for all $w \in F_i$.

We now show that $L(w) = \{a \in A : f(a) \leq w(c)\}$ for each $w \in F_i$. Since for $a \in L(w)$ we have

$$\perp = g(j_A(a) \wedge j_i(\bar{w})) = f(a) \wedge g j_i(\bar{w}) = f(a) \wedge \bar{w}(c) = f(a) \wedge \overline{w(c)},$$

it follows that $f(a) \leq w(c)$. Conversely, if $f(a) \leq w(c)$ then

$$\perp = f(a) \wedge \overline{w(c)} = f(a) \wedge \bar{w}(c) = g(j_A(a) \wedge j_i(\bar{w})),$$

i.e., $j_A(a) \wedge j_i(\bar{w}) \in J$. That means that there are finite subsets $\{a_k : k \in K\} \subseteq A$ and $\{w_k : k \in K\} \subseteq F_i$ such that $a_k \in L(w_k)$ for all $k \in K$, and such that

$$\begin{aligned} j_A(a) \wedge j_i(\bar{w}) &\leq \bigvee_K (j_A(a_k) \wedge j_i(\overline{w_k})) \\ &= \bigwedge_{K_1 \biguplus K_2 = K} \left(j_A\left(\bigvee_{K_1} a_k\right) \vee j_i\left(\bigvee_{K_2} \overline{w_k}\right)\right). \end{aligned}$$

Now for each partition $K_1 \biguplus K_2 = K$ it follows from properties of the coproduct ([1, VII 1(ii)]) that either $a \leq \bigvee_{K_1} a_k$ or $\bar{w} \leq \bigvee_{K_2} \overline{w_k}$. In the latter case we get $w \geq \bigwedge_{K_2} w_k$, from which the second defining property of i-systems implies that

$$\bigwedge_{K_2} a_k \in \bigcap_{K_2} L(w_k) \subseteq L(w).$$

Therefore $a \in L(w)$ by Lemma 4.2.4.

Finally, identify each element of A with its image under f. This identification makes C an extension of A because f must be one-to-one. The reason that f must be one-to-one is that by taking $w = \perp$ we get

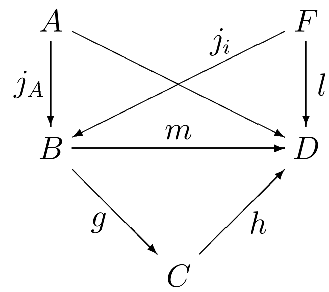
$$f(a) = \perp = w(c) \Longrightarrow a \in L(w) = L(\perp) = \{\perp\}$$

for any $a \in A$. □

The particular morphism f and element c constructed in the proof of Theorem 4.2.5 are universal with respect to their properties.

Theorem 4.2.6. *Suppose $S = \{L(w) : w \in F_i\}$ is an i-system of ideals of A for some $i \in I$. Let $C \geq A$ be the extension and c the element constructed in the proof of Theorem 4.2.5. Then for any other extension $D \geq A$ having an element d realizing S there is a unique morphism $h : C \to D$ over A taking c to d.*

Proof. Let l be the morphism of Proposition 3.4.1 from F_i into D taking s_i to d. The coproduct property of B applied to this map, together with the insertion of A in D, produces a unique morphism m making the top part of the diagram commute. We claim that m factors through



g, i.e., that there is a morphism h such that $hg = m$. This is because for any $b \in B$ such that $g(b) = \bot$, i.e., $b \in J$, there are finite subsets $\{a_k : k \in K\} \subseteq A$ and $\{w_k : k \in K\} \subseteq F_i$ such that $a_k \in L(w_k)$ for all $k \in K$ and

$$b \leq \bigvee_K (j_A(a_k) \wedge j_i(\overline{w_k})).$$

But $a_k \in L(w_k)$ implies $a_k \leq w_k(d) = l(w_k)$, hence $a_k \wedge l(\overline{w_k}) = \bot$, with the result that $mj_A(a_k) \wedge mj_i(\overline{w_k}) = \bot$. Therefore

$$m(b) \leq \bigvee_K m(j_A(a_k) \wedge j_i(\overline{w_k})) = \bot.$$

Finally, the uniqueness of h is a consequence of the uniqueness of m and the fact that $\langle f(A) \cup \{c\}\rangle = C$, where $f = gj_A$. □

We have finally assembled the tools we need to characterize injective T-algebras.

Theorem 4.2.7. *A T-algebra A is injective if and only if for every $i \in I$ and for every i-system S there is an element of A realizing S.*

Proof. Suppose A is injective, $i \in I$, and S is an i-system. Let $c \in C \geq A$ be the items constructed in the proof of Theorem 4.2.5. By the injectivity of A there is some morphism $j : C \to A$ such that j is the identity map on A. Set $a_0 \equiv j(c)$, and observe that a_0 is of type at most i because morphisms preserve type by Proposition 2.5.4. And a_0 realizes S because

$$a \in L(w) \Longrightarrow a \leq w(c) \Longrightarrow a = j(a) \leq jw(c) = w(j(c)) = w(a_0).$$

Now suppose that for every $i \in I$ and for every i-system S there is an element of A realizing S. To test the injectivity of A consider a morphism $f : B \to A$ and superalgebra $C \geq B$ having element $c \in C$. It is sufficient to extend f to a morphism $\hat{f} : \langle B, c\rangle \to A$, since a continuation of this process by transfinite induction results in an extension of f to all of C. We use Proposition 3.7.2 to achieve the extension by one element as follows. First let i be the type of c, and for each $w \in F_i$ let

$$L(w) = \{a_1 \in A : \exists b \in B \ (b \le w(c) \text{ and } f(b) \ge a_1)\}.$$

Then $S \equiv \{L(w) : w \in F_i\}$ is an i-system by Proposition 4.2.2, and is therefore realized by some $a \in A$. We claim that this setup satisfies the hypotheses of Proposition 3.7.2. Condition (1a) is satisfied by virtue of the fact that a is of type at most i. To establish condition (1b) suppose that $ct = b$ for some $t \in T$ and $b \in B$, and let $s_i t \equiv w \in F_i$. Then

$$b \le w(c) \Longrightarrow f(b) \in L(w) \Longrightarrow f(b) \le w(a) = at,$$

and $\bar{c}t = \bar{b}$ implies $f\left(\bar{b}\right) \le \bar{a}t$ in similar fashion, with the result that $f(b) = at$. To verify condition (2) consider $b \in B$ and finite subsets $T', T'' \subseteq T$, and let $w = \bigwedge_{T'} s_i t' \wedge \bigwedge_{T''} \overline{s_i} t''$. Then

$$\begin{aligned} b \wedge w(c) = \bot &\Longrightarrow b \le \bar{w}(c) \Longrightarrow f(b) \in L(\bar{w}) \\ &\Longrightarrow f(b) \le \bar{w}(a) \Longrightarrow f(b) \wedge w(a) = \bot. \end{aligned}$$

This completes the proof. □

4.3. **Maximal systems of ideals.** We need to consider the partitions of a given finite set $T_0 \subseteq T$ into two parts, T_1 and T_2. As before, we use the symbol $\uplus$ for disjoint union, and refer to the partition by writing $T_1 \uplus T_2 = T_0$.

Lemma 4.3.1. *Suppose c is an element of type at most i in some extension $C \ge A$, and fix $w \in F_i$ and $a \in A$. If there is a finite subset $T_0 \subseteq T$ and an element $\bot < a_0 \in A$ such that for all partitions $T_1 \uplus T_2 = T_0$ we have*

$$a_0 \wedge \overline{\bigvee_{T_1} at_1} \le \bigvee_{T_2} \bar{w}(c)\, t_2,$$

then $a \not\le w(c)$. Conversely, if $a \not\le w(c)$ then such a subset T_0 and element a_0 exist, provided that C is an essential extension of A.

Proof. If $a \not\le w(c)$ then $a \wedge \overline{w(c)} = a \wedge \bar{w}(c) > \bot$. If C is an essential extension of A there must be a finite subset $T_0 \subseteq T$ and element $\bot < a_0 \in A$ such that

$$a_0 \le \bigvee_{T_0} (a \wedge \bar{w}(c))\, t_0 = \bigwedge_{T_1 \uplus T_2 = T_0} \left(\bigvee_{T_1} at_1 \vee \bigvee_{T_2} \bar{w}(c)\, t_2 \right),$$

which is to say that $a_0 \wedge \overline{\bigvee_{T_1} at_1} \le \bigvee_{T_2} \bar{w}(c)\, t_2$ for all partitions $T_1 \uplus T_2 = T_0$.

On the other hand suppose that $T_0 \subseteq T$ is a finite subset and $\bot < a_0 \in A$ an element for which every partition of T_0 satisfies the inequality displayed in the lemma. Then a reversal of the preceding argument leads to the conclusion that $a \wedge \overline{w(c)} > \bot$, i.e., $a \nleq w(c)$. □

Observe that i-systems are ordered by containment, i.e., $\{L(w)\} \leq \{M(w)\}$ if and only if $L(w) \subseteq M(w)$ for all $w \in F_i$. Observe that in this case, any element (in any extension) which realizes $\{M(w)\}$ also realizes $\{L\{w\}\}$. Finally, observe also that the union of a tower of i-systems is an i-system, so that every i-system is contained in a maximal i-system by Zorn's Lemma. We characterize maximal i-systems in Theorem 4.3.2.

Theorem 4.3.2. *The following conditions are equivalent for an i-system $S = \{L(w)\}$ of ideals of A.*

(1) *For every $a \in A$ and $w \in F_i$ with $a \notin L(w)$ there is some finite subset $T_0 \subseteq T$ and element $\bot < a_0 \in A$ such that*

$$a_0 \wedge \overline{\bigvee_{T_1} at_1} \in L\left(\bigvee_{T_2} \bar{w}t_2\right)$$

for all partitions $T_1 \uplus T_2 = T_0$.

(2) *Every element which realizes S in an extension of A does so exactly.*

(3) *Every element which realizes S in an essential extension of A does so exactly.*

(4) *Every element which realizes S in the injective hull of A does so exactly.*

(5) *S is maximal among i-systems of ideals of A.*

Proof. Suppose (1) holds, let c be an element realizing S in some extension $C \geq A$, and consider $a \in A$ such that $a \leq w(c)$. Then by the first part of Lemma 4.3.1 there can be no finite $T_0 \subseteq T$ and $\bot < a_0 \in A$ such that

$$a_0 \wedge \overline{\bigvee_{T_1} at_1} \leq \bigvee_{T_2} \bar{w}(c)\, t_2$$

for all partitions $T_1 \uplus T_2 = T_0$. Hence there can be no finite $T_0 \subseteq T$ and $\bot < a_0 \in A$ such that

$$a_0 \wedge \overline{\bigvee_{T_1} at_1} \in L\left(\bigvee_{T_2} \bar{w}t_2\right),$$

for all partitions $T_1 \uplus T_2 = T_0$. It follows from (1) that $a \in L(w)$, i.e.,

$$L(w) = \{a \in A : a \leq w(c)\},$$

meaning c exactly realizes S.

The implications from (2) to (3) and (3) to (4) are obvious. Assume (4), and to prove (1) consider $w \in F_i$ and $a \notin L(w)$. Let $C \geq A$ be the injective hull of A. Then

$$S_C = \{\{c \in C : c \leq a_1 \text{ for some } a_1 \in L(w)\} : w \in F_i\}$$

is an i-system of ideals of C by Proposition 4.2.2, and thus is realized by an element $c_0 \in C$ by Theorem 4.2.7. Now c_0 clearly also realizes S, and does so exactly by (4), hence $a \not\leq w(c_0)$. From Lemma 4.3.1 we then get a finite subset $T_0 \subseteq T$ and an element $\bot < a_0 \in A$ which, in light of the exactness of the realization of S by c_0, satisfy (1).

If S were properly smaller than another i-system S' then by Theorem 4.2.5 we could find an extension $C \geq A$ with an element c realizing S' exactly. But then c would realize S inexactly. This proves that (5) follows from (2). On the other hand, if $C \geq A$ is any extension having an element c which realizes S then

$$S \leq S' \equiv \{L(w(c)) : w \in F_i\}.$$

Hence if S is maximal it follows that $S = S'$, which is to say that c realizes S exactly. This shows that (2) follows from (5), and completes the proof. □

Corollary 4.3.3. *Every i-system is contained in an i-system satisfying Theorem 4.3.2.*

Corollary 4.3.4. *A T-algebra A is injective if and only if for every $i \in I$ and for every maximal i-system S of ideals of A there is an element of A which realizes S exactly.*

It is tempting to speculate that the injective hull of a T-algebra A could be constructed as the algebra of maximal i-systems. Unfortunately, such a construction cannot be straightforward, since i-systems are not in one-to-one correspondence with the elements of the injective hull. Indeed, many different elements of the injective hull can give rise to the same maximal i-system. For example, if $T = Z_2 + Z_2$ and $A = \mathbf{2} \equiv \{\bot, \top\}$ is acted on trivially by T, then the injective hull of A is $\mathbf{2}^T$, and the atoms of this algebra all generate the same system of ideals of A.

4.4. **Checking only large types.** In this subsection we point out that, in order to verify the condition for injectivity of Theorem 4.2.7, we need not check all i-systems, but can confine the verification to i-systems for the larger (finer) i's in I.

Let us fix notation to be used throughout the rest of this subsection. Suppose we are given $i \geq j$ in I, and let $\rho^i_j : R_i \to R_j$ be the canonical

pSpT surjection of Proposition 2.3.1, i.e., $\rho_j^i(s_i t) = s_j t$ for all $t \in T$. Let $p_j^i : F_i \to F_j$ be the T-morphism induced by ρ_j^i, where F_i and F_j are the free T-algebras over R_i and R_j, respectively.

It may be helpful to describe the action of p_j^i more concretely. As we mentioned in Subsection 3.4, an element $w \in F_i$ may be expressed in the form

$$w = \bigvee_{\Theta} \bigwedge_{S} (s_i t_s)^{\theta(s)},$$

for $S \equiv \operatorname{supp} w$ and $\Theta \subseteq \{\pm 1\}^S$. Therefore

$$p_j^i(w) = \bigvee_{\Theta} \bigwedge_{S} \left(p_j^i(s_i)\, t_s\right)^{\theta(s)} = \bigvee_{\Theta} \bigwedge_{S} \left(\rho_j^i(s_i)\, t_s\right)^{\theta(s)} = \bigvee_{\Theta} \bigwedge_{S} (s_j t_s)^{\theta(s)}.$$

Finally, consider a j-system $S_j = \{L_j(w) : w \in F_j\}$ of ideals on a given T-algebra A, and let $S_i = \{L_i(w) : w \in F_i\}$ be defined by the rule $L_i(w) \equiv L_j\left(p_j^i(w)\right)$ for all $w \in F_i$. Then it is easy to check that S_i is an i-system of ideals of A.

Proposition 4.4.1. *If S_j is maximal among j-systems then S_i is maximal among i-systems.*

Proof. Consider $a \in A$ and $w \in F_i$ such that $a \notin L_i(w) = L_j\left(p_j^i(w)\right)$. Then the maximality of S_j implies the existence of a finite subset $T_0 \subseteq T$ and an element $\bot < a_0 \in A$ such that

$$a_0 \wedge \overline{\bigvee_{T_1} a t_1} \in L_j\left(\bigvee_{T_2} \overline{p_j^i(w)} t_2\right)$$

for all partitions $T_1 \biguplus T_2 = T_0$. But $\bigvee_{T_2} \overline{p_j^i(w)} t_2 = p_j^i\left(\bigvee_{T_2} \bar{w} t_2\right)$ since p_j^i commutes with the Boolean operations and the actions, hence

$$a_0 \wedge \overline{\bigvee_{T_1} t_1 a} \in L_i\left(\bigvee_{T_2} \bar{w} t_2\right)$$

for all partitions $T_1 \biguplus T_2 = T_0$. That is, S_i is maximal as well. □

Corollary 4.4.2. *If S_j is maximal among j-systems then any element of A which realizes S_i also realizes S_j, and does both exactly.*

Proof. Suppose $a_0 \in A$ realizes S_i. To show that a_0 has type at most j, consider actions $t', t'' \in T$ such that $s_j t' = s_j t''$. Since S_i is maximal by Proposition 4.4.1, a_0 realizes it exactly by Theorem 4.3.2. Hence

$$\begin{aligned}\{a \in A : a \le a_0 t'\} &= L_i(s_i t') = L_j\left(p_j^i(s_i t')\right) = L_j\left(\rho_j^i(s_i t')\right) = L_j(s_j t') \\ &= L_j(s_j t'') = L_j\left(\rho_j^i(s_i t'')\right) = L_j\left(p_j^i(s_i t'')\right) = L_i(s_i t'') \\ &= \{a \in A : a \le a_0 t''\},\end{aligned}$$

from which it follows that $a_0t' = a_0t''$. That is, a_0 has type at most j.

To demonstrate that a_0 realizes S_j, consider $w_j \in F_j$, and write w_j in the form

$$w_j = \bigvee_{\Theta} \bigwedge_{S} (s_j t_s)^{\theta(s)},$$

where $S = \operatorname{supp} w_j$ and $\Theta \subseteq \{\pm 1\}^S$. Put

$$w_i \equiv \bigvee_{\Theta} \bigwedge_{S} (s_i t_s)^{\theta(s)} \in F_i.$$

We are not claiming that w_i is uniquely determined by w_j, as indeed it may vary with the choice of t_s for each $s \in S$. Nevertheless it is still true that $p^i_j(w_i) = w_j$, and that

$$w_i(a_0) = \bigvee_{\Theta} \bigwedge_{S} (a_0 t_s)^{\theta(s)} = w_j(a_0).$$

Therefore

$$\begin{aligned} L_j(w_j) &= L_j\left(p^i_j(w_i)\right) = L_i(w_i) = \{a \in A : a \le w_i(a_0)\} \\ &= \{a \in A : a \le w_j(a_0)\}. \end{aligned}$$

That is, a_0 exactly realizes S_j. □

We summarize our characterizations of injective T-algebras.

Theorem 4.4.3. *The following are equivalent for a T-algebra A.*

(1) *A is injective.*
(2) *For every $i \in I$, every i-system is realized in A.*
(3) *For every $i \in I$, every maximal i-system is realized in A.*
(4) *For every $j \in I$ there is some $i \in I$ with $i \ge j$ such that every maximal i-system is realized in A.*

In particular, if T admits a finest pointed antiflow R_i, i.e., if I contains a largest element i, then a T-algebra A is injective if and only if every maximal i-system is realized in A.

Corollary 4.4.4. *If T is connected then a T-algebra A is injective if and only if every maximal T-ideal of A is principal.*

5. When all pointed antiflows are finite

Throughout this section we assume that all pointed antiflows R_i, $i \in I$, are finite. This means, in particular, that orbits of elements of T-algebras are finite. The assumption of finite pointed antiflows is equivalent to the assumption that $\hat{T}$ is compact (Remarks 2.2.1(3) and 2.4.1(2)). Since for all intents and purposes we may replace T

by $\hat{T}$ (Theorems 2.4.2 and 2.6.1), the working hypothesis is that T is compact.

5.1. **When $\hat{T}$ is a compact group.** In the presence of our running hypothesis that all antiflows are finite, either the surjectivity or the injectivity of all actions on all antiflows implies that $\hat{T}$ is a compact group (Lemma 5.1.1). In this case things are particularly simple: every T-algebra essentially extends its stationary subalgebra (Proposition 5.1.3); every T-algebra A satisfies

$$A_s \leq A \leq E,$$

where E is the injective hull of A_s and the embeddings are essential (Proposition 5.1.5); the injective objects have a particularly simple structure (Theorem 5.1.6).

Lemma 5.1.1. *The following are equivalent for a T-algebra A.*

(1) *For each $i \in I$ and $t \in T$, the action of t_i on R_i is one-to-one.*
(2) *For each $i \in I$ and $t \in T$, t_i maps R_i onto itself.*
(3) *For each $i \in I$, T_i is a group.*
(4) *$\bar{T}$ is a compact group.*
(5) *$\hat{T}$ is a compact group.*

Proof. The equivalence of the first three conditions is clear. If (3) holds then $\prod_I T_i$ is a compact group because each T_i is, and so the closed submonoid

$$\bar{T} = \left\{ \bar{t} \in \prod_I T_i : \forall i \geq j \ \left(\phi_j^i \left(\bar{t}\,(i) \right) = \bar{t}\,(j) \right) \right\}$$

is clearly a compact group as well. And this implies in turn that $\hat{T} = \bar{T}$. The implication from (5) to (1) is Remark 2.2.1(3). □

Definition 5.1.2. For a T-algebra A, the *stationary subalgebra of* A is

$$A_s \equiv \{ a \in A : at = a \text{ for all } t \in T \}.$$

We regard A_s to be a T-subalgebra of A acted upon trivially by T.

Proposition 5.1.3. *If $\hat{T}$ is a topological group then any T-algebra is an essential extension of its stationary subalgebra.*

Proof. Consider a T-algebra A with element $a > \bot$, let $a_0 = \bigvee aT$. Each action must map $a_0 T$ into itself, and because $a_0 T$ is finite and the action one-to-one, it must map $a_0 T$ onto itself. Therefore each action fixes a_0, i.e., $a_0 \in A_s$. This makes A an essential extension of A_s by Proposition 3.1.7. □

Here is an example to show that the Proposition 5.1.3 need not hold, even when both monoid and algebra are finite.

Example 5.1.4. *Consider the flow* $X = \{x_1, x_2, x_3, x_4\}$ *with actions* $\{1, f_1, f_2\}$ *defined as follows.*

fx	x_1	x_2	x_3	x_4
1	x_1	x_2	x_3	x_4
f_1	x_4	x_2	x_2	x_4
f_2	x_2	x_2	x_4	x_4

Let A *be the Stone space* $\mathbf{2}^X$ *of* X*, and let* T *be* $\{1, t_1, t_2\}$*, where* t_i *is the dual of* f_i*, i.e.,* $at_i = f_i^{-1}\{a\}$ *for subsets* $a \subseteq X$*. Let* $a_1 = \{x_1, x_2\}$ *and* $a_2 \equiv \{x_2, x_3\}$*, so that*

$$\bigvee a_1 T = a_1 1 \vee a_1 t_1 \vee a_1 t_2 = a_1 \vee a_2 \vee a_1 = \{x_1, x_2, x_3\}\,.$$

However, $A_s = \{\emptyset, X\}$*, and so* A *is not an essential extension of* A_s*.*

Proposition 5.1.5. *If* $\hat{T}$ *is a topological group then any* T*-algebra* A *satisfies*

$$A_s \leq A \leq E,$$

where the embedding $A_s \leq E$ *is the injective hull of* A_s *and the embedding* $A \leq E$ *is the injective hull of* A*.*

Proof. Let $A \leq E$ be the injective hull of A_s (Theorem 4.1.5). By Proposition 4.1.4 $A \leq E$ is the maximal essential extension of Proposition 4.1.3, and so there is an embedding of A into E over A_s. It is in this sense that A is intermediate between A_s and E. It follows that $A \leq E$ is the injective hull of A. □

Proposition 5.1.6. *Suppose that* $\hat{T}$ *is a topological group. Then for any complete algebra* B*,*

$$E \equiv \left\{a \in B^T : a \text{ is constant on } \sim_i \text{ classes for some } i \in I\right\}$$

is an injective T*-algebra, and every injective* T*-algebra has this form.*

Proof. Given the complete algebra B, we see by Theorem 3.3.4 that E is the cofree T-algebra over B. Thus E is injective by Lemma 4.1.1. On the other hand, suppose we are given an arbitrary injective object A. We first claim that A_s is a complete algebra. For if we regard both A_s and its completion (without actions) B to be T-algebras with trivial action, then the embedding $A_s \leq B$ is a T-injection which, by virtue of the injective property of A, lifts to an injection of B into A over A_s. It follows that $A_s = B$, i.e., A_s is complete.

Now A is the injective hull of $B \equiv A_s$ by Proposition 5.1.3, so to finish the proof we argue that

$$E \equiv \left\{a \in B^T : a \text{ is constant on } \sim_i \text{ classes for some } i \in I\right\},$$

is also the injective hull of B. Since E is the cofree T-algebra over B of Theorem 3.3.4, it is injective by Lemma 4.1.1. Furthermore, by regarding B once again as a T-algebra with trivial action, we can lift the identity map to a T-injection from B into E, under which each element of B maps to the corresponding constant function. We identify each element of B with the corresponding constant function, so that we have $B \leq E$. All that remains is to show that this extension is essential.

Consider an arbitrary $\bot < a \in E$. Then a is constant on $\sim_i$ classes for some $i \in I$, and in particular there is at least one such class $[t]_i$ on which $a(t') = b > 1$ for all $t' \in [t]_i$. But since T_i acts transitively on the finite set $T/{\sim_i}$, there is a finite subset $T' \subseteq T$ such that $\left(\bigvee_{T'} at'\right)(t) \geq b$ for all $t \in T$, i.e., $\bigvee_{T'} at'$ dominates the constantly b function. It follows from Proposition 3.1.7 that E is an essential extension of B. □

6. Projectives

6.1. **Projectives in general.** We now turn to projectivity in **BaT**. Let us recall the definition.

Definition 6.1.1. A T-algebra A is *projective* if and only if for each morphism $f : A \to B$ and each epimorphism $g : C \to B$ there is a morphism $h : A \to C$ such that $gh = f$.

Recall that the T-epimorphisms are precisely the surjective morphisms by Proposition 1.2.2. Also note that we always have the trivial projective $\mathbf{2} \equiv \{\bot, \top\}$.

We can already state the first characterization of projectivity in **BaT**. Recall that a retraction is an epimorphism $g : C \to A$ with a right inverse, i.e., there exists a morphism $h : A \to C$ such that $gh = 1_A$.

Lemma 6.1.2. *The following are equivalent for a T-algebra A.*

(1) *A is projective.*
(2) *Every epimorphism onto A is a retraction.*
(3) *Every epimorphism onto A out of a coproduct of T-algebras of the form F_i, $i \in I$, is a retraction.*

Proof. It is clear that (1) implies (2) and that (2) implies (3). Assume (3), and in order to prove (1) consider a given epimorphism $g : C \to B$ and a given homomorphism $f : A \to B$. For each $a \in A$ choose $c_a \in C$ such that $g(c_a) = f(a)$. Let $i_a \equiv \operatorname{type} a \vee \operatorname{type} c_a$, let (R_a, s_a) be a copy of (R_{i_a}, s_{i_a}), let F_a be a copy of F_{i_a}, the free T-algebra over (R_a, s_a), and let $l_a : F_a \to A$ and $k_a : F_a \to C$ be the unique morphisms given by Theorem 3.4.1 such that $l_a(s_a) = a$ and $k_a(s_a) = c_a$. Let F be the coproduct of the family $\{F_a : a \in A\}$, and let $l : F \to A$ and $k : A \to C$ be the unique maps induced by the l_a's and k_a's respectively. That is, $l(s_a) = a$ and $k(s_a) = c_a$ for all $a \in A$. Since gk and fl agree on $\{s_a : a \in A\}$, and since this set generates F, it follows that $gk = fl$. Now apply (3) to l to get a morphism $m : A \to F$ such that $lm = 1_A$. Set $h \equiv km$. Then we have

$$gh = gkm = flm = f,$$

as desired. □

6.2. **When do projectives exist?** We propose to do now for projective T-algebras what we did for free T-algebras in Subsection 3.6. We need a little notation in addition to that of Subsection 3.4. For $i \geq j$ in I let $p^i_j : F_i \to F_j$ be the unique T-morphism given by Theorem 3.4.1 such that $p^i_j(s_i) = s_j$. Note that the restriction of p^i_j to the generating set $R_i \subseteq F_i$ is just the **pSpT** surjection ρ^i_j of Definition 2.1.2. We use 1_j to designate the identity morphism on F_j.

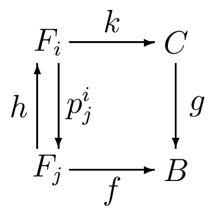
Theorem 6.2.1. *For $j \in I$, the first five conditions are equivalent and imply the sixth. The first six conditions are equivalent if the suitable relations on T correspond to the source stabilizers. All seven conditions are equivalent if T is a topological group.*

(1) *For every $i \geq j$ in I there is some $k \geq i$ and some $w \in F_k$ of type at most j such that $p^k_j(w) = s_j$.*
(2) *For every $i \geq j$ in I there is some $w \in F_i$ of type at most j such that $p^i_j(w) = s_j$.*
(3) *For every $i \geq j$ in I there is a T-morphism $h : F_j \to F_i$ such that $p^i_j h = 1_j$.*
(4) *F_j is projective.*
(5) *For every T-epimorphism $f : A \to B$ and every $b \in B$ of type at most j there is some $a \in A$ of type at most j such that $f(a) = b$.*
(6) *For every $i \geq j$ in I there is a finite subset $R \subseteq \left(\rho^i_j\right)^{-1}(s_j)$ such that $Rt = R$ for all $t \in \operatorname{stab} s_j$.*
(7) *For every $i \geq j$ in I, $\left(\rho^i_j\right)^{-1}(s_j)$ is finite.*

Proof. To show that (2) follows from (1), consider $i \geq j$ in I and find $k \geq i$ and $w \in F_k$ for which $p^k_j(w) = s_j$. Then $w' \equiv p^k_i(w)$, which is of type at most j by Proposition 2.5.4 and which lies in F_i, satisfies (2) because

$$p^i_j(w') = p^i_j p^k_i(w) = p^k_j(w) = s_j.$$

If (2) holds then Theorem 3.4.1 provides a unique T-morphism $h : F_j \to F_i$ such that $h(s_j) = w$. Since $p^i_j h$ agrees with 1_j on s_j, the two T-morphisms must be the same by the uniqueness clause of Theorem 3.4.1. That is, (3) holds. To prove that (3) implies (4), consider a T-morphism f and a T-epimorphism g, choose $c \in C$ such that $f(s_j) = g(c)$, and set $i \equiv j \vee \operatorname{type} c$. Let k be the T-morphism given



by Theorem 3.4.1 such that $k(s_i) = c$, and let h be the T-morphism whose existence is asserted in (3). Now gk and fp^i_j agree on s_i, and because T-morphisms out of F_i are determined by their values at s_i, we conclude that $gk = fp^i_j$. Therefore

$$gkh(s_j) = fp^i_j h(s_j) = f(s_j),$$

and we likewise conclude that $gkh = f$. This shows that F_j is projective.

To show that (4) implies (5), consider a given T-epimorphism $f : A \to B$ and element $b \in B$ of type at most j, choose $a_0 \in A$ such that $f(a_0) = b$, and set $i \equiv j \vee \operatorname{type} a_0$. Let $k : F_j \to B$ and $g : F_i \to A$ be the T-morphisms given by Theorem 3.4.1 such that $k(s_j) = b$ and $g(s_i) = a_0$. Let $h : F_j \to F_i$ be a T-morphism produced by the projectivity of F_j such that $p^i_j h = 1_j$. Now kp^i_j and fg agree at s_i and are therefore identical, with the consequence that

$$fgh(s_j) = kp^i_j h(s_j) = k(s_j) = b.$$

The desired element is $a \equiv gh(s_j)$. This works because $\operatorname{type} a \leq \operatorname{type} s_j = j$ by Proposition 2.5.4. Finally, to deduce (1) from (5) simply apply (5) to the T-epimorphism p^i_j and the element $s_j \in F_j$. We have proven the first five conditions equivalent.

To show that (2) implies (6) fix $i \geq j$ in I and use (2) to get $w \in F_i$ of type at most j such that $p^i_j(w) = s_j$. Put

$$R \equiv \operatorname{supp} w \cap \left(\rho^i_j\right)^{-1}(s_j).$$

Now $\operatorname{stab} s_j \subseteq \operatorname{stab} w$ because $\operatorname{type} w \leq j$, so any $t \in \operatorname{stab} s_j$ actually permutes the elements of $\operatorname{supp} w$ by Proposition 3.4.3. Since t maps $\left(\rho^i_j\right)^{-1}(s_j)$ into itself, it follows that t also permutes R, i.e., $Rt = R$.

Now assume that the suitable relations on T correspond to the source stabilizers. To show that (6) implies (2), consider $i \geq j$ in I, let R be the finite subset of $\left(\rho^i_j\right)^{-1}(s_j)$ such that $Rt = R$ for all $t \in \operatorname{stab} s_j$, and put $w \equiv \bigvee R \in F_i$. Then for any $t \in \operatorname{stab} s_j$ we have

$$wt = \bigvee Rt = \bigvee R = w,$$

so that $\operatorname{type} w \leq j$ by Remark 2.5.2(2). And clearly $p^i_j(w) = s_j$. This completes the proof that (6) implies (2).

Finally, if T is a topological group then the equivalence of (6) and (7) is a consequence of the fact that if $i \geq j$ in I then $\operatorname{stab} s_j$ acts transitively by right multiplication on the cosets of $\operatorname{stab} s_i$. The proof is done. □

Definition 6.2.2. We say that an element $j \in I$ is *almost maximal* if it satisfies the first three conditions of Theorem 6.2.1.

Remark 6.2.3. Let i be a maximal element of I.

(1) Then i is a maximum element because I is a lattice. By Remark 2.4.1(4), this happens if and only if $\hat{T}$ is discrete.
(2) i is almost maximal.
(3) Any element $j \in I$ is almost maximal if and only if there is some $w \in F_i$ such that $p^i_j(w) = s_j$ and $\operatorname{type} w \leq j$.

Proposition 6.2.4. *Suppose that the suitable relations on T correspond to the source stabilizers.*

(1) *Any element of I above an almost maximal element is itself almost maximal.*
(2) *If I contains an almost maximal element, then every element is dominated by an almost maximal element.*

Proof. Suppose T is a topological group. If $i \geq j \geq k$ in I and k is almost maximal then there must be some subset R of $\left(\rho^i_k\right)^{-1}(s_k)$ such that $Rt = R$ for all $t \in \operatorname{stab} s_k$. But then $S \equiv \rho^i_j(R)$ is a subset of $\left(\rho^j_k\right)^{-1}(s_k)$ such that satisfies $St = S$ for all $t \in \operatorname{stab} s_k$. □

It turns out that the type of a nontrivial element of a projective T-algebra is almost maximal, provided that the suitable relations on T correspond to the source stabilizers.

Theorem 6.2.5. *Suppose the suitable relations on T correspond to the source stabilizers, and that A is a projective T-algebra with element $a \neq \bot, \top$. Then* $\operatorname{type} a$ *is almost maximal in I.*

Proof. Let $j \equiv \operatorname{type} a$ and consider $i \geq j$ in I. For each $b \in A$ let (R_b, s_b) be a copy of (R_k, s_k), where k is $i \vee \operatorname{type} b$, and let F_b be a copy of F_k. Let C be the coproduct of the family $\{F_b : b \in A\}$, and let $p : C \to A$ be the unique T-morphism such that $p(s_b) = b$ for all

$b \in A$. Since A is projective, there is a T-morphism $h : A \to C$ such that $ph = 1_A$. Put $c \equiv h(a)$. Note that $\operatorname{type} c = j$ by Proposition 2.5.4. Now C is freely generated as a naked algebra by $\bigcup_A R_b$, and $c \in C \setminus \{\perp, \top\}$, so c has nonempty support $S \subseteq \bigcup_A R_b$. Note that $S = St$ for any $t \in \operatorname{stab} s_j$ by Proposition 3.4.3. Let $b \in A$ be such that $S_b \equiv S \cap R_b \neq \emptyset$, and let $k \equiv i \vee \operatorname{type} b$. Since for any $t \in T$ it is true that $R_b t \subseteq R_b$, it follows that $S_b t = S_b$ for all $t \in \operatorname{stab} s_j$. Finally, put

$$R \equiv \{s_i t : s_b t \in S_b\} \subseteq S_i.$$

It follows from the fact that $k \geq i$ that R is finite, for S_b is finite and for all $t, t' \in T$ we have

$$s_b t = s_b t' \Longleftrightarrow s_k t = s_k t' \Longrightarrow s_i t = s_i t'.$$

Now consider $t \in \operatorname{stab} s_i$ and $r \in R$, say $r = s_i t_r$ for some $t_r \in T$ such that $s_b t_r \in S_b$. Then $rt = s_i t_r t$ lies in Rt because $s_b t_r t$ lies in $S_b t \subseteq S_b$. This shows that $Rt \subseteq R$. On the other hand, since $S_b t \supseteq S_b$ there is some $s_r \in S_b$ for which $s_r t = s_b t_r$, say $s_r = s_b t'$ for some $t' \in T$. We have

$$s_b t' t = s_r t = s_b t_r \Longleftrightarrow s_k t' t = s_k t_r \Longrightarrow s_i t' t = s_i t_r = r.$$

Now $s_i t' \in R$ because $s_b t' = s_r \in S_b$, and this shows that $r \in Rt$, i.e., that $Rt \supseteq R$. This completes the proof of the theorem. □

We summarize our results.

Theorem 6.2.6. *Suppose that the suitable relations on T correspond to the source stabilizers. Then nontrivial projective T-algebras exist if and only if I contains an almost maximal element. Furthermore, the projective objects are precisely the retracts of coproducts of T-algebras of the form F_i for i almost maximal in I.*

Proof. If T contains the almost maximal element i then F_i is a nontrivial projective T-algebra by Theorem 6.2.1. And if nontrivial projective T-algebras exist then I contains an almost maximal element by Theorem 6.2.5. Now on general principles, any retract of a coproduct of projectives is projective. And if A is any projective T-algebra then the first few sentences of the proof of Theorem 6.2.5 show that A is a retract of a coproduct of T-algebras of the form F_i for i almost maximal in I. □

This immediately yields a characterization of those topological groups T for which nontrivial projective T-algebras exist. Recall that a topological group H is *totally bounded* if for any nonempty open subset U of H there is a finite set $F \subseteq H$ such that $UF = H$.

Theorem 6.2.7. *Let T be a topological group. Nontrivial projective T-algebras exist if and only if T has an open subgroup H such that all open subgroups of H have finite index in H. If the identity element of T has a neighborhood base consisting of open subgroups, this is the same as to say that T has an open subgroup H which is totally bounded.*

Proof. Just note that $i \in I$ is almost maximal if and only if every open subgroup of $\operatorname{stab} s_i$ has finite index in $\operatorname{stab} s_i$. □

Here is an example which shows that the hypothesis that the suitable relations on T correspond to the source stabilizers cannot be omitted from Theorems 6.2.5 or 6.2.6, or from Proposition 6.2.4(1). This example also violates the implication from (7) to (2) in Theorem 6.2.1.

Example 6.2.8. *Let T be the five element monoid whose multiplication table is below.*

row× column	1	t_1	t_2	t_3	t_4
1	1	t_1	t_2	t_3	t_4
t_1	t_1	t_1	t_2	t_3	t_4
t_2	t_2	t_1	t_2	t_3	t_4
t_3	t_3	t_4	t_3	t_2	t_1
t_4	t_4	t_4	t_3	t_2	t_1

Here are four suitable relations on T whose types are almost maximal. In the right column are elements $w \in F_i$ which witness the almost maximality of each type as in Remark 6.2.3(3). (Here i designates the top element of I, the type corresponding to the identity suitable relation. Therefore F_i is the free algebra on the generating set T.)

suitable relation	*witness*
$\{\{1\},\{t_1\},\{t_2\},\{t_3\},\{t_4\}\}$	1
$\{\{1\},\{t_1,t_4\},\{t_2,t_3\}\}$	$1 \wedge t_1 \wedge t_4$
$\{\{1,t_1,t_4\},\{t_2,t_3\}\}$	$t_1 \wedge t_4$
$\{\{t_1,t_4\},\{1,t_2,t_3\}\}$	$t_2 \wedge t_3$

For example, if $w = t_1 \wedge t_4$ then

$$w1 = wt_1 = wt_4 = w, \quad wt_2 = wt_3 = t_2 \wedge t_3,$$

so type w *corresponds to the third suitable relation in the table. Of the many other types, the authors believe only those in the table are almost maximal. In particular, the element $t_1 \wedge t_2^{-1}$ of the projective T-algebra F_i has type corresponding to the suitable relation $\{\{1\},\{t_1,t_2,t_3,t_4\}\}$, and this type is not almost maximal.*

Example 6.2.8 raises the question of whether every nontrivial projective T-algebra contains a nontrivial element of almost maximal type, i.e., whether Theorem 6.2.5 holds without the hypothesis that the suitable relations correspond to the source stabilizers. The authors are willing to conjecture that the answer to this question is yes.

We finish with an application to an almost finite case. Suppose that T is a topological group with only finite pointed antiflows. Then the group action can be replaced by the action of a compact group, namely $\bar{T}$, and all elements of I are almost maximal. Hence all T-algebras F_i, $i \in I$, are projective. In this case there is a very easy characterization of the finite projective T-algebras.

Corollary 6.2.9. *Let T be a topological group which has only finite pointed antiflows. Then any finite T-algebra A is projective if and only if it has an atom that is fixed by every element of the group, i.e., if and only if the Stone space of A has a fixed point with respect to the induced group action.*

Proof. First note that any finite coproduct B of a family of T-algebras of the form F_i, $i \in I$, has at least two atoms which are fixed by the action, namely $\bigwedge_S s$ and $\bigwedge_S s^{-1}$, where S is the set which freely generates B. This means that the Stone space Y of B, which can be identified with the set of atoms of B and which we regard as a Boolean flow as in Theorem 1.2.1, also has at least two fixed points. Now any finite projective T-algebra A is a retract of such a coproduct B; say $f : B \to A$ is a T-surjection and $h : A \to B$ a T-injection such that $fh = 1_A$. Let X be the Stone space of A and $f' : X \to Y$ and $h' : Y \to X$ be the flow maps dual to f and h, so that f' is injective and h' is surjective and $h'f' = 1_X$. Then any fixed point of Y is taken to a fixed point of X by h', so we conclude that X has at least one fixed point.

Now suppose that A is a finite T-algebra with Stone space X having fixed point x. Let B be the coproduct of the family $\{F_a : a \in A\}$, where F_a is a copy of $F_{\text{type}\, a}$ for each $a \in A$, and let f be the epimorphism which takes the source s_a of F_a to a for each $a \in A$. B is finite and projective. Let Y be the Stone space of B and let $f' : X \to Y$ be the injective flow map dual to f. We will be done if we can show that f has a right inverse $h : A \to B$, or equivalently that there is a flow map $h' : Y \to X$ such that $h'f' = 1_X$. But since Y is finite, the continuity of h' is automatic. Define h' as follows: for each $y \in f'(X)$ let $h'(y)$ be the preimage of y under f'. For every $y \notin f'(X)$ let $h'(y) = x$. Since f' is a flow map, so is h'. Clearly h' is as required. □

Here is an example which shows that Corollary 6.2.9 is false without the group hypothesis.

Example 6.2.10. *Let T be the monoid of Example 6.2.8, acting on the Boolean flows X and Y as follows.*

tx	x_0	x_1	x_2
1	x_0	x_1	x_2
t_1	x_0	x_1	x_2
t_2	x_0	x_1	x_2
t_3	x_0	x_2	x_1
t_4	x_0	x_2	x_1

ty	y_0	y_1	y_2	y_3
1	y_0	y_1	y_2	y_3
t_1	y_0	y_1	y_2	y_1
t_2	y_0	y_1	y_2	y_2
t_3	y_0	y_2	y_1	y_1
t_4	y_0	y_2	y_1	y_2

Both X and Y have fixed points x_0 and y_0, respectively. However, the clopen algebra of X is a T-algebra which is demonstrably not projective, for the flow map $f' : X \to Y$ which takes x_k to y_k, $k = 0, 1, 2$, has no left inverse.

If T is a topological group with only finite pointed antiflows we can get a sufficient condition for a (possibly infinite) T-algebra being projective which is internal, i.e., which only involves the structure of the T-algebra. A slight weakening of this sufficient condition turns out to be necessary. Unfortunately those conditions are very technical, and we do not have a complete characterization yet. That is why those conditions are not treated here.

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