A NATURAL EXTENSION OF THE CONTINUOUS FUNCTIONS WITH COMPACT SUPPORT

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Abstract. Let \( Y \) be a locally compact space, \( C_K(Y) \) the collection of real-valued continuous functions with compact support, and \( B_L(Y) \) the set of all Baire functions with Lindelöf cozero-set. We show that the embedding \( C_K(Y) \subseteq B_L(Y) \) of archimedean \( \ell \)-groups (or vector lattices) has this universal mapping property: any homomorphism \( C_K(Y) \xrightarrow{\varphi} A \), where \( A \) has the property

\begin{equation}
(*) \quad \text{is divisible and both conditionally and laterally } \sigma \text{-complete,}
\end{equation}

has a unique extension \( B_L(Y) \xrightarrow{\varphi^*} A \); also, \( B_L(Y) \) has property \((*)\). Our perspective on this is as follows.

In a category \( C \) an object \( G \) is epicomplete if the only epic monics out of \( G \) are isomorphisms, epic or monic meant in the categorical sense of right or left cancellable. For each of the categories \( \text{Arch} \), archimedean \( \ell \)-groups with \( \ell \)-homomorphisms, and its companion category \( W \), \( \text{Arch} \)-objects with distinguished weak unit and unit-preserving \( \ell \)-homomorphisms, (and for the corresponding categories of vector lattices) epicompleteness has been characterized as divisible and conditionally and laterally \( \sigma \)-complete, and it has been shown to be monoreflective. Denote the reflecting functors by \( \beta \) and \( \beta^W \), respectively.

What are they? For \( W \) the Yosida representation has been used to realize \( \beta^W A \) as a certain quotient of \( B(YA) \), the Baire functions on the Yosida space of \( A \). For \( \text{Arch} \) very little has been known. Here we give a general representation theorem, Theorem A, for \( \beta G \) as a certain subdirect product of \( W \)-epicomplete objects derived from \( G \). That result, some \( W \)-theory, and the relation between epicity and relative uniform density are then employed to show Theorem B: \( \beta C_K(Y) = B_L(Y) \); i.e., the result of the first paragraph.

1. Introduction

Here we set some terminology and context, by way of sketching the proof of Theorem B in the Abstract. This reverses the order of events in the Abstract and in subsequent sections. We do this because otherwise the point of Theorem A might not be so visible.

Principal references will be: for category theory, [22]; for topology, [13] and [15]; for basics on \( \ell \)-groups (lattice-ordered groups), [1] and [10]; for vector lattices, [25];
for details on epics and epicompletions, various papers in the literature, especially [3]–[7], [24] and [18].

1.1 Let \( C \) be a category, \( R \) a full subcategory. A \( C \)-morphism \( \alpha : A \to B \) is \( R \)-extendable if for each \( \varphi \in C(A, R) \) there is \( \tilde{\varphi} \) with \( \tilde{\varphi}\alpha = \varphi \). \( R \) is said to be monoreflective if for each object \( G \) there is an \( R \)-extendable \( G \xrightarrow{rG} rG \), which is monic and epic with \( rG \in R \).

When this obtains, since the maps \( rG \) are epic, the extensions \( \tilde{\varphi} \) are unique. Also, given \( G \), a pair \((rG, rG)\) is essentially unique, and any pair with its properties is called an \( R \)-monoreflection of \( G \). Also, the situation defines a functor \( r : C \to R \): We thus have the definition of “\( C \)-ec is monoreflective in \( C \),” where \( C \)-ec denotes the full subcategory of \( C \)-epicomplete objects. This occurs frequently, and when it does, \( C \)-ec is the smallest monoreflective subcategory, with the largest reflections, and is thus of especial significance. A familiar example is the category \( C \) of Tychonoff spaces, where epicomplete means compact and the monoreflection is Stone-Čech compactification. See [17] for further examples and discussion.

We are concerned here with the categories \( \text{Arch} \) and \( \text{W} \); \( \text{Arch} \)-epicomplete will be denoted just epicomplete, with functor \( \beta \), and \( \text{Arch-ec} \)-extendable will be shortened to ec-extendable. For \( \text{W} \), we write \( \text{W} \)-epicomplete with functor \( \beta^W \).

1.2 Let us focus, and expand, on the situation \( C_K(Y) \leq B_L(Y) \) in the Abstract. \( Y \) is a locally compact Hausdorff space, and

\[
\begin{align*}
C(Y) &= \{ f \in \mathbb{R}^Y : f \text{ is continuous} \}, \\
C_K(Y) &= \{ f \in C(Y) : \text{coz} f \text{ is compact} \}, \\
C_0(Y) &= \{ f \in C(Y) : f \text{ vanishes at } \infty \}, \\
B(Y) &= \{ f \in \mathbb{R}^Y : f \text{ is Baire} \}, \\
B_L(Y) &= \{ f \in B(Y) : \text{coz} f \text{ is Lindelöf} \}.
\end{align*}
\]

Here \( \text{coz} f = \{ y \in Y : f(y) \neq 0 \} \), and \( f \) vanishes at \( \infty \) provided that for all \( \varepsilon > 0 \) there exists a compact set \( K \) with \( |f(y)| \leq \varepsilon \) for all \( y \notin K \). Also, \( B(Y) \) is the least subset of \( \mathbb{R}^Y \) containing \( C(Y) \) and closed under pointwise convergence of sequences, and a space is Lindelöf if each open cover has a countable subcover.

Now any \( \mathbb{R}^Y \) is an archimedean \( \ell \)-group in the pointwise operations and order, and so, too, is each of the \( C \)'s and \( B \)'s above, as \( \ell \)-subgroups of \( \mathbb{R}^Y \).

In \( C(Y), B(Y), \) and \( \mathbb{R}^Y \), we designate as weak unit the constant function \( 1 \), and we have the \( \text{W} \)-inclusions \( C(Y) \leq B(Y) \leq \mathbb{R}^Y \). In general, the other \( B \)'s and \( C \)'s have no weak units, and we have the \( \text{Arch} \)-inclusions (suppressing \( Y \)):

\[
C_K \leq C_0 \leq B_L.
\]

Ultimately we shall show in 4.6 and 5.3 below:

**Theorem B.** For \( G = C_K \) and \( G = C_0, G \leq B_L \) is an ec-monoreflection of \( G \) in Arch.
The case $G = C_K$ is the central one, as we shall see. Here, the theorem states exactly:

B1 $B_L$ is epicomplete.
B2 $C_K \leq B_L$ is epic.
B3 $C_K \leq B_L$ is ec-extendable.

We shall prove B1 and B2 more-or-less directly in 4.1 and 4.2; B1 is routine, B2 uses relative uniform density and is more complicated, but most of the work resides in previous knowledge of $C(Y) \leq B(Y)$.

We don’t know how to prove B3 directly, but rather employ the Theorem A mentioned in the Abstract. Here, given $G$, we find a (or several) $P$ which is epicomplete and a product of $\mathbf{W}$-ec objects, and an embedding $G \hookrightarrow P$ so that $\beta G$ is the “epiclosure” of $G$ in $P$, discussed below in 2.1. For $G = C_K$, we construct $P$ so that visibly $B_L$ embeds in $P$ over $C_K$. (All of that uses “$\mathbf{W}$-decomposition of $\mathbf{Arch}$-morphisms,” and knowledge of $\beta^{\mathbf{W}}$’s.) Then, using B1 and B2, $B_L = \beta C_K$ follows.

2. Some Preliminaries for Theorem A

Theorem A (3.2 below) results from a combination of the categorical features of (epi, extremal-monic)-factorizations, the method of “reduction by principal perps” from $\mathbf{Arch}$ to $\mathbf{W}$, the properties of epicompleteness, and the idea of a coessential subset. We explain these first. The reader who finds this unnecessary or tedious might go on to Section 3, referring to this section as needed.

2.1 Factorization, etc.

In a category $C$, the morphism $A \xrightarrow{m} B$ is called extremal monic if $m$ is monic, and if $m = ge$, with $e$ epic, implies $e$ is an isomorphism. In this circumstance $A$ is called an extremal subobject of $B$. By identifying $A$ with $m(A)$, one thinks of $A$ as a subobject which admits no epic enlargement within $B$.

If $C$ is “sufficiently complete” then:

(a) Any morphism $f$ has an essentially unique (epi, extremal monic)-factorization $f = me$, with $e$ epic and $m$ extremal monic. When $f$ is monic, we call the codomain of $e$, which is the domain of $m$, the epiclosure of the domain of $f$.

(b) Any square $fe = mg$ with $e$ epic and $m$ extremal monic “diagonalizes”: there is a unique $d$ with $de = g$ and $md = f$.

(c) The full subcategory $\mathbf{R}$ is epireflective in $C$ if and only if $\mathbf{R}$ is closed under formation of products and extremal subobjects in $C$.

For the meaning of “sufficiently complete,” and details of the results above, see [22]. We don’t need to go into this here.

2.2 Arch and $\mathbf{W}$.

(a) $\mathbf{Arch}$ and $\mathbf{W}$ are “sufficiently complete,” so 2.1 (a), (b), and (c) are valid in $\mathbf{Arch}$ and $\mathbf{W}$.
(b) Let $G$ be a $\mathbf{W}$-object (resp., $\mathbf{Arch}$-object). $G$ is $\mathbf{W}$-epicomplete (resp., $\mathbf{Arch}$-epicomplete) if and only if $G$ is divisible and both conditionally and laterally $\sigma$-complete. We denote the corresponding categories $\mathbf{W}$-ec (resp., $\mathbf{Arch}$-ec).

(c) The class $\mathbf{W}$-ec (resp., $\mathbf{Arch}$-ec) is monoreflective in $\mathbf{W}$ (resp., $\mathbf{Arch}$).

(d) Let $G \twoheadrightarrow H$ be a surjection of $\mathbf{W}$ (resp., $\mathbf{Arch}$). If $G$ is $\mathbf{W}$-epicomplete (resp., $\mathbf{Arch}$-epicomplete), then so is $H$.

(a) is discussed in [4] and [16]. (b) is from [4]. (c) for $\mathbf{W}$ was first shown in [26], using the theory of $\sigma$-frames. Moments later, (c) was derived from (b) and 2.1(c) in [4]. [14] and [27] show that a surjection preserves the algebraic properties in b), so d) follows. Note that (d) implies this stronger form of epicompleteness: each epic out of $G$ is a surjection.

We remark now, and then no more, that all of the considerations of this paper apply to the (less general) corresponding categories of vector lattices because: (1) any $\mathbf{Arch}$- (or $\mathbf{W}$-) object with property (*) “is” a vector lattice, and (2) it is a result of Bleier and Conrad (see the discussion in [3]) that the category of archimedean vector lattices can be identified with a monoreflective subcategory of $\mathbf{Arch}$ by forgetting scalar multiplication, and likewise for $\mathbf{W}$. In this general vein, we note also that certain aspects of the property (*) have been studied by Fremlin in [14], where (*) is called “universally $\sigma$-complete.”

2.3 Reduction of $\mathbf{Arch}$ to $\mathbf{W}$.

(a) Let $G \in \mathbf{Arch}$ and $u \in G^+$. Then $u^\perp = \{ g \in G : |g| \wedge u = 0 \}$ is an ideal (convex sub-$\ell$-group); also, $G / u^\perp \in \mathbf{Arch}$ - see [10] - and the quotient map $G \rightarrow G / u^\perp$ is an $\mathbf{Arch}$-morphism.

(b) Let $\varphi \in \mathbf{Arch} (G,H)$, and let $u \in G^+$. Then $\varphi (u^\perp) \subseteq \varphi (u)^\perp$, so

$$\varphi_u (g + u^\perp) \equiv \varphi (g) + \varphi (u)^\perp$$

defines a homomorphism $\varphi_u : G / u^\perp \rightarrow H / \varphi (u)^\perp$ for which the $\mathbf{Arch}$-square below commutes.

\[
\begin{array}{ccc}
G & \rightarrow & G / u^\perp \\
\varphi \downarrow & & \varphi_u \\
H & \rightarrow & H / \varphi (u)^\perp \\
\end{array}
\]

(c) Now $u \in G^+$ is called a weak unit in $G$ when $u^\perp = \{0\}$. In any event, the coset $u + u^\perp$ is a weak unit in $G / u^\perp$. Any element larger than a weak unit is a weak unit. So if $u_1 \geq u$ in $G$ then $u_1 + u^\perp$ is a weak unit in $G / u^\perp$, and for $\varphi \in \mathbf{Arch} (G,H)$, $\varphi (u_1) + \varphi (u)^\perp$ is a weak unit in $H / \varphi (u)^\perp$. This makes the $\varphi_u$ of (b) into a $\mathbf{W}$-morphism indicated by

\[
(G / u^\perp, u_1 + u^\perp) \xrightarrow{\varphi_u} (H / \varphi (u)^\perp, \varphi (u_1) + \varphi (u)^\perp) \in \mathbf{W}.
\]
(d) Now let \( U \subseteq G^+ \) and \( U^\perp = \bigcap U u^\perp \). Again we have an ideal, and \( G/U^\perp \in \text{Arch} \). Let \( \varphi \in \text{Arch}(G, H) \). From the basic properties of the product in \( \text{Arch} \) (which is just the \( \ell \)-group product; i.e., the cartesian product with + and \( \leq \) defined coordinate-wise), we have the commuting \( \text{Arch} \)-square below,

\[
\begin{array}{ccc}
G & \xrightarrow{\gamma} & \prod G/u^\perp \\
\varphi & \downarrow & \prod \varphi_u \\
H & \xrightarrow{\rho} & \prod H/\varphi(u)^\perp
\end{array}
\]

where the products are indexed by \( U \), \( \gamma(g)_u = (g + u^\perp) \) for all \( u \in U \), \( \rho \) is defined likewise, \( \ker \gamma = U^\perp \), and \( \ker \rho = \varphi(U)^\perp \). So in the event (to materialize shortly) that \( U^\perp = \{0\} \) in \( G \) and \( \varphi(U)^\perp = \{0\} \) in \( H \), \( \gamma \) and \( \rho \) will be embeddings.

2.4 Coessential subsets.

(a) **Definition.** Let \( G \in |\text{Arch}| \). An archimedean kernel in \( G \) is an ideal \( I \) of \( G \) for which \( G/I \) is archimedean.

For \( S \subseteq G \), \( \text{ak}_G S \) denotes the least archimedean kernel of \( G \) containing \( S \); this kernel exists because \( \text{Arch} \) is an SP-class in abelian \( \ell \)-groups. See Section 4 for more on archimedean kernels.

(b) **Definition.** Let \( G \in |\text{Arch}| \) and \( U \subseteq G^+ \). We say that \( U \) is coessential in \( G \) if the only morphism \( \alpha \in \text{Arch}(G, H) \) such that \( \alpha(u) = 0 \) for all \( u \in U \) is the zero map. \( \varphi \in \text{Arch}(G, H) \) is called coessential if \( \varphi(G^+) \) is coessential in \( H \).

We record some simple features of coessentiality.

(c) \( U \) is coessential in \( G \) if and only if the embedding of the generated \( \ell \)-group ideal \( \langle U \rangle \leq G \) is a coessential morphism.

(d) An epic is coessential.

(e) The composition of coessential morphisms is coessential.

(f) If \( U \) is coessential in \( G \) and \( G \xrightarrow{\varphi} H \) is coessential then \( \varphi(U) \) is coessential in \( H \).

(g) For \( U \subseteq G^+ \), these are equivalent.

\((g1)\) \( U \) is coessential in \( G \).

\((g2)\) \( \text{ak}_G U = G \).

\((g3)\) Whenever \( \varphi \in \text{Arch}(G, H) \) is coessential (or is epic, or is surjective), then \( \varphi(U)^\perp = \{0\} \).

**Proofs.**\( (c), (d), (e), (f), \) and \((g1) \iff (g2)\) are routine. \((g2) \implies (g3)\). Suppose \((g2)\) holds, and hence also \((g1)\). Let \( \varphi \in \text{Arch}(G, H) \) be coessential. By \((g1)\) and \((f)\), \( \varphi(U) \) is coessential in \( H \). This means that \( \text{ak} \varphi(U) = H \). Since for any \( V, V^\perp \) is always an archimedean kernel \((|10|)\), we have \( H = \text{ak} \varphi(U) \subseteq \varphi(U)^\perp \). But \( V^\perp = H \) iff \( V = \{0\} \). \((g3) \implies (g2)\). If \( \text{ak}_G U \neq G \) then the quotient map \( G \xrightarrow{\varphi} G/\text{ak}_G U \equiv H \) is surjective and has \( \varphi(U) = \{0\} \), so \( \varphi(U)^\perp = H \neq \{0\} \).
We quote the following just by way of context; we won’t explicitly use it here. This generalizes a crucial result from [3], and the proof there works here.

(h) Let $U$ be coessential in $G$, and $\varphi \in \text{Arch}(G,H)$. Then $\varphi$ is Arch-epic if and only if $\varphi$ is coessential and for each $u \in U$, the $W$-morphism

$$\varphi_u : (G/u^\perp, u + u^\perp) \to (H/\varphi(u)^\perp, \varphi(u) + \varphi(u)^\perp)$$

is $W$-epic.

3. Theorem A

The following situation and notation will obtain throughout this section, and will be referred to later.

3.1 Let $G \in \text{Arch}$ and let $U \subseteq G^+$ with $U^\perp = \{0\}$. For each $u \in U$ fix $u_1 \in G$ with $u_1 \geq u$. We have, for each $u \in U$, the quotient $G \twoheadrightarrow G/u^\perp$ in Arch, the $W$-object $(G/u^\perp, u_1 + u^\perp)$, and the $W$-ec monoreflection

$$b_u : (G/u^\perp, u_1 + u^\perp) \to \beta^W(G/u^\perp, u_1 + u^\perp) \equiv P_u.$$

We construe this last in Arch, and just write $b_u : G/u^\perp \leq P_u$.

Thus we have

$$\begin{array}{ccc}
G & \xrightarrow{\gamma} & \prod G/u^\perp \\
\downarrow & & \downarrow b \\
\Delta & \xrightarrow{\epsilon} & K \\
\downarrow & & \downarrow \mu \\
\epsilon & \xrightarrow{\mu} & K \\
\end{array}$$

where the notation is as in 2.3(d), $b = \prod_U b_u$, $\Delta \equiv b\gamma$, and $\Delta = \mu\epsilon$ is the (epi, extremal monic)-factorization of $\Delta$.

By 2.1 and 2.2, each $P_u$ is Arch-epicomplete, and so then are $P$ and $K$. Also $\epsilon$ is monic as a first factor of the monic $\Delta$. So $G \xrightarrow{\epsilon} K$ is an epic embedding into an epicomplete object, an “epicompletion” of $G$, and this will be an ec-monoreflection of $G$ if and only if $\epsilon$ is ec-extendable. Relatively simple examples show that, at this level of generality, ec-extendability can fail; see 3.3(b) below.

3.2 Theorem A. Assume the notation of 3.1. If $U$ is coessential in $G$ then $G \xrightarrow{\epsilon} K$ is ec-extendable, and thus is an ec-monoreflection of $G$.

Proof. Let $\Phi \in \text{Arch}(G,L)$ with $L$ epicomplete, and let $G \xrightarrow{\varphi} H \xrightarrow{m} L$ be the (epi, extremal monic)-factorization of $\Phi$. We have the following diagram in Arch. Note the following.

1. $H$ is an extremal subobject of the epicomplete object $L$, so is epicomplete by 2.1 and 2.2. By 2.2, $H/\varphi(u)^\perp$ is also epicomplete.
2. We may view $\varphi_u$ as a $W$-morphism, obtaining the following diagram in $W$.
3. Now view $\overline{\varphi_u}$ as an Arch-morphism, and construct $\psi = \prod \overline{\varphi_u}$. 
(4) $\rho$ is the standard map into the product, defined by $\rho(h) = (\rho(h)_u)$, where $\rho(h)_u = h + \varphi(u)^\perp$. Because $U$ is coessential in $G$ and $\varphi$ is epic, $\varphi(U)$ is coessential in $H$, from which it follows from 2.4 that $\varphi(U)^\perp = \bigcap \varphi(u)^\perp = \{0\}$. Hence $\rho$ is one-to-one. Since $H$ is epicomplete, $\rho$ is extremal monic.

(5) That $(\psi\mu)\varepsilon = \rho\varphi$ is easily checked. So by 2.1 there is $K \xrightarrow{\delta} H$ with $\delta\varepsilon = \varphi$ and $\rho\delta = \psi\mu$.

(6) $m\delta\varepsilon = \Phi$ follows.

\[
(G/u^\perp, u_1 + u^\perp) \xrightarrow{b_u} P_u = \beta^W(G/u^\perp, u_1 + u^\perp)
\]

(3.3) **Comments.**

(a) The point of 3.2 is that for certain $G$ the $u$'s and $u_1$'s can be chosen so that one knows what the $P_u$'s are, and can compute $K$. This is done for $G = C_K(Y)$ in the next sections. It is to be noted that, in general, the calculation of $K$ is not so straightforward; see 6.1.

(b) The following is shown in [24]. Let $u \in G^+$ be a weak unit, so $(G, u) \in |W|$. Then $\beta G = \beta^W(G, u)$ iff $\{u\}$ is coessential, in which case $u$ is called a “near unit.” 3.2 is a generalization of the “if” portion of this result: take $U = \{u\}$ with $u_1 = u$. A corresponding converse of 3.2 can be formulated; see comment (c) below.

Examples are presented in [24] of $(G, u) \in |W|$ with $\{u\}$ not coessential, and these are then examples alluded to at the end of 3.1, where $G \xrightarrow{\varepsilon} K$ fails to be ec-extendable.

(c) While we have no use for it, it is interesting and not hard to prove that for any $u \in G^+$, $\beta^W(G/u^\perp, u + u^\perp)$ is naturally $W$-isomorphic to $(\beta G/u^\perp, u + u^\perp)$. Thus, when $U^\perp = \{0\}$ in $\beta G$, which follows from coessentiality, $\beta G$ embeds in $P = \prod P_u$. Note that this is exactly the use of coessentiality in the proof of 3.2, to make $\rho$ an embedding.
(d) The following comparison of 3.2 with a standard construction is curious. “Everybody knows” that epireflections are constructed by embedding in products via 2.1. In a suitable category \( C \), with subcategory \( A \), given \( R(A) \), the epireflective hull of \( A \), given \( G \), one takes \( S \) to be a “skeleton-set” of epis in \( C(G, A) \). Then

\[
\begin{array}{ccc}
G & \xrightarrow{\Delta} & P = \prod S A_s \\
\downarrow \epsilon & & \downarrow \mu \\
K & \xrightarrow{\mu} & \\
\end{array}
\]

where \( A_s \) is the codomain of \( s \), \( \pi_s \circ \Delta = s \) for each \( s \in S \), and \( \Delta = \mu \epsilon \) is the (epi-extremal monic)-factorization. Here, \( \Delta \) is constructed so that \( G \xrightarrow{\Delta} P \) is \( A \)-extendable, and it follows that \( G \xrightarrow{\mu} K \) is also \( A \)-extendable. (See 37.1 of [22].)

In our present situation, \( \text{Arch-ec} \) is \( R(W \text{-ec}) \), meaning the epireflective hull in \( \text{Arch} \) generated by \( W \text{-ec} \), by [4]. But we see no reason why, for the construct \( G \xrightarrow{\Delta} P \) in 3.2, \( \Delta \) should be \( W \text{-ec} \)-extendable (in \( \text{Arch} \)). The point may be “not all is one.”

(e) We note a cardinal generalization of 3.2. For \( \alpha \) a regular cardinal or the symbol \( \infty \), let \( \text{Arch}_\alpha \) (resp., \( W_\alpha \)) have object class \( |\text{Arch}| \) (resp., \( |W| \)), but the morphisms are \( \alpha \)-complete \( \text{Arch-} \) (resp., \( W \)-) morphisms. Then (see [16]) \( \text{Arch-ec} \) (resp., \( W \)-ec) is monoreflective in \( \text{Arch}_\alpha \) (resp., \( W_\alpha \)). Denote the reflecting functors by \( \beta_\alpha \) and \( \beta_\alpha^W \). (“\( \alpha \)-complete” is defined as “\( < \alpha \)-complete,” so \( \text{Arch}_\omega = \text{Arch} \) and \( \beta_\omega = \beta \); likewise for \( W \).”) 3.2 generalizes to show that, for \( U \) coessential, etc., \( \beta_\alpha G \) is to be found in \( \prod \beta_\alpha^W(G/u^+, u_1 + u^+) \). Note that [5] describes \( \beta_\omega^W \) and \( \beta_\infty^W \), but for other \( \alpha \), \( \beta_\alpha^W \) is a mystery.

(f) Of course, 3.2 is describing a connection between \( W \text{-ec} \) as a reflective subcategory of \( W \) and \( \text{Arch-ec} \) as a reflective subcategory of \( \text{Arch} \). Perhaps in that vein a generalization is possible, which might, in the most general terms, go as follows. Let \( S \) be a mono-, or perhaps merely epi-, reflective subcategory of \( W \), and let \( R \) be the reflective subcategory of \( \text{Arch} \) generated in some way from \( S \). Then the \( R \)-reflection of \( G \) is to be found somehow inside some product of \( S \)-reflections of \( G/u' \)’s. But, the details escape us; too many crucial features of epicomplete objects are used in 3.2.

(g) To recognize a \( \beta G \) from 3.2, in some specific cases there may be a virtue in minimizing the cardinal number \( |U| \). The case \( |U| = 1 \) is mentioned in (b) above and there is information about the case \( |U| = \omega \) in [9]. For \( G = C_K(Y) \), which is discussed in the next section, it is not hard to see that the minimum \( |U| \) is the so-called Lindelöf degree \( L(Y) \) (see [13]), but we see no value in this observation.

Of course, we have here a cardinal invariant of archimedean \( \ell \)-groups which one might study, the “coessentiality character,” \( \epsilon G = \min \{|U| : U \text{ is coessential in } G\} \).

(h) Likewise, there may be a virtue in having a pairwise disjoint \( U \), but frequently such does not exist: it’s easy to see from 4.10 below that \( C_K(Y) \) has a disjoint
coessential subset if and only if $Y$ is a topological sum $\sum_i Y_i$ of compact $Y_i$. Then $U = \{ \psi_i : i \in I \}$, where $\psi_i$ is the characteristic function of $Y_i$. We will return to this topic in [8].

4. $B_L(Y) = \beta C_K(Y)$

Let $Y$ be a fixed locally compact space, $C_K = C_K(Y)$ as in the first section, “$Y$” usually being suppressed. The proof of “$B_L = \beta C_K” was outlined in Section 1, and we proceed to the details. The terms epi, epicomplete, etc., refer to Arch.

4.1 Proposition. $B_L$ is epicomplete.

Proof. A $\sigma$-ideal in $G$ is an ideal $I$ in $G$ with this property.

$$(\sigma) \quad \{ g_n \} \subseteq I^+, \bigvee_{n=1}^G g_n = g \implies g \in I.$$

The result follows from (1), (2), and (3).

(1) If $G$ is epicomplete then so is each of its $\sigma$-ideals.
(2) $B(Y)$ is epicomplete.
(3) $B_L$ is a $\sigma$-ideal in $B(Y)$.

Recall that an Arch object is epicomplete if and only if it is divisible and both conditionally and laterally $\sigma$-complete (2.2). Using that, (1) is routine, and (2) follows because existing countable suprema in $B(Y)$ are pointwise. See [5] for further discussion.

For (3), note first that for $U \subseteq V \subseteq Y$, if $U$ is Baire in $Y$ and $V$ is Lindelöf then $U$ must also be Lindelöf. For by covering $V$ with open sets having compact closure and then taking a countable subcover, we get $U \subseteq \bigcup K_n$ for countably many compact $K_n \subseteq Y$. Since $U$ is Baire in $Y$, each $U \cap K_n$ is Baire in $K_n$, and a Baire set in a compact space is Lindelöf. Thus $U = \bigcup (U \cap K_n)$ is Lindelöf as well.

To show that $B_L$ is a sub-$\ell$-group of $B(Y)$, observe that, since $\text{coz } (f) = \text{coz } (-f) = \text{coz } |f|$, we have for $f_i \in B_L$ that

$$\text{coz } (f_1 + f_2) = \text{coz } |f_1 + f_2| \subseteq \text{coz } (|f_1| + |f_2|) = \text{coz } f_1 \cup \text{coz } f_2,$$

$$\text{coz } (f_1 \lor f_2) = \text{coz } |f_1 \lor f_2| \subseteq \text{coz } (|f_1| \lor |f_2|) = \text{coz } f_1 \cup \text{coz } f_2,$$

$$\text{coz } (f_1 \land f_2) = \text{coz } |f_1 \land f_2| \subseteq \text{coz } (|f_1| \land |f_2|) = \text{coz } f_1 \cup \text{coz } f_2.$$

Since these are Baire subsets of the Lindelöf set $\text{coz } f_1 \cup \text{coz } f_2$, all are Lindelöf and $B_L$ is closed under the $\ell$-group operations. Furthermore, $f \geq g \geq 0$ for $f \in B_L$ and $g \in B(Y)$ implies $\text{coz } f \supseteq \text{coz } g$ and $\text{coz } g$ Lindelöf, hence $g \in B_L$. Finally, $B_L$ has property ($\sigma$) in $B(Y)$ because, for non-negative functions $f_n$,

$$f = \bigvee_{n=1}^B f_n \implies f(x) = \bigvee_{n=1}^B f_n(x) \quad \forall x \in Y \implies \text{coz } f = \bigcup \text{coz } f_n.$$

If all $\text{coz } f_n$ are Lindelöf, so is $\text{coz } f$. \qed
4.2 Proposition. $C_K \leq B_L$ is epic.

We prove this by showing relative uniform density, which we explain briefly. ([25] is the best general reference; see also [6].) In an archimedean ℓ-group $B$, choose $0 \leq u \in B$. The sequence $\{a_n\}$ converges to $b$ relatively uniformly regulated by $u$ if

$$\forall k \in \mathbb{N} \exists n(k) \in \mathbb{N} \langle n \geq n(k) \implies k |a_n - b| \leq u \rangle;$$

we write $a_n \rightarrow b(u)$. For $B$ divisible, this is the more familiar

$$\forall \epsilon = \frac{1}{k} \exists n(\epsilon) \in \mathbb{N} \langle n \geq n(\epsilon) \implies |a_n - b| \leq \epsilon u \rangle.$$ 

Ordinary uniform convergence of real-valued functions is this with $u$ the constant function $1$.

For $A \subseteq B$, the iterated pseudo-closures are

$$r_0 (A, B) = A,$$

$$r_1 (A, B) = \{ b \in B : a_n \rightarrow b(u) \text{ for some } (a_n) \subseteq A \text{ and some } u \in B^+ \},$$

$$r_\alpha (A, B) = \begin{cases} r_1 (r_\gamma (A, B), B) \text{ if } \alpha = \gamma + 1 \\ \bigcup_{\gamma < \alpha} r_\alpha (A, B) \text{ if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Then, $r_{\omega_1+1} (A, B) = r_{\omega_1} (A, B)$, and this is denoted $r (A, B)$ and called the relative uniform closure of $A$ in $B$.

4.3 Theorem ([6]), (a) If $A \subseteq B$ is relatively uniformly dense then it is epic.
(b) For any Tychonoff space $X$, $C (X) \leq B (X)$ is relatively uniformly dense.
(c) Suppose $B$ is epicomplete. If $A \leq B$ is epic then it is relatively uniformly dense.

We don’t need (c) for the proof of 4.2, but it’s worth noting, as is the fact that the hypothesis “$B$ is epicomplete” cannot be dropped.

We also need the following.

4.4 Proposition. Suppose $S$ is locally compact and $\sigma$-compact. Then $C_K (S)$ is relatively uniformly dense in $C (S)$. In fact, $r_1 (C_K (S), C (S)) = C (S)$.

Proof. Write $S = \bigcup_n K_n$, where each $K_n$ is compact and each $K_n \subseteq \text{int } K_{n+1}$. For each $n$ choose $u_n \in C (S)$ with $0 \leq u_n \leq 1$ such that $u_n$ is 1 on $K_n$ and 0 on $S \setminus K_{n+1}$. This is possible since $S$ is normal.

For any $f \in C_0 (S)$ (recall that these are the functions vanishing at infinity), $u_n f \rightarrow f (1)$. This means that $u_n f \rightarrow f$ uniformly on $S$. We explain. For $\epsilon > 0$, there is compact $K \subseteq S$ with $|f (x)| \leq \epsilon$ for $x \notin K$. Since $K \subseteq S = \bigcup \text{int } K_n$, there is $n = n (\epsilon)$ with $K \subseteq K_n$. If $m \geq n$, then $|u_m f - f|$ is 0 on $K_m$ and at most $\epsilon$ off $K_m$.

Now let $g \in C (S)$. Since each $g$ is bounded on $K_n$, it is easy to build a function $u \in C_0 (S)$ with $u (x) > 0$ for each $x \in S$ and $u g \in C_0 (S)$. As in the previous
paragraph, \( u_n(ug) \longrightarrow ug(1) \). Now multiply by \( \frac{1}{u} \in C(S) \):
\[
 u_ng = \frac{1}{u} (u_n(ug)) \longrightarrow \frac{1}{u} (ug) \left( \frac{1}{u} \right),
\]
so \( u_ng \longrightarrow g \left( \frac{1}{u} \right) \). (Checking simple inequalities justifies the notational sleight-of-hand.) Finally, note that each \( u_ng \in C_K(S) \). \( \square \)

**4.5 Proof of 4.2.**

By 4.3, it suffices to show that \( B_L = r(C_K, B_L) \). Suppose \( f \in B_L \); since \( Y \) is locally compact and Tychonoff, we can cover \( coz f \) by cozero sets \( U \) of \( Y \) which have compact closure. Each \( U \) is cozero in \( \overline{U} \), so is locally compact and \( \sigma \)-compact. Since \( coz f \) is Lindelöf, countably many \( U \)’s cover, too, and their union \( S \) is locally compact and \( \sigma \)-compact. Note that \( S \) is a cozero set in \( Y \) and \( f|S \in B(S) \).

We embed \( B(S) \hookrightarrow B_L(Y) \) by \( g \hookrightarrow g' \), where \( g' = g \) on \( S \) and 0 off \( S \). Here \( g' \in B(Y) \) because \( S \) is zero-set embedded (because \( S \) is a cozero set, or because \( S \) is Lindelöf ([11])), and thus Baire-set embedded. In fact \( g' \in B_L \) since \( coz g' = coz g \), and \( coz g \) is Lindelöf since it is a Baire (in \( Y \)) subset of the Lindelöf set \( S \). This is an embedding as an \( \ell \)-group, so that \( a_n \longrightarrow b(u) \) in \( B(S) \) implies \( a'_n \longrightarrow b'(u') \). Thus for any \( A \subseteq B(S) \),
\[
 r(A, B(S))' \subseteq r(A', B(S))' \subseteq r(A', B_L).
\]
Finally, observe that \( C_K(S)' \subseteq C_K(Y) \).

Recall that \( f \in B_L \), so that \( f|S \in B(S) \); note that \( f = (f|S)' \). By 4.3(b), \( C(S) \) is relatively uniformly dense in \( B(S) \); by 4.4, \( C_K(S) \) is relatively uniformly dense in \( C(S) \); since the composition of relatively uniformly dense embeddings is relatively uniformly dense (see 1.1 of [6] if needed), \( C_K(S) \) is relatively uniformly dense in \( B(S) \). So
\[
 f = (f|S)' \in r(C_K(S), B(S))' \subseteq r(C_K, B_L).
\]
This completes the proof of 4.2.

We now use 4.1, 4.2, and Theorem A to prove the result in the title of this section.

**4.6 Theorem.** For any locally compact \( Y \), \( C_K(Y) \leq B_L(Y) \) is an ec-monorefraction of \( C_K(Y) \).

**4.7 Outline of the proof.** We shall select coessential \( U \subseteq C_K^+ \), and correspondingly \( V \), so that the map \( \Delta \), as in 3.1 and 3.2, has \( C_K \longrightarrow K \) as ec-monorefraction,

\[
 C_K \longrightarrow \Delta \longrightarrow K \longrightarrow P
\]
and so that there is an embedding \( B_L \longrightarrow P \) over \( C_K \) with \( \nu \alpha = \Delta \), where \( C_K \leq B_L \). Since \( B_L \) is epicomplete (4.1), \( \nu \) is extremal monic. But then we have two expressions for the (essentially unique) (epi,extremal monic)-factorization of \( \Delta \): \( \nu \alpha = \Delta = \mu \epsilon \).
Hence there is an isomorphism $K \xrightarrow{d} B_L$ with $\epsilon d = dv$. Thus $C_K \leq B_L$ is an ec-monoreflection.

The proof of 4.6 will be completed — by selecting appropriate $U$, $U_1$ and $\nu$ — in 4.13, after the necessary facts and tools are assembled. First, we need to know what it means that $U$ be coessential in $C_K$.

4.8 Proposition. Let $G \in \text{Arch}$.

(a) If $I \subseteq G$ then $I$ is an archimedean kernel of $G$ if and only if $I$ is an ideal which is relatively uniformly closed.

(b) If $J$ is an ideal in $G$, then so is $r (J, G)$.

(c) For $S \subseteq G$, $\operatorname{ak}_G S = r (\langle S \rangle, G)$ where $\langle S \rangle$ denotes the ideal in $G$ generated by $S$:

$$\langle S \rangle = \left\{ g \in G : |g| \leq \sum_{1 \leq i \leq n} |s_i|, \ s_i \in S, n \in \mathbb{N} \right\}.$$

(d) If $U \subseteq G^+$ then $U$ is coessential in $G$ if and only if $r (\langle U \rangle, G) = G$.

Proof. For (a) see [25]. (b) is easy, and (c) follows. Then (d) follows using 2.4(e). □

We use the usual notation from [15]: for $s \in \mathbb{R}^X$, $Z (s) = \{ x \in X : s (x) = 0 \}$ and $\operatorname{coz} s = X \setminus Z (s)$, and for $S \subseteq \mathbb{R}^X$, $Z (S) = \bigcap_S Z (s)$ and $\operatorname{coz} S = \bigcup_S \operatorname{coz} s$.

4.9 Proposition. In the $\ell$-group $\mathbb{R}^X$:

(a) If $f_n \rightarrow f$ (u) then $f_n \rightarrow f$ pointwise.

(b) Suppose $G \subseteq \mathbb{R}^X$ and $S \subseteq G$. Then $Z (S) = Z (\operatorname{ak}_G S)$.

(c) Suppose $G \subseteq \mathbb{R}^X$, $U \subseteq G^+$, and $\operatorname{coz} G = X$. If $U$ is coessential in $G$ then $\operatorname{coz} U = X$.

Proof. (a) is easy and is Lemma 2 of [6]. For (b), $Z (S) \supseteq Z (\operatorname{ak}_G S)$ just because $S \subseteq \operatorname{ak}_G S$. For the reverse inclusion, suppose $x \in Z (S)$. Then $x \in Z (\langle S \rangle)$ from the description of $\langle S \rangle$ in 4.8. By (a), $f (x) = 0$ for every $f \in r_1 (\langle S \rangle, G)$; by induction, $f (x) = 0$ for every $f \in r (\langle S \rangle, G)$, which is $\operatorname{ak}_G S$ by 4.8. (c) follows by 2.4(g). □

4.10 Proposition. Consider $G = C_K (Y)$, and let $U \subseteq C_K^+$. $U$ is coessential in $C_K$ if and only if $\operatorname{coz} U = Y$.

Proof. Suppose $\operatorname{coz} U = Y$. We show $\langle U \rangle = C_K$; a fortiori, $U$ is coessential. If $f \in C_K$, then $\overline{\operatorname{coz} f}$ is compact, so there are finitely many $u_i \in U$ with $\overline{\operatorname{coz} f} \subseteq \bigcup \operatorname{coz} u_i$. Let $u = \sum u_i \in \langle U \rangle$, so $\operatorname{coz} u = \bigcup \operatorname{coz} u_i$. Since $\overline{\operatorname{coz} f}$ is compact, there is $r > 0$ with $u \geq r$ on $\operatorname{coz} f$. Since $f$ is bounded, there is $n \in \mathbb{N}$ with $nu \geq |f|$, hence $f \in \langle U \rangle$.

The converse is 4.9(c). □

4.11 Theorem. ([5] and [26]) Let $K$ be compact. Then, $\beta^w (C (K), 1) = (B (K), 1)$.

4.12 Proposition. Let $G \leq \mathbb{R}^X$ and $u \in G^+$.
(a) \( u^+ = \{ g \in G : \text{coz} \, g \cap \text{coz} \, u = \emptyset \} \).

(b) If \( u_1 \in G^+ \) has \( \text{coz} \, u_1 \supseteq \text{coz} \, u \) then \( u_1 + u^+ \) is a weak unit in \( G/u^+ \), and the restriction map effects a \( W \)-isomorphism from \( (G/u^+, u_1 + u^+) \) onto \( (G|_{\text{coz} \, u}, u_1|_{\text{coz} \, u}) \), where \( G|_{\text{coz} \, u} \) denotes the \( \ell \)-group of restrictions

\[ \{ g|_{\text{coz} \, u} : g \in G \} \].

(c) If \( u_1 \) is 1 on \( \text{coz} \, u \) then \( (G/u^+, u_1 + u^+) \) is \( W \)-isomorphic to \( (G|_{\text{coz} \, u}, 1) \).

(Here (a) is well-known and easily proved, and (b) and (c) follow easily.)

4.13 Proof of 4.6. We keep in mind the outline in 4.7.

Take any \( U \subseteq C_K \) with \( \text{coz} \, U = Y \) and \( 0 \leq u \leq 1 \) for each \( u \in U \). By 4.10, \( U \) is coessential.

For each \( u \in U \), \( \text{coz} \, u \) is compact, so there is \( u_1 \in C_K^+ \) with \( u_1 \, (x) = 1 \) for each \( x \in \text{coz} \, u \). We elaborate. For each \( x \in \text{coz} \, u \) there is open \( L_x \) with \( x \in L_x \) and \( L_x \) compact. Finitely many of the \( L_x \)'s cover \( \text{coz} \, u \); their union \( L \) is open with \( L \subseteq K \) compact, so \( \text{coz} \, u \subseteq \text{int} \, K \subseteq K \). There is \( u_1 \in C(Y)^+ \) with \( u_1 \) being 1 on \( \text{coz} \, u \) and 0 off \( K \), because a compact set and a disjoint closed set are always completely separated (3.11 of [15]). But \( u_1 \in C_K \).

Now \( C_K|_{\text{coz} \, u} = C(\text{coz} \, u) \). (Any compact set is \( C \)-embedded (3.11 of [15]), so any \( f \in C(\text{coz} \, u) \) extends to \( f' \in C(Y) \). Then \( u_1 \, f \in C_K \) is an extension of \( f \). The other inclusion is clear.) Apply 4.12(c): \( (C_K/u^+, u_1 + u^+) \) is \( W \)-isomorphic to \( (C(\text{coz} \, u), 1) \). By 4.11, \( P_u \equiv \beta_W \, (C_K/u^+, u_1 + u^+) \) “is” \( (B(\text{coz} \, u), 1) \).

Consequently, the \( C_K \to P = \prod_U \, P_u \) in 4.7 is (represented as) \( P = \prod_U \, B(\text{coz} \, u) \), with \( \Delta \, (g) = (\Delta \, (g)_u) = (g|_{\text{coz} \, u}) \). Then \( \Delta \) is lifted over \( C_K \to B_L \) to \( \nu : B_L \to P \) defined, obviously, by \( \nu \, (f) = (\nu \, (f)_u) = (f|_{\text{coz} \, u}) \). (The restriction of a Baire function to a subspace is a Baire function on the subspace.)

According to 4.7, the proof of 4.6 is now complete.

4.14 Comments. (a) Referring to the discussion in Section 1 and 3.3(d), 4.6 is equivalent to (4.1, 4.2, and)

\[ (*) \quad C_K \, (Y) \leq B_L \, (Y) \text{ is } W\text{-ec-extendable in Arch.} \]

And 4.11 says, among other things, that \( C \, (K) \leq B \, (K) \) is \( W\text{-ec-extendable in Arch.} \) and also in Arch by 3.3(b). On the other hand, in [2] we find a locally compact \( Y \) for which \( C \, (Y) \leq B \, (Y) \) fails to be \( W\text{-ec-extendable in Arch.} \); i.e., is not a \( W\text{-ec} \) reflection. This, in light of (*), is somewhat surprising.

(b) The process used here for showing \( \beta \, C_K = B_L \) can surely be applied to any \( G \) which is presented as a group of functions in an understandable way, e.g., as in [23], because Theorem A is general and 4.11 and 4.12 are specializations of much more general facts; see [5].
5. \( B_L(Y) = \beta C_0(Y) \)

We derive this Theorem as a corollary of 4.6 and two simple facts, whose proofs we defer for the moment.

**5.1 Proposition.** Let \( C \) be a category, \( R \) a full subcategory, and suppose that \( G \overset{\alpha}{\to} B \) is an epireflection of \( G \) into \( R \). If \( \alpha = se \), with \( e \) epi, as \( G \overset{e}{\to} A \overset{s}{\to} B \), then \( A \overset{s}{\to} B \) is a reflection of \( A \) into \( R \).

**5.2 Proposition.** Let \( Y \) be locally compact. If \( f \in C_0(Y)^+ \), then \( \sqrt{f} \in C_0(Y) \), and there is a sequence \( \{g_n\} \subseteq C_K(Y) \) with \( g_n \to f(\sqrt{f}) \). So \( C_K(Y) \) is relatively uniformly dense and thus epically embedded in \( C_0(Y) \).

**5.3 Corollary.** \( B_L(Y) \) is an ec-monoreflection of \( C_0(Y) \).

*Proof.* By 4.6, \( C_K \overset{\alpha}{\leq} B_L \) is an ec-reflection, which factors as \( C_K \overset{e}{\leq} \overset{s}{\leq} B_L \), with \( e \) epi, by 5.2. The result follows by 5.1. \( \square \)

**5.4 Proof of 5.1.** It suffices to show that

\[ \varphi \in C(A,R) \implies \exists \gamma \text{ with } \varphi = \gamma s. \]

Since \( \alpha \) is a reflection, there is \( \gamma \) with \( \gamma \alpha = \varphi e \). Then \( \varphi e = \gamma \alpha = \gamma (se) \), whence \( \gamma s = \varphi \), since \( e \) is epi.

**5.5 Proof of 5.2.** (This is somewhat similar to the argument in 4.4.) First, if \( h \in C_0^+ \), there is a sequence \( \{h_n\} \subseteq C_K \) with \( \text{coz } h_n \subseteq \text{coz } h \) such that \( h_n \to h^2(h) \). We elaborate. Let \( K_n = \{x \in Y : h(x) \geq \frac{1}{n}\} \); these are compact, since \( h \in C_0 \). Then choose \( u_n \in C(Y), 0 \leq u_n \leq 1 \), such that \( u_n \) is 1 on \( K_n \) and 0 off \( K_{n+1} \) (by 3.11 of [15]). Then \( u_n \) and \( u_n h \in C_K \), and \( u_n h \to h(1) \). Set \( h_n = u_n h^2 \).

To prove 5.2 let \( f \in C_0^+ \), and note that \( \sqrt{f} \in C_0^+ \) also. Now apply the result in the paragraph above, using \( h = \sqrt{f} \), to find a sequence \( \{h_n\} \) in \( C_K \) with \( h_n \to (\sqrt{f})^2(\sqrt{f}) \).

**5.6 Remark.** Let \( C_L(Y) = \{f \in C(Y) : \text{coz } f \text{ is Lindelöf}\} \). It seems likely that \( C_K \leq C_L \) is relatively uniformly dense by some argument like 5.5 or 4.4 and thus that \( \beta C_L = B_L \), just as 5.3 is proved. But we don’t see how to prove that.

6. Concluding Remarks

**6.1 Factorization.** In view of 4.3 and the central role here of the (epi, extremal monic)-factorization, one might wonder if that factorization of an embedding \( A \leq B \), with \( A \) divisible and \( B \) epicomplete, is

\[ A \overset{e}{\leq} r(A,B) \overset{m}{\leq} B. \]

“\( A \) divisible” can’t be dropped: consider \( \mathbb{Z} \leq \mathbb{R} \). “\( B \) epicomplete” can’t be dropped: there are examples at the end of [6] and in [20]. Here \( r(A,B) \) is epicomplete (proof below), so \( m \) is extremal monic. So by 4.3, \( e \) is epi if and only if \( A \leq r(A,B) \) is
relatively uniformly dense. But there seems to be no obvious reason for that, because of the loss of regulators upon descent from $B$ to $r(A, B)$.

**Proof that $r(A, B)$ is epicomplete.** This follows from the more general statement: if $A \leq B$, with $A$ divisible, $B$ epicomplete, and $A$ relatively uniformly closed in $B$, then $A$ is epicomplete. By 2.2(b), we could show that $A$ is conditionally and laterally $\sigma$-complete; it seems easier to use [18] and show that $A$ is relatively uniformly complete and laterally $\sigma$-complete. (See [18], also [7] and [25], for discussions of relative uniform completeness.) Since $B$ is epicomplete it is relatively uniformly complete; then, since $A$ is relatively uniformly closed, it is relatively uniformly complete. Now let $\{a_n\}$ be a countable disjoint family in $A^+$. Then $b = \bigvee a_n$ exists in $B$. Let $b_n = \bigvee_{i \leq n} a_i$ and $u = \bigvee n a_n$ (in $B$). 4.3 of [4] says $b_n \to b(u)$. So $b \in r(A, B) = A$ since $\{b_n\} \subseteq A$.

**6.2 Absolutely relatively uniformly closed.** We repeat a question from [6], p. 125. Call $A$ absolutely relatively uniformly closed if $A \leq B$ in $\text{Arch}$ implies $r(A, B) = A$.

This implies that $A$ is epicomplete, since $A \leq \beta A$ is relatively uniformly dense (4.3). The question is the converse.

**6.3 Representation of epicomplete objects** It is shown in [5] that $(A, e)$ is $W$-epicomplete if and only if it is $W$-isomorphic to a $(D(X), 1)$, for $X$ compact and basically disconnected. One asks if there is a similar representation of epicomplete objects in $\text{Arch}$. (This is, of course closely related to, and perhaps more basic than, the problem treated here of representing $G \leq \beta G$.) More specifically, consider first $B_L(Y)$. Now $B_L(Y) \subseteq B(Y)$ and $B(Y) \approx D(X)$ as above. Viewing $B_L(Y) \leq D(X)$, $P = Z(B_L(Y))$ is what is called a $P$-set in $X$, meaning that the intersection of countably many neighborhoods of $P$ is again a neighborhood, and if we form a quotient of $X$ by collapsing $P$ to a point $p$, we find that the resulting space $X'$ is still basically disconnected, $p$ is a $P$-point in $X'$, and

$$B_L(Y) = \{ f \in D(X') : f(p) = 0 \} \equiv D(X', p).$$

It’s fairly easy to see that if $X$ is a compact basically disconnected space with $P$-point $p$, then $D(X, p)$ is epicomplete, more-or-less generalizing 4.1. The question is the converse: does $G$ epicomplete imply $G \approx D(X, p)$ for some compact basically disconnected $X$ with $P$-point $p$?

We don’t know the answer. A sequel to this paper, [8], will consider the issue.

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