FIXED POINTS OF QUANTUM OPERATIONS

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Abstract. Quantum operations frequently occur in quantum measurement theory, quantum probability, quantum computation and quantum information theory. If an operator A is invariant under a quantum operation ϕ we call A a ϕ-fixed point. Physically, the ϕ-fixed points are the operators that are not disturbed by the action of ϕ. Our main purpose is to answer the following question. If A is a ϕ-fixed point, is A compatible with the operation elements of ϕ? We shall show in general that the answer is no and we shall give some sufficient conditions under which the answer is yes. Our results will follow from some general theorems concerning completely positive maps and injectivity of operator systems and von Neumann algebras.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. We use the notation

$$\mathcal{B}(\mathcal{H})^+ = \{A \in \mathcal{B}(\mathcal{H})\colon A \geq 0\}\,, \quad \mathcal{E}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H})\colon 0 \leq A \leq I\}\,,$$

that is, $\mathcal{B}(\mathcal{H})^+$ is the positive cone for $\mathcal{B}(\mathcal{H})$ and $\mathcal{E}(\mathcal{H})$ is the set of **quantum effects** [2, 6, 8, 10]. Quantum effects correspond to yes-no quantum measurements that may be unsharp. Denoting the set of trace class operators on $\mathcal{H}$ by $\mathcal{T}(\mathcal{H})$, the set of **states** (or **density operators**) of a quantum system is described by

$$\mathcal{D}(\mathcal{H}) = \left\{\rho \in \mathcal{T}(\mathcal{H})^+\colon\ \operatorname{tr}(\rho) = 1\right\}.$$

The probability that an effect A occurs (has a yes answer) in the state ρ is given by $P_\rho(A) = \operatorname{tr}(\rho A)$.

General quantum measurements that have more than two values (not just yes-no) are described by effect-valued measures. In this paper we shall only consider discrete effect-valued measures. These are described by a sequence $E_i \in \mathcal{E}(\mathcal{H})$, $i = 1, 2, \ldots$, satisfying $\sum E_i = I$ where the sum converges in the strong operator topology. In this case the probability that outcome i occurs in the state ρ is $P_\rho(E_i)$ and the post-measurement state given that i occurs is $E_i^{1/2}\rho E_i^{1/2}/\operatorname{tr}(\rho E_i)$. Moreover, the resulting state after the measurement is executed but no observation is performed is given by

$$\phi(\rho) = \sum E_i^{1/2}\rho E_i^{1/2}.$$

Notice that ϕ is an affine map from $\mathcal{D}(\mathcal{H})$ into $\mathcal{D}(\mathcal{H})$. Also, ϕ extends to a unital, trace preserving, completely positive map on $\mathcal{B}(\mathcal{H})$. (Detailed definitions will be given subsequently.) An important physical question is whether the measurement disturbs

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the state ρ. The fact that the measurement does not disturb ρ is given mathematically by the equation $\phi(\rho) = \rho$. It is shown in [3] that $\phi(\rho) = \rho$ if and only if ρ commutes with every E_i, $i = 1, 2, \ldots$. We then say that ρ is **compatible** with E_i, $i = 1, 2, \ldots$, and this result is called the generalized Lüders theorem.

In the dual picture, the probability that an effect A occurs in the state ρ given that the measurement was performed is

$$P_{\phi(\rho)}(A) = \operatorname{tr}\left[A \sum E_i^{1/2} \rho E_i^{1/2}\right] = \operatorname{tr}\left(\sum E_i^{1/2} A E_i^{1/2} \rho\right).$$

If A is not disturbed by the measurement in any state we have

$$\sum E_i^{1/2} A E_i^{1/2} = A. \tag{1.1}$$

Again, defining $\phi(A) = \sum E_i^{1/2} A E_i^{1/2}$, Eqn. (1.1) reduces to $\phi(A) = A$. But now A may not be in $\mathcal{T}(\mathcal{H})^+$ and the previous proof of the generalized Lüders theorem does not go through. In fact, we shall show that $\phi(A) = A$ does not necessarily imply that A is compatible with E_i, $i = 1, 2, \ldots$. (This solves an open problem posed in [3, 8].) Another way that (1.1) comes about is from the law of total probability which is given by

$$\begin{aligned}\operatorname{tr}(\rho A) = P_\rho(A) &= \sum P_\rho(E_i) P_\rho(A \mid E_i) = \sum \operatorname{tr}(\rho E_i) \frac{\operatorname{tr}(E_i^{1/2} \rho E_i^{1/2} A)}{\operatorname{tr}(\rho E_i)} \\ &= \operatorname{tr}\left(\rho \sum E_i^{1/2} A E_i^{1/2}\right).\end{aligned}$$

If this law holds for every $\rho \in \mathcal{D}(\mathcal{H})$ we again obtain $\phi(A) = A$.

An application of our result can be found in axiomatic quantum field theory. Suppose a measurement $\{E_i \colon i = 1, 2, \ldots\}$ is performed in a bounded spacetime region X and $A \in \mathcal{E}(\mathcal{H})$ is a measurement performed in another spacetime region Y that is spacelike separated from X. According to Einstein causality, the measurement in X should not disturb A so that $\phi(A) = A$. But applying our result, A may not be compatible with E_i, $i = 1, 2, \ldots$. Thus, the axiom of local commutativity does not follow from Einstein causality. We conclude that this axiom may be too strong and it should be replaced by a weaker axiom.

More general measurements are frequently considered in quantum dynamics, quantum computation and quantum information theory [2, 6, 11]. Let $A_i \in \mathcal{B}(\mathcal{H})$, $i = 1, 2, \ldots$, and let $\mathcal{A} = \{A_i, A_i^* \colon i = 1, 2, \ldots\}$. Assume for now that $\sum A_i^* A_i = I$ (trace preserving) and that $\sum A_i A_i^* = I$ (unitality). (The unitality condition is sometimes omitted and the trace preserving condition is sometimes relaxed to $\sum A_i^* A_i \leq I$.) For $B \in \mathcal{B}(\mathcal{H})$, define $\phi_{\mathcal{A}}(B) = \sum A_i B A_i^*$. It can be shown that $\phi_{\mathcal{A}} \colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a normal completely positive map. Moreover, $\phi_{\mathcal{A}}(I) = I$ (unital) and $\operatorname{tr}[\phi_{\mathcal{A}}(B)] = \operatorname{tr}(B)$ for all $\mathcal{B} \in \mathcal{T}(\mathcal{H})$ (trace preserving). In fact, if $\mathcal{H}$ is separable, then any map satisfying these conditions has the form $\phi_{\mathcal{A}}$ for some $\mathcal{A}$ [6]. There are various interpretations for $\phi_{\mathcal{A}}$. For example, $\phi_{\mathcal{A}}$ can describe a quantum measurement, an interaction of a quantum system with an environment followed by a unitary evolution, a noisy quantum channel or a quantum error correction map. We call $\phi_{\mathcal{A}}$ a **quantum operation** and we call $\mathcal{A}$ the set of **operation elements** for $\phi_{\mathcal{A}}$. Note that our previous examples are a special type of quantum operations. This can be seen by letting $A_i = E_i^{1/2}$.

We say that $B \in \mathcal{B}(\mathcal{H})$ is a $\phi_{\mathcal{A}}$ **fixed point** if $\phi_{\mathcal{A}}(B) = B$ and denote the set of $\phi_{\mathcal{A}}$ fixed points by $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}$. It is clear that the commutant $\mathcal{A}' \subseteq \mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}$. The main purpose of this paper is to study $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}$ and the question whether $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} = \mathcal{A}'$. We shall show that in general $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} \neq \mathcal{A}'$ and shall give sufficient conditions under which equality holds. For example, in quantum computation it is assumed that $\dim \mathcal{H} < \infty$. For this case we shall show that $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} = \mathcal{A}'$. Thus, a noisy quantum channel does not disturb a state ρ if and only if ρ is compatible with the operation elements A_i, $i = 1, 2, \ldots$. Related results and methods may be found in [1].

2. Completely Positive Maps

This section studies completely positive maps on von Neumann algebras. Such maps give a unifying generalization of quantum operations and many of our results will follow from these general considerations.

An **operator system** is a linear subspace of $\mathcal{B}(\mathcal{H})$ that is closed under the involution * and contains the identity operator. Let $\mathcal{M}_k$ be the C^*-algebra of $k \times k$ complex matrices which we identify with $\mathcal{B}(\mathbb{C}^k)$. For an operator system $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ we consider $\mathcal{S} \otimes \mathcal{M}_k$ embedded in the C^*-algebra $\mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_k$. Then $\mathcal{S} \otimes \mathcal{M}_k$ carries the natural operator norm and the natural operator order. Given operator systems $\mathcal{V}$ and $\mathcal{W}$ and a linear map $\phi\colon \mathcal{V} \to \mathcal{W}$, for any integer $k \geq 1$, there is defined a linear map $\phi_k\colon \mathcal{V} \otimes \mathcal{M}_k \to \mathcal{W} \otimes \mathcal{M}_k$ given by

$$\phi_k(v) = [\phi(v_{ij})] \text{ where } v = [v_{ij}] \in \mathcal{V} \otimes \mathcal{M}_k, \quad i, j = 1, \ldots, k.$$

We then have a nondecreasing sequence of operator norms

$$\|\phi\| = \|\phi_1\| \leq \|\phi_2\| \leq \|\phi_3\| \leq \cdots .$$

The map ϕ is called **completely bounded** if

$$\|\phi\|_{\mathrm{cb}} = \sup_{k \geq 1} \|\phi_k\| < \infty.$$

It follows that $\|\cdot\|_{\mathrm{cb}}$ is a norm on the linear space $\mathcal{CB}(\mathcal{V}, \mathcal{W})$ of completely bounded maps from $\mathcal{V}$ into $\mathcal{W}$. If $\|\phi\|_{\mathrm{cb}} \leq 1$, then ϕ is called **completely contractive**. If ϕ_k is positive for all k, then ϕ is called **completely positive**. Any completely positive map ϕ is completely bounded and

$$\|\phi(I)\| = \|\phi\| = \|\phi\|_{\mathrm{cb}}.$$

In particular, if $\phi(I) = I$ then ϕ is completely contractive [12].

Now let $\mathcal{V} \subseteq \mathcal{B}(\mathcal{H})$ be an operator system. A map $\psi\colon \mathcal{V} \to \mathcal{V}$ is **idempotent** if $\psi \circ \psi = \psi$. A completely contractive idempotent map ψ from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{V}$ is called a **projection** onto $\mathcal{V}$. If there exists a projection from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{V}$ then $\mathcal{V}$ is **injective**. A linear map ψ from a C^*-subalgebra $\mathcal{M}$ into itself is a **conditional expectation** if ψ is a positive idempotent, its range is a C^*-subalgebra $\mathcal{N}$ of $\mathcal{M}$ and $\psi(CB) = \psi(C)B$ and $\psi(BC) = B\psi(C)$ for every $B \in \mathcal{N}$ and $C \in \mathcal{M}$. A C^*-algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is injective if and only if there exists a conditional expectation on $\mathcal{B}(\mathcal{H})$ with range $\mathcal{M}$ [15]. A von Neumann algebra $\mathcal{M}$ is injective if and only if $\mathcal{M}'$ is injective [9, 17]. A **state** on a C^*-algebra $\mathcal{M}$ is a positive linear functional $\omega\colon \mathcal{M} \to \mathbb{C}$ with norm $\|\omega\| = 1$.

(If $\mathcal{M}$ has a unit I, the condition $\|\omega\| = 1$ is equivalent to $\omega(I) = 1$.) We say that ω is **faithful** if $\omega(A^*A) = 0$ implies $A = 0$.

In the sequel we shall need the following theorem of M.-D. Choi [4].

Theorem 2.1. *Suppose that ϕ is a completely contractive and completely positive map from a unital C^*-algebra $\mathcal{C}$ into $\mathcal{B}(\mathcal{H})$. Then $\phi(C)^*\phi(C) \leq \phi(C^*C)$ for every $C \in \mathcal{C}$. Moreover, if $\phi(C)^*\phi(C) = \phi(C^*C)$ for some $C \in \mathcal{C}$, then for all $B \in \mathcal{C}$ we have $\phi(BC) = \phi(B)\phi(C)$, $\phi(CB) = \phi(C)\phi(B)$.*

If $\phi\colon \mathcal{M} \to \mathcal{M}$ is a unital completely positive map and ω is a state on $\mathcal{M}$, then $\omega \circ \phi$ is again a state on $\mathcal{M}$. We say that ω is **ϕ-invariant** of $\omega \circ \phi = \omega$. We say that $A \in \mathcal{M}$ is a **fixed point** of ϕ if $\phi(A) = A$ and denote the set of fixed points of ϕ by $\mathcal{M}^\phi$ [1]. Notice that $\mathcal{M}^\phi$ is an operator system. In general, $\mathcal{M}^\phi$ is not an algebra [1]. It is easy to check that

$$\mathcal{I}(\phi) = \{A \in \mathcal{M}\colon \phi(AB) = A\phi(B), \phi(BA) = \phi(B)A \text{ for every } B \in \mathcal{M}\}$$

is a C^*-algebra in $\mathcal{M}$ and $\mathcal{I}(\phi) \subseteq \mathcal{M}^\phi$. Moreover, if ϕ is weakly continuous, then $\mathcal{I}(\phi)$ is a von Neumann subalgebra of $\mathcal{M}$ and $\mathcal{M}^\phi$ is a weakly closed operator system.

Lemma 2.2. *The following statements are equivalent.* (a) *$\mathcal{M}^\phi = \mathcal{I}(\phi)$.* (b) *$\mathcal{M}^\phi$ is a C^*-algebra.* (c) *If $A \in \mathcal{M}^\phi$, then $A^*A \in \mathcal{M}^\phi$.*

Proof. (a)$\Rightarrow$(b)$\Rightarrow$(c) is clear. To prove that (c) implies (a) suppose (c) holds and $A \in \mathcal{M}^\phi$. Then $A^*A \in \mathcal{M}^\phi$ so that

$$\phi(A)^*\phi(A) = A^*A = \phi(A^*A).$$

It follows from Theorem 2.1 that for every $B \in \mathcal{M}$ we have

$$\phi(AB) = \phi(A)\phi(B) = A\phi(B)$$

and

$$\phi(BA) = \phi(B)\phi(A) = \phi(B)A.$$

Hence, $A \in \mathcal{I}(\phi)$. □

Theorem 2.3. *If ϕ admits a faithful invariant state ω, then $\mathcal{M}^\phi = \mathcal{I}(\phi)$.*

Proof. Suppose that $A \in \mathcal{M}^\phi$. By Theorem 2.1 we have

$$A^*A = \phi(A)^*\phi(A) \leq \phi(A^*A)$$

so that $\phi(A^*A) - A^*A \in \mathcal{M}^+$. Since

$$\omega\left[\phi(A^*A) - A^*A\right] = \omega \circ \phi(A^*A) - \omega(A^*A) = 0$$

we conclude that $\phi(A^*A) = A^*A$. Hence, $A^*A \in \mathcal{M}^\phi$. Applying Lemma 2.2 we have that $\mathcal{M}^\phi = \mathcal{I}(\phi)$. □

For $\phi\colon \mathcal{M} \to \mathcal{M}$ we denote the map obtained by composing ϕ with itself n times by ϕ^n. If $\psi\colon \mathcal{M} \to \mathcal{M}$ we denote the composition $\psi \circ \phi$ by $\psi\phi$.

Theorem 2.4. *Let $\phi\colon \mathcal{M} \to \mathcal{M}$ be a weakly continuous, unital, completely positive map.* (a) *There exists an idempotent, unital, completely positive map $\psi\colon \mathcal{M} \to \mathcal{M}$ with range* $\operatorname{ran}(\psi) = \mathcal{M}^\phi$*.* (b) *$\mathcal{M}^\phi = \mathcal{I}(\phi)$ if and only if ψ is a conditional expectation.* (c) *If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\mathcal{M}^\phi = \mathcal{I}(\phi)$, then $\mathcal{I}(\phi)$ is an injective von Neumann algebra.*

Proof. (a) Let $\psi_n\colon \mathcal{M} \to \mathcal{M}$ be the sequence of Cesàro means

$$\psi_n = \frac{1}{n}\sum_{i=1}^{n} \phi^i.$$

Then ψ_n, $n = 1, 2, \ldots$, are unital completely positive (and hence completely contractive) maps. Since $\mathcal{M}$ has a predual, it follows from the Alaoglu theorem that the closed unit ball of $\mathcal{M}$ is compact in the weak * topology. Hence, the closed unit ball of $\mathcal{CB}(\mathcal{M})$ is compact in the point-weak topology. It follows that there exists a subsequence ψ_{n_k} and a unital completely positive map $\psi\colon \mathcal{M} \to \mathcal{M}$ such that

$$w\text{-}\lim_{k\to\infty} \psi_{n_k}(A) = \psi(A) \tag{2.1}$$

for every $A \in \mathcal{M}$. For any integer $n \geq 1$ we have

$$\|\psi_n - \phi\psi_n\| = \frac{1}{n}\,\|\phi - \phi^{n+1}\| \leq \frac{2}{n} \tag{2.2}$$

and hence $\psi_n - \phi\psi_n$ converges uniformly to 0. For any $k \geq 1$ we have

$$\phi\psi - \psi = (\phi\psi - \phi\psi_{n_k}) + (\phi\psi_{n_k} - \psi_{n_k}) + (\psi_{n_k} - \psi).$$

Note that the point-weak limit of the expression of the right side is 0 as $n_k \to \infty$. Indeed, $\psi_{n_k} - \psi \to 0$ by the definition of $\psi, \psi_{n_k} - \phi\psi_{n_k} \to 0$ by (2.2) and

$$\phi\psi - \phi\psi_{n_k} = \phi(\psi - \psi_{n_k}) \to 0$$

because ϕ is weakly continuous. In a similar way we have $\psi\phi = \psi$ and thus,

$$\phi\psi = \psi\phi = \psi. \tag{2.3}$$

By induction we see that $\phi^k\psi = \psi\phi^k = \psi$ for all $k \geq 1$ and hence

$$\psi_{n_k}(\psi(A)) = \psi(A) \tag{2.4}$$

for all $A \in \mathcal{M}$. Taking the weak limit in (2.4) as $k \to \infty$ we conclude that ψ is idempotent. In particular, $\operatorname{ran}(\psi) = \mathcal{M}^\psi$. Applying (2.3) we have $\operatorname{ran}(\psi) \subseteq \mathcal{M}^\phi$. To prove the converse inclusion, let $A \in \mathcal{M}^\phi$. Then $\psi_{n_k}(A) = A$ for all $k \geq 1$ and by (2.1) we have $\psi(A) = A$. Hence, $\operatorname{ran}(\psi) = \mathcal{M}^\phi$.

(b) If $\mathcal{M}^\phi = \mathcal{I}(\phi)$, then $\operatorname{ran}(\psi) = \mathcal{M}^\phi$ is a von Neumann subalgebra of $\mathcal{M}$. For $A \in \operatorname{ran}(\psi)$ we have

$$\psi(A)^*\psi(A) = A^*A = \psi(A^*A).$$

Hence, by Theorem 2.1 we obtain

$$\psi(AB) = \psi(A)\psi(B) = A\psi(B)$$

and

$$\psi(BA) = \psi(B)\psi(A) = \psi(B)A$$

for all $B \in \mathcal{M}$. Thus, ψ is a conditional expectation. Conversely, if ψ is a conditional expectation, then $\mathcal{M}^\phi = \operatorname{ran}(\psi)$ is a C^*-subalgebra of $\mathcal{M}$ so applying Lemma 2.2, we conclude that $\mathcal{M}^\phi = \mathcal{I}(\phi)$.
(c) If $\mathcal{M}^\phi = \mathcal{I}(\phi)$, then by (b), ψ is a conditional expectation with $\operatorname{ran}(\psi) = \mathcal{I}(\phi)$ so $\mathcal{I}(\phi)$ is an injective von Neumann algebra. $\square$

Corollary 2.5. *Let $\phi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a weakly continuous, unital, completely positive map. If $\mathcal{I}(\phi)$ is not injective, then $\mathcal{B}(\mathcal{H})^\phi \neq \mathcal{I}(\phi)$.*

3. Quantum Operations

Let $\mathcal{A}$ be a set of operators $\mathcal{A} = \{A_i, A_i^*\colon i = 1, 2, \dots\}$ where $A_i \in \mathcal{B}(\mathcal{H})$ satisfy $\sum A_i A_i^* \leq I$. A map $\phi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ of the form $\phi_\mathcal{A}(B) = \sum A_i B A_i^*$ is called a **quantum operation**. If $B \geq 0$ then this series gives an increasing sequence of positive operators that is bounded above by $\|B\| I$ so it converges in the strong operator topology. It follows that the series converges in the strong operator topology for any $B \in \mathcal{B}(\mathcal{H})$. It is easy to see that a quantum operation is completely positive. If $\phi_\mathcal{A}(I) = I$ or equivalently if $\sum A_i A_i^* = I$, then $\phi_\mathcal{A}$ is **unital**. If $\sum A_i^* A_i = I$ then $\phi_\mathcal{A}$ is **trace preserving**. If the A_i are self-adjoint then $\phi_\mathcal{A}$ is **self-adjoint**. An important example of a self-adjoint quantum operation is **Lüders operation**

$$L_\mathcal{A}(B) = \sum A_i^{1/2} B A_i^{1/2}$$

where $A_i \geq 0$ and $\sum A_i = I$. A quantum operation $\phi_\mathcal{A}$ is **faithful** if $\phi_\mathcal{A}(B^*B) = 0$ implies that $B = 0$.

Theorem 3.1. *Let $\phi_\mathcal{A}$ be a quantum operation.* (a) *$\phi_\mathcal{A}$ is a weakly continuous completely positive map.* (b) *If $\phi_\mathcal{A}$ is trace preserving then $\phi_\mathcal{A}$ is faithful and* $\operatorname{tr}(\phi_\mathcal{A}(B)) = \operatorname{tr}(B)$ *for every $B \in \mathcal{T}(\mathcal{H})$.*

Proof. (a) Let ℓ_2 denote the Hilbert space of square summable complex sequences and let $\ell_2(\mathcal{H}) = \ell_2 \otimes \mathcal{H}$ be the Hilbert space of square summable sequences with elements in $\mathcal{H}$. Let $V\colon \mathcal{H} \to \ell_2(\mathcal{H})$ be the linear operator defined by

$$Vh = (A_1^* h, A_2^* h, \dots).$$

Now $V \in \mathcal{B}(\mathcal{H}, \ell_2(\mathcal{H}))$ because

$$\|Vh\|^2 = \sum \langle A_i^* h, A_i^* h\rangle = \sum \langle A_i A_i^* h, h\rangle = \left\langle \sum A_i A_i^* h, h \right\rangle \leq \|h\|^2.$$

The adjoint $V^* \in \mathcal{B}(\ell_2(\mathcal{H}), \mathcal{H})$ is given by

$$V^*(h_1, h_2, \dots) = \sum A_i h_i.$$

It follows that $\phi_\mathcal{A}(B) = V^*(I \otimes B)V$ for all $B \in \mathcal{B}(\mathcal{H})$. Since the map $B \mapsto I \otimes B$ from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\ell_2(\mathcal{H}))$ is weakly continuous [5] and completely positive, it follows that $\phi_\mathcal{A}$ is weakly continuous and completely positive.
(b) Suppose that $\phi_\mathcal{A}$ is trace preserving. To show $\phi_\mathcal{A}$ is faithful, suppose $\phi_\mathcal{A}(B^*B) = 0$. Then for every $h \in \mathcal{H}$ we have

$$0 = \langle \phi_\mathcal{A}(B^*B)h, h\rangle = \left\langle \sum A_i B^* B A_i^* h, h \right\rangle = \sum \langle B A_i^* h, B A_i^* h\rangle.$$

Hence, $BA_i^*h = 0$ for every $h \in \mathcal{H}$ so that $BA_i^* = 0$. But then

$$B = B \sum A_i^* A_i = 0.$$

Finally, let $B \in \mathcal{T}(\mathcal{H})^+$. Then the operators $A_iBA_i^*$ and $A_i^*A_iB$ are trace class and

$$\operatorname{tr}\left(\sum_{i=1}^n A_iBA_i^*\right) = \operatorname{tr}\left(\sum_{i=1}^n A_i^*A_iB\right) \le \left\|\sum_{i=1}^n A_i^*A_i\right\| \operatorname{tr}(B) \le \operatorname{tr}(B).$$

Since $\sum_{i=1}^n A_iBA_i^*$ is a nondecreasing sequence of positive operators converging strongly to $\phi_{\mathcal{A}}(B)$ and since the trace is continuous with respect to such sequences, we have

$$\operatorname{tr}\left(\phi_{\mathcal{A}}(B)\right) = \lim_{n\to\infty} \operatorname{tr}\left(\sum_{i=1}^n A_iBA_i^*\right) \le \operatorname{tr}(B).$$

Hence, $\phi_{\mathcal{A}}(B) \in \mathcal{T}(\mathcal{H})^+$. Again by the continuity of trace on bounded nondecreasing sequences we have

$$\operatorname{tr}\left(\phi_{\mathcal{A}}(B)\right) = \lim_{n\to\infty} \operatorname{tr}\left(\sum_{i=1}^n A_i^*A_iB\right) = \operatorname{tr}\left(\sum_{i=1}^\infty A_i^*A_iB\right) = \operatorname{tr}(B).$$

The result for arbitrary $B \in \mathcal{T}(\mathcal{H})$ now follows. □

Let $\phi_{\mathcal{A}}$ be a unital quantum operation and define the fixed point set $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}$ as before. We then have the von Neumann algebras $\mathcal{I}(\phi_{\mathcal{A}})$ and $\mathcal{A}'$ as well as the weakly closed operator system $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}$ and it is clear that

$$\mathcal{A}' \subseteq \mathcal{I}(\phi_{\mathcal{A}}) \subseteq \mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}.$$

We are now interested in when these sets coincide; that is, when $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} \subseteq \mathcal{A}'$. The next theorem generalizes a result in [3] and has essentially the same proof.

Theorem 3.2. *Let $\phi_{\mathcal{A}}$ be a self-adjoint quantum operation. If $B \in \mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}$ is positive and has pure point spectrum which can be totally ordered in decreasing order, then $B \in \mathcal{A}'$.*

Proof. Let h be a unit eigenvector of B corresponding to the largest eigenvalue $\lambda_1 = \|B\|$. Then $\phi_{\mathcal{A}}(B) = B$ implies that

$$\lambda_1 = \sum \langle BA_ih, A_ih\rangle \le \|B\| \sum \|A_ih\|^2 = \lambda_1 \sum \left\langle A_i^2h, h\right\rangle \le \lambda_1.$$

Since $\langle BA_ih, A_ih\rangle \le \lambda_1\langle A_i^2h, h\rangle$, it follows that

$$\langle(\lambda_1 I - B)A_ih, A_ih\rangle = 0.$$

Hence, $(\lambda_1 I - B)A_ih = 0$ for every eigenvector h corresponding to λ_1. Thus, A_i leaves the λ_1-eigenspace invariant. Letting P_1 be the corresponding spectral projection of B we have $P_1A_iP_1 = A_iP_1$ which implies that $A_iP_1 = P_1A_i$, $i = 1, 2, \ldots$. Now $B = \lambda_1P_1 + B_1$ where B_1 is a positive operator with a largest eigenvalue. Since

$$\lambda_1P_1 + B_1 = B = \phi_{\mathcal{A}}(B) = \lambda_1\phi_{\mathcal{A}}(P_1) + \phi_{\mathcal{A}}(B_1) = \lambda_1P_1 + \phi_{\mathcal{A}}(B_1)$$

we have $\phi_{\mathcal{A}}(B_1) = B_1$. Proceeding by induction, $B \in \mathcal{A}'$. □

We shall show later that Theorem 3.2 cannot be extended to an arbitrary positive $B \in \mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}}$. Moreover, it cannot be extended to a non-self-adjoint $\phi_\mathcal{A}$ even in the case where B is positive with finite spectrum and $\mathcal{A}$ contains only two operation elements. The next result follows from Lemmas 3.1 and 3.3 of [1]. However, we present a simpler and more algebraic proof.

Lemma 3.3. *If $\phi_\mathcal{A}$ is a unital quantum operation, then the following statements are equivalent.* (a) $\mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}} = \mathcal{A}'$. (b) $\mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}}$ *is a von Neumann algebra.* (c) $\mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}} = \mathcal{I}(\phi_\mathcal{A})$. (d) *If $B \in \mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}}$, then $B^*B \in \mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}}$.*

Proof. (a)⇒(b) is clear, (b)⇒(c) follows from Lemma 2.2 and (c)⇒(d) is clear. To show that (d) implies (a) assume that (d) holds and $B \in \mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}}$. Then $B^*B \in \mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}}$. Notice that

$$\begin{aligned} 0 \le [B, A_i]\,[B, A_i]^* &= (BA_i - A_iB)(A_i^*B^* - B^*A_i^*) \\ &= BA_iA_i^*B^* + A_iBB^*A_i^* - A_iBA_i^*B^* - BA_iB^*A_i^*. \end{aligned}$$

Summing over i yields

$$\begin{aligned} 0 \le \sum [B, A_i]\,[B, A_i]^* &= BB^* + \phi_\mathcal{A}(BB^*) - \phi_\mathcal{A}(B)B^* - B\phi_\mathcal{A}(B^*) \\ &= \phi_\mathcal{A}(BB^*) - BB^* = 0. \end{aligned}$$

Hence, $[B, A_i] = 0$ for all $i = 1, 2, \ldots$. In a similar way we have $[B, A_i^*] = 0$. Hence, $B \in \mathcal{A}'$ so that $\mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}} = \mathcal{A}'$ □

Corollary 3.4. *Let $\phi_\mathcal{A}$ be a unital quantum operation. If $B \in \mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}}$ then $B \in \mathcal{A}'$ if and only if $B^*B, BB^* \in \mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}}$.*

An operator $W \in \mathcal{T}(\mathcal{H})$ is **faithful** if for any $A \in \mathcal{B}(\mathcal{H})$, $\mathrm{tr}(W^*A^*AW) = 0$ implies $A = 0$.

Theorem 3.5. *Let $\phi_\mathcal{A}$ be a trace preserving, unital, quantum operation.*
(a) *If $\dim(\mathcal{H}) < \infty$, then $\mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}} = \mathcal{A}'$.* (b) *If there exists a faithful operator $W \in \mathcal{T}(\mathcal{H}) \cap \mathcal{A}'$, then $\mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}} = \mathcal{A}'$.* (c) *If $B \in \mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}}$ and $B = C + D$ for $C \in \mathcal{A}'$, $D \in \mathcal{T}(\mathcal{H})$, then $B \in \mathcal{A}'$.* (d) $\mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}} \cap \mathcal{T}(\mathcal{H}) = \mathcal{A}' \cap \mathcal{T}(\mathcal{H})$.

Proof. (a) If $\dim(\mathcal{H}) = n < \infty$, then $\omega(B) = \mathrm{tr}(B)/n$ is a faithful ϕ-invariant state. The result follows from Theorem 2.3 and Lemma 3.3. (b) By the proof of Lemma 3.3, if $B \in \mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}}$ then

$$\phi_\mathcal{A}(B^*B) - B^*B \ge 0.$$

Since $W \in \mathcal{T}(\mathcal{H}) \cap \mathcal{A}'$, by Theorem 3.1(b) we have

$$\mathrm{tr}\,[W^*(\phi_\mathcal{A}(B^*B) - B^*B)W] = \mathrm{tr}\,[\phi_\mathcal{A}(W^*B^*BW)] - \mathrm{tr}(W^*B^*BW) = 0.$$

Hence, $\phi_\mathcal{A}(B^*B) = B^*B$ so that $B^*B \in \mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}}$. Applying Lemma 3.3 gives $\mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}} = \mathcal{A}'$ (c) Since

$$C + D = \phi_\mathcal{A}(C + D) = \phi_\mathcal{A}(C) + \phi_\mathcal{A}(D) = C + \phi_\mathcal{A}(D)$$

we have $D \in \mathcal{B}(\mathcal{H})^{\phi_\mathcal{A}}$. Now $D^*D \in \mathcal{T}(\mathcal{H})$ and by the proof of Lemma 3.3,

$$\phi_\mathcal{A}(D^*D) - D^*D \ge 0.$$

Since $\phi_{\mathcal{A}}$ is trace preserving, we have

$$\operatorname{tr}\left[\phi_{\mathcal{A}}(D^*D) - D^*D\right] = 0.$$

Hence, $\phi_{\mathcal{A}}(D^*D) = D^*D$ so that $D^*D \in \mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}$. Similarly, $DD^* \in \mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}$ so by Corollary 3.4, $D \in \mathcal{A}'$. Hence, $B \in \mathcal{A}'$. (d) follows from (c). □

The next result follows from Theorem 2.4 and Lemma 3.3.

Theorem 3.6. *Let $\phi_{\mathcal{A}}$ be a unital quantum operation.* (a) *There exists an idempotent, unital, completely positive map $\psi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ with* $\operatorname{ran}(\psi) = \mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}$. (b) *$\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} = \mathcal{A}'$ if and only if ψ is a conditional expectation.* (c) *If $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} = \mathcal{A}'$ then $\mathcal{A}'$ is an injective von Neumann algebra.*

4. Example

It follows from Theorem 3.6(c) that if $\mathcal{A}'$ is not injective then $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} \neq \mathcal{A}'$. We can apply this observation to obtain examples for various conjectures. For instance, the following counterexample shows: (a) If $\phi_{\mathcal{A}}(B) = \sum A_i^{1/2} B A_i^{1/2}$, $A_i \geq 0$, $\sum A_i = I$, is a Lüders operation, then $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} \neq \mathcal{A}'$ in general. (This answers a question posed in [3, 8].) (b) If $\phi_{\mathcal{A}}(B) = A_1 B A_1^* + A_2 B A_2^*$ is a trace preserving, unital, quantum operation, then $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} \neq \mathcal{A}'$ in general. (It has been shown that if A_1 and A_2 are positive, then $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} = \mathcal{A}'$ [3, 8].)

Example. Let $\mathbb{F}_2$ be the free group on two generators g_1, g_2 with identity e. It is clear that $\mathbb{F}_2$ is countable. Let $\mathcal{H} = \ell_2(\mathbb{F}_2)$ be the separable complex Hilbert space

$$\mathcal{H} = \ell_2(\mathbb{F}_2) = \left\{ f\colon \mathbb{F}_2 \to \mathbb{C} : \sum |f(x)|^2 < \infty \right\}.$$

For $x \in \mathbb{F}_2$ define $\delta_x\colon \mathbb{F}_2 \to \mathbb{C}$ by

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

Then $\{\delta_x \colon x \in \mathbb{F}_2\}$ is an orthonormal basis for $\mathcal{H}$. Define the unitary operators U_1, U_2 on $\mathcal{H}$ by $U_1\delta_x = \delta_{g_1 x}$, $U_2\delta_x = \delta_{g_2 x}$. The von Neumann algebra generated by U_1 and U_2 is denoted by $\mathcal{N} = L(\mathbb{F}_2)$. It is known that $\mathcal{N}$ and hence $\mathcal{N}'$ are not injective [13, 16].

Lemma 4.1. *Suppose $B \in \mathcal{B}(\mathcal{H})$ has the form $B\delta_x = \lambda_x \delta_x$, $0 \leq \lambda_x \leq 1$. Then $B \in \mathcal{E}(\mathcal{H})$ and if $B \in \mathcal{N}'$, then $B = \lambda_e I$.*

Proof. It is clear that $B \in \mathcal{E}(\mathcal{H})$. Now suppose that $B \in \mathcal{N}'$. Then

$$\lambda_{g_1}\delta_{g_1} = B\delta_{g_1} = BU_1\delta_e = U_1 B\delta_e = \lambda_e U_1 \delta_e = \lambda_e \delta_{g_1}.$$

Hence, $\lambda_{g_1} = \lambda_e$ and in a similar way

$$\lambda_{g_1^{-1}} = \lambda_{g_2} = \lambda_{g_2^{-1}} = \lambda_e.$$

Now suppose $x \in \mathbb{F}_2$ has the form $x = g_1 y$ for some $y \in \mathbb{F}_2$. Then

$$\lambda_x \delta_x = B\delta_x = B\delta_{g_1 y} = BU_1\delta_y = U_1 B \delta_y = \lambda_y U_1 \delta_y = \lambda_y \delta_{g_1 y} = \lambda_y \delta_x.$$

Hence, $\lambda_{g_1 y} = \lambda_y$ for every $y \in \mathbb{F}_2$. Similarly,

$$\lambda_{g_1^{-1}y} = \lambda_{g_2 y} = \lambda_{g_2^{-1}y} = \lambda_y$$

for every $y \in \mathbb{F}_2$. Continuing by induction, we conclude that $\lambda_x = \lambda_e$ for every $x \in \mathbb{F}_2$. Hence, $B = \lambda_e I$. $\square$

Let $A_1 = 2^{-1/2} U_1$ and $A_2 = 2^{-1/2} U_2$ and let $\mathcal{A} = \{A_1, A_2\}$. Then $\mathcal{A}' = \mathcal{N}'$ and

$$A_1 A_1^* + A_2 A_2^* = A_1^* A_1 + A_2^* A_2 = I.$$

Thus, $\phi_{\mathcal{A}}$ is a trace preserving, unital quantum operation. Now a B of the form in Lemma 4.1 satisfies $B \in \mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}$ if and only if

$$\frac{1}{2}\lambda_{g_1^{-1}x} + \frac{1}{2}\lambda_{g_2^{-1}x} = \lambda_x \tag{4.1}$$

for all $x \in \mathbb{F}_2$. Define $B \in \mathcal{B}(\mathcal{H})$ by $B\delta_x = \lambda_x \delta_x$ where $\lambda_x = 0$ if x ends in g_2^{-1}, $\lambda_x = 1$ if x ends in g_1^{-1} and $\lambda_x = 1/2$ otherwise. Then it is easy to check that (4.1) is satisfied. Hence, $B \in \mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}$ and by Lemma 4.1 $B \notin \mathcal{A}'$. Thus, $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} \neq \mathcal{A}'$.

Theorem 4.2. *There exists a Lüders operation $\phi_{\mathcal{A}}$ on $\mathcal{H} = \ell_2(\mathbb{F}_2)$ such that $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} \neq \mathcal{A}'$. More precisely, there exists a $B \in \mathcal{E}(\mathcal{H})$ such that $B \in \mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}$ but $B \notin \mathcal{A}'$.*

Proof. By taking the real and imaginary parts of $U_1 = V_1 + \mathrm{i}V_2$, $U_2 = V_3 + \mathrm{i}V_4$ we see that $\mathcal{N}$ is generated by four self-adjoint operators V_1, V_2, V_3, V_4. Moreover, $\mathcal{N}$ is generated by the four positive operators $C_i = \|V_i\| I - V_i$, $i = 1, 2, 3, 4$. Let $A_i = C_i / 4\|C_i\|$, $i = 1, 2, 3, 4$, and let $A_5 = I - \sum_{i=1}^4 A_i$. Then $A_i \in \mathcal{E}(\mathcal{H})$, $i = 1, \ldots, 5$, $\sum A_i = I$ and $\mathcal{A} = \{A_i \colon i = 1, \ldots, 5\}$ generates $\mathcal{N}$. Now $\phi_{\mathcal{A}}$ is a Lüders operation and since $\mathcal{N}' = \mathcal{A}'$ is not injective we have $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} \neq \mathcal{A}'$. Hence, there exists a $B \in \mathcal{B}(\mathcal{H})$ such that $B \in \mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} \setminus \mathcal{A}'$. Now the real part or the imaginary part B_1 of B also satisfies $B_1 \in \mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} \setminus \mathcal{A}'$. Letting $B_2 = \|B_1\| I - B_1$ we see that $B_2 \geq 0$ and that $B_3 = B_2 / \|B_2\| \in \mathcal{E}(\mathcal{H})$. Moreover $B_3 \in \mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} \setminus \mathcal{A}'$. $\square$

Although the B in Theorem 4.2 exists, it appears to be quite difficult to construct a concrete example of such a B.

5. Concluding Remarks

We have seen that the injectivity of the commutant of the set of operation elements $\mathcal{A}'$ of a Lüders operation $\phi_{\mathcal{A}}$ plays a role in deciding whether the set of fixed points $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}$ coincides with $\mathcal{A}'$ or not. However, the following question remains: if $\mathcal{A}'$ is injective, does this imply that $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} = \mathcal{A}'$? On the other hand, it has been proved that for a Lüders operation with only two operation elements, we have $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}} = \mathcal{A}'$ [3, 8]. In this case, $\mathcal{A}$ is commutative and it follows that $\mathcal{A}$ and $\mathcal{A}'$ are injective. It is then natural to ask whether the result is true for any commutative set of operation elements $\mathcal{A}$. Finally, we ask the following general question. Is it true that $\mathcal{B}(\mathcal{H})^{\phi_{\mathcal{A}}}$ is an injective envelope of either $\mathcal{A}'$ or $\mathcal{I}(\phi_{\mathcal{A}})$ [12].

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