Almost Sharp Quantum Effects

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Abstract

Quantum effects are represented by operators on a Hilbert space satisfying $0 \leq A \leq I$, and sharp quantum effects are represented by projection operators. We say that an effect A is almost sharp if $A = PQP$ for projections P and Q. We give simple characterizations of almost sharp effects. We also characterize effects that can be written as longer products of projections. For generality we first work in the formalism of von Neumann algebras. We then specialize to the full operator algebra $B(H)$ and to finite dimensional Hilbert spaces.

1 Introduction

Let H be a complex Hilbert space that represents the state space of a quantum system $\mathcal{S}$. The set of *effects* $\mathcal{E}(H)$ for $\mathcal{S}$ is the set of operators on H satisfying $0 \leq A \leq I$. Effects represent yes-no measurements that may be unsharp (imprecise, fuzzy). It is interesting that many of the important classes of quantum operators are given by subsets of $\mathcal{E}(H)$. For example, the *sharp* yes-no measurements are represented by the set of projection operators $\mathcal{P}(H) \subset \mathcal{E}(H)$. A *state* for $\mathcal{S}$ is represented by an operator $W \in \mathcal{E}(H)$ satisfying $tr(W) = 1$. We call W a *density operator* and denote the set of density operators by $\mathcal{D}(H)$. The *pure states* for $\mathcal{S}$ are given by $\mathcal{D}(H) \cap \mathcal{P}(H)$.

The probability that $A \in \mathcal{E}(H)$ has values yes (or is true) in the state $W \in \mathcal{D}(H)$ is given by $p_W(A) = tr(WA)$. If W happens to be a pure state corresponding to the unit vector ψ, then $p_W(A) = \langle A\psi, \psi\rangle$. In particular, sharp effects are called *quantum events* and the probability that event $Q \in \mathcal{P}(H)$ occurs in the state $W \in \mathcal{D}(H)$ is $p_W(Q) = tr(WQ)$. For $P, Q \in \mathcal{P}(H)$ we define the *conditional probability of Q given P* in the state W by

$$p_W(Q|P) = \frac{tr(PWPQ)}{tr(WP)} = \frac{tr(WPQP)}{tr(WP)}$$

from which we obtain

$$p_W(PQP) = tr(WPQP) = p_W(P)\, p_W(Q|P). \tag{1.1}$$

Now (1.1) is analogous to the traditional probability formula

$$p(A \cap B) = p(A)\, p(B|A)$$

so in some sense PQP corresponds to an intersection of events. However, in general $p_W(PQP) \neq p_W(QPQ)$ so the order of measurements is relevant. In fact, $P \circ Q = PQP$ corresponds to a sequential measurement in which we measure P first and Q second. We call $P \circ Q$ the *sequential product* of P and Q (see [2] and [3]). More generally, we define the *sequential product* $A \circ B = B^{1/2}AB^{1/2} \in \mathcal{E}(H)$ for any $A, B \in \mathcal{E}(H)$ but our main interest is in sequential products of sharp effects. The next result gives some of the important properties of the sequential product.

Theorem 1 ([3]) *Let $A, B \in \mathcal{E}(H)$, $P, Q \in \mathcal{P}(H)$. Then*

1. *$A \circ B = B \circ A$ if and only if $AB = BA$.*
2. *If $A \circ B \in \mathcal{P}(H)$ then $AB = BA$.*
3. *$P \circ Q \in \mathcal{P}(H)$ if and only if $PQ = QP$.*

If $AB = BA$ we say that A and B are *compatible.* Physically, compatible effects correspond to effects that are simultaneously measurable. Thus, A and B are simultaneously measurable if and only if their order of measurement is irrelevant. A state $W \in \mathcal{D}(H)$ is *faithful* if $tr(WA) = 0$ for $A \in \mathcal{E}(H)$ implies that $A = 0$. The next result gives an interesting probabilistic characterization of compatible sharp effects.

Corollary 2 *For $P, Q \in \mathcal{P}(H)$ and faithful $W \in \mathcal{D}(H)$, $PQ = QP$ if and only if $p_W(Q \circ P) \leq p_W(Q \circ (P \circ Q))$.*

Proof. Suppose that $p_W(Q \circ P) \leq p_W(Q \circ (P \circ Q))$. Then

$$tr[W(QPQ - QPQPQ)] \leq 0.$$

It is easy to check that $PQP \leq P$ and it follows that $QPQPQ \leq QPQ$. Hence, $QPQ - QPQPQ \in \mathcal{E}(H)$ so that

$$tr[W(QPQ - QPQPQ)] = 0.$$

Since W is faithful, we conclude that

$$QPQ = QPQPQ = (QPQ)^2.$$

Thus, $Q \circ P = QPQ \in \mathcal{P}(H)$ so by part 3 of Theorem 1, $PQ = QP$. The converse is trivial. ■

The next Corollary solves Problem 11028 in the American Mathematical Monthly (vol 110, p. 636 (2003)).

Corollary 3 *Let* $\dim(H) = n < \infty$ *and let* $P, Q \in \mathcal{P}(H)$. *Then* $PQ \in \mathcal{P}(H)$ *if and only if* $tr(PQ) = tr(PQPQ)$.

Proof. Notice that $W = \frac{1}{n}I \in \mathcal{D}(H)$ is faithful and

$$p_W(Q \circ P) = tr(WQPQ) = \frac{1}{n}tr(QPQ) = \frac{1}{n}tr(PQ).$$

Assuming that $tr(PQ) = tr(PQPQ)$ we have that

$$p_W(Q \circ P) = \frac{1}{n}tr(QPQP) = p_W(Q \circ (P \circ Q)).$$

By Corollary 1.2 we have $PQ = QP$ so that $PQ \in \mathcal{P}(H)$. The converse is trivial ■

One of our main concerns is to characterize effects of the form $A = P \circ Q$ for $P, Q \in \mathcal{P}(H)$. Such effects are called *almost sharp* because they may be obtained by measuring two sharp effects. In a sense, almost sharp effects are "close" to being sharp and we shall present a simple characterization of such effects. Defining the *negation* of $A \in \mathcal{E}(H)$ by $A' = I - A$, we see immediately that $A' \in \mathcal{E}(H)$. We say that A is *nearly sharp* if both A and A' are almost sharp. We shall show that the set of nearly sharp effects has the structure of an orthocomplemented partially ordered set.

Letting $\mathcal{P}_1(H) = \mathcal{P}(H)$ and $\mathcal{P}_2(H)$ be the set of almost sharp elements we see that

$$\mathcal{P}_2(H) = \{A \in \mathcal{E}(H) : A = P_1 \circ P_2,\ P_1, P_2 \in \mathcal{P}(H)\}$$

and it follows from part 3 of Theorem 1 that $\mathcal{P}_2(H)$ strictly contains $\mathcal{P}_1(H)$. We shall show that

$$\mathcal{P}_3(H) = \{A \in \mathcal{E}(H) : A = P_1 \circ (P_2 \circ P_3), P_1, P_2, P_3 \in \mathcal{P}(H)\}$$

strictly contains $\mathcal{P}_2(H)$ if $\dim(H)$ is sufficiently large. This suggests the natural problem of when $A \in \mathcal{E}(H)$ has the form

$$A = P_1 \circ (P_2 \circ \cdots \circ (P_{n-1} \circ P_n)). \tag{1.2}$$

Writing (1.2) in terms of operator products gives

$$A = P_1 P_2 \cdots P_{n-1} P_n P_{n-1} \cdots P_2 P_1.$$

We shall also characterize effects that have the form (1.2)

For generality we shall first work in the formalism of a von Neumann algebra. We then consider the full operator algebra $B(H)$ for H separable. Finally, we show that simplifications and further insights can be obtained from considering finite dimensional Hilbert spaces.

2 Effects on von Neumann Algebras

Let M be a von Neumann algebra on a Hilbert space H. The set of effects in M is

$$\mathcal{E}(M) = \{A \in M : 0 \leq A \leq I\}$$

and the set of projections or sharp effects in M is

$$\mathcal{P}(M) = \left\{P \in M : P = P^* = P^2\right\} \subset \mathcal{E}(M).$$

For $P, Q \in \mathcal{P}(M)$, according to the usual comparison of projections we define $P \preceq Q$ if there exists a partial isometry $U \in M$ such that $U^*U = P$ and $UU^* \leq Q$. Notice that UU^* is a projection whose range is contained in the range of Q.

For $A \in \mathcal{E}(M)$ we define P_A to be the projection onto the closure of the range of A. It can be shown that

$$P_A = \lim_{n\to\infty} A^{\frac{1}{n}}$$

in the strong operator topology so that $P_A \in M$. Moreover, $P_A A = AP_A = A$. Letting N_A be the projection onto the null space of A we have that

$$N_A = I - P_A = (P_A)'.$$

It is easy to check that P_A is the smallest projection satisfying $A \leq P_A$, $N_{A'}$ is the largest projection satisfying $N_{A'} \leq A$ and that

$$N_{A'} = \lim_{n\to\infty} A^n.$$

It follows that $N_A = \lim_{n\to\infty} (A')^n$ and hence

$$N_{A'}A = AN_{A'} = N_{A'}.$$

Notice that if $A \in \mathcal{E}(M)$ has the form $A = PQP$ for some $P, Q \in \mathcal{P}(M)$, then we also have that $A = P_A Q P_A$.

Lemma 4 *For $A \in \mathcal{E}(M)$ we have that $P_{AA'} = P_A - N_{A'}$.*

Proof. It is clear that $AA' \in \mathcal{E}(M)$, $AA' \leq A$ and $AA' \leq A'$. Hence, $P_{AA'} \leq P_A$, $P_{AA'} \leq P_{A'}$, and it follows that $N_A \leq N_{AA'}$ and $N_{A'} \leq N_{AA'}$. Since N_A and $N_{A'}$ are mutually orthogonal we conclude that $N_A + N_{A'} \leq N_{AA'}$. To prove the reverse inequality, let $x \neq 0$ satisfying $N_{AA'}x = x$. Write $x = Ax + A'x$ and notice that $A'(Ax) = 0$ and $A(A'x) = 0$. Then $Ax = N_{A'}(Ax)$ and $A'x = N_A(A'x)$. Since $N_{A'}A = N_{A'}$ and $N_A A' = N_A$, we have that $x = N_A x + N_{A'}x$. We conclude that $N_A + N_{A'} = N_{AA'}$. Therefore

$$P_{AA'} = I - N_{AA'} = I - N_A - N_{A'} = P_A - N_{A'}.$$

■

In the process of proving Lemma 4 we also obtained the interesting result $N_A + N_{A'} = N_{AA'}$.

The motivation for the next Theorem is the following: If $0 \leq A \leq I$, then

$$\begin{pmatrix} A & \sqrt{AA'} \\ \sqrt{AA'} & A' \end{pmatrix}$$

is a projection whose compression to the (1,1) component is A.

Theorem 5 *An effect $A \in \mathcal{E}(M)$ is almost sharp if and only if $P_{AA'} \preceq N_A$.*

Proof. Suppose that $P_{AA'} \preceq N_A$. Then there exists a partial isometry $U \in M$ such that $U^*U = P_{AA'}$ and $UU^* \leq N_A$. Then U maps the range of $P_{AA'}$ into the range of N_A and U^* maps the range of N_A onto the range of $P_{AA'}$. Notice that $N_{A'} = P_A - P_{AA'}$, $P_{AA'}$ and N_A are mutually orthogonal projections satisfying

$$N_A + N_{A'} + P_{AA'} = I. \tag{2.1}$$

Define Q_1 by the formula

$$Q_1 = P_{AA'}AP_{AA'} + P_{AA'}\sqrt{AA'}U^*N_A + N_AU\sqrt{AA'}P_{AA'} + N_AUA'U^*N_A.$$

It is clear that $Q_1 = Q_1^*$ and to show that $Q_1 \in \mathcal{P}(M)$ we have

$$\begin{aligned} Q_1^2 &= P_{AA'}\left(A^2 + \sqrt{AA'}U^*N_AU\sqrt{AA'}\right)P_{AA'} \\ &\quad + P_{AA'}\left(AP_{AA'}\sqrt{AA'}U^* + \sqrt{AA'}U^*N_AUA'U^*\right)N_A \\ &\quad + N_A\left(U\sqrt{AA'}P_{AA'}A + UA'U^*N_AU\sqrt{AA'}\right)P_{AA'} \\ &\quad + N_A\left(U\sqrt{AA'}P_{AA'}\sqrt{AA'}U^* + UA'U^*N_AUA'U^*\right)N_A. \end{aligned} \tag{2.2}$$

Notice that

$$U = N_AU = UP_{AA'} = N_AUP_{AA'} \tag{2.3}$$

and hence

$$P_{AA'} = U^*N_AU. \tag{2.4}$$

By (2.4) the first term in (2.2) becomes

$$P_{AA'}\left(A^2 + AA'\right)P_{AA'} = P_{AA'}AP_{AA'}.$$

By (2.4) again, the second term in (2.2) becomes

$$P_{AA'}\left(A\sqrt{AA'}U^* + A'\sqrt{AA'}U^*\right)N_A = P_{AA'}\sqrt{AA'}U^*N_A.$$

In a similar way, the third term in (2.2) becomes

$$N_A\left(UA\sqrt{AA'} + UA'\sqrt{AA'}\right)P_{AA'} = N_AU\sqrt{AA'}P_{AA'}.$$

Finally, by (2.3) and (2.4) the fourth term in (2.2) becomes

$$N_A U\left(AA' + P_{AA'}A'A'\right)U^* N_A = N_A U\left(AA' + A'A'\right)U^* N_A = N_A U A' U^* N_A.$$

We conclude that $Q_1 \in \mathcal{P}(M)$. To show that $Q_1 \leq P_{AA'} + N_A$ we have that

$$Q_1\left(P_{AA'} + N_A\right) = \left(P_{AA'} + N_A\right)Q_1 = Q_1.$$

Since $N_{A'}$ is orthogonal to $P_{AA'}$ and N_A we see that

$$Q = N_{A'} + Q_1 \in \mathcal{P}(M).$$

Since $N_{A'} \leq A \leq P_A$, $P_{AA'} \leq P_A$, and $P_A N_A = 0$ we have by Lemma 4 that

$$\begin{aligned} P_A Q P_A &= P_A N_{A'} P_A + P_A Q_1 P_A = N_{A'} + P_{AA'} Q_1 P_{AA'} \\ &= N_{A'} + P_{AA'} A P_{AA'} = N_{A'} + \left(P_A - N_{A'}\right) A \left(P_A - N_{A'}\right) \\ &= N_{A'} + \left(P_A - N_{A'}\right) A = N_{A'} + P_A A - N_{A'} A \\ &= N_{A'} + A - N_{A'} = A. \end{aligned}$$

Conversely, suppose that there exists a $Q \in \mathcal{P}(M)$ such that $A = P_A Q P_A$. Letting $B = P_A Q N_A$ we have that $BB^* \geq 0$ and

$$\begin{aligned} BB^* &= P_A Q N_A Q P_A = P_A Q (I - P_A) Q P_A \\ &= P_A Q P_A - P_A Q P_A Q P_A = A - A^2 = AA'. \end{aligned}$$

Using the polar decomposition of B we find a partial isometry $U \in M$, with initial space P_{B^*}, the range of B^*, and final space P_B, the range of B, such that $B = \sqrt{AA'}U$. Now

$$P_B = P_{BB^*} = P_{AA'}$$

and since $P_{B^*} \leq N_A$ we obtain that $UU^* = P_{AA'}$ and $U^*U \leq N_A$. Hence, $P_{AA'} \preceq N_A$. ■

We can gain an intuitive feeling for Theorem 5 as follows. Since $AA' = A - A^2$ we see that A is sharp iff $AA' = 0$ or equivalently $P_{AA'} = 0$. Now Theorem 5 states that A is almost sharp if and only if $P_{AA'}$ is not too big in the sense that $P_{AA'}$ is dominated by N_A. Recall that a *factor* is a von Neumann algebra with trivial center and that in this case all projections are comparable relatively to $\preceq$. It follows that projections have a well-defined dimension $\dim(P)$ in a factor.

Corollary 6 *If M is a factor, then an effect $A \in \mathcal{E}(M)$ is almost sharp if and only if* $\dim\left(P_{AA'}\right) \leq \dim\left(N_A\right)$.

Corollary 6 is not true if M is not a factor even when there is a well defined dimension function. To show this, let M_n be the matrix algebra of $n \times n$ complex matrices and let M be the von Neumann algebra $M = M_n \oplus M_n$. Let $I_n \in M_n$ be the identity matrix and let

$$A = \frac{1}{2} I_n \oplus 0 = \begin{bmatrix} \begin{bmatrix} 1/2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/2 \end{bmatrix} & \\ & \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \end{bmatrix} \in M_n \oplus M_n.$$

Then $\dim(P_{AA'}) = \dim(N_A) = n$ but A cannot be written in the form $P_A Q P_A$ for some projection Q in $M = M_n \oplus M_n$.

Applying Corollary 6, a moment thought shows that $A \in \mathcal{E}(M)$ is almost sharp if and only if A^n is almost sharp for every positive integer n. This gives the nontrivial result that if $A = P_A Q P_A$ for $Q \in \mathcal{P}(M)$, then $A^n = P_A Q_n P_A$ for some $Q_n \in \mathcal{P}(M)$.

In a Type III factor, all proper projections are equivalent (have the same dimension). Recall that if A is not invertible, then A is *singular*.

Corollary 7 *Let M be a Type III factor and let $A \in \mathcal{E}(M)$ with $A \neq I$. Then A is almost sharp if and only if A is singular.*

Proof. Suppose A is invertible and $A = P_A Q P_A$ for some $Q \in \mathcal{P}(M)$. Since $A \leq P_A$, P_A is also invertible which implies that $P_A = I$. Hence, $A = Q$ which implies that $A = I$. Conversely, suppose that $A \neq 0$ and that A is singular. Then $P_{AA'} \neq I$ and $N_A \neq 0$. The result follows from Corollary 6. ■

Recall that an effect $A \in \mathcal{E}(M)$ is *nearly sharp* if A and A' are both almost sharp.

Corollary 8 *An effect $A \in \mathcal{E}(M)$ is nearly sharp if and only if $P_{AA'} \preceq N_A$ and $P_{AA'} \preceq N_{A'}$. If M is a factor, then $A \in \mathcal{E}(M)$ is nearly sharp if and only if* $\dim(P_{AA'}) \leq \dim(N_A)$ *and* $\dim(P_{AA'}) \leq \dim(N_{A'})$.

An *orthoposet* is a system $(\mathcal{P}, \leq, ', 0, 1)$ where $(\mathcal{P}, \leq, 0, 1)$ is a bounded poset and $' : \mathcal{P} \to \mathcal{P}$ satisfies $a'' = a$, $a \leq b$ implies $b' \leq a'$ and $a \wedge a' = 0$. An orthoposet $\mathcal{P}$ is *orthomodular* if $a \leq b'$ implies $a \vee b$ exists and $a \leq b$ implies $b = a \vee (b \wedge a')$. It is well known that $\mathcal{P}(M)$ forms an orthomodular lattice [1].

Corollary 9 *If $\mathcal{E}(M)_{ns}$ is the set of nearly sharp elements in $\mathcal{E}(M)$, then $(\mathcal{E}(M)_{ns}, \leq, ', 0, 1)$ is an orthoposet.*

Proof. All the properties of an orthoposet are clear except for the condition $A \wedge A' = 0$ for every $A \in \mathcal{E}(M)_{ns}$. To verify this condition suppose that $B \in \mathcal{E}(M)_{ns}$ with $B \leq A$ and $B \leq A'$. Then $B \leq \frac{A + A'}{2} = \frac{1}{2} I$. Since $\left(\frac{1}{2} I\right)' = \frac{1}{2} I$, it follows that $\frac{1}{2} I \leq B'$ so that $N_{B'} = 0$. Since $B \in \mathcal{E}(M)_{ns}$, by Theorem 5, $P_{BB'} \preceq N_{B'}$ so that $P_{BB'} = 0$. Hence, $BB' = 0$. Since $B' \geq \frac{1}{2} I$ we conclude that $B = 0$ and it follows that $A \wedge A' = 0$. ■

3 Effects on $B(H)$

Let H be a separable Hilbert space and let M be the factor $B(H)$ consisting of all bounded linear operators on H. As in Section 1, we use the notation $\mathcal{E}(H)$, $\mathcal{P}(H)$ for $\mathcal{E}(M)$, $\mathcal{P}(M)$ respectively. For $P \in \mathcal{P}(H)$ we define

$$[0,P] = \{A \in \mathcal{E}(H) : 0 \leq A \leq P\}.$$

Theorem 10 *1. If $P \in \mathcal{P}(H)$ with $\dim(P') = \infty$, then*

$$[0,P] = \{P_1QP_1 : Q, P_1 \in \mathcal{P}(H), P_1 \leq P\}.$$

2. If $P \in \mathcal{P}(H)$ with $\dim(P) = \dim(P') = \infty$, then $A \in \mathcal{E}(H)$ satisfies $AP = PA$ if and only if $A = P_1QP_1 + P_2RP_2$ with $P_1, P_2, Q, R \in \mathcal{P}(H)$ and $P_1 \leq P$, $P_2 \leq P'$.

Proof. (1) If $A = P_1QP_1$, then $0 \leq A \leq P_1 \leq P$ so $A \in [0,P]$. Conversely, if $0 \leq A \leq P$, then $P_A \leq P$ and hence $N_A \geq P'$. Thus $\dim(N_A) = \infty$ and the result follows from Corollary 6.

(2) If $A = P_1QP_1 + P_2RP_2$ with the given properties, it is clear that $AP = PA$. Conversely, suppose that $AP = PA$. Then we can write $A = PAP + P'AP'$. Since $PAP \in [0,P]$ and $P'AP' \in [0,P']$ the result follows from part (1). ∎

A projection P is an example of a simple *superselection rule* and an $A \in \mathcal{E}(H)$ satisfying $AP = PA$ is said to satisfy the superselection rule P. Part 2 of Theorem 10 states that if $\dim(P) = \dim(P') = \infty$, then $A \in \mathcal{E}(H)$ satisfies the superselection rule P if and only if A is the sum of two nearly sharp elements, $A = A_1 + A_2$, where A_1 is contained in the *superselection sector* $[0,P]$ and A_2 is contained in the superselection sector $[0,P']$.

Suppose $P \in \mathcal{P}(H)$ with $\dim(P) = \dim(P') = \infty$. If $A \in [0,P]$ then by Part 1 of Theorem 10, A is almost sharp. If $A, B \in [0,P]$ and $A + B \in \mathcal{E}(H)$, then $A + B \in [0,P]$. It follows that $[0,P]$ is an *effect algebra* with *unit* P (see [1] and [2]). If $A \in [0,P]$, then $\lambda A \in [0,P]$ for every $\lambda \in [0,1]$ and it follows that $[0,P]$ is a *convex* effect algebra [1]. Finally, if $A, B \in [0,P]$ then $A \circ B \leq A \leq P$ where $A \circ B = A^{1/2}BA^{1/2}$. Hence, $A \circ B \in [0,P]$ and we conclude that $[0,P]$ is a *sequential effect algebra* [2] of almost sharp effects. Notice however, that if $A \in [0,P]$ then $A = P_AQP_A$ where $P_A \in [0,P]$ but $Q \notin [0,P]$ in general.

We now consider the question of when $A \in \mathcal{E}(H)$ has the form

$$A = P_1P_2 \cdots P_nQP_n \cdots P_2P_1 \tag{3.1}$$

for $P_i, Q \in \mathcal{P}(H)$ for $i = 1, \ldots n$. To answer this question we shall need some preliminary results. If $\dim(N_A) = \infty$, then by Corollary 6, A is almost sharp so that A certainly has the form (3.1). We therefore assume that $\dim(N_A) < \infty$.

Lemma 11 *1. Suppose that $A, B \in \mathcal{E}(H)$ satisfy $A = P_ABP_A$, $\dim(N_A) < \infty$, and $\dim(P_{AA'}) = \infty$. Then $\dim(N_B) < \infty$ and $\dim(P_{BB'}) = \infty$.*

2. *If $A \in \mathcal{E}(H)$ satisfies $\dim(N_A) < \infty$ and $\dim(P_{AA'}) = \infty$, then A does not have the form (3.1).*

Proof. (1) We first show that $\dim(N_B) \leq \dim(N_A)$. If this is not true, then there exists a nonzero vector $x \in \ker(B) \ominus \ker(A)$. But then $P_A x = x$ and we have that $Ax = P_A B P_A x = P_A B x = 0$. Hence $x \in \ker(A)$ which is a contradiction. We now show that $\dim(P_{BB'}) = \infty$. If $\dim(P_{BB'}) < \infty$, then by (2.1) we have that

$$\dim(P_{B'}) = \dim(I - N_{B'}) = \dim(N_B + P_{BB'}) < \infty.$$

Hence, B' is a finite rank operator and it follows that $B = I - B'$ is the identity plus a finite rank map. Since $P_A = I - N_A$ and N_A has finite rank, we conclude that $A = P_A B P_A$ is also the identity plus a finite rank operator. But this contradicts the fact that $\dim(P_{AA'}) = \infty$.

(2) Suppose on the contrary that A has the form (3.1) for some integer n. For each $i = 1, 2, \ldots, n-1$, let

$$B_i = P_{i+1} P_{i+2} \cdots P_n Q P_n \ldots P_{i+2} P_{i+1}.$$

Then $A = P_1 B P_1$, $B_i = P_{i+1} B_{i+1} P_{i+1}$ for $i = 1, \ldots, n-2$, and $B_{n-1} = P_n Q P_n$. By Corollary 6

$$\dim\left(P_{B_{n-1}B'_{n-1}}\right) \leq \dim\left(N_{B_{n-1}}\right).$$

On the other hand, by successive applications of part 1 of this Theorem we deduce that $\dim\left(N_{B_{n-1}}\right) < \infty$ and $\dim\left(P_{B_{n-1}B'_{n-1}}\right) = \infty$. ■

For the remaining case, assume that $\dim(N_A) = k < \infty$ and $\dim(P_{AA'}) = m < \infty$.

Lemma 12 *Suppose that $A, B \in \mathcal{E}(H)$, that $\dim(N_A) = k < \infty$, $\dim(P_{AA'}) = m > k$ and $A = P_A B P_A$. Then $\dim(N_B) \leq k$ and $\dim(P_{BB'}) \geq m - k$. Moreover, there exists a $B \in \mathcal{E}(H)$ such that $A = P_A B P_A$, $\dim(N_B) = k$, and $\dim(P_{BB'}) = m - k$.*

Proof. That $\dim(N_B) \leq k$ follows from part 1 of Lemma 11. To show that $\dim(P_{BB'}) \geq m - k$, define

$$\begin{aligned} \widetilde{A} &= (N_A + P_{AA'}) A (N_A + P_{AA'}) = P_{AA'} A P_{AA'} \\ \widetilde{B} &= (N_A + P_{AA'}) B (N_A + P_{AA'}), \end{aligned}$$

and think of them as being defined in the range of $N_A + P_{AA'}$ which is a $k + m$ dimensional Hilbert space. It is easy to check that $P_{\widetilde{A}} = P_{AA'}$ and that $P_{\widetilde{A}} \widetilde{B} P_{\widetilde{A}} = P_{AA'} B P_{AA'} = P_{AA'} A P_{AA'} = \widetilde{A}$. The eigenvalues of $\widetilde{A}$ are

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0,$$

where 0 has multiplicity k and $1 > \lambda_1$. The eigenvalues of $\widetilde{B}$ are

$$s_1 \geq \cdots \geq s_k \geq s_{k+1} \geq \cdots \geq s_m \geq s_{m+1} \geq \cdots \geq s_{m+k},$$

where $1 \geq s_1$ and $s_{m+k} \geq 0$. Since $\widetilde{A}$ is the compression of $\widetilde{B}$ to a k-codimensional space, it follows from Cauchy's Interlacing Theorem [4] that

$$s_j \geq \lambda_j \geq s_{j+k} \qquad \text{for } j = 1, \ldots, m.$$

We then obtain that $0 < \lambda_m \leq s_m$ and $s_{k+1} \leq \lambda_1 < 1$. This implies that $1 > s_{k+1} \geq \cdots \geq s_m > 0$ and it follows that $\dim \left(P_{\widetilde{B}\widetilde{B}'}\right) \geq m - k$. To see that $\dim \left(P_{BB'}\right) = \dim \left(P_{\widetilde{B}\widetilde{B}'}\right)$, consider $\widetilde{A}$ and $\widetilde{B}$ as operators on $N_A + P_{AA'}$ and represent A and B by the block matrices

$$A = \begin{pmatrix} I_{N_{A'}} & 0 \\ 0 & \widetilde{A} \end{pmatrix} \text{ and } B = \begin{pmatrix} I_{N_{A'}} & 0 \\ 0 & \widetilde{B} \end{pmatrix}.$$

Then notice that

$$BB' = \begin{pmatrix} I_{N_{A'}} & 0 \\ 0 & \widetilde{B} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{B}' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{B}\widetilde{B}' \end{pmatrix}.$$

To check the second assertion, consider the effect defined on $P_{AA'} + N_A$

$$\widetilde{B} = \begin{pmatrix} \lambda_1 & \cdots & 0 & \cdots & 0 & \sqrt{\lambda_1 (1-\lambda_1)} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k & \cdots & 0 & 0 & \cdots & \sqrt{\lambda_k (1-\lambda_k)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & \lambda_m & 0 & \cdots & 0 \\ \sqrt{\lambda_1 (1-\lambda_1)} & \cdots & 0 & \cdots & 0 & 1-\lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\lambda_k (1-\lambda_k)} & \cdots & 0 & 0 & \cdots & 1-\lambda_k \end{pmatrix}.$$

It is easy to check that $\widetilde{A} = P_{\widetilde{A}} \widetilde{B} P_{\widetilde{A}}$. Since $\begin{pmatrix} \lambda_i & \sqrt{\lambda_i (1-\lambda_i)} \\ \sqrt{\lambda_i (1-\lambda_i)} & I - \lambda_i \end{pmatrix}$ is unitarily equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for every $i \leq k$, it follows that $\widetilde{B}$ has eigenvalues

$$0, \ldots 0, \lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_m, 1, \ldots 1,$$

where the multiplicity of 0 and 1 are each k. Since $A = N_{A'} + \widetilde{A}$ we define $B = N_{A'} + \widetilde{B}$ and we are finished. ■

Lemma 13 *Suppose that* $\dim(N_A) = k > 0$, $\dim(P_{AA'}) = m$ *and* A *has the form (3.1). Then* $n \geq \frac{m}{k}$.

Proof. The case $n = 1$ follows from Corollary 6. Assume that the result is true for $n-1$ and suppose that A has the form (3.1). Then

$$B = P_2 \cdots P_n Q P_n \cdots P_2$$

satisfies the hypothesis and we conclude that

$$n - 1 \geq \frac{\dim(P_{BB'})}{\dim(N_B)}.$$

By Lemma 12 we have that $\dim(N_B) \leq k$ and $\dim(P_{BB'}) \geq m - k$. Hence, $n - 1 \geq \frac{m-k}{k}$ so that $n \geq \frac{m}{k}$. ■

Lemma 14 *Suppose that $A \in \mathcal{E}(H)$ with $\dim(N_A) = k > 0$ and $\dim(P_{AA'}) = m$. Then for any $n \geq \frac{m}{k}$, A has a representation of the form* (3.1).

Proof. If $m \leq k$, by Corollary 6 we can write $A = PQP$ and we are finished. We now assume that $m > k$. By Lemma 12 we can find $B_1 \in \mathcal{E}(H)$ such that $A = P_A B_1 P_A$, $\dim(N_{B_1}) = k$ and $\dim\left(P_{B_1 B_1'}\right) = m - k$. If $m - k \leq k$, by Corollary 6 we can write $B_1 = PQP$ which implies that

$$A = P_A PQP P_A.$$

In this case $\frac{m}{k} \leq 2$ and we again are finished. If on the other hand, $m - k > k$, by Lemma 12 we find $B_2 \in \mathcal{E}(H)$ such that $B_1 = P_{B_1} B_2 P_{B_1}$, $\dim(N_{B_2}) = k$ and $\dim\left(P_{B_2 B_2'}\right) = m - 2k$. If $m - 2k \leq k$ we are again finished, otherwise there exists $B_3 \in \mathcal{E}(H)$ such that $B_2 = P_{B_2} B_3 P_{B_2}$, $\dim(N_{B_3}) = k$ and $\dim\left(P_{B_3 B_3'}\right) = m - 3k$. Proceeding this way, we can find n such that $m - (n-1)k \leq k$ and we prove the result. ■

Applying these lemmas, we obtain the following

Theorem 15 *Let $A \in \mathcal{E}(H)$ with H separable and $A \neq I$. Then A has the form (3.1) if and only if $\dim(N_A) = \infty$ or $0 < \dim(N_A) < \infty$ and $\dim(P_{AA'}) < \infty$.*

For $A \in \mathcal{E}(H)$ we define the *fuzzy index* $f(A)$ to be the smallest integer n so that A has the form (3.1). Then A is sharp if and only if $f(A) = 0$ and A is almost sharp if and only if $f(A)$ is 0 or 1. By convention, if no such n exists then $f(A) = \infty$. For m, k positive integers we denote the smallest integer greater than or equal $\frac{m}{k}$ by $\left\lceil \frac{m}{k} \right\rceil$. We extend this definition to include values 0 and ∞ for m and k by defining

$$\left\lceil \frac{m}{k} \right\rceil = \begin{cases} \infty & \text{if} \quad k = 0 \text{ and } m \neq 0 \\ 0 & \text{if} \quad m = 0 \\ 1 & \text{if} \quad k = \infty \text{ and } m \neq 0 \\ \infty & \text{if} \quad m = \infty \text{ and } k \neq \infty \end{cases}.$$

The next result again follows from the previous lemmas

Theorem 16 *Let $A \in \mathcal{E}(H)$ with H separable. Then*

$$f(A) = \left\lceil \frac{\dim(N_A)}{\dim(P_{AA'})} \right\rceil.$$

4 Finite Dimensional Effects

We now show that we can obtain further insights and some simpler proofs for the set $\mathcal{E}(H)$ on a finite dimensional Hilbert space. Although finite dimensional Hilbert spaces may seem restrictive for quantum systems, there are important fields such as quantum computation and information theory that are based on such spaces [5].

Let $A \in \mathcal{E}(H)$ where $\dim(H) < \infty$. We define the following nonnegative integers

$$\begin{aligned} n_0(A) &= \dim(N_A), \\ n_1(A) &= \dim(N_{A'}), \text{ and} \\ n(A) &= \dim(H) - n_0(A) - n_1(A). \end{aligned}$$

Notice that $n_0(A)$ is the multiplicity of the eigenvalue 0, $n_1(A)$ is the multiplicity of the eigenvalue 1, and $n(A)$ is the number of eigenvalues λ with $0 < \lambda < 1$ including multiplicity.

Lemma 17 *If* $A \in \mathcal{E}(H)$ *with* $\dim(H) < \infty$, *then* $n(A) = \dim(P_{AA'}) = \mathit{rank}(P_A - A)$.

Proof. By diagonalizing A we can assume without loss of generality that A has the form

$$A = \mathit{diag}(\lambda_1, \ldots, \lambda_m, 1, \ldots, 1, 0, \ldots, 0), \tag{4.1}$$

where $0 < \lambda_i < 1$ for $i = 1, \ldots, m$. We then have that

$$\begin{aligned} AA' &= \mathit{diag}(\lambda_1(1-\lambda_1), \ldots, \lambda_m(1-\lambda_m), 0, \ldots, 0, 0, \ldots, 0), \text{ and} \\ P_A - A &= \mathit{diag}(1-\lambda_1, \ldots, 1-\lambda_m, 0, \ldots, 0, 0, \ldots, 0). \end{aligned}$$

It follows that $\dim(P_{AA'}) = \mathit{rank}(P_A - A) = n(A) = m$. ■

The next result follows from Corollary 6 and Lemma 17. However, we give a much shorter proof which relies on the fact that $\dim(H) < \infty$.

Theorem 18 *Let* $A \in \mathcal{E}(H)$ *with* $\dim(H) < \infty$.

1. *A is almost sharp if and only if* $n(A) \le n_0(A)$.
2. *A is nearly sharp if and only if* $n(A) \le \min\{n_0(A), n_1(A)\}$.

Proof. (1) Suppose $A = P_A Q P_A$ for $Q \in \mathcal{P}(H)$. Then by Lemma 17 we have that

$$\begin{aligned} n(A) &= \mathit{rank}(P_A - A) = \mathit{rank}(P_A(I-Q)P_A) \le \mathit{rank}(I-Q) \\ &= \dim(N_Q) \le \dim(N_{P_A Q P_A}) = \dim(N_A) = n_0(A). \end{aligned}$$

Conversely, suppose that $m = n(A) \leq n_0(A)$. Without loss of generality we can represent A as a diagonal matrix (4.1). Let

$$\Lambda = diag(\lambda_1, \ldots, \lambda_m) = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_m \end{pmatrix} \in M_m,$$

and represent A by the block matrix

$$A = \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & I_{n_1(A)} & 0 \\ 0 & 0 & 0_{n_0(A)} \end{pmatrix}.$$

Since $m \leq n_0(A)$, we can split the last block into a $m \times m$ block and an $l \times l$ block, where $l = n_0(A) - m$. Let Q be the block matrix

$$Q = \begin{pmatrix} \Lambda & 0 & \sqrt{\Lambda(I_m - \Lambda)} & 0 \\ 0 & I_{n_1(A)} & 0 & 0 \\ \sqrt{\Lambda(I_m - \Lambda)} & 0 & I_m - \Lambda & 0 \\ 0 & 0 & 0 & 0_l \end{pmatrix}.$$

It is easy to check that $P_A Q P_A = A$ and that $Q \in \mathcal{P}(H)$.

The proof of (2) follows directly from (1). ■

We now consider some examples in $\mathcal{E}(\mathbb{C}^3)$. Let $0 < \lambda_i < 1$ for $i = 1, 2, 3$, and consider the following effects

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \ B = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } E = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then A and B are not almost sharp, D is almost sharp but it is not nearly sharp and E is nearly sharp. The fuzzy indexes are $f(A) = \infty$, $f(B) = 2$, $f(D) = f(E) = 1$. The decomposition $E = P_E Q P_E$ has the following form

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \sqrt{\lambda_1(1-\lambda_1)} \\ 0 & 1 & 0 \\ \sqrt{\lambda_1(1-\lambda_1)} & 0 & 1-\lambda_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Although E is almost sharp, λE for $0 < \lambda < 1$ is not almost sharp. Our last result follows from Theorem 18.

Theorem 19 *Let* $A \in \mathcal{E}(H)$, $A \neq I$, *with* $\dim(H) < \infty$. *Then* A *has the form (3.1) if and only if* A *is singular.*

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