FUZZY QUANTUM PROBABILITY THEORY

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Abstract

This paper surveys some of the recent results that have been obtained in fuzzy quantum probability theory. In this theory fuzzy events are represented by quantum effects and fuzzy random variables are represented by quantum measurements. Probabilities, conditional probabilities and sequential products of effects are discussed. The relationship between the law of total probability and compatibility of measurements is treated. Results concerning independent effects are given. Finally, properties of the sequential product, almost sharp effects and nearly sharp effects are discussed.

1 Introduction

Quantum mechanics is intrinsically a probabilistic theory because we cannot predict the outcome of a measurement exactly even if we know the state of the system precisely. All we can do is give a statistical distribution for the outcome values. The statistical nature of quantum mechanics even occurs for absolutely precise measurements. But such measurements do not exist in practice. All measurements are imprecise (unsharp, fuzzy). This imprecision causes another level of uncertainty which is explained and studied in fuzzy quantum probability theory.

In this paper we shall discuss how the concepts of traditional Hilbert space quantum mechanics must be altered to accommodate this new level of fuzziness. In traditional quantum mechanics, the events are represented by projection operators. The basic change in fuzzy quantum probability is that events are replaced by the more general effects which are positive operators bounded above by the identity operator. Measurements are now represented by effect-valued measures instead of projection-valued measures.

We begin with a discussion of probabilities and conditional probabilities of effects. The conditional probability motivates a definition of the sequential product of effects. This sequential product is crucial for the remaining studies of the paper. For example, the sequential product is employed to formulate a quantum law of total probability. The relationship between the law of total probability and compatibility of measurements is treated. We also use sequential products to define and discuss the independence of effects.

We next consider properties of the sequential product and introduce the concepts of almost sharp and nearly sharp effects. The almost sharp effects are characterized in both infinite dimensional and finite dimensional Hilbert spaces. We then define the fuzziness index of an effect and give a simple formula for its value. Finally, it is shown that the set of nearly sharp effects has the structure of an othocomplemented partially ordered set. In order to give the reader a flavor of the subject we present a few of the simpler proofs. However, for most of the proofs we refer to the literature.

2 Notation and Definitions

Let H be a complex Hilbert space that represents the state space of a quantum system $\mathcal{S}$. The set of **effects** $\mathcal{E}(H)$ for $\mathcal{S}$ is the set of operators on H satisfying $0 \leq A \leq I$. More precisely, $A \in \mathcal{E}(H)$ if and only if $0 \leq \langle Ax, x\rangle \leq \langle x, x\rangle$ for all $x \in H$. Effects represent yes-no measurements that may be unsharp (imprecise, fuzzy). From a probabilistic viewpoint, effects can be thought of as fuzzy quantum events. It is interesting that many of the important classes of quantum operators are given by subsets of $\mathcal{E}(H)$. For example, **sharp** yes-no measurements are represented by the set of projection operators $\mathcal{P}(H) \subseteq \mathcal{E}(H)$. In traditional quantum mechanics projections correspond to (sharp) quantum events. A **state** for $\mathcal{S}$ is represented by an operator $W \in \mathcal{E}(H)$ satisfying $\mathrm{tr}(W) = 1$. We call W a **density operator** and denote the set of density operators by $\mathcal{D}(H)$. The density operators correspond to the probability measures of classical probability theory. The **pure states** of $\mathcal{S}$ are given by $\mathcal{D}(H)\cap\mathcal{P}(H)$.

A more general measurement (not just two-valued) is represented by a positive operator-valued (POV) or effect-valued measure $X\colon\mathcal{B}(\mathbb{R})\to\mathcal{E}(H)$. That is $X(\mathbb{R})=I$ and $X(\cup\Delta_i)=\sum X(\Delta_i)$ for $\Delta_i\cap\Delta_j=\emptyset$, $i\neq j$, and convergence of the summation is in the strong operator topology. In this case, $X(\Delta)$ represents the effect observed when the measurement X has a value in the Borel set Δ. A projection-valued (PV) measure $X\colon\mathcal{B}(\mathbb{R})\to\mathcal{P}(H)$ corresponds to a sharp measurement and by the spectral theorem gives a single self-adjoint operator called an **observable**. In this paper we shall mainly consider **discrete measurements** which are given by a countable set $\{A_i\}\subseteq\mathcal{E}(H)$ such that $\sum A_i=I$. Then A_i corresponds to the effect observed when the measurement results in outcome i.

For $A\in\mathcal{E}(H)$, the **negation** of A is defined by $A'=I-A$. Of course, $A\in\mathcal{E}(H)$ and if $P\in\mathcal{P}(H)$ then $P'\in\mathcal{P}(H)$. For $A,B\in\mathcal{E}(H)$, if $A+B\leq I$ we define $A\oplus B=A+B$. Then $A\oplus B$ is defined if and only if $A\leq B'$. Roughly speaking, $A\oplus B$ corresponds to a parallel measurement of A and B. The algebraic system $(\mathcal{E}(H),\oplus,0,I)$ forms an **effect algebra**. Effect algebras are important in foundational studies of quantum mechanics [4, 6, 7, 8].

An **orthoposet** is an algebraic system $(\mathcal{P},\leq,{}',0,1)$ where $(\mathcal{P},\leq,0,1)$ is a bounded partially ordered set and ${}'\colon\mathcal{P}\to\mathcal{P}$ is an **orthocomplementation** which satisfies $a''=a$, $a\leq b$ implies $b'\leq a'$ and $a\wedge a'=0$. An orthoposet $\mathcal{P}$ is **orthomodular** if $a\leq b'$ implies that $a\vee b$ exists and $a\leq b'$ implies that $b=a\vee(b\wedge a')$. It is well known that $(\mathcal{P}(H),\leq,{}',0,I)$ forms an orthomodular lattice [3].

It is instructive to notice the analogy between our previous concepts and those of a fuzzy set system $[0,1]^Y$. An effect corresponds to a fuzzy set in $[0,1]^Y$ and a projection corresponds to a subset of Y. Thus, $\mathcal{E}(H)$ corresponds to a noncommutative fuzzy set theory and $\mathcal{P}(H)$ corresponds to a noncommutative set theory. For $W\in\mathcal{D}(H)$ we can think of $(\mathcal{E}(H),W)$ as a noncommutative fuzzy probability space. A measurement in our framework is a noncommutative analog of a fuzzy random variable.

3 Probability and Conditional Probability

One of the main axioms of quantum mechanics says that the probability that the quantum event $P\in\mathcal{P}(H)$ occurs when the system $\mathcal{S}$ is in state

$W \in \mathcal{D}(H)$ is

$$p_W(P) = \mathrm{tr}(WP) \tag{3.1}$$

We generalize (3.1) by postulating that the probability that $A \in \mathcal{E}(H)$ is observed (has value yes) when the system is in state $W \in \mathcal{D}(H)$ is

$$p_W(A) = \mathrm{tr}(WA) \tag{3.2}$$

In particular, for a discrete measurement $X = \{A_i\}$, the probability that X results in outcome i is

$$p_W(X = i) = p_W(A_i) = \mathrm{tr}(WA_i) \tag{3.3}$$

Another axiom of quantum mechanics says that if the initial state of $\mathcal{S}$ is $W \in \mathcal{D}(H)$ and event $P \in \mathcal{P}(H)$ occurs then the final state is

$$PWP/\mathrm{tr}(WP) \tag{3.4}$$

assuming that $\mathrm{tr}(WP) \neq 0$. This axiom is called the **projection postulate**. Now the projection postulate is somewhat controversial and we shall take the viewpoint that it is valid for certain ideal measurements which, in the sequel, we assume are operative. We now give a heuristic motivation for (3.4) that will be useful when we generalize it to the fuzzy case.

Let us visualize $P \in \mathcal{P}(H)$ as a filter for particle beams and suppose the incoming beam is in the pure state $x \in H$, $\|x\| = 1$. After the beam impinges the filter P the outgoing beam is in the unnormalized state Px. Using Dirac notation, the incoming state is given by the density operator $W_{\mathrm{in}} = |x\rangle\langle x|$ and the unnormalized outgoing state becomes

$$|Px\rangle\langle Px| = P|x\rangle\langle x|P^* = P|x\rangle\langle x|P = PW_{\mathrm{in}}P \tag{3.5}$$

Moreover, the probability that a particle is transmitted becomes, by (3.1)

$$p_{W_{\mathrm{in}}}(P) = \mathrm{tr}(PW_{\mathrm{in}}P) \tag{3.6}$$

Since mixed states are convex combinations of pure states, we can assume that (3.5) and (3.6) hold for any $W_{\mathrm{in}} \in \mathcal{D}(H)$. Moreover, normalizing (3.5) gives

$$W_{\mathrm{out}} = \frac{PW_{\mathrm{in}}P}{\mathrm{tr}(W_{\mathrm{in}}P)}$$

which is (3.4).

Now suppose that $A \in \mathcal{E}(H)$ is an imprecise filter for particle beams and as before the incoming beam is in the pure state x. Since A is unsharp we cannot assume that after the beam impinges A the outgoing beam is in the unnormalized state Ax. However, we do assume that the outgoing beam is in the unnormalized state Cx where C is a positive operator depending on A. As before the unnormalized outgoing state becomes

$$|Cx\rangle\langle Cx| = C|x\rangle\langle x|C^* = C|x\rangle\langle x|C = CW_{\text{in}}C \tag{3.7}$$

and the probability that a particle is transmitted becomes

$$p_{W_{\text{in}}}(A) = \text{tr}(CW_{\text{in}}C) \tag{3.8}$$

Applying (3.2) and (3.8) gives

$$\text{tr}(W_{\text{in}}A) = \text{tr}(W_{\text{in}}C^2) \tag{3.9}$$

Since (3.9) holds for all $W_{\text{in}} \in \mathcal{D}(H)$ we conclude that $C^2 = A$ so $C = A^{1/2}$ the unique positive square root of A. Applying (3.7), the normalized outgoing state is

$$W_{\text{out}} = \frac{A^{1/2}W_{\text{in}}A^{1/2}}{\text{tr}(W_{\text{in}}A)}$$

Motivated by the previous heuristic argument we postulate that if the initial state is W and effect A is observed then the final state is

$$A^{1/2}WA^{1/2}/\text{tr}(AW) \tag{3.10}$$

For readers who object to this argument we may regard (3.10) as a postulate that only applies to certain ideal measuring apparatuses.

Now suppose a discrete measurement $X = \{A_i\}$, $A_i \in \mathcal{E}(H)$, is performed. If outcome i occurs then by (3.10) the post-measurement state is

$$\frac{A_i^{1/2}WA_i^{1/2}}{\text{tr}(A_iW)} = \frac{A_i^{1/2}WA_i^{1/2}}{p_W(A_i)} \tag{3.11}$$

If X is performed but the outcome is not observed, the post-measurement state becomes

$$W_{\text{after}} = \sum A_i^{1/2}WA_i^{1/2} \tag{3.12}$$

Notice that (3.12) is a consequence of (3.11) because W_{after} should be a convex combination of the states in (3.11) with coefficients $p_W(A_i)$. Observe that W_{after} is indeed a state because

$$\operatorname{tr}(W_{\text{after}}) = \operatorname{tr}\left(\sum A_i^{1/2} W A_i^{1/2}\right) = \operatorname{tr}\left(\sum A_i W\right) = \operatorname{tr}(W) = 1$$

For $A, B \in \mathcal{E}(H)$ the **conditional probability** that B is observed given that A is observed in the state W is by (3.10)

$$\begin{aligned} p_W(B \mid A) &= \frac{\operatorname{tr}(A^{1/2} W A^{1/2} B)}{\operatorname{tr}(WA)} = \frac{\operatorname{tr}(W A^{1/2} B A^{1/2})}{\operatorname{tr}(WA)} \\ &= \frac{p_W(A^{1/2} B A^{1/2})}{p_W(A)} \end{aligned}$$

We then have that

$$p_W(A^{1/2} B A^{1/2}) = p_W(A) p_W(B \mid A) \tag{3.13}$$

Equation (3.13) reminds us of the classical formula

$$p(A \cap B) = p(A) p(B \mid A)$$

so $A^{1/2} B A^{1/2}$ is analogous to an intersection $A \cap B$. We use the notation $A \circ B = A^{1/2} B A^{1/2}$ and unlike $A \cap B$ we see that $A \circ B \neq B \circ A$ in general. In fact, $A \circ B$ is the **sequential product** of A and B in which A is measured first and B is second. That $A \circ B \neq B \circ A$ in general is due to quantum interference. For example, let A and B be polarization filters in planes perpendicular to a photon beam, where A polarizes vertically and B at a 45° angle. If the incoming beam is prepared in a state of horizontal polarization, then $A \circ B$ transmits no particles, while $B \circ A$ will transmit particles so $A \circ B \neq B \circ A$. Notice that $A \circ B \in \mathcal{E}(H)$ because

$$\begin{aligned} 0 &\leq \left\langle B A^{1/2} x, A^{1/2} x \right\rangle = \langle A \circ B x, x \rangle \leq \left\langle A^{1/2} x, A^{1/2} x \right\rangle \\ &= \langle Ax, x \rangle \leq \langle x, x \rangle \end{aligned}$$

for all $x \in H$.

The operation $A \oplus B$ describes a parallel combination of A and B while $A \circ B = A^{1/2} B A^{1/2}$ describes a series combination in A is measured first and B second. We have seen that the probability law

$$p_W(A \circ B) = p_W(A) p_W(B \mid A)$$

holds for all $A, B \in \mathcal{E}(H)$, $W \in \mathcal{D}(H)$. Now let $X = \{A_i\}$, $A_i \in \mathcal{E}(H)$, be a discrete measurement. The law of total probability would say that for any $B \in \mathcal{E}(H)$

$$p_W(B) = \sum p_W(A_i) p_W(B \mid A_i) = \sum p_W(A_i \circ B)$$

If this law holds for all $W \in \mathcal{D}(H)$, then it follows that

$$B = \sum A_i \circ B = \sum A_i^{1/2} B A_i^{1/2} \tag{3.14}$$

We call (3.14) the **quantum law of total probability**. In general, this law (3.14) does not hold. If $AB = BA$ we say that A and B are **compatible**. Compatible effects have classical properties relative to each other. For example, if $BA_i = A_i B$ for all i, then (3.14) holds. It can be shown that (3.14) does not imply that B is compatible with all A_i [1]. Compatible effects are sometimes called simultaneously measurable. The next result gives sufficient conditions under which the quantum law of total probability implies compatibility [5, 10].

Theorem 3.1. *Equation (3.14) implies $BA_i = A_i B$ for all i if any of the following conditions hold:* (a) $i = 1, 2$. (b) $\dim H < \infty$. (c) *B has pure point spectrum that can be totally ordered in decreasing order.*

Equation (3.14) that $B = \sum A_i \circ B$ can also be interpreted as saying that B is not disturbed by the measurement $X = \{A_i\}$. This suggests an application to axiomatic quantum field theory. Suppose the measurement $X = \{A_i\}$ is performed in a bounded spacetime region Δ_1 and $B \in \mathcal{E}(H)$ is performed in another spacetime region Δ_2 that is spacelike separated from Δ_1. According to Einstein causality, the measurement in Δ_1 should not disturb B so $B = \sum A_i \circ B$. But as we said earlier this does not imply that $BA_i = A_i B$ for all i. Thus, the axiom of local commutativity does not follow from Einstein causality. This indicates that the axiom of local commutativity may be too strong and should be replaced by a weaker axiom.

4 Independence

Independence is one of the most important concepts in classical probability theory. However, it does not seem to be well developed in quantum probability theory. We say that $A, B \in \mathcal{E}(H)$ are **independent in the state**

$W \in \mathcal{D}(H)$ if $p_W(A \circ B) = p_W(A)p_W(B)$. By (3.13) this is equivalent to $p_W(B \mid A) = p_W(B)$ whenever $P_W(A) \neq 0$. We now give a simple example in which

$$p_W(A \circ B) = p_W(A)p_W(B) \neq p_W(B \circ A)$$

Recall that a pure state is a one-dimensional projection P_x onto the subspace spanned by a unit vector x. For the pure state $W = P_x$ we have that

$$p_W(A) = \operatorname{tr}(WA) = \langle Ax, x\rangle$$

Notice that $A, B \in \mathcal{E}(H)$ are independent in the pure state $W = P_x$ if

$$\langle A \circ Bx, x\rangle = \langle Ax, x\rangle\langle Bx, x\rangle$$

For our example, let $A, B \in \mathcal{P}(\mathbb{C}^2)$ be the projections

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and let $x = (1, 0) \in \mathbb{C}^2$. Then

$$A \circ B = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}A$$

and

$$B \circ A = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}B$$

Since $\langle Ax, x\rangle = 1$ and $\langle Bx, x\rangle = 1/2$ we have that

$$\langle A \circ Bx, x\rangle = \frac{1}{2}\langle Ax, x\rangle = \langle Ax, x\rangle\langle Bx, x\rangle$$

but

$$\langle B \circ Ax, x\rangle = \frac{1}{2}\langle Bx, x\rangle \neq \langle Bx, x\rangle\langle Ax, x\rangle$$

This example shows that unlike the classical case, independence is not a symmetric relation. However, at the intuitive level this is not completely surprising. For example, my heart is independent of my little finger but my little finger is not independent of my heart.

Recall that two bounded random variables f and g are **uncorrelated** relative to a probability measure μ if $\int fg\,d\mu = \int f\,d\mu \int g\,d\mu$. Of course, two independent random variables are uncorrelated. The next theorem is a classical result of probability theory.

Theorem 4.1. *Two bounded random variables f, g are uncorrelated relative to every probability measure on their sample space if and only if f or g is constant.*

We now generalize Theorem 4.1 to fuzzy quantum probability theory [10]

Theorem 4.2. *For $A, B \in \mathcal{E}(H)$ the following statements are equivalent.*
(a) *A and B are independent in every state $W \in \mathcal{D}(H)$.*
(b) *A and B are independent in every pure state in $\mathcal{D}(H)$.*
(c) *$A = cI$ or $B = cI$ for some $0 \leq c \leq 1$.*

We now give some examples of independent effects. Any two effects of the form $A \otimes I, I \otimes B \in \mathcal{E}(H \otimes H)$ are independent in a pure tensor state $x \otimes y$. If x is a unit eigenvector of $A \in \mathcal{E}(H)$ then A and B are independent in the pure state P_x for every $B \in \mathcal{E}(H)$. The next result generalizes this situation [10].

Theorem 4.3. *The following statements are equivalent.* (a) $p_W(A) = 1$. (b) $WA = AW = W$. (c) $p_W(A \circ B) = p_W(B)$ *for every* $B \in \mathcal{E}(H)$.

Notice that Theorem 4.3 says that if $p_W(A) = 1$, then A and B are independent in W for every $B \in \mathcal{E}(H)$.

5 Properties of Sequential Product

This section summarizes some of the properties of the sequential product. For proofs we refer the reader to [10].

Theorem 5.1. *Let $A, B \in \mathcal{E}(H)$.* (a) *If $A \circ B = B \circ A$ then $AB = BA$.* (b) *If $A \circ B \in \mathcal{P}(H)$ then $AB = BA$.*

Theorem 5.1(a) says that if the measurement order of A and B is irrelevant, then A and B are compatible. Of course, the converse is trivial. Theorem 5.1(b) says that even if A and B are both sharp, their sequential product is not sharp unless A and B are compatible.

To give the reader a flavor of the subject we now present the simple proof of Theorem 5.1(a). Suppose that $A \circ B = B \circ A$. We then have that

$$(A^{1/2}B^{1/2})(B^{1/2}A^{1/2}) = (B^{1/2}A^{1/2})(A^{1/2}B^{1/2})$$

This shows that the operators $M = A^{1/2}B^{1/2}$ and $N = B^{1/2}A^{1/2}$ are normal. Letting $T = A^{1/2}$ we conclude that $MT = TN$. It follows from Fuglede's theorem [11] that $M^*T = TN^*$. Hence, $B^{1/2}A = AB^{1/2}$ and we conclude that $BA = AB$.

It is clear that the distributive law

$$A \circ (B \oplus C) = A \circ B \oplus A \circ C$$

always holds. However, as we have seen in Section 4 in general we have that

$$(A \oplus B) \circ C \neq A \circ C \oplus B \circ C$$

The next result shows that the associative law need not hold.

Theorem 5.2. *For $A, B \in \mathcal{E}(H)$ we have that $A \circ (B \circ C) = (A \circ B) \circ C$ for every $C \in \mathcal{E}(H)$ if and only if $AB = BA$.*

The fact that $A \circ (B \circ C) \neq (A \circ B) \circ C$ in general is puzzling because physically one would expect the two sides to be equal. For example, suppose A, B and C represent beam filters. This would say that if we first apply A and then $B \circ C$ considered as a single filter then this would be different than first applying $A \circ B$ considered as a single filter and then applying C. Mathematically this can be written as

$$A^{1/2}B^{1/2}CB^{1/2}A^{1/2} \neq (A^{1/2}BA^{1/2})^{1/2}C(A^{1/2}BA^{1/2})^{1/2}$$

For a further discussion on this point we refer the reader to [10]. The next theorem gives even stronger results for other types of associativity.

Theorem 5.3. *For $A, B \in \mathcal{E}(H)$ the following statements are equivalent.* (a) *$A \circ (C \circ B) = (A \circ C) \circ B$ for every $C \in \mathcal{E}(H)$.* (b) *$C \circ (A \circ B) = (C \circ A) \circ B$ for every $C \in \mathcal{E}(H)$.* (c) *$A = cI$ or $B = cI$ for some $0 \leq c \leq 1$.*

The system $(\mathcal{E}(H), \oplus, \circ, 0, I)$ is a special case of a mathematical structure called a **sequential effect algebra**. This structure is important for foundational studies in quantum mechanics [9, 10]. Other examples of sequential effect algebras are boolean algebras and fuzzy set systems with their usual operations.

6 Almost Sharp Effects

We say that $A \in \mathcal{E}(H)$ is **almost sharp** if $A = PQP = P \circ Q$ for $P, Q \in \mathcal{P}(H)$. This terminology stems from the fact that A may be obtained by measuring two sharp effects. In a sense, almost sharp effects are "close" to being sharp and we shall present a simple characterization of such effects. It follows from Theorem 5.1(b) that the set of almost sharp effects is strictly larger than $\mathcal{P}(H)$. Our characterization holds for the case $\dim(H) < \infty$. Generalizations to $\dim(H) = \infty$ are possible but are a little more complicated to state. We refer the reader to [2] for generalizations and omitted proofs. We say that $A \in \mathcal{E}(H)$ is **nearly sharp** if A and A' are almost sharp.

For $A \in \mathcal{E}(H)$ with $\dim(H) < \infty$ define the following nonnegative integers:

$$n_0(A) = \text{multiplicity of the eigenvalue 0 for } A$$

$$n_1(A) = \text{multiplicity of the eigenvalue 1 for } A$$

$$n(A) = \dim(H) - n_0(A) - n_1(A)$$

Notice that $n(A)$ is the sum of the multiplicities of the eigenvalues λ for A satisfying $0 < \lambda < 1$. We call such eigenvalues **fuzzy eigenvalues**. The next theorem shows that A is almost (nearly) sharp if and only if the number of fuzzy eigenvalues is smaller than the number of sharp eigenvalues in a certain sense. For $A \in \mathcal{E}(H)$ we denote the projection onto the range of A by P_A.

Lemma 6.1. *If $A \in \mathcal{E}(H)$ with $\dim(H) < \infty$, then $n(A) = \dim(P_{AA'}) = \operatorname{rank}(P_A - A)$*

Proof. By diagonalizing A we can assume without loss of generality that A has the form

$$A = \operatorname{diag}(\lambda_1, \dots, \lambda_m, 1, \dots, 1, 0, \dots, 0) \tag{6.1}$$

where $0 < \lambda_i < 1$ for $i = 1, \dots, m$. We then have that

$$\begin{aligned} AA' &= \operatorname{diag}\left(\lambda_1(1-\lambda_1), \dots, \lambda_m(1-\lambda_m), 0, \dots, 0\right) \\ P_A - A &= \operatorname{diag}(1-\lambda_1, \dots, 1-\lambda_m, 0, \dots, 0) \end{aligned}$$

It follows that

$$\dim(P_{AA'}) = \operatorname{rank}(P_A - A) = n(A) = m$$

□

Theorem 6.2. *Let $A \in \mathcal{E}(H)$ with* $\dim(H) < \infty$ (a) *A is almost sharp if and only if $n(A) \leq n_0(A)$.* (b) *A is nearly sharp if and only if $n(A) \leq$* $\min\{n_0(A), n_1(A)\}$.

Proof. (a) Suppose that A is almost sharp so that $A = PQP$ for $P, Q \in \mathcal{P}(H)$. Since $A \leq P$ it follows that $P_A \leq P$. Since $AP_A = P_A A = A$ we conclude that $A = P_A Q P_A$. Letting N_B be the projection onto the nullspace of $B \in \mathcal{E}(H)$, by Lemma 6.1 we have

$$\begin{aligned} n(A) &= \operatorname{rank}(P_A - A) = \operatorname{rank}\left(P_A(I - Q)P_A\right) \leq \operatorname{rank}(I - Q) \\ &= \dim(N_Q) \leq \dim(N_{P_A Q P_A}) = \dim(N_A) = n_0(A) \end{aligned}$$

Conversely, suppose that $m = n(A) \leq n_0(A)$. Without loss of generality we can represent A as a diagonal matrix (6.1). Let

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$$

and represent A by the block matrix

$$A = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & I_{n_1(A)} & 0 \\ 0 & 0 & 0_{n_0(A)} \end{bmatrix}$$

Since $M \leq n_0(A)$ we can split the last block into an $m \times m$ block and an $\ell \times \ell$ block where $\ell = n_0(A) - m$. Let Q be the block matrix

$$A = \begin{bmatrix} \Lambda & 0 & \sqrt{\Lambda(I_m - \Lambda)} & 0 \\ 0 & I_{n_1(A)} & 0 & 0 \\ \sqrt{\Lambda(I_m - \Lambda)} & 0 & I_m - \Lambda & 0 \\ 0 & 0 & 0 & 0_\ell \end{bmatrix}$$

It is easy to check that $P_A Q P_A = A$ and that $Q \in \mathcal{P}(H)$.

The proof of (b) follows directly from (a). □

We now consider effects that are longer sequential products of sharp effects. The **fuzziness index** $f(A)$ of $A \in \mathcal{E}(H)$ is the smallest integer n such that A has the form

$$A = P_1 \cdots P_n Q P_n \cdots P_1$$

for $Q, P_i \in \mathcal{E}(H)$, $i = 1, \ldots, n$. Thus, A is sharp if and only if $f(A) = 0$ and A is almost sharp and not sharp if and only if $f(A) = 1$. If no such n exists, we write $f(A) = \infty$.

Theorem 6.3. *For $A \in \mathcal{E}(H)$ with $\dim(H) < \infty$ we have that*

$$f(A) = \left\lceil \frac{n(A)}{n_0(A)} \right\rceil$$

We now give some examples in the Hilbert space $H = \mathbb{C}^3$. In $\mathcal{E}(\mathbb{C}^3)$ define $A = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, $B = \operatorname{diag}(\lambda_1, \lambda_2, 0)$, $C = \operatorname{diag}(\lambda_1, 1, 0)$, $D = \operatorname{diag}(\lambda_1, 0, 0)$ where $0 < \lambda_i < 1$. $i = 1, 2, 3$. We have that $f(A) = \infty$ $f(B) = 2$, $f(C) = f(D) = 1$. Notice that C is nearly sharp and D is almost sharp but not nearly sharp. We now write $C = PQP$, $P, Q \in \mathcal{P}(H)$.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \sqrt{\lambda_1(1-\lambda_1)} \\ 0 & 1 & 0 \\ \sqrt{\lambda_1(1-\lambda_1)} & 0 & 1-\lambda_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

7 Structure of Nearly Sharp Effects

The sets of almost sharp effects and nearly sharp effects have very weak structures. For example, $C = \operatorname{diag}(\lambda_1, 1, 0)$ is nearly sharp, but λC is not even almost sharp for $0 < \lambda < 1$. Also, if A and B are nearly sharp and $A + B \in \mathcal{E}(H)$, then $A + B$ may not be almost sharp. However, if A is almost sharp, then A^n is almost sharp for all $n \in \mathbb{N}$. The next result gives the structure of the set $\mathcal{E}(H)_{ns}$ of nearly sharp elements in $\mathcal{E}(H)$ when $\dim(H) < \infty$. This result also holds in the von Neumann algebra setting [2].

Theorem 7.1. *If $\dim(H) < \infty$, then $(\mathcal{E}(H)_{ns}, \leq, ', 0, I)$ is an orthoposet.*

Proof. All the properties of an orthoposet are clear except for the condition $A \wedge A' = 0$ for every $A \in \mathcal{E}(H)_{ns}$. To verify this condition suppose that $B \in \mathcal{E}(H)_{ns}$ with $B \leq A$ and $B \leq A'$. Then

$$B \leq \frac{A + A'}{2} = \frac{1}{2} I$$

Since $(I/2)' = I/2$, it follows that $I/2 \leq B'$ so that $N_{B'} = 0$. Hence, $n_0(B') = 0$ and by Lemma 6.1 and Theorem 6.2 we conclude that

$$\dim(P_{BB'}) = n(B') = 0$$

Hence, $BB' = 0$ so that $B \in \mathcal{P}(H)$. Since $B \leq I/2$, we have that $B = 0$. It follows that $A \wedge A' = 0$. □

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