

TOPOLOGICAL REALIZATIONS OF FAMILIES OF ERGODIC AUTOMORPHISMS, MULTITOWERS AND ORBIT EQUIVALENCE

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ABSTRACT. We study minimal topological realizations of families of ergodic measure preserving automorphisms (e.m.p.a.'s). Our main result is the following theorem.

Theorem. Let $\{T_p : p \in \mathcal{I}\}$, be arbitrary finite or countable collection of e.m.p.a.'s on nonatomic Lebesgue probability spaces (Y_p, ν_p) . Let S be a Cantor minimal system such that the cardinality of the set \mathcal{E}_S of all ergodic S -invariant Borel probability measures is at least the cardinality of \mathcal{I} . Then for any collection $\{\mu_p : p \in \mathcal{I}\}$, of distinct measures from \mathcal{E}_S there is a Cantor minimal system S' in the topological orbit equivalence class of S such that, as a measure preserving system, (S', μ_p) is isomorphic to T_p for every $p \in \mathcal{I}$. Moreover, S' can be chosen strongly orbit equivalent to S if and only if all finite topological factors of S are measure-theoretic factors of T_p for all $p \in \mathcal{I}$.

This result shows, in particular, that there are no restrictions at all for the topological realizations of countable families of e.m.p.a.'s in Cantor minimal systems.

Furthermore, since realizations are taking place within orbital classes of a given Cantor minimal system, our results generalize the strong orbit realization theorem and the orbit realization theorem of [21]. Those theorems are now special cases of our result where the collections $\{T_p\}$, $\{\mu_p\}$ consist of just one element each.

1. INTRODUCTION

The problem of constructing topological models for ergodic measure preserving transformations has been studied for quite a long time. For a single transformation T of a Lebesgue probability space the natural class of topological systems in which one should try to find a model for T is the class of uniquely ergodic homeomorphisms of compact metric spaces. The famous Jewett-Krieger theorem ([17], [18]) states that an *arbitrary* ergodic automorphism of a nonatomic Lebesgue space has a topological model in the class of Cantor minimal and uniquely ergodic systems (strictly ergodic homeomorphisms of a Cantor set).

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More recently, a number of results have been obtained concerning a more general problem of topological realizations of *families* of ergodic automorphisms within a single topological system (say, within a minimal, but not uniquely ergodic Cantor system S) with respect to its different invariant measures. More precisely, given a family $\{T_\alpha : \alpha \in \mathcal{I}\}$ of e.m.p.a.'s, one wants to decide whether or not there is a Cantor minimal system S with a collection of ergodic invariant measures $\{\mu_\alpha : \alpha \in \mathcal{I}\}$ in 1-to-1 correspondence with $\{T_\alpha\}$ such that measure-theoretically the system (S, μ_α) is isomorphic to T_α for every $\alpha \in \mathcal{I}$.

A result of T. Downarowicz [5] shows how dramatically non-uniquely ergodic such systems S can be. Namely, an arbitrary Choquet metrizable simplex can be considered, up to affine homeomorphism, the simplex of all Borel probability invariant measures (in the weak*-topology) of a certain Cantor minimal system. B. Weiss [24] gave an example of a *universal minimal system*, i.e., a minimal homeomorphism $S : X \rightarrow X$ of a Cantor set X such that given any aperiodic automorphism T of a Lebesgue probability space (Y, ν) , there exists a S -invariant Borel probability measure μ such that (X, S, μ) and (Y, T, ν) are measurably isomorphic. We also mention the result of T. Downarowicz and J. Serafin [9], which characterized the class of functions that can occur as the entropy functions defined on the space of all Borel probability invariant measures of a topological (Cantor) dynamical system. In the papers of Downarowicz and Lacroix [8] and Downarowicz and Durand [7] the results on minimal realizations of families of e.m.p.a.'s have been obtained under some restrictions (existence of certain nontrivial common factors).

All these results, however, left open even the following natural question, which was one of the motivations for us to start this work. Given a family of two e.m.p.a.'s (or more concretely, of two irrational rotations of a circle) is it possible to realize it in a Cantor minimal system with exactly two ergodic invariant probability measures? We show here that the answer is positive even for arbitrary finite or countable families of e.m.p.a.'s. In particular, we show the following.

Theorem 1. *Let $\{T_p : p \in \mathcal{I}\}$, be arbitrary finite or countable collection of e.m.p.a.'s on nonatomic Lebesgue probability spaces (Y_p, ν_p) . Let S be a Cantor minimal system such that the cardinality of the set \mathcal{E}_S of all ergodic S -invariant Borel probability measures is at least the cardinality of \mathcal{I} . Then for any collection $\{\mu_p : p \in \mathcal{I}\}$, of distinct measures from \mathcal{E}_S there is a Cantor minimal system S' with the same orbits as S such*

that, as a measure preserving system, (S', μ_p) is isomorphic to T_p for every $p \in \mathcal{I}$.

Moreover, S' can be chosen strongly orbit equivalent to S (by the identity map) if and only if all finite topological factors of S are measure-theoretic factors of T_p for all $p \in \mathcal{I}$.

Recently, after a preliminary version of this paper was written, we received a preprint [6] of T. Downarowicz, whose results overlap with ours. For arbitrary finite (but not countably infinite) families of e.m.p.a.'s, the positive answer to the realization problem follows also from [6]. The method of [6] is very different from ours, and the problem is considered not in the orbit theory setting. Roughly speaking, it is shown in [6] that the condition of minimality of S imposes only the same (and no additional) restrictions on realizability of arbitrary (finite, countable, or uncountable) families of e.m.p.a.'s (or endomorphisms) as the condition of aperiodicity of S . This reduction of “minimal realizability” to “aperiodic realizability” can be applied to finite families where (due to, say, Jewett-Krieger theorem) the restrictions for the aperiodic realizability are void, but cannot be applied to infinite families, for which the precise restrictions for aperiodic realizability are unknown and, as was noted in [6], seem somewhat mysterious.

When dealing with realizations of an (infinitely) countable family $\{T_p\}$ of e.m.p.a.'s, it is natural to expect *a priori* that the topology on the space of ergodic invariant measures for a homeomorphism S may impose additional restrictions for realizability (minimal, aperiodic, etc). Therefore, the fact that there are no restrictions at all in the countable case looks, at least for us, a little bit counterintuitive, even modulo the positive result in the finite case. It should be noted, however, that a supporting argument for the positive countable result comes from [9], where the necessary entropy function condition for realizability of arbitrary families is established, and this necessary condition turns out to be void in the countable case. Based on that, the question whether or not there are any restrictions which are non-void in the countable case was asked in [6]. Specifically, Downarowicz conjectures that there are no restrictions in the countable case, and our work proves this conjecture.

In the present paper the topological models for the collections of e.m.p.a.'s will be constructed within the topological orbit equivalence classes of given homeomorphisms. In this connection, we need to mention some facts on measurable and topological orbit equivalence. The first fundamental result here is the theorem of H. Dye (see [10],

[11]) saying that any two e.m.p.a.'s of nonatomic Lebesgue probability spaces are orbit equivalent. The next striking step was done by W.Krieger [19], where a deep connection between the (measurable) orbit equivalence and the isomorphism of von Neumann algebras was established. This circle of ideas was, in a sense, finalized in the work of A. Connes [3], U. Haagerup [14] and A. Connes - J. Feldman - B. Weiss [4].

In the topological setting, the parallel theory of topological orbit equivalence, with C^* -algebras playing the role similar to the role of von Neumann algebras in the measurable setting, was initiated much later. It started in the work of A. Vershik [22] [23] on adic realizations of e.m.p.a.'s, and then was developed in the series of remarkable results of T. Giordano, R. Herman, I. Putnam, C. Skau, E. Glasner and B. Weiss [13], [16], [15]. In the relatively recent paper of N. Ormes [21], the orbit realization theorem (ORT) solves in a positive way the problem of constructing the topological models of an individual e.m.p.a. within the topological orbit equivalence class of a given Cantor minimal system. Both the Jewett-Krieger theorem and Dye's theorem are special cases of the results of ORT. Also in [21] is the strong orbit realization theorem (SORT) which shows that except for a necessary condition on periodic factors, a realization as in ORT can take place within a *strong* orbit equivalence class. In a strong orbit equivalence, the related orbit cocycle functions are required to have at most one point of discontinuity each. Thus SORT stands in stark contrast to a result of Boyle which states that two minimal systems related by continuous cocycles are conjugate or conjugate to the inverse of the other [2]. The results here demonstrate even more dramatically how time changes which are discontinuous only at a single point can alter the measure-theoretic properties of a minimal homeomorphism.

From the orbital point of view, the main results of [21] are the starting point of our study, and the technique of [21] is an essential ingredient in our proofs. Because of this, the reader is strongly advised to be familiar with [21]. Our goal is to extend the results of [21] from the case of a single e.m.p.a. to the case of arbitrary countable families of transformations.

The main step in [21] is, roughly speaking, an inductive construction of a nested sequence of multitowers for an e.m.p.a. T and a compatible nested sequence for a given Cantor minimal system S with respect to some S -invariant ergodic measure μ . The floors of the multitowers for S are then rearranged so that the rearranged towers define a new homeomorphism S' , for which μ is still invariant, and such that S' considered as a μ -preserving system is measure-theoretically isomorphic to

T . This inductive construction of [21] is the prototype of the inductive construction in our paper, where, instead of a single measure μ , we are dealing with a family $\{\mu_p\}$ of measures.

The idea of the present paper (again, very roughly speaking) is the following. Given a finite family $\{T_1, T_2, \dots, T_m\}$ (instead of a single transformation T), and a Cantor minimal system S having at least m ergodic invariant Borel probability measures, we construct the nested multitowers, in a sense, simultaneously for all T_p 's, $1 \leq p \leq m$, and do the rearrangements of their floors in the same way for each T_p . In the case of a countable family $\{T_1, T_2, \dots, T_m, \dots\}$, at every step $(m+1)$ of the inductive construction we bring, along with the measures μ_1, \dots, μ_m considered at the previous m -th step, one more measure μ_{m+1} . This becomes possible because the mutual singularity of the distinct ergodic S -invariant measures μ_p 's implies that, for each m , the range of the vector measure (μ_1, \dots, μ_m) is the entire m -dimensional unit cube (this is a simple special case of the Lyapunov theorem on the range of the vector measure [20] though we do not need the full strength of that theorem). This, in turn, gives us enough flexibility to construct, on each step $(m+1)$ of the construction, the necessary multitowers in such a way that not only the previous measures μ_1, \dots, μ_m can be taken care of with "better accuracy" than in the previous step, but in addition, the measure μ_{m+1} can join them.

The organization of this paper is the following. In §2 we introduce some necessary notions and give basic definitions. In §3 we state several lemmas which reduce the proof of the main theorem to the construction of certain sequences of multitowers for a homeomorphism S and for e.m.p.a.'s T_p . This section also contains the proofs of all these lemmas except the central Lemma 6. The proof of Lemma 6 is given in §4. Finally, §5 contains some concluding remarks.

2. PRELIMINARIES

Now let us introduce some necessary definitions. For definitions that are not given here, see [21].

Let S be a minimal homeomorphism of the Cantor set X . Denote by \mathcal{E}_S the set of all ergodic S -invariant Borel probability measures.

Let $\{(Y_p, T_p, \mathcal{B}_p, \nu_p) : p \in \mathcal{I}\}$ be a countable collection of e.m.p.a.'s of nonatomic Lebesgue probability spaces $\{Y_p\}$. We will primarily work with the case where \mathcal{I} is countably infinite, as opposed to finite, and note when the situation is different (simpler) for the finite case. Let Y be the disjoint union of the Y_p 's. The set Y becomes a measurable space if we introduce a σ -algebra \mathcal{B} on it, namely the one generated

by all σ -algebras \mathcal{B}_p 's on Y_p 's. Each measure ν_p , $p \in \mathcal{I}$, can be pulled on Y as a measure concentrated on Y_p . With respect to each measure ν_p the space Y is a probability measure space. Let $T : Y \rightarrow Y$ be the transformation which is equal to T_p when restricted to Y_p . Then T is a measure preserving transformation of Y with respect to each ν_p . Let $\mathcal{F}_T = \{\nu_p : p \in \mathcal{I}\}$.

We will assume throughout that the cardinality of \mathcal{E}_S is greater than or equal to the cardinality of \mathcal{F}_T , a necessary condition for the realization of the family of transformations in the orbit equivalence class of S . Enumerate any subcollection $\{\mu_p : p \in \mathcal{I}\}$ of distinct elements of \mathcal{E}_S . Our goal is to find S' orbit equivalent to S with (S', μ_p) measurably conjugate to (T, ν_p) for all $p \in \mathcal{I}$.

A *clopen multitower* for S is a collection of disjoint clopen sets $\sigma = \{A(i, j) \mid 1 \leq i \leq I, 0 \leq j < H(i)\}$ for some positive integers $I, H(1), H(2), \dots, H(I)$ where $SA(i, j) = A(i, j + 1)$ for $0 \leq j < (H(i) - 1)$ and $\cup_{i=1}^I \cup_{j=0}^{H(i)-1} A(i, j) = X$. We use the notation $\sigma = \langle A(i, j); H(i); I \rangle$ to denote such a multitower. Define $\text{base}(\sigma) = \cup_{i=1}^I A(i, 0)$ and $\text{top}(\sigma) = \cup_{i=1}^I A(i, H(i) - 1)$. If Σ and σ are towers for S and every floor of σ is the union of floors of Σ we say that Σ *refines* σ . We say (σ, Σ) is a *nested pair*, or $\sigma \prec \Sigma$, if Σ refines σ and $\text{base}(\Sigma) \subseteq \text{base}(\sigma)$ (or equivalently, $\text{top}(\Sigma) \subseteq \text{top}(\sigma)$). A sequence of towers $(\sigma_1, \sigma_2, \dots)$ is called a *nested sequence* if $\sigma_m \prec \sigma_{m+1}$ for all m .

We will construct clopen multitowers for S , but also consider them measure-theoretically. For a clopen multitower σ for S and any $\mu \in \mathcal{M}_S$, we will consider the ‘‘architecture’’ of σ with respect to μ . That is, in addition to the number of towers I , and the heights of the towers $H(i)$, we will consider the measures of the floors $\mu(A(i, 0))$.

A *measurable multitower* $\tau = \langle B(i, j); H(i); I \rangle$ for T is defined analogously to a multitower for S , where the requirement that floors are clopen is replaced with the requirement that the floors are measurable (i.e., elements of \mathcal{B}), and the requirement that the union of floors be equal to X is replaced by the requirement that the union of floors is a set of full measure for every measure in \mathcal{F}_T .

Definition 2. Let $\sigma = \langle A(i, j); H(i); I \rangle$ be a clopen multitower for S and $\tau = \langle B(i, j); H'(i); I' \rangle$ a measurable multitower for T . Let $\mu \in \mathcal{E}_S$ and $\nu \in \mathcal{F}_T$. We say $(\sigma; \mu) \approx (\tau; \nu)$ if

- (1) $\nu \left[\cup_{i=1}^I \cup_{j=0}^{H'(i)-1} B(i, j) \right] = 1$
- (2) $H(i) = H'(i)$ for $1 \leq i \leq I$,
- (3) $\mu A(i, 0) = \nu B(i, 0)$ for $1 \leq i \leq I$.

Intuitively, the relation $(\sigma; \mu) \approx (\tau; \nu)$ means that σ and τ have the same architecture with respect to the chosen measures.

Definition 3. Let $(\sigma_1, \sigma_2, \dots, \sigma_M)$ be a finite nested sequence of clopen multitowers for S . Set $\sigma_m = \langle A_m(i, j); H_m(i); I_m \rangle$.

Let $(\tau_1, \tau_2, \dots, \tau_M)$ be a finite nested sequence of measurable multitowers for T . Set $\tau_m = \langle B_m(i, j); H'_m(i); I'_m \rangle$.

We say $(\sigma_1, \sigma_2, \dots, \sigma_M)$ and $(\tau_1, \tau_2, \dots, \tau_M)$ are compatible with respect to measures $\mu \in \mathcal{E}_T$ and $\nu \in \mathcal{F}_T$ if for all $1 \leq m \leq M$,

- (1) $(\sigma_m; \mu) \approx (\tau_m; \nu)$,
- (2) for all $1 \leq i \leq I_m$ and $1 \leq k \leq I_{m-1}$,
 $\#\{A_m(i, j) \mid A_m(i, j) \subseteq A_{m-1}(k, 0), 0 \leq j < H_m(i)\}$
 $= \#\{B_m(i, j) \mid B_m(i, j) \subseteq B_{m-1}(k, 0), 0 \leq j < H_m(i)\}$,
- (3) for all $1 \leq i \leq I_m$ and $1 \leq k \leq I_{m-1}$,
 $A_m(i, 0) \subseteq A_{m-1}(k, 0) \iff B_m(i, 0) \subseteq B_{m-1}(k, 0)$,
- (4) for all $1 \leq i \leq I_m$ and $1 \leq k \leq I_{m-1}$,
 $A_m(i, H_m(i) - 1) \subseteq A_{m-1}(k, H_{m-1}(k) - 1)$
 $\iff B_m(i, H_m(i) - 1) \subseteq B_{m-1}(k, H_{m-1}(k) - 1)$.

Notation 4. We will use the notation

$$(\sigma_1, \sigma_2, \dots, \sigma_M; \mu) \approx (\tau_1, \tau_2, \dots, \tau_M; \nu)$$

if the two sequences of towers are compatible with respect to measures μ and ν .

Note that conditions (2) - (4) in Definition 3 are void when $M = 1$, so there is no conflict between this notation and Definition 2.

The point of the definition of compatible multitowers is demonstrated in the following lemmas.

3. REDUCTION LEMMAS

The notion of compatibility allows us to recursively define a sequence of bijections of floors of towers.

3.1. Correspondence of Floors. Suppose $(\sigma_1, \sigma_2, \dots, \sigma_M)$ and $(\tau_1, \tau_2, \dots, \tau_M)$ are two finite nested sequences of multitowers for S and T which are compatible with respect to $\mu \in \mathcal{E}_T$ and $\nu \in \mathcal{F}_T$. We may then define a collection of bijections $\{h_m : 1 \leq m \leq M\}$ where h_m is a bijection from the set of floors of σ_m to the set of floors of τ_m intersected with a set of ν measure 1 in the following way.

Define $h_1(A_1(i, j)) = B_1(i, j)$.

Then for $m > 1$, we define $h_m(A_m(i, 0)) = B_m(i, 0)$. Condition (3) of compatibility insures

$$A_m(i, 0) \subset A_{m-1}(k, l) \implies h_m(A_m(i, 0)) \subset h_{m-1}(A_{m-1}(k, l)).$$

We define $h_m(A_m(i, j))$ recursively for $j > 0$. Having defined $h_m(A_m(i, \hat{j}))$, for $\hat{j} < j$, we define $h_m(A_m(i, j))$ to be the floor $B_m(i, j')$ where j' is the minimum floor height such that $B_m(i, j')$ is not already the image of a floor and $B_m(i, j') \subset h_{m-1}(A_{m-1}(k, l))$ where (k, l) is the index with $A_m(i, j) \subset A_{m-1}(k, l)$. Condition (2) insures that there are exactly the right number of floors of each type to make this assignment.

In particular, we have recursively defined a collection of bijections which

- (1) respects the subset relation, i.e., if $A_m(i, j) \subset A_{m-1}(k, l)$ then $h_m(A_m(i, j)) \subset h_{m-1}(A_{m-1}(k, l))$,
- (2) is measure-preserving, i.e., $\nu(h_m(A_m(i, j))) = \mu(A_m(i, j))$,
- (3) respects columns, i.e., for all i, j , $h_m(A_m(i, j)) = B_m(i, l)$ for some l
- (4) respects bases, i.e., for all i , $h_m(A_m(i, 0)) = B_m(i, 0)$,
- (5) respects tops, i.e., for all i , $h_m(A_m(i, H_m(i)-1)) = B_m(i, H_m(i)-1)$.

Furthermore, if $A_m(i, j)$ is a non-top floor of σ_m then $h_m(A_m(i, j))$ is a non-top floor of τ_m . Therefore, $h_m^{-1}Th_m(A_m(i, j)) = A_m(i, j')$ for some $0 \leq j' < H_m(i)$. Set $n_m(i, j) = j' - j$.

We have insured in our definition of h_m that if $A_m(i, j) \subset A_{m-1}(k, l)$ and $A_{m-1}(k, l)$ is a non-top floor, then $n_{m-1}(k, l) = n_m(i, j)$.

The point that the above maps h_m and n_m are recursively defined is more than a remark. It is important to our construction that we can fix all of the above for the first m multitowers then add another $(m+1)$ st multitower and extend all of the maps above.

As we see below, if we replace the finite sequences of towers with an infinite sequence of compatible multitowers satisfying separation properties, then we will be able to conclude that e.m.p.a.'s (S, μ) and (T, ν) are measurably conjugate.

Lemma 5 (Reduction 1). *Suppose there exists an infinite sequence of nested clopen multitowers $\sigma_1 \prec \sigma_2 \prec \dots$ for S , and an infinite sequence of nested measurable multitowers $\tau_1 \prec \tau_2 \prec \dots$ for T such that*

- (1) $(\sigma_1, \sigma_2, \dots, \sigma_m; \mu_p) \approx (\tau_1, \tau_2, \dots, \tau_m; \nu_p)$ for all $m, p \geq 1$,
- (2) $\bigcap_m \text{base}(\sigma_m)$ is one point set $\{x_0\}$ and $\bigvee_m \sigma_m$ generates the topology of X ,
- (3) the partitions $\{\tau_m\}_{m \in \mathbb{N}}$ separate points in Y on a set $Y_0 \subset Y$ with $\nu_p(Y_0) = 1$ for all p .

Then there is a minimal homeomorphism S' such that S and S' are strongly orbit equivalent by the identity map and for every $\mu_p \in \mathcal{E}_S$, (S', μ_p) is measurably conjugate to (T_p, ν_p) .

Proof. Given the above hypotheses, define the countable collection of bijections h_m as in Section 3.1. Since the sequence of clopen towers $\{\sigma_m\}$ separates points, every point $x \in X$ is uniquely identified according to the sequence of floors in $\{\sigma_m : m \geq 1\}$ which contain it.

Note that $\bigcap_m \text{top}(\sigma_m) = \{S^{-1}(x_0)\}$. If $x \neq S^{-1}(x_0)$, then there is a tower σ_m such that $x \notin \text{top}(\sigma_m)$. Let $A_m(i, j)$ denote the floor of σ_m containing x . Define $S'(x) = S^{n_m(i, j)}(x)$. Because the maps n_m respect subsets, the definition of $S'(x)$ does not depend on the choice of m . If $x = S^{-1}(x_0)$, set $S'(x) = S(x)$. The function $n(x)$ which satisfies $S'(x) = S^{n(x)}(x)$ has at most one point of discontinuity (namely, $S^{-1}(x_0)$) since $n(x)$ is constant on any (clopen) floor of σ_m which is not a subset of $\text{top}(\sigma_m)$. The same holds for the function $m(x)$ satisfying $S(x) = (S')^{m(x)}(x)$.

That S' is continuous on $X \setminus \{S^{-1}(x_0)\}$ follows from the fact that n is constant on clopen sets (floors of σ_m). That S' is continuous at $S^{-1}(x_0)$ follows from the fact that $S'(\text{top}(\sigma_m)) = \text{base}(\sigma_m) \ni x_0$ and $\bigcap_m \text{base}(\sigma_m) = \{x_0\}$. The minimality of S' follows from the fact that S and S' have the same orbits, and S is minimal. Therefore, S and S' are minimal homeomorphisms of the Cantor set, strongly orbit equivalent by the identity map.

Now fix $\mu_p \in \mathcal{E}_S$. Since the multitowers $\{\sigma_m\}$ separate points in X and $\{\tau_m\}$ separates points on a set of ν_p -measure 1, the sequence of maps $\{h_m\}$ can be extended to a measure-space isomorphism $h : (X, \mu_p) \rightarrow (Y, \nu_p)$. For every non-top floor $A(i, j) \in \sigma_m$, we have $S'(A(i, j)) = h^{-1}Th(A(i, j))$. We also have $\mu_p(\bigcap_m \text{top}(\sigma_m)) = 0$. Therefore the map h gives a measurable conjugacy between (S', μ_p) and (T_p, ν_p) . \square

Our task then is to construct multitowers as in the previous lemma. This will be done by an inductive process, which is essentially described in Lemma 6. This lemma, as well as Proposition 7 below, will be used in the proof of the realization theorem for (infinitely) countable families of e.m.p.a.'s. In the case of a finite family $\{T_p\}$ the inductive process is slightly different (simpler), since there is no need to bring an additional measure μ_{m+1} on the $(m+1)$ -st step, if $m+1$ is bigger than the number of the automorphisms in the family $\{T_p\}$. Because of this, some insignificant changes (replacement of $m+1$ by m) are needed in the statement of Lemma 6 in the finite case. We leave these details to the reader.

Lemma 6 (Reduction 2). *Suppose every finite (periodic) factor of S is a finite (periodic) factor of T_p for all $p \in \mathcal{I}$.*

Let $\epsilon > 0$. Let \mathcal{P} be a clopen partition of X , and let $x_0 \in X$. Let \mathcal{Q} be a finite partition of $\cup_{p=1}^{m+1} Y_p$ which is measurable with respect to ν_p for $1 \leq p \leq m+1$.

Suppose there exist a clopen multitower σ for S , and τ a measurable multitower for T such that $(\sigma; \mu_p) \approx (\tau; \nu_p)$ for every $1 \leq p \leq m$, and $x_0 \in \text{base}(\sigma)$. Then there exists a clopen multitower Σ for S and measurable multitowers τ' and \mathcal{T} for T such that

- (1) $(\sigma, \Sigma; \mu_p) \approx (\tau', \mathcal{T}; \nu_p)$ for $1 \leq p \leq m+1$,
- (2) $\|\tau - \tau'\|_{\nu_p} < \epsilon$ for $p = 1, 2, \dots, m$,
- (3) $x_0 \in \text{base}(\Sigma)$, $\text{diam}(\text{base}(\Sigma)) < \epsilon$,
- (4) Σ refines \mathcal{P} ,
- (5) \mathcal{T} refines \mathcal{Q} on a set E with $\nu_p(E) > 1 - \epsilon$ for $p = 1, 2, \dots, m+1$.

Before we turn our attention to the proof of the lemma above, let us indicate how the realization theorem follows from it.

Proposition 7. *Lemma 6 implies Theorem 1.*

Proof. Let S and T be as in the hypotheses of Theorem 1, and assume Lemma 6 is true.

First we explain that if our interest is orbit equivalence, as opposed to strong orbit equivalence, we may assume without loss of generality that S has no finite factors, and thus assume that S and T satisfy the hypotheses of Lemma 6. By Lemma 7.1 of [21], there is a unital ordered group $\mathcal{G} = (G, G_+, u)$ where (G, G_+) is an acyclic simple dimension group, $\mathcal{G}/\text{Inf}(\mathcal{G}) \cong \mathcal{G}^S/\text{Inf}(\mathcal{G}^S)$ and G has trivial rational subgroup. Further, by [16], there is a minimal homeomorphism \widehat{S} of the Cantor set with $\mathcal{G}^{\widehat{S}} \cong \mathcal{G}$. By [13], \widehat{S} and S are orbit equivalent. Since \widehat{S} and S are orbit equivalent, \mathcal{M}_S and $\mathcal{M}_{\widehat{S}}$ are affinely homeomorphic. Thus, by replacing S with \widehat{S} we have the hypotheses of both Theorem 1 and of Lemma 6. This replacement does not change the orbit equivalence class in which we are working.

For the strong orbit equivalence statement of Theorem 1, we note that the necessity of the condition on finite factors for S and T_p is already known (see [21], p. 110), so we only need to prove the sufficiency part.

Now let $\{\epsilon_m\}$ be a summable sequence of positive numbers. Let $\{\mathcal{P}_m\}_{m \in \mathbb{N}}$ be a refining sequence of (finite) clopen partitions in X which generate the topology. Let \mathcal{P}_0 and σ_0 be a trivial partition of X and multitower for S , respectively. Let $\{\mathcal{Q}_m\}_{m \in \mathbb{N}}$ be a refining sequence of finite measurable partitions of Y which separate points on a set of full measure in Y with respect to all $\nu \in \mathcal{F}_T$. Let \mathcal{Q}_0 and $\tau_0^{(0)}$ be a trivial partition of Y and multitower for T , respectively. Assume that

for some $M \geq 0$, we have a finite nested sequence of clopen multitowers $\sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_M$ for S and a finite nested sequence of multitowers $\tau_1^{(M)} \prec \tau_2^{(M)} \prec \cdots \prec \tau_M^{(M)}$ satisfying

- (1) $(\sigma_1, \sigma_2, \dots, \sigma_M; \mu_p) \approx (\tau_1^{(M)}, \tau_2^{(M)}, \dots, \tau_M^{(M)}; \nu_p)$ for $1 \leq p \leq M$,
- (2) $x_0 \in \bigcap_{p=1}^M \text{base}(\sigma_p)$, $\text{diam}(\bigcap_{p=1}^M \text{base}(\sigma_p)) < \epsilon_M$, σ_M refines \mathcal{P}_M ,
- (3) $\tau_M^{(M)}$ refines \mathcal{P}_M on a set E_M with $\nu_p(E_M) > 1 - \epsilon_M$.

Now apply Lemma 6 with $\epsilon = \epsilon_M$, $\mathcal{P} = \mathcal{P}_{M+1}$, $\mathcal{Q} = \mathcal{Q}_{M+1}$, $\sigma = \sigma_M$, $\tau = \tau_M^{(M)}$ and the collections of measures $\mu_1, \mu_2, \dots, \mu_{M+1}$ and $\nu_1, \nu_2, \dots, \nu_{M+1}$. Set $\sigma_{M+1} = \Sigma$, $\tau_{M+1}^{(M+1)} = \mathcal{T}$ and $\tau_M^{(M+1)} = \tau'$.

We define the multitowers $\tau_m^{(M+1)}$ for $0 \leq m < M$ in the following way. When intersected with $\bigcup_{p=1}^M Y_p$, every floor F of $\tau_{M-1}^{(M)}$ is a union over some index set of floors of $\tau_M^{(M)}$. We let F' be the union over the same index set of floors of $\tau_{M-1}^{(M+1)}$, but now intersected with $\bigcup_{p=1}^{M+1} Y_p$. Thus we obtain a new multitower $\tau_{M-1}^{(M+1)}$ for T where

$$(\tau_{M-1}^{(M+1)}; \nu_p) \approx (\tau_{M-1}^{(M)}; \nu_p)$$

and

$$\|\tau_{M-1}^{(M)} - \tau_{M-1}^{(M+1)}\|_{\nu_p} < \epsilon_M$$

for $1 \leq p \leq M$. Proceeding in this way, we define $\tau_m^{(M+1)}$ for $m = M-1, M-2, \dots, 1$.

Clearly $\sigma_1 \prec \sigma_2 \prec \cdots$ forms an infinite sequence of nested clopen multitowers where $\{x_0\} = \bigcap_{m=1}^{\infty} \text{base}(\sigma_m)$, and $\bigvee \sigma_m$ generates the topology of X .

Fix $p \geq 1$. For $M \geq m, p$, we have

$$\|\tau_m^{(M)} - \tau_m^{(M+1)}\|_{\nu_p} < \epsilon_M.$$

Since $\sum_{M \geq 0} \epsilon_M < \infty$ the Borel-Cantelli Lemma implies that there are ν_p -measurable multitowers $\tau_m = \lim_{M \rightarrow \infty} \tau_m^{(M)}$. Since $(\sigma_m; \mu_p) \approx (\tau_m^{(M)}; \nu_p)$ for $M \geq m, p$, it is also true that $(\sigma_m; \mu_p) \approx (\tau_m; \nu_p)$. Similarly,

$$(\sigma_1, \sigma_2, \dots, \sigma_m; \mu_p) \approx (\tau_1, \tau_2, \dots, \tau_m; \nu_p).$$

If $m \geq p$, the multitower $\tau_m^{(m)}$ refines \mathcal{Q}_m on a set E_m with $\nu_p(E_m) > 1 - \epsilon_m$. Since $\|\tau_m - \tau_m^{(m)}\|_{\nu_p} < \sum_{M \geq m+1} \epsilon_M$, we have that τ_m refines \mathcal{P}_m on a set F_m where $\nu_p(F_m) > 1 - \sum_{M \geq m} \epsilon_M$. Therefore, as partitions, the collection of multitowers $\{\tau_m\}$ separate points on a set of ν_p measure one for every $p \geq 1$. Therefore, the conditions of Lemma 5 are satisfied and Theorem 1 holds. \square

It remains then to verify Lemma 6. Its proof follows to a great extent the scheme of the proof of Lemma 5.1 of [21]. We reproduce here the relevant definitions of [21] and include the proofs of all statements that are not precisely the same as those in [21].

4. PROOF OF LEMMA 6

Let us assume that the towers σ and τ are as in the hypothesis of Lemma 6. We must construct three multitowers Σ , τ' and \mathcal{T} . To do this requires us to construct several intermediate multitowers.

4.1. Step 1 - Getting Started. The following is a simple lemma to begin the construction.

Lemma 8. *Given any integer $Z > 0$ we can create a clopen multitower $\Sigma_1 = \langle A(i, j); \widehat{H}(i); \widehat{I} \rangle$ for S where $\Sigma_1 \succ \sigma$ and Σ_1 satisfies the following properties:*

- $x_0 \in \text{base}(\Sigma_1)$
- $\text{diam}(\text{base}(\Sigma_1)) < \epsilon$,
- $\text{base}(\Sigma_1)$ is a subset of the floor of σ which contains x_0 ,
- $\text{top}(\Sigma_1)$ is a subset of the floor of σ which contains $S^{-1}(x_0)$,
- $\widehat{H}(i) > Z$ for all i ,
- Σ_1 refines \mathcal{P} .

To achieve all of the above, it is only necessary to select a clopen multitower (Kakutani skyscraper) whose base is a clopen neighborhood of x_0 with sufficiently small diameter and afterwards to refine the columns so that the resulting multitower refines both σ and \mathcal{P} . The value of Z which we will use in this particular construction will be specified later after several definitions and notations are established (Section 4.4.1). For now we remark that the choice is based on the Ergodic Theorem applied to indicator functions of the floors in the towers σ and τ and measures μ_p, ν_p with $1 \leq p \leq m + 1$. Since we currently have all of these elements at our disposal, let us assume that we have created $\Sigma_1 = \langle A(i, j); \widehat{H}(i); \widehat{I} \rangle$.

4.2. Step 2 - The Copying Lemma. Next we create a multitower \mathcal{T}_1 for T satisfying $(\mathcal{T}_1; \nu_p) \approx (\Sigma_1; \mu_p)$ for $p = 1, 2, \dots, m + 1$. This requires a version of the Alpern Multitower Lemma below [1]. See [12] for a simple proof which follows Kakutani's idea of the proof of classical Rokhlin Lemma.

Lemma 9 (Alpern Multitower Lemma). *Let (Y, T, ν) be an arbitrary e.m.p.a. of a nonatomic Lebesgue probability space.*

Let $H(1), H(2), \dots, H(I)$ be a set of positive integers with $\gcd\{H(i) :$

$1 \leq i \leq I\} = g$, and let $\alpha(1), \alpha(2), \dots, \alpha(I)$ be positive real numbers with $\sum_{i=1}^I H(i)\alpha(i) = 1$. If T has a finite factor of cardinality g (i.e., a factor consisting of one orbit of cardinality g), then there is a multitower $\mathcal{T} = \langle B(i, j); H(i); I \rangle$ for T with $\nu(B(i, 0)) = \alpha_i$.

Our statement of the copying lemma is slightly different from the standard one (in which it is assumed that $\gcd\{H(i) : 1 \leq i \leq I\} = 1$), but it follows immediately from the standard version.

Let $g = \gcd\{\widehat{H}(i)\}$. Since the orbit of period g is a factor of S , it is also a factor of T_p for all $p \in \mathcal{I}$. Therefore, we may apply the above lemma to each of the e.m.p.a.'s (Y_p, T_p, ν_p) for $1 \leq p \leq m + 1$, with $I = \widehat{I}$, $H(i) = \widehat{H}(i)$ and $\alpha_i = \mu_p(A(i, 0))$. Then by unioning the floors of the individual $m + 1$ different multitowers together, we can create a multitower \mathcal{T}_1 for T satisfying $(\mathcal{T}_1; \nu_p) \approx (\Sigma_1; \mu_p)$ for $1 \leq p \leq m + 1$. We may assume $(\mathcal{T}_1; \nu_p)$ is the trivial multitower for $p > m + 1$.

Next we refine the towers of \mathcal{T}_1 by the partition \mathcal{Q} and by τ . Call the resulting multitower $\mathcal{T}_2 = \langle B(i, j); H(i); I \rangle$. Note that although \mathcal{T}_2 refines τ as a partition, the pair (τ, \mathcal{T}_2) is not nested since the base of \mathcal{T}_2 need not be a subset of the base of τ .

4.3. Step 3 - Labellings. We now consider two labellings on \mathcal{T}_2 . For us, a *labelling* \mathcal{L} on a multitower is a map from the set of floors of the multitower to a finite subset of $\mathbb{N} \times \mathbb{N}$. The labellings we have in mind are generally those defined by a refining multitower. For example, for $\tau = \langle b(k, l); h(k); K \rangle$, we have a labelling \mathcal{L} on \mathcal{T}_2 where \mathcal{L} is the map from floors of \mathcal{T}_2 to pairs of the form $\{(k, l) : 1 \leq k \leq K, 0 \leq l < h(k)\}$ where $\mathcal{L}(B(i, j)) = (k, l)$ if $B(i, j) \subset b(k, l)$.

The point of defining labellings is to work in the other direction. That is, given the multitower \mathcal{T}_2 we wish to define a labelling which gives rise to a multitower τ' with $\tau' \prec \mathcal{T}$, $(\tau'; \nu_p) \approx (\sigma; \mu_p)$ and $\|\tau' - \tau\|_{\nu_p} < \epsilon$ for $1 \leq p \leq m$. In this vein, we define (h, β) -labellings ([21], Def. 5.2, p. 120).

Definition 10. Let $\mathcal{T} = \langle B(i, j); H(i); I \rangle$ be a multitower for an e.m.p.a. (Y, T, ν) . Let $h = (h(1), h(2), \dots, h(K))$ be an element of \mathbb{N}^K and let $\beta = (\beta(1), \beta(2), \dots, \beta(K))$ be a K -tuple of positive real numbers such that $\sum_{k=1}^K h(k)\beta(k) = 1$. We say a labelling \mathcal{L} of \mathcal{T} is an (h, β) -labelling with respect to ν if it has the following properties:

- (1) $\text{Image}(\mathcal{L}) = \{(k, l) \in \mathbb{N}^2 \mid 1 \leq k \leq K, 0 \leq l < h(k)\}$
- (2) for $0 \leq j < (H(i) - 1)$,
if $\mathcal{L}(B(i, j)) = (k, l)$ with $l < (h(k) - 1)$ then $\mathcal{L}(B(i, j + 1)) = (k, l + 1)$,

if $\mathcal{L}(B(i, j)) = (k, h(k) - 1)$ then $\mathcal{L}(B(i, j + 1)) = (k', 0)$ for some k' ,

(3) $\nu(\mathcal{L}^{-1}\{(k, l)\}) = \beta(k)$ for all (k, l) .

Thus for any $1 \leq p \leq m$, setting $\beta_p(k) = \nu_p(b(k, 0))$, the labelling \mathcal{L} defined on \mathcal{T}_2 via the relationship $\tau \prec \mathcal{T}_2$ is an (h, β_p) -labelling with respect to ν_p . Conversely, to define τ' with $(\tau'; \nu_p) \approx (\tau; \nu_p)$ we need to define a new labelling on \mathcal{T}_2 which is an (h, β_p) -labelling with respect to ν_p . In addition, we would like (τ', \mathcal{T}_2) to be a nested pair.

Definition 11. An (h, β) -labelling on a multitower $\mathcal{T} = \langle B(i, j); H(i); I \rangle$ is nested if $\mathcal{L}(B(i, 0)) \in \{(k, 0) \mid k \in \mathbb{N}\}$ for all i .

Recall that the important feature of the pair of multitowers for T that we need to arrange is that they be compatible with the pair of S multitowers. In this vein we consider a second labelling on \mathcal{T}_2 which we call \mathcal{L}' . This one is induced by the relationships $\sigma \prec \Sigma_1$, $(\Sigma_1; \mu_p) \approx (\mathcal{T}_1; \nu_p)$ and $\mathcal{T}_1 \prec \mathcal{T}_2$ for $1 \leq p \leq m$. That is, the relation $\sigma \prec \Sigma_1$ gives rise to a labelling \mathcal{K} on Σ_1 . Since $(\Sigma_1; \mu_p) \approx (\mathcal{T}_1; \nu_p)$ there is a measure-preserving bijection of floors (Section 3.1) which pushes the labelling \mathcal{K} onto a labelling on \mathcal{T}_1 . Finally, if floor of \mathcal{T}_1 has a label (k, l) , then we may assign that same label to all subset floors in \mathcal{T}_2 . This defines a nested labelling \mathcal{L}' . For $1 \leq p \leq m$, since $(\sigma; \mu_p) \approx (\tau; \nu_p)$, the labelling \mathcal{L}' is also an (h, β) -labelling of \mathcal{T}_2 with respect to ν_p with $\beta(k) = \nu_p(b(k, 0))$.

With the assistance of the following notations, we are able to restate our goal in terms of labellings.

Notation 12. For the labelling \mathcal{L} on $\mathcal{T}_2 = \langle b(k, l); h(k); K \rangle$, let $N_k[m, n]_i$ be the number of floors in the i th column whose height index is between m and $(n - 1)$ and whose label is $(k, 0)$, i.e.

$$N_k[m, n]_i = \#\{B(i, j) \mid \mathcal{L}(B(i, j)) = (k, 0) \text{ and } j \in [m, n]\}$$

We will use the notation $N'_k[m, n]_i$ and $N''_k[m, n]_i$ similarly for labellings \mathcal{L}' and \mathcal{L}'' of \mathcal{T}_2 .

Definition 13. For two labellings of the same multitower \mathcal{L} and \mathcal{L}' , let $\|\mathcal{L} - \mathcal{L}'\|_\nu$ denote the ν -measure of the collection of floors which have different images under \mathcal{L} and \mathcal{L}' .

We seek a labelling \mathcal{L}'' of $\mathcal{T}_2 = \langle B(i, j); H(i); I \rangle$ such that

- (a) for $1 \leq p \leq m$, \mathcal{L}'' is an (h, β_p) -labelling with respect to ν_p where $\beta_p(k) = \nu_p(b(k, 0))$,
- (b) $N''_k[0, H(i)]_i = N'_k[0, H(i)]_i$ for all i and k ,
- (c) $\mathcal{L}''(B(i, 0)) = \mathcal{L}'(B(i, 0))$ for all i , (therefore insuring that the labelling is nested)
- (d) $\mathcal{L}''(B(i, H(i) - 1)) = \mathcal{L}'(B(i, H(i) - 1))$ for all i ,

(e) $\|\mathcal{L}'' - \mathcal{L}\|_{\nu_p} < \epsilon/2$ for $1 \leq p \leq m$.

4.4. Step 4 - Relabelling. To prove that a labelling of the required form exists, we will invoke the Ergodic Theorem. If the columns of \mathcal{T}_2 are sufficiently tall, the Ergodic Theorem gives us control over the label frequencies.

Definition 14. Let $\mathcal{T} = \langle B(i, j); H(i); I \rangle$ be a measurable multitower for an e.m.p.a. (Y, T, ν) . Let \mathcal{L} be a (h, β) -labelling of \mathcal{T} for some (h, β) . For all i , let $C(i) = \bigcup_{j=0}^{H(i)-1} B(i, j)$. We say \mathcal{L} satisfies the δ -frequency condition with respect to ν if there is a collection of columns $G = \{C(i_1), C(i_2), \dots, C(i_L)\}$ such that $\nu(G) > (1 - \delta)$ and for $C(i) \in G$

$$2\delta H(i) \leq m \leq H(i) \quad \text{implies} \quad \left| \frac{1}{m} N_k[0, m)_i - \beta(k) \right| < \delta \text{ for all } k.$$

The δ -frequency condition is guaranteed in a multitower with sufficiently tall columns.

Lemma 15. ([21], Lemma 5.6, p. 124) Let (Y, T, ν) be an arbitrary e.m.p.a. and let τ be a measurable multitower for T and let $0 < \delta < \frac{1}{2}$ be given. There is an integer $Z_2(\delta, T, \nu)$ such that if a multitower \mathcal{T} refines the multitower τ and the column heights of \mathcal{T} all exceed $Z_2(\delta, T, \nu)$, then the labelling on \mathcal{T} defined by τ satisfies the δ -frequency condition with respect to ν .

See [21] for the proof, an application of the Pointwise Ergodic Theorem.

Given any $\delta > 0$, for each $1 \leq p \leq m$, we can apply the above lemma to (X, S, μ_p) and σ and to (X, T, ν_p) and τ . By taking maximum Z_2 value, we can insure the conclusion of the lemma for all of these measures simultaneously. Note that this value of Z_2 depends only on δ , σ , τ , and the measures μ_p and ν_p where $1 \leq p \leq m$.

Next we prove the main lemma for the proof of our main theorem, the Labelling Lemma, which establishes the existence of a labelling satisfying the desired properties. This is the finite family version of [21, Lemma 5.5, p. 121] We reproduce the proof of with the appropriate changes. It is striking how little needs to change in the argument.

Lemma 16 (Labelling Lemma). Let $h = (h(1), h(2), \dots, h(K))$ be such that $\gcd(h(1), h(2), \dots, h(K)) = 1$ and for $1 \leq p \leq m$, let $\beta_p = (\beta_p(1), \beta_p(2), \dots, \beta_p(K))$ be positive real numbers such that $\sum_k h(k)\beta_p(k) = 1$.

Let $\mathcal{T} = \langle B(i, j); H(i); I \rangle$ is a measurable tower for T and let $\mathcal{L}, \mathcal{L}'$ be (h, β_p) -labellings of \mathcal{T} with respect to ν_p .

For any $0 < \epsilon < 1$, there exist numbers $0 < \delta < 1/2$ and $Z_1(\delta)$ such that if $H(i) \geq Z_1(\delta)$ for all i , $\mathcal{L}, \mathcal{L}'$ satisfy the δ -frequency condition with respect to all measures ν_p , $1 \leq p \leq m$, and \mathcal{L}' is a nested labelling, then there exists a nested labelling \mathcal{L}'' of \mathcal{T} such that

- (a) \mathcal{L}'' is an (h, β_p) -labelling with respect to ν_p ,
- (b) $N_k''[0, H'(i)]_i = N_k'[0, H'(i)]_i$ for all i and k ,
- (c) $\mathcal{L}''(B(i, 0)) = \mathcal{L}'(B(i, 0))$ for all i ,
- (d) $\mathcal{L}''(B(i, H'(i) - 1)) = \mathcal{L}'(B(i, H'(i) - 1))$ for all i ,
- (e) $\|\mathcal{L}'' - \mathcal{L}\|_{\nu_p} < \epsilon/2$ for $p = 1, 2, \dots, m$.

Proof. Let J be an integer such that for any $j \geq J$, there exist non-negative integers m_1, m_2, \dots, m_K where $j = \sum_{k=1}^K m_k h(k)$. Choose δ and $Z_1(\delta)$ such that:

$$0 < \delta < \epsilon \beta_p(k)/6 \text{ for all } k, p, \quad \text{and} \quad Z_1(\delta) > (J + 3 \max h(k))/\delta.$$

Assume $\mathcal{T} = \langle B(i, j); H(i); I \rangle$ is a multitower with $H(i) \geq Z_1(\delta)$ for all i . Let \mathcal{L} and \mathcal{L}' be two labellings of \mathcal{T} which satisfy the hypotheses of the lemma.

Because the labellings $\mathcal{L}, \mathcal{L}'$ both satisfy the δ -frequency condition with respect to all measures ν_p for $1 \leq p \leq m$, there is a collection of columns G_p, G'_p , as in Definition 14, for each $1 \leq p \leq m$. Let $G = \cup_{p=1}^m (G_p \cap G'_p)$. From here the proof from [21] essentially carries over.

From the definition of G , if $C(i) \in G$ then there is a p , $1 \leq p \leq m$ such that for all k and for all m , $2\delta H(i) \leq m \leq H(i)$ we have

$$\left| \frac{1}{m} N_k[0, m]_i - \beta_p(k) \right| < \delta \quad \text{and} \quad \left| \frac{1}{m} N'_k[0, m]_i - \beta_p(k) \right| < \delta.$$

It is the fact that $\frac{1}{m} N_k[0, m]_i$ and $\frac{1}{m} N'_k[0, m]_i$ are close to one another that is important here, not that either is close to $\beta_p(k)$. This is why the proof for one measure carries over so easily to a finite collection of measures. Also take note that $\nu_p(\cup_{C(i) \in G} C(i)) > (1 - 2\delta)$ for any $1 \leq p \leq m$.

For any floor $B(i, j)$ where $C(i) \notin G$, let $\mathcal{L}''(B(i, j)) = \mathcal{L}'(B(i, j))$. In sections A-D below we will define \mathcal{L}'' on a fixed column $C(i) \in G_p \subset G$. The index i is fixed throughout these sections and will be suppressed. We will consider the labellings as maps from $\{j \mid 0 \leq j < H\}$ to $\{(k, l) \mid 1 \leq k \leq K, 0 \leq l < h(k)\}$.

A. Base and top floors

Let k_1, k_2 be such that $\mathcal{L}'(0) = (k_1, 0)$, and $\mathcal{L}'(H - 1) = (k_2, h(k_2) - 1)$.

Let $\mathcal{L}''(j) = \mathcal{L}'(j)$ for $j \in [0, h(k_1))$ and $j \in [H - h(k_2), H)$.

B. Floors $h(k_1)$ through $(\bar{J} - 1)$

Let \bar{J} be the smallest integer such that $\bar{J} > (J + h(k_1))$ and $\mathcal{L}(\bar{J}) = (k, 0)$ for some k . Notice that $\bar{J} \leq (J + 2 \max h(k)) < \delta H$.

There are coefficients $m_k \in \mathbb{N}$ such that $(\bar{J} - h(k_1)) = \sum m_k h(k)$. For each k , put the label $(k, 0)$ on m_k of the floors in $[h(k_1), \bar{J})$, filling in labels on the remaining such that:

- if $\mathcal{L}''(j) = (k, l)$ with $l < (h(k) - 1)$ then $\mathcal{L}''(j + 1) = (k, l + 1)$,
- if $\mathcal{L}''(j) = (k, h(k) - 1)$ then $\mathcal{L}''(j + 1) = (k', 0)$ for some k' .

C. Floors \bar{J} through $(\bar{L} - 1)$

We will define $\mathcal{L}''(j) = \mathcal{L}(j)$ for $j \in [\bar{J}, \bar{L})$. We will choose \bar{L} to be as large as possible under the condition that with this definition, $(N_k''[0, \bar{L}] + N_k''[H - h(k_2), H])$ does not exceed $N_k'[0, H)$ for any k .

For all k and for $\bar{J} \leq L \leq H$, define

$$E_k(L) := N_k'[0, H) - (N_k''[0, \bar{J}] + N_k''[H - h(k_2), H] + N_k[\bar{J}, L]).$$

Notice that for all k , $E_k(L)$ is a non-increasing function of L . We will let \bar{L} be the largest integer L such that $E_k(L) \geq 0$ for all k . (One can show that $E_k(\bar{J}) > 0$ for all k , and $E_k(H) < 0$ for some k , so \bar{L} is well-defined. The proof that $E_k(\bar{J}) > 0$ for all k is contained in the following.)

Claim: $\bar{L} > (1 - \epsilon/2)H$.

Proof of Claim: We first estimate the summands of $E_k(\bar{L})$. Since our column is in G_p we have

$$(1) \quad N_k'[0, H) > (\beta_p(k) - \delta)H$$

for all k .

Also for all k , we have

$$(2) \quad N_k''[0, \bar{J}] + N_k''[H - h(k_2), H] \leq (J + 3 \max h(k)) < \delta H.$$

Notice that if $\bar{J} \leq L < 2\delta H$ then for all k we have

$$\begin{aligned} E_k(L) &> (\beta_p(k) - \delta)H - \delta H - N_k[\bar{J}, L] > (\beta_p(k) - 2\delta)H - 2\delta H \\ &= (\beta_p(k) - 4\delta)H \\ &> \beta_p(k)(1 - 4\epsilon/6)H > 0. \end{aligned}$$

Therefore $\bar{L} \geq 2\delta H$, and again since our column is in G we have

$$(3) \quad N_k[\bar{J}, \bar{L}] \leq N_k[0, \bar{L}] < (\beta_p(k) + \delta)\bar{L}$$

for all k .

For some k , we have $E_k(\bar{L}) = 0$. Using 1, 2 and 3 we see that for this k ,

$$0 = E_k(\bar{L}) > (\beta_p(k) - \delta)H - \delta H - (\beta_p(k) + \delta)\bar{L}.$$

Rearranging, we get

$$\begin{aligned} \bar{L} &> [(\beta_p(k) - 2\delta)/(\beta_p(k) + \delta)] H > (1 - 3\delta/\beta_p(k))H \\ &> (1 - \epsilon/2)H \end{aligned}$$

End of proof of Claim. \square

D. Floors \bar{L} through $(H - h(k_2))$

We have $E_k(\bar{L}) = N'_k[0, H] - N''_k[0, \bar{L}] - N''_k[H - h(k_2), H]$ for all k . Multiplying both sides of this equation by $h(k)$, then summing over k , we see that $\sum_k E_k(\bar{L})h(k) = (H - \bar{L} - h(k_2))$. Therefore, there are exactly enough unlabelled floors left to label $E_k(\bar{L})$ of the floors in $[\bar{L}, H - h(k_2))$ with $(k, 0)$ for each k , and fill in labels on the remaining floors such that:

- if $\mathcal{L}''(j) = (k, l)$ with $l < (h(k) - 1)$, then $\mathcal{L}''(j + 1) = (k, l + 1)$,
- if $\mathcal{L}''(j) = (k, h(k) - 1)$ then $\mathcal{L}''(j + 1) = (k', 0)$ for some k' .

After labelling the remaining floors in this way we see that

$N''_k[0, H] = E_k(\bar{L}) + N''_k[0, \bar{L}] + N''_k[H - h(k), H] = N'_k[0, H]$, for all k .

We define \mathcal{L}'' in this manner for all $C(i) \in G$. We get $N''_k[0, H(i)]_i = N'_k[0, H(i)]_i$ for all i and k , and $\mathcal{L}'' = \mathcal{L}'$ on $\text{base}(\mathcal{T})$ and $\text{top}(\mathcal{T})$. All that remains is to show that \mathcal{L}'' and \mathcal{L} differ on a set of measure less than ϵ with respect to all of the measures ν_p with $1 \leq p \leq m$. For $C(i) \in G$ we have $\mathcal{L} = \mathcal{L}''$ on $[\bar{J}(i), \bar{L}(i)]$. We know that $\bar{J}(i) < \delta H(i) < (\epsilon/6)H(i)$ and $(H(i) - \bar{L}(i)) < (\epsilon/2)H(i)$. Therefore,

$$\begin{aligned} \|\mathcal{L} - \mathcal{L}''\|_{\nu_p} &\leq \sum_{C(i) \in G} \left[\sum_{j=0}^{\bar{J}(i)-1} \nu_p(B(i, j)) + \sum_{j=\bar{L}(i)}^{H(i)-1} \nu_p(B(i, j)) \right] + \sum_{C(i) \notin G} \nu_p(C(i)) \\ &< \sum_{C(i) \in G} [(\bar{J}(i) + H(i) - \bar{L}(i))\nu_p(B(i, 0))] + 2\delta \\ &< \sum_{C(i) \in G} [(\epsilon/6 + \epsilon/2)H(i)\nu_p(B(i, 0))] + \epsilon/3 \\ &\leq (\epsilon/6 + \epsilon/2) + \epsilon/3 = \epsilon. \end{aligned}$$

\square

4.4.1. *Selection of δ and H .* Let $0 < \epsilon < 1$ be given. Let $\sigma = \langle b(k, l); h(k); K \rangle$ and $\tau = \langle b(k, l); h(k); K \rangle$ be as in the statement of Lemma 6.

From ϵ , we obtain a $0 < \delta < 1/2$ and an integer $Z_1(\delta)$ from the Labelling Lemma. Using this δ , we obtain $Z_2(\delta, S, \mu_p), Z_2(\delta, T, \nu_p)$ for each $1 \leq p \leq m$ from the above lemma. Let Z be the maximum of $\{Z_1(\delta), \max_p Z_2(\delta, S, \mu_p), \max_p Z_2(\delta, T, \nu_p)\}$. Then by the Labelling Lemma, we get the desired labelling \mathcal{L}'' and a multitower for T , τ'_1 . It is by this process that the labelling \mathcal{L}'' is constructed, and hence a multitower τ'_1 for T such that (τ'_1, \mathcal{T}_2) is a nested pair, $(\tau'_1; \nu_p) \approx (\sigma; \mu_p)$ and $\|\tau'_1 - \tau\|_{\nu_p} < \epsilon/2$ for all $1 \leq p \leq m$.

4.5. **Step 5 - Cleaning Up.** Note that we are not quite done since it is not true that $(\Sigma_1; \mu_p) \approx (\mathcal{T}_2; \nu_p)$, instead, \mathcal{T}_2 is a refined copy of Σ_1 .

Repairing this is a step that may look just technical, but, in fact, it plays a crucial role in the entire proof. The reader may have noticed that up to this moment we have never used the fact that the ergodic S -invariant measures μ_p , $p = 1, 2, \dots$ are distinct (hence, mutually singular). It is obvious, however, that without using this fact the proof cannot be done. Further, one may also note that until this point, the floors of \mathcal{T}_2 could have the property that they have positive measure with respect to only one measure.

Each base set $A(i, 0)$ in Σ_1 corresponds to a collection of base sets $B(k_i, 0), B(k_i+1, 0), \dots, B(k_{i+1}-1, 0)$ in \mathcal{T}_2 where $\mu_p(A(i, 0)) = \nu_p\left(\bigcup_{j=k_i}^{k_{i+1}-1} B(j, 0)\right)$. We will fix this, as in [21, p. 126], by partitioning each $A(i, 0)$ into clopen sets $A'(k_i, 0), A'(k_i+1, 0), \dots, A'(k_{i+1}-1, 0)$. However (and this is the main difference between the situation in [21] and the situation here), we now need to control the values of all measures μ_1, \dots, μ_m when we do this partitioning. Since the measures μ_p are mutually singular, the range of the vector measure (μ_1, \dots, μ_m) is the entire m -dimensional cube. This allows us to choose the clopen sets $A'(k, 0)$ such that

$$\sum_k |\mu_p(A'(k, 0)) - \nu_p(B(k, 0))| H'(k) < \epsilon/2$$

for any $1 \leq p \leq m$. Let $A'(i, j) = S^j A'(i, 0)$ for $0 \leq j < H'(i)$, and let $\Sigma = \langle A(i, j); H'(i); I' \rangle$. Now choose measurable sets $B'(i, 0) \subseteq \bigcup_{k=k_i}^{k_{i+1}-1} B(k, 0)$ such that

$$\nu_p(B'(i, 0)) = \mu_p(A'(i, 0)) \text{ and } \sum_i |\nu_p(B'(i, 0) \Delta B(i, 0))| H'(i) < \epsilon/2.$$

Let $B'(i, j) = T^j B'(i, 0)$ for $0 \leq j < H'(i)$ and $\mathcal{T} = \langle B'(i, j); H'(i); I' \rangle$. Finally, consider the labelling \mathcal{L}'' on \mathcal{T} , where $\mathcal{L}''(B'(i, j)) = \mathcal{L}''(B(i, j))$. This defines a new multitower τ' such that $(\tau'; \nu_p) \approx (\sigma; \mu_p)$ and $\|\tau' - \tau\|_{\nu_p} < \epsilon/2$ for $1 \leq p \leq m$. These are the final versions of the towers, and this completes the proof of Lemma 6 and, therefore, the proof of the main theorem.

5. CONCLUDING REMARKS

The realization theorem for countable families of e.m.p.a.'s cannot be fully generalized to arbitrary (uncountable) families. The result of [9] shows that there are some descriptive conditions of the semicontinuity type that are necessary for topological, even nonminimal, realization of a family of e.m.p.a.'s, and the complete list of these conditions is unclear. What looks plausible is that the recent result of T. Downarowicz [6], which shows that the conditions for minimal realization and conditions for aperiodic realization are the the same, can be strengthened “in the orbital direction”. Namely, it may be true that every family that can be realized aperiodically can also be realized minimally *in the orbit equivalence class of any Cantor minimal system whose simplex of invariant measures is affinely homeomorphic to the simplex corresponding to the aperiodic realization*.

A natural approach to this conjecture can be to apply first the countable realization theorem to a w^* -dense countable set of ergodic measures in the aperiodic realization. The difficulty is to prove, for every w^* -convergent sequence of measures ν_p , the convergence of the corresponding multitowers to the multitower associated with the limiting measure ν . This requires constructing multitowers such that every floor has the boundary of zero measure with respect to all invariant measures simultaneously. It turns out that for Cantor minimal systems such “regular” towers can be constructed. It also seems possible that such towers exist for Cantor aperiodic systems. There are, however, other types of unresolved convergence difficulties in this approach, and because of this we suspend further discussion.

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