A STRUCTURE FOR QUANTUM MEASUREMENTS

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Abstract

This paper presents an additive and product structure for quantum measurements. The additive structure generalizes the orthosum of effects in effect algebras and preserves sums of expectations. The additive structure also provides a natural order for measurements and it is shown that an initial interval of measurements forms an effect algebra. In certain cases, this effect algebra retains the properties of the original effect algebra on which the measurements are defined. Various sequential products of measurements are introduced and compared. Conditional measurements are also studied.

1 Introduction

The simplest and most basic type of measurement is a one-zero (or yes-no) measurement. We call such two-valued measurements effects. Quantum effects may be unsharp (imprecise) and they generalize the sharp (precise) quantum events that have been studied for over 75 years. In the Hilbert space formulation of quantum mechanics, effects are represented by positive operators bounded above by the identity operator, while sharp effects are represented by projection operators [2, 3, 11, 12, 13]. In the last fifteen years, researchers in the foundations of quantum mechanics have studied effects in the more general context of an effect algebra [4, 5, 7]. An effect algebra encapsulizes the structure of the orthosum $a \oplus b$ for effects a and b. Roughly speaking $a \oplus b$ corresponds to a parallel measurement of a and b. More recently a sequential product $a \circ b$ has been introduced to form an abstract structure called a sequential effect algebra [6, 8, 9, 10]. The product $a \circ b$ corresponds to a series measurement in which a is performed first and b second.

Although effect algebras and sequential effect algebras have provided insights for a better understanding of quantum measurements, their applicability has been limited to two-valued measurements. It is important that we attain a deeper understanding of more general measurements, say measurements with a finite or even infinite number of real values. In this paper we study ways in which the orthosum and sequential product can be extended from effects to more general measurements. Just as a sequential effect algebra describes a mathematical structure for quantum effects we would like to describe a mathematical structure for measurements. Although these structures are defined on effect algebras, they apply just as well to the special case of Hilbert space quantum mechanics.

We first point out in Section 3 that a general measurement is described by a normalized effect-valued measure. We then consider the special case of measurements with a finite number of values. It is noted that there is a one-to-one correspondence between elements of an effect algebra and $(1,0)$-measurements. We denote the set of measurements on an effect algebra $\mathcal{E}$ by $\mathcal{M}(\mathcal{E})$. We define the partial binary operation $\oplus$ on $\mathcal{M}(\mathcal{E})$. It is shown that $\oplus$ extends the orthosum on $\mathcal{E}$ and provides additive expectations in the sense that $E_s(X \oplus Y) = E_s(X) + E_s(Y)$ where E_s denotes the expectation in the state s. We prove that $\mathcal{M}(\mathcal{E})$ is a generalized effect algebra and introduce the natural order $X \leq Y$ on $\mathcal{M}(\mathcal{E})$. It follows that $(\mathcal{M}(\mathcal{E}, \leq)$ is a partially ordered set. We then show that for any nontrivial $X \in \mathcal{M}(\mathcal{E})$, the interval

$$[\widehat{0}, X] = \{Y \in \mathcal{M}(\mathcal{E}) \colon Y \leq X\}$$

is an effect algebra. In many cases, these intervals preserve the properties of $\mathcal{E}$. The concepts of local and global sharpness of measurements are introduced.

In Section 4 we study measurements with finitely many real values which we call finite measurements. Denoting the set of finite measurements by $\mathcal{M}_F(\mathcal{E})$ we show that any $X \in \mathcal{M}_F(\mathcal{E})$ is the orthosum of $(1,0)$-measurements. It is also shown that if $X \in \mathcal{M}_F(\mathcal{E})$ then $[\widehat{0}, X]$ is isomorphic to a cartesian product of intervals in $\mathcal{E}$. The relationship between divisor effect algebras and finite measurements on chains is discussed.

Section 5 considers measurements on a sequential effect algebra $\mathcal{E}$. For $X \in \mathcal{M}_F(\mathcal{E})$ and $Y \in \mathcal{M}(\mathcal{E})$ we define the conditional measurement $[Y \mid X]$. The related concepts of conditional probability and conditional expectation are discussed. It is shown that

$$[(Y \oplus Z) \mid X] = [Y \mid X] \oplus [Z \mid X]$$

Finally, various types of sequential products of measurements are introduced and compared. For simplicity we have phrased our previous discussion in terms of effect algebras and sequential effect algebras. However, when we arrive at our rigorous presentation in the text we shall mainly consider σ-effect algebras and σ-sequential effect algebras. These σ-structures are commonly employed in quantum mechanics when measures and states are studied [2, 3, 11, 12, 13]

2 Definitions and Notation

An **effect algebra** is a system $(\mathcal{E}, 0, 1, \oplus)$ where 0 and 1 are distinct elements of the set $\mathcal{E}$ and $\oplus$ is a partially defined binary operation on $\mathcal{E}$ called an **orthosum** that satisfies the following conditions.

(EA1) If $a \oplus b$ is defined then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.

(EA2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(EA3) For every $a \in \mathcal{E}$ there exists a unique $a' \in \mathcal{E}$ such that $a \oplus a' = 1$.

(EA4) If $a \oplus 1$ is defined then $a = 0$.

We call the elements of $\mathcal{E}$ **effects** and view them as quantum events that may be imprecise or fuzzy. Whenever we write $a \oplus b$ we are implicitly assuming that this element is defined. In accordance with (EA2) we can omit parentheses and write $a \oplus b \oplus c$. We define $a \leq b$ if there exists a $c \in \mathcal{E}$ such that $a \oplus c = b$. It is easy to show that $(\mathcal{E}, \leq)$ is a partially ordered set with $0 \leq a \leq 1$ for every $a \in \mathcal{E}$ and that $a \oplus b$ is defined if and only if $a \leq b'$.

When $a \le b'$ we say that a and b are **orthogonal** and write $a \perp b$. It is also easy to show that $a'' = a$ and that $a \le b$ implies $b' \le a'$.

An element $a \in \mathcal{E}$ is **sharp** if the greatest lower bound $a \wedge a' = 0$. We denote the set of sharp elements of $\mathcal{E}$ by $\mathcal{E}_S$. An **orthoalgebra** is an effect algebra in which $a \perp a$ implies that $a = 0$. It can be shown that an effect algebra $\mathcal{E}$ is an orthoalgebra if and only if $\mathcal{E} = \mathcal{E}_S$. A σ-**effect algebra** is an effect algebra in which $a_1 \le a_2 \le \cdots$ implies that the least upper bound $\vee a_i$ exists. If a_i, $i = 1, 2, \ldots$, is a sequence in a σ-effect algebra for which $a_1 \oplus \cdots \oplus a_n$ exists for every $n \in \mathbb{N}$, then it follows that $\vee_n(a_1 \oplus \cdots \oplus a_n)$ exists and we write this element as $\oplus_{i=1}^{\infty} a_i$.

An **orthomodular poset** is an orthoalgebra $\mathcal{E}$ in which $a \perp b$ implies that $a \oplus b = a \vee b$. An effect algebra is **lattice ordered** if it is a lattice under its usual order. An **orthomodular lattice** is a lattice ordered orthomodular poset. An MV-**effect algebra** is a lattice ordered effect algebra in which $a \wedge b = 0$ implies that $a \perp b$ (and it then follows that $a \oplus b = a \vee b$). A distributive orthomodular lattice is a **Boolean algebra**. For more details about these algebraic structures we refer the reader to [4, 5, 7].

If $\mathcal{E}$ is an effect algebra and $a \in \mathcal{E}$ with $a \ne 0$ then the interval $[0, a] = \{b \in \mathcal{E} : b \le a\}$ becomes an effect algebra with unit a and orthosum $\oplus_a$ defined as follows. For $b, c \in [0, a]$, $b \oplus_a c$ is defined if $b \oplus c$ is defined and $b \oplus c \le a$ in which case $b \oplus_a c = b \oplus c$. If $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are effect algebras, their cartesian product $\mathcal{E}_1 \times \cdots \times \mathcal{E}_n$ becomes an effect algebra under their coordinate orthosum

$$(a_1, \ldots, a_n) \oplus (b_1, \ldots, b_n) = (a_1 \oplus b_1, \ldots, a_n \oplus b_n)$$

All the previous definitions and results apply to the corresponding σ-structures in the natural way.

If $\mathcal{E}$ and $\mathcal{F}$ are effect algebras a map $\phi\colon \mathcal{E} \to \mathcal{F}$ is **additive** if $a, b \in \mathcal{E}$ with $a \perp b$ implies that $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. An additive map $\phi\colon \mathcal{E} \to \mathcal{F}$ satisfying $\phi(a) = 1$ is called a **morphism**. A **monomorphism** is a morphism $\phi\colon \mathcal{E} \to \mathcal{F}$ such that $\phi(a) \perp \phi(b)$ implies $a \perp b$. A surjective monomorphism is an **isomorphism**. It can be shown that a morphism $\phi\colon \mathcal{E} \to \mathcal{F}$ is an isomorphism if and only if ϕ is bijective and ϕ^{-1} is a morphism. If $\mathcal{E}$ and $\mathcal{F}$ are σ-effect algebras, a morphism $\phi\colon \mathcal{E} \to \mathcal{F}$ is a σ-**morphism** if $a_1 \le a_2 \le \cdots$ implies that $\phi(\vee a_i) = \vee \phi(a_i)$. We define a σ-isomorphism in a similar way. A simple example of a σ-effect algebra is the unit interval $[0, 1] \subseteq \mathbb{R}$. For $a, b \in [0, 1]$ we say that $a \oplus b$ is defined if

$a+b \leq 1$ and in this case $a \oplus b = a+b$. A σ-morphism $\phi\colon \mathcal{E} \to [0,1]$ is called a **state** on the σ-effect algebra $\mathcal{E}$. The set of states on $\mathcal{E}$ is denoted by $\Omega(\mathcal{E})$.

We now give examples of the most common σ-effect algebras. For $n \in \mathbb{N}$, the n-**chain** $C_n = \{0, 1, 2a, \ldots, na\}$ where $na = 1$ becomes a totally ordered σ-effect algebra where $ra \oplus sa$ is defined if $r+s \leq n$ and in this case $ra \oplus sa = (r+s)a$. The only sharp elements in C_n are 0 and 1. A Boolean σ-algebra $\mathcal{B}$ is a σ-effect algebra in which $a \oplus b$ is defined if $a \wedge b = 0$ in which case $a \oplus b = a \vee b$. All the elements of $\mathcal{B}$ are sharp. For a nonempty set X, the set of fuzzy subsets $[0,1]^X$ of X is a σ-effect algebra in which $f \oplus g$ is defined if $f(x) + g(x) \leq 1$ for all $x \in X$ and in this case $f \oplus g = f + g$. An element of $[0,1]^X$ is sharp if and only if it is a characteristic function. Thus, the sharp fuzzy sets correspond to the subsets of X.

For quantum mechanics the most important example is a Hilbert space σ-effect algebra $\mathcal{E}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. The elements of $\mathcal{E}(\mathcal{H})$ are the positive operators on $\mathcal{H}$ that are bounded above by the identity operator I. For $A, B \in \mathcal{E}(\mathcal{H})$ we say that $A \oplus B$ is defined if $A + B \leq I$ and in this case $A \oplus B = A + B$. The set of sharp elements of $\mathcal{E}(\mathcal{H})$ is the set of projection operators $\mathcal{P}(\mathcal{H})$ on $\mathcal{H}$. By Gleason's theorem, if $\dim(\mathcal{H}) \geq 3$, then s is a state on $\mathcal{E}(\mathcal{H})$ if and only if there exists a density operator W such that $s(A) = \mathrm{tr}(WA)$ for all $A \in \mathcal{E}(\mathcal{H})$. Also recall that any self-adjoint operator S on $\mathcal{H}$ corresponds to a spectral measure P^S which is a σ-morphism from the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ into $\mathcal{P}(\mathcal{H})$. More generally, a σ-morphism from $\mathcal{B}(\mathbb{R})$ into $\mathcal{E}(\mathcal{H})$ is called a **normalized positive operator-valued measure** (POVM) [2, 3, 11].

Effect algebras are limited by the fact that they only describe what is roughly speaking a parallel combination $a \oplus b$ of effects. In order to describe a series combination of effects we introduce a sequential product $a \circ b$. For a binary operation $a \circ b$ if $a \circ b = b \circ a$ we write $a \mid b$. A **sequential effect algebra** (SEA) is a system $(\mathcal{E}, 0, 1, \oplus, \circ)$ where $(\mathcal{E}, 0, 1, \oplus)$ is an effect algebra and $\circ$ is a binary operation on $\mathcal{E}$ that satisfies the following conditions.

(SEA1) The map $\phi(b) = a \circ b$ is additive for every $a \in \mathcal{E}$.

(SEA2) $1 \circ a = a$.

(SEA3) If $a \circ b = 0$, then $a \mid b$.

(SEA4) If $a \mid b$ then $a \mid b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$ for every $c \in \mathcal{E}$.

(SEA5) If $c \mid a$ and $c \mid b$, then $c \mid a \circ b$ and $c \mid (a \oplus b)$.

A σ-**SEA** $\mathcal{E}$ is a SEA that is a σ-effect algebra satisfying the following conditions.

(σ-SEA1) If $a_1 \leq a_2 \leq \cdots$, then $b \circ (\vee a_i) = \vee(b \circ a_i)$ for all $b \in \mathcal{E}$.

(σ-SEA2) If $a_1 \leq a_2 \leq \cdots$ and $b \mid a_i$ then $b \mid \vee a_i$.

For details concerning SEA's we refer the reader to [8, 9]. We shall be content here with some examples. In the σ-effect algebra $[0,1] \subseteq \mathbb{R}$ the unique sequential product is $a \circ b = ab$ so $[0,1]$ becomes a σ-SEA. In the σ-effect algebra $[0,1]^X$ the unique sequential product is $f \circ g = fg$ so $[0,1]^X$ becomes a σ-SEA. A Boolean σ-algebra is a σ-SEA under the unique sequential product $a \circ b = a \wedge b$. A Hilbert space σ-effect algebra $\mathcal{E}(\mathcal{H})$ is a σ-SEA under the sequential product $A \circ B = A^{1/2} B A^{1/2}$. It is unknown whether this sequential product is unique. Finally, it is easy to show that there does not exist a sequential product on a chain C_n for $n \geq 2$ [8].

3 Orthosums of Measurements

In the sequel, $\mathcal{E}$ will denote a σ-effect algebra and $\mathcal{B}(\mathbb{R})$ will denote the σ-effect algebra of Borel subsets of $\mathbb{R}$. A **measurement** X on $\mathcal{E}$ is a normalized effect-valued measure on $\mathcal{B}(\mathbb{R})$. That is, $X \colon \mathcal{B}(\mathbb{R}) \to \mathcal{E}$ is a σ-morphism in the sense that $X(\mathbb{R}) = 1$ and

$$X\left(\bigcup_{i=1}^{\infty} \Delta_i\right) = \bigoplus_{i=1}^{\infty} X(\Delta_i)$$

whenever $\Delta_i \cap \Delta_j = \emptyset$, $i \neq j$. We may think of $X(\Delta)$ as the effect that is observed when X has a value in the Borel set Δ. We denote the set of measurements on $\mathcal{E}$ by $\mathcal{M}(\mathcal{E})$. A particularly simple class of measurements are the **finite measurements** which have a finite set of values. For such a measurement X there exists a finite set

$$\Lambda(X) = \{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{R}$$

such that $X(\{\lambda_i\}) = a_i \in \mathcal{E}$ where $a_i \neq 0$, $i = 1, \ldots, n$ and $\oplus a_i = 1$. Then for any $\Delta \in \mathcal{B}(\mathbb{R})$ we have

$$X(\Delta) = \oplus \{a_i, : \lambda_i \in \Delta\}$$

where by convention $\oplus(\emptyset)=0$. We denote the set of finite measurements by $\mathcal{M}_F(\mathcal{E})$. Special cases are the **constant** measurements satisfying $X(\{\lambda\})=1$ for some $\lambda\in\mathbb{R}$. In particular we define $\widehat{0},\widehat{1}\in\mathcal{M}_F(\mathcal{E})$ where $\widehat{0}(\{0\})=1$ and $\widehat{1}(\{1\})=1$. Other important examples are the $(1,0)$-measurements $\widehat{a}$ for $a\in\mathcal{E}$, $a\neq 0,1$, where $\widehat{a}(\{1\})=a$, $\widehat{a}(\{0\})=a'$.

For $\Delta\in\mathcal{B}(\mathbb{R})$ our notation implies that Δ' is the complement $\mathbb{R}\smallsetminus\Delta$ of Δ. For $X,Y\in\mathcal{M}(\mathcal{E})$ we say that $X\oplus Y$ **exists** if

$$p(X,Y)=X\left(\{0\}'\right)\oplus Y\left(\{0\}'\right)$$

is defined and in this case $(X\oplus Y)(\Delta)=X(\Delta)\oplus Y(\Delta)$ if $0\notin\Delta$ and if $0\in\Delta$ then

$$(X\oplus Y)(\Delta)=(X\oplus Y)\left(\Delta\smallsetminus\{0\}\right)\oplus p(X,Y)'$$

We call $X\oplus Y\colon\mathcal{B}(\mathbb{R})\to\mathcal{E}$ the **orthosum** of X and Y and if $X\oplus Y$ exists we write $X\perp Y$. Notice that $X\perp Y$ if and only if $X\left(\{0\}'\right)\leq Y(\{0\})$. If we call $X\left(\{0\}'\right)$ the **support** of $X\in\mathcal{M}(\mathcal{E})$ we see that $X\perp Y$ if and only if the supports of X and Y are orthogonal.

Theorem 3.1. *If $X\perp Y$ then $X\oplus Y\in\mathcal{M}(\mathcal{E})$.*

Proof. First we have

$$(X\oplus Y)(\mathbb{R})=(X\oplus Y)\left(\{0\}'\right)\oplus p(X,Y)'=p(X,Y)\oplus p(X,Y)'=1$$

To show that $X\oplus Y$ is countably additive, suppose that $\Delta_i\in\mathcal{B}(\mathbb{R})$ with $\Delta_i\cap\Delta_j=\emptyset$, $i\neq j$. If $0\notin\cup\Delta_i$ we have

$$\begin{aligned}(X\oplus Y)(\cup\Delta_i)&=X(\cup\Delta_i)\oplus Y(\cup\Delta_i)=[\oplus X(\Delta_i)]\oplus[\oplus Y(\Delta_i)]\\&=\oplus\left[X(\Delta_i)\oplus Y(\Delta_i)\right]=\oplus(X\oplus Y)(\Delta_i)\end{aligned}$$

Now suppose that $0\in\cup\Delta_i$. We can assume without loss of generality that $0\in\Delta_1$ and $0\notin\Delta_i$, $i\neq 1$. We then have

$$\begin{aligned}(X\oplus Y)(\cup\Delta_i)&=(X\oplus Y)\left(\cup\Delta_i\smallsetminus\{0\}\right)\oplus p(X,Y)'\\&=(X\oplus Y)\left[(\Delta_1\smallsetminus\{0\})\cup\bigcup_{i\neq 1}\Delta_i\right]\oplus p(X,Y)'\\&=(X\oplus Y)\left(\Delta_1\smallsetminus\{0\}\right)\oplus\bigoplus_{i\neq 1}(X\oplus Y)(\Delta_i)\oplus p(X,Y)'\\&=(X\oplus Y)\left(\Delta_1\smallsetminus\{0\}\right)\ \oplus p(X,Y)'\bigoplus_{i\neq 1}(X\oplus Y)(\Delta_i)\\&=\oplus(X\oplus Y)(\Delta_i)\end{aligned}$$

□

For $s \in \Omega(\mathcal{E})$, $X \in \mathcal{M}(\mathcal{E})$ we interpret $s\left[X(\Delta)\right]$ as the probability that X has a value in Δ when the system is in the state s. It is then natural to call the probability measure $s \circ X \colon \mathcal{B}(\mathbb{R}) \to [0,1]$ the **distribution of X in the state s**. The **expectation of X in the state s** is defined by

$$E_s(X) = \int_{\mathbb{R}} \lambda s\left[X(d\lambda)\right]$$

if the integral exists. The next lemma shows that the orthosum on $\mathcal{M}(\mathcal{E})$ has two important properties; it extends the orthosum on $\mathcal{E}$ and it provides additive expectations.

Lemma 3.2. (i) *If $a \perp b$ then $(a \oplus b)^\wedge = a^\wedge \oplus b^\wedge$.* (i) *If $X \perp Y$ and $E_s(X)$, $E_s(Y)$ exist, then $E_s(X \oplus Y) = E_s(X) + E_s(Y)$.*

Proof. (i) If $a \perp b$ then

$$p(\widehat{a}, \widehat{b}) = \widehat{a}\left(\{1\}\right) \oplus \widehat{b}\left(\{1\}\right) = a \oplus b$$

is defined. Moreover,

$$(\widehat{a} \oplus \widehat{b})\left(\{1\}\right) = \widehat{a}\left(\{1\}\right) \oplus \widehat{b}\left(\{1\}\right) = a \oplus b$$

and

$$(\widehat{a} \oplus \widehat{b})\left(\{0\}\right) = (\widehat{a} \oplus \widehat{b})(\emptyset) \oplus p(\widehat{a}, \widehat{b})' = (a \oplus b)'$$

Hence, $\widehat{a} \oplus \widehat{b} = (a \oplus b)^\wedge$. (ii) We have that

$$\begin{aligned} E_s(X \oplus Y) &= \int_{\mathbb{R}} \lambda s\left[(X \oplus Y)(d\lambda)\right] = \int_{\{0\}'} \lambda s\left[(X \oplus Y)(d\lambda)\right] \\ &= \int_{\{0\}'} \lambda \left\{s\left[X(d\lambda)\right] + s\left[Y(d\lambda)\right]\right\} \\ &= \int_{\{0\}'} \lambda s\left[X(d\lambda)\right] + \int_{\{0\}'} \lambda s\left[Y(d\lambda)\right] \\ &= \int_{\mathbb{R}} \lambda s\left[X(d\lambda)\right] + \int_{\mathbb{R}} \lambda s\left[Y(d\lambda)\right] = E_s(X) + E_s(Y) \end{aligned}$$

□

Example 1. Let $\mathcal{E} = \mathcal{E}(\mathcal{H})$ be a Hilbert space σ-effect algebra. Let $X, Y \in \mathcal{M}_F(\mathcal{E})$ be defined by $X(\{1/2\}) = I$ and $Y(\{1\}) = I/2$, $Y(\{0\}) = I/2$. Then $E_s(X) = 1/2 = E_S(Y)$ for every $s \in \Omega(\mathcal{E})$ and yet $X \neq Y$. This shows that expectations do not determine measurements even when there is a rich supply of states.

We say that $\Omega(\mathcal{E})$ is **order determining** if $s(a) \leq s(b)$ for every $s \in \Omega(\mathcal{E})$ implies that $a \leq b$. A measurement X is **bounded** if there exists an $M > 0$ such that $X([-M, M]) = 1$. If $X, Y \in \mathcal{M}(\mathcal{E})$ are bounded and $X \perp Y$ then $E_s(X \oplus Y) = E_s(X) + E_s(Y)$ for every $s \in \Omega(\mathcal{E})$. However, $X \oplus Y$ need not be the unique measurement satisfying this equation even when $\Omega(\mathcal{E})$ is order determining. For instance in Example 1, let $X_1(\{1\}) = I/4$, $X_1(\{0\}) = 3I/4$. Then $E_s(X) = 1/2 = E_s(X_1 \oplus X_1)$ for every $s \in \Omega(\mathcal{E})$ but $Y = X_1 \oplus X_1$.

In the sequel we shall repeatedly use the fact that for $X, Y \in \mathcal{M}(\mathcal{E})$ if $X(\Delta) = Y(\Delta)$ for every $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$ then $X = Y$. The next result summarizes the important properties of orthosums.

Theorem 3.3. *The set $\mathcal{M}(\mathcal{E})$ forms a generalized effect algebra. That is, the following properties are satisfied.*
(i) *If $X \perp Y$, then $Y \perp X$ and $Y \oplus X = X \oplus Y$*
(ii) *If $Y \perp Z$ and $X \perp (Y \oplus Z)$ then $X \perp Y$, $Z \perp (X \oplus Y)$ and $(X \oplus Y) \oplus Z = X \oplus (Y \oplus X)$*
(iii) *$\widehat{0} \perp X$ and $\widehat{0} \oplus X = X$ for all $X \in \mathcal{M}(\mathcal{E})$.*
(iv) *If $X \oplus Y = X \oplus Z$ then $Y = Z$.*
(v) *If $X \oplus Y = \widehat{0}$, then $X = Y = \widehat{0}$.*

Proof. (i) This follows directly from the definition. (ii) If $X \perp (Y \oplus Z)$, then

$$p(X, Y \oplus Z) = X\left(\{0\}'\right) \oplus Y\left(\{0\}'\right) \oplus Z\left(\{0\}'\right)$$

is defined. Since $p(X \oplus Y, Z) = p(X, Y \oplus Z)$ it follows that $X \perp Y$ and $Z \perp (X \oplus Y)$. Moreover, if $0 \notin \Delta$ then

$$\begin{aligned}[(X \oplus Y) \oplus Z](\Delta) &= [X(\Delta) \oplus Y(\Delta)] \oplus Z(\Delta) = X(\Delta) \oplus (Y(\Delta) \oplus Z(\Delta)) \\ &= [X \oplus (Y \oplus Z)](\Delta)\end{aligned}$$

It follows that $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$. (iii) Since $p(\widehat{0}, X) = X\left(\{0\}'\right)$ we have that $\widehat{0} \oplus X$ exists. If $0 \notin \Delta$ then

$$(\widehat{0} \oplus X)(\Delta) = \widehat{0}(X) \oplus X(\Delta) = 0 \oplus X(\Delta) = X(\Delta)$$

so that $\widehat{0} \oplus X = X$. (iv) If $0 \notin \Delta$, then

$$X(\Delta) \oplus Y(\Delta) = X(\Delta) \oplus Z(\Delta)$$

so by cancellation, $Y(\Delta) = Z(\Delta)$. Hence, $Y = Z$. (v) If $X \oplus Y = \widehat{0}$ then

$$X\left(\{0\}'\right) \oplus Y\left(\{0\}'\right) = 0$$

Hence, $X(\{0\}) = Y\left(\{0\}'\right) = 0$ so that $X = Y = \widehat{0}$. □

For $X, Y \in \mathcal{M}(\mathcal{E})$ we write $X \leq Y$ if there is a $Z \in \mathcal{M}(\mathcal{E})$ such that $X \oplus Z = Y$. For $a, b \in \mathcal{E}$ with $a \leq b$ we denote by $b \ominus a$ the unique element c that satisfies $a \oplus c = b$.

Theorem 3.4. *For $X, Y \in \mathcal{M}(\mathcal{E})$, $X \leq Y$ if and only if $X(\Delta) \leq Y(\Delta)$ for every $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$.*

Proof. Suppose that $X \leq Y$ so there exists a $Z \in \mathcal{M}(\mathcal{E})$ such that $X \oplus Z = Y$. Then for any $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$ we have

$$X(\Delta) \oplus Z(\Delta) = (X \oplus Z)(\Delta) = Y(\Delta)$$

Hence, $X(\Delta) \leq Y(\Delta)$. Conversely, suppose $X(\Delta) \leq Y(\Delta)$ for every $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$. If $0 \in \Delta$, since $X(\Delta') \leq Y(\Delta')$ we have

$$X(\Delta) = Y(\Delta) \oplus [Y(\Delta') \ominus X(\Delta')] \geq Y(\Delta)$$

Hence, $Y(\Delta) \oplus X(\Delta)'$ is defined whenever $0 \in \Delta$. Define $Z(\Delta) = Y(\Delta) \ominus X(\Delta)$ for every $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$ and if $0 \in \Delta$ define $Z(\Delta) = Y(\Delta) \oplus X(\Delta)'$. To show that $Z \in \mathcal{M}(\mathcal{E})$ we have

$$Z(\mathbb{R}) = Y(\mathbb{R}) \oplus X(\mathbb{R})' = 1$$

Suppose $\Delta_i \cap \Delta_j = \emptyset$, $i \neq j$ and $0 \notin \cup\Delta_i$. Then

$$\begin{aligned} Z(\cup\Delta_i) = Y(\cup\Delta_i) \ominus X(\cup\Delta_i) &= \oplus Y(\Delta_i) \ominus [\oplus(\Delta_i)] \\ &= \oplus [Y(\Delta_i) \ominus X(\Delta_i)] = \oplus Z(\Delta_i) \end{aligned}$$

If $0 \in \cup\Delta_i$, then we can assume without loss of generality that $0 \in \Delta_1$ and $0 \notin \Delta_i$ for $i \neq 1$. We then have

$$\begin{aligned}
Z(\cup\Delta_i) &= Y(\cup\Delta_i) \oplus X(\cup\Delta_i)' = \oplus Y(\Delta_i) \oplus [\oplus X(\Delta_i)]' \\
&= \oplus Y(\Delta_i) \oplus \left[X(\Delta_1)' \ominus \left(\bigoplus_{i\neq 1} X(\Delta_i)\right)\right] \\
&= [Y(\Delta_1) \oplus X(\Delta_1)'] \oplus \bigoplus_{i\neq 1}[Y(\Delta_i) \ominus X(\Delta_i)] \\
&= Z(\Delta_1) \oplus \bigoplus_{i\neq 1} Z(\Delta_i) = \oplus Z(\Delta_i)
\end{aligned}$$

Hence, $Z \in \mathcal{M}(\mathcal{E})$. To show that $X \perp Z$ we have

$$Z(\{0\}) = Y(\{0\}) \oplus X(\{0\})' \geq X(\{0\})' = X\left(\{0\}'\right)$$

Hence, $X\left(\{0\}'\right) \perp Z\left(\{0\}'\right)$ so $X \perp Z$. If $0 \notin \Delta$, then

$$(X \oplus Z)(\Delta) = X(\Delta) \oplus Z(\Delta) = Y(\Delta)$$

Therefore, $Y = X \oplus Z$ so that $X \leq Y$. □

Lemma 3.5. *If $a_1 \leq a_2 \leq \cdots$, $b_1 \leq b_2 \leq \cdots$, and $a_i \perp b_i$, $i = 1, 2, \ldots$, then $\vee a_i \perp \vee b_i$.*

Proof. If $i \leq j$, we have that $a_i \leq a_j \leq b_j'$. Therefore,

$$a_i \leq \bigwedge_{i\leq j} b_j' = \left(\bigvee_{i\leq j} b_j\right)'$$

Hence

$$a_i' \geq \bigvee_{i\leq j} b_j = \bigvee b_j$$

so that $\wedge a_i' \geq \vee b_j$. We conclude that $\vee a_i \leq (\vee b_j)'$ so that $\vee a_i \perp \vee b_j$. □

Theorem 3.6. *If $X_i \in \mathcal{M}(\mathcal{E})$ satisfy $X_1 \leq X_2 \leq \cdots$, then $\vee X_i$ exists.*

Proof. For $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$, by Theorem 3.4 we have that $X_1(\Delta) \leq X_2(\Delta) \leq \cdots$ so $\vee X_i(\Delta)$ exists. Define $X(\Delta) = \vee X_i(\Delta)$. To show that X is additive, suppose $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2 = \emptyset$. Then

$$X(\Delta_1 \cup \Delta_2) = \vee X_i(\Delta_1 \cup \Delta_2) = \vee [X_i(\Delta_1) \oplus X_i(\Delta_2)]$$

By Lemma 3.5 we have $\vee X_i(\Delta_1) \perp \vee X_i(\Delta_2)$ and since

$$X_i(\Delta_1) \oplus X_i(\Delta_2) \leq \vee X_i(\Delta_1) \oplus \vee X_i(\Delta_2)$$

we have

$$\begin{aligned} X(\Delta_1 \cup \Delta_2) &= \vee \left[X_i(\Delta_1) \oplus X_i(\Delta_2)\right] \leq \vee X_i(\Delta_1) \oplus \vee X_i(\Delta_2) \\ &= X(\Delta_1) \oplus X(\Delta_2) \end{aligned}$$

Since

$$\vee \left[X_i(\Delta_1) \oplus X_i(\Delta_2)\right] \geq X_i(\Delta_1) \oplus X_j(\Delta_2)$$

for all i, j, we have

$$\vee \left[X_i(\Delta_1) \oplus X_i(\Delta_2)\right] \geq \vee_i \left[X_i(\Delta_2) \oplus X_j(\Delta_2)\right] = \left[\vee_i X_i(\Delta_1)\right] \oplus X_j(\Delta_2)$$

Similarly,

$$\vee \left[X_i(\Delta_2) \oplus X_i(\Delta_2)\right] \geq \vee_i X_i(\Delta_1) \oplus \vee_j X_j(X_2) = X(\Delta_1) \oplus X(\Delta_2)$$

It follows that $X(\Delta_1 \cup \Delta_2) = X(\Delta_1) \oplus X(\Delta_2)$. Moreover, it is clear that if $\Delta_1 \subseteq \Delta_2 \subseteq \cdots$, $0 \notin \cup \Delta_i$ then $X(\cup \Delta_i) = \vee X(\Delta_i)$. This shows that X is countably additive on $\{0\}'$. Finally, if we define $X(\{0\}) = \left[X\left(\{0\}'\right)\right]'$ it is straightforward to show that X extends to an element $Y \in \mathcal{M}(\mathcal{E})$ and by Theorem 3.4 we have that $Y = \vee X_i$. □

Corollary 3.7. *The set $\mathcal{M}(\mathcal{E})$ forms a generalized σ-effect algebra.*

It follows from Theorem 3.4 that $(\mathcal{M}(\mathcal{E}), \leq)$ is a partially ordered set and $\widehat{0} \leq X$ for every $X \in \mathcal{M}(\mathcal{E})$. For $X \in \mathcal{M}(\mathcal{E})$ with $X \neq \widehat{0}$, by Corollary 3.7 the system $\left([\widehat{0}, X], \widehat{0}, X, \oplus_1\right)$ forms a σ-effect algebra where for $Y, Z \in [\widehat{0}, X]$ $Y \oplus_1 Z$ is defined if $Y \oplus Z$ is defined and $Y \oplus Z \leq X$ in which case $Y \oplus_1 Z = Y \oplus Z$. In the sequel, we shall omit the subscript in $\oplus_1$. We say that $X \in \mathcal{M}(\mathcal{E})$ is **maximal** if $X(\{0\}) = 0$. If X is maximal then $X \leq Y$ implies that $X = Y$ so X is maximal in the order sense. In this case, $[\widehat{0}, X]$ is a maximal σ-effect algebra in $\mathcal{M}(\mathcal{E})$. The next result shows that $[\widehat{0}, \widehat{1}]$ is essentially $\mathcal{E}$.

Lemma 3.8. *The interval $[\widehat{0}, \widehat{1}] \subseteq \mathcal{M}(\mathcal{E})$ is σ-isomorphic to $\mathcal{E}$.*

Proof. The map $\phi(a) = \widehat{a}$ is a bijection from $\mathcal{E}$ onto $[\widehat{0}, \widehat{1}]$ that preserves the unit. If $a \perp b$ then by Lemma 3.2, $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. Hence, ϕ is a morphism. If $\widehat{a} \perp \widehat{b}$ then by definition $a \perp b$. Hence, ϕ is a monomorphism so ϕ is an isomorphism. Finally, suppose that $a_1 \leq a_2 \leq \cdots$. Then $\widehat{a}_1 \leq \widehat{a}_2 \leq \cdots$ and by the proof of Theorem 3.6 we have that

$$(\vee \widehat{a}_i)(\{1\}) = \vee \widehat{a}_i(\{1\}) = \vee a_i$$

Moreover,

$$(\vee \widehat{a}_i)\left(\{0\}'\right) = \vee \widehat{a}_i\left(\{0\}'\right) = \vee a_i$$

so that

$$(\vee \widehat{a}_i)(\{0\}) = (\vee \widehat{a}_i)'$$

It follows that $\vee \widehat{a}_i = (\vee a_i)^\wedge$. Hence, ϕ is a σ-isomorphic. □

The next result shows that $[\widehat{0}, X]$ preserves the structure of $\mathcal{E}$ for orthoalgebras, orthomodular posets and Boolean algebras.

Lemma 3.9. *Let $X \in \mathcal{M}(\mathcal{E})$ with $X \neq \widehat{0}$.* (i) *If $\mathcal{E}$ is an orthoalgebra then so is $[\widehat{0}, X]$.* (ii) *If $\mathcal{E}$ is an orthomodular poset then so is $[\widehat{0}, X]$.* (iii) *If $\mathcal{E}$ is a Boolean algebra then so is $[\widehat{0}, X]$.*

Proof. (i) Suppose that $\mathcal{E}$ is an orthoalgebra. For $Y \in [\widehat{0}, X]$ suppose that $Y \oplus Y$ exists in $[\widehat{0}, X]$. Then $Y\left(\{0\}'\right) \oplus Y\left(\{0\}'\right)$ is defined so $Y\left(\{0\}'\right) = 0$. Hence, $Y(\{0\}) = 1$ so $Y = \widehat{0}$. (ii) Suppose that $\mathcal{E}$ is a orthomodular poset. For $Y, Z \in [\widehat{0}, X]$ assume that $Y \oplus Z$ exists in $[\widehat{0}, X]$. If $0 \notin \Delta$ then $Y(\Delta) \oplus Z(\Delta) = Y(\Delta) \vee Z(\Delta)$. It follows that $Y \oplus Z = Y \vee Z$ in $[\widehat{0}, X]$ so $[\widehat{0}, X]$ is an orthomodular poset. (iii) Suppose that $\mathcal{E}$ is a Boolean algebra. For $Y, Z \in [\widehat{0}, X]$ define $Y_1, Z_1, W \in [\widehat{0}, X]$ as follows. For $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$ define $W(\Delta) = Y(\Delta) \wedge Z(\Delta)$, $Y_1(\Delta) = Y(\Delta) \ominus W(\Delta)$, $Z_1(\Delta) = Z(\Delta) \ominus W(\Delta)$. Extend Y_1, Z_1 and W to $\mathcal{B}(\mathbb{R})$ as we have done before. It is straightforward to show that $Y_1 \oplus Z_1 \oplus W \in [\widehat{0}, X]$ and that $Y = Y_1 \oplus W$, $Z = Z_1 \oplus W$. It follows that $[\widehat{0}, X]$ is a Boolean algebra. □

It is an open question whether Lemma 3.9 holds for lattice ordered effect algebras, MV-effect algebras and orthomodular lattices. We shall show later that these results hold for $X \in \mathcal{M}_F(\mathcal{E})$.

A measurement $X \in \mathcal{M}(\mathcal{E})$ is **globally sharp** if $X(\Delta) \in \mathcal{E}_S$ for all $\Delta \in \mathcal{B}(\mathbb{R})$. We call X **locally sharp** in $[\widehat{0}, Y]$ if $X \in [\widehat{0}, Y]_S$. Of course, X can be locally sharp in $[\widehat{0}, Y]$, $X \in [\widehat{0}, Z]$ but not locally sharp in $[\widehat{0}, Z]$. Indeed, $Y \in [\widehat{0}, Y]_S$ but in simple examples $Y \in [\widehat{0}, Z] \smallsetminus [\widehat{0}, Z]_S$. In a similar way local sharpness does not imply global sharpness. The next result shows that global sharpness implies local sharpness.

Lemma 3.10. *If X is globally sharp and $X \in [\widehat{0}, Y]$, then $X \in [\widehat{0}, Y]_S$.*

Proof. Suppose $Z \le X$ and $Z \le Y \ominus X$. Then $Z\left(\{0\}'\right) \le X\left(\{0\}'\right)$ and

$$Z\left(\{0\}'\right) \le Y\left(\{0\}'\right) \ominus X\left(\{0\}'\right) \le 1 - X\left(\{0\}'\right)$$

Since $X\left(\{0\}'\right) \in \mathcal{E}_S$ we have that $Z\left(\{0\}'\right) = 0$. Hence, $Z\left(\{0\}\right) = 1$ so that $Z = \widehat{0}$. It follows that $X \in [\widehat{0}, Y]_S$. $\square$

If $X \in \mathcal{M}(\mathcal{E})$ and $f \colon \mathbb{R} \to \mathbb{R}$ is a Borel function, we define $f(X) \in \mathcal{M}(\mathcal{E})$ by $f(X)(\Delta) = X\left(f^{-1}(\Delta)\right)$ for every $\Delta \in \mathcal{B}(\mathbb{R})$. In the usual way we have that

$$E_S\left[f(X)\right] = \int f(\lambda) s\left[X(d\lambda)\right]$$

if the integral exists.

4 Finite Measurements

We may think of a finite measurement $X \in \mathcal{M}_F(\mathcal{E})$ as a set of ordered pairs $\{(\lambda_i, a_i)\}_{i=1}^n$ where $\lambda_i \in \mathbb{R}$ satisfy $\lambda_i \ne \lambda_j$, $i \ne j$, and $a_i \in \mathcal{E}$ satisfy $a_i \ne 0$ and $\oplus a_i = 1$. We call $\Lambda(X) = \{\lambda_1, \ldots, \lambda_n\}$ the **value set** of X and think of X as a function $X \colon \Lambda(X) \to \mathcal{E}$ given by $X(\lambda_i) = a_i$, $i = 1, \ldots, n$. It follows from our previous results that for $X = \{(\lambda_i, a_i)\}$, $Y = \{(\mu_j, b_j)\} \in \mathcal{M}_F(\mathcal{E})$, $X \oplus Y$ exists if

$$p(X, Y) = \bigoplus_{\lambda_i \ne 0} a_i \bigoplus \bigoplus_{\mu_j \ne 0} b_j$$

is defined and in this case $(X \oplus Y)(\lambda_i) = a_i$ if $\lambda_i \in \Lambda(X) \smallsetminus \Lambda(Y)$, $\lambda_i \ne 0$, $(X \oplus Y)(\mu_j) = b_j$ if $\mu_j \in \Lambda(Y) \smallsetminus \Lambda(X)$, $\mu_j \ne 0$, and $(X \oplus Y)(\lambda_i) = a_i \oplus b_j$

if $\lambda_i = \mu_j \neq 0$. Finally, $(X \oplus Y)(0) = p(X,Y)'$ if $p(X,Y) \neq 1$. We thus see that

$$\Lambda(X \oplus Y) = [\Lambda(X) \cup \Lambda(Y)] \smallsetminus \{0\}$$

if $p(X,Y) = 1$ and

$$\Lambda(X \oplus Y) = \Lambda(X) \cup \Lambda(Y) \cup \{0\}$$

if $p(X,Y) \neq 1$. Moreover, by Theorem 3.4 we have that $X \leq Y$ if and only if $\Lambda(X) \subseteq \Lambda(Y) \cup \{0\}$ and $X(\lambda) \leq Y(\lambda)$ for all $\lambda \in \Lambda(X) \smallsetminus \{0\}$. For $\lambda \in \mathbb{R}$ let $f_\lambda \colon \mathbb{R} \to \mathbb{R}$ be defined by $f_\lambda(x) = \lambda x$. For $X \in \mathcal{M}_F(\mathcal{E})$ we use the notation $\lambda X = f_\lambda(X)$. Then $0X = \widehat{0}$ and if $\lambda \neq 0$ we have that $\lambda X = \{(\lambda\lambda_i, a_i)\}$. The next result shows that $X \in \mathcal{M}_F(\mathcal{E})$ if and only if X is a "linear combination" of $(1,0)$-measurements.

Lemma 4.1. *An $X \in \mathcal{M}(\mathcal{E})$ is finite if and only if $X = \lambda_1\widehat{a}_1 \oplus \cdots \oplus \lambda_n\widehat{a}_n$ where $\oplus a_i = 1$.*

Proof. If X has the above form, then X is clearly finite. Conversely, suppose X is finite. If X is a constant $X(\{\lambda\}) = 1$ then $X = \lambda\widehat{1}$. Suppose $X = \{(\lambda_i, a_i)\}_{i=1}^n$ where $\lambda_i \neq 0$, $i = 1, \ldots, n$. Then $X = \lambda_1\widehat{a}_1 \oplus \cdots \oplus \lambda_n\widehat{a}_n$. Otherwise, $X = \{(\lambda_i, a_i)\}_{i=1}^n$ where $\lambda_n = 0$ and $\lambda_i \neq 0$, $i = 1, \ldots, n-1$. Then $X = \lambda_1\widehat{a}_1 \oplus \cdots \oplus \lambda_{n-1}\widehat{a}_{n-1}$. □

If the λ_i are distinct and $a_i \neq 0$, $i = 1, \ldots, n$, then the decomposition in Lemma 4.1 is unique up to order. Elements $a_1, \ldots, a_n \in \mathcal{E}$ **coexist** if $a_1 \oplus \cdots \oplus a_n$ is defined.

Theorem 4.2. *If $X \in \mathcal{M}_F(\mathcal{E})$ with $X \neq \widehat{0}$, then there exist coexisting elements $a_1, \ldots, a_n \in \mathcal{E}$ with $a_i \neq 0$ such that*

$$[\widehat{0}, X] \approx [0, a_1] \times \cdots \times [0, a_n] \tag{4.1}$$

Conversely, if $a_1, \ldots, a_n$ coexist with $a_i \neq 0$, then there exists an $X \in \mathcal{M}_F(\mathcal{E})$ such that (4.1) holds where $\approx$ denotes σ-isomorphic.

Proof. Let $X(\lambda_i) = b_i$ with $\oplus b_i = 1$. If $\lambda_i \neq 0$ for every i, let $a_i = b_i$ and otherwise we can assume without loss of generality that $\lambda_{n+1} = 0$ and we let $a_i = b_i$, $i = 1, \ldots, n$. Then $Y \in [\widehat{0}, X]$ if and only if $Y(\lambda_i) \leq a_i$ $i = 1, \ldots, n$.

It is easy to check that $Y \mapsto (Y(\lambda_1, \dots, Y(\lambda_n))$ is σ-isomorphism from $[\widehat{0}, X]$ onto $[0, a_1] \times \cdots \times [0, a_n]$. Conversely, if $a_1, \dots, a_n$ coexist, let

$$\Lambda(X) = \{\lambda_1, \dots, \lambda_n\} \subseteq \mathbb{R} \setminus \{0\}$$

and define $X \colon \Lambda(X) \to \mathcal{E}$ by $X(\lambda_i) = a_i$. If $\oplus a_i \neq 1$, extend X to $\Lambda(X) \cup \{0\}$ by defining $X(0) = (\oplus a_i)'$. Then $[\widehat{0}, X]$ is σ-isomorphic to $[0, a_1] \times \cdots \times [0, a_n]$. □

Corollary 4.3. *Let $X \in \mathcal{M}_F(\mathcal{E})$ with $X \neq \widehat{0}$. If $\mathcal{E}$ has any of the following properties, then so does $[\widehat{0}, X]$.* (i) *Lattice ordered,* (ii) *MV-effect algebra,* (iii) *orthoalgebra,* (iv) *orthomodular poset,* (v) *orthomodular lattice,* (vi) *Boolean algebra.*

Proof. (i) If $\mathcal{E}$ is lattice ordered, $[0, a]$ is also lattice ordered and the result follows from Theorem 4.2. (ii) If $\mathcal{E}$ is an MF-effect algebra, $[0, a]$ is lattice ordered by (i). If $b, c \in [0, a]$ and $b \wedge c = 0$ then $b \oplus c = b \vee c$ in $[0, a]$ so $[0, a]$ is an MV-effect algebra and the result follows from Theorem 4.2. (iii), (iv) and (vi) were proved in Lemma 3.9 and also follow from Theorem 4.2. (v) follows from (iv) and (i). □

Example 2. If $X \in \mathcal{M}_F([0, 1])$ has n nonzero values then $[\widehat{0}, X] \approx [0, 1]^n$. In this case $Y \in [\widehat{0}, X]$ can be locally sharp but not globally sharp. For example, if Y corresponds to $(1, 0, \dots, 0) \in [0, 1]^n$ then Y is locally sharp in $[\widehat{0}, X]$.

Example 3. Let $X \in \mathcal{M}_F(\mathcal{E}(\mathcal{H}))$ be sharp. Suppose that $X = \{(\lambda_i, P_i)\}_{i=1}^n$ where $\lambda_i \neq 0$ and $\sum P_i = I$. Then $P_i\mathcal{H}$ is a Hilbert space whose dimension is not greater than that of $\mathcal{H}$ and we have

$$[\widehat{0}, X] \approx \mathcal{E}(P_1\mathcal{H}) \times \cdots \times \mathcal{E}(P_n\mathcal{H})$$

In particular, if the P_i are one-dimensional projections then $[\widehat{0}, X] \approx [0, 1]^n$.

Example 4. Let S be a finite set and let 2^S have its Boolean algebra structure. For $A \in 2^S$ we can identify $\widehat{A}$ with χ_A. If $X = \{(\lambda_i, A_i)\} \in \mathcal{M}_F(2^S)$ we can identify X with the function $f \colon S \to \mathbb{R}$ given by $f(s) = \lambda_i$ if $s \in A_i$. We then have that $X = f^{-1}$. In this way we identify $\mathcal{M}_F(2^S)$ with $\mathbb{R}^S$. For $f \in \mathbb{R}^S$ define the **support** of f by

$$\text{supp}(f) = \{s \in S \colon f(s) \neq 0\}$$

For $f, g \in \mathbb{R}^S$ we see that $f \perp g$ if and only if $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$ and in this case $f \oplus g = f + g$. Also, $f \leq g$ if and only if $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$ and $f(s) = g(s)$ for all $s \in \mathrm{supp}(f)$. If $f \in \mathbb{R}^S$ then $[\widehat{0}, f] \approx 2^{|\mathrm{supp}(f)|}$ where $|\mathrm{supp}(f)|$ is the cardinality of $\mathrm{supp}(f)$.

Let N be a positive integer with prime factorization

$$N = p_1^{k_1} \ldots p_j^{k_j}$$

where the p_i are distinct primes and $k_i \geq 1$, $i = 1, \ldots, j$. Of course, N has $(k_1 + 1) \cdots (k_j + 1)$ positive integer divisors. Let $\mathcal{D}(N)$ be the set of all positive integer divisors of N and for $m, n \in \mathcal{D}(N)$ if m divides n we write $m \mid n$. It is well known that $(\mathcal{D}(N), |)$ is a lattice. For $m, n \in \mathcal{D}(N)$ we say that $m \oplus n$ is defined if $mn \mid N$ in which case $m \oplus n = mn$. Then $(\mathcal{D}(N), 1, N, \oplus)$ is an effect algebra called a **divisor effect algebra**. Notice that the effect algebra order $m \leq n$ is the usual order $m \mid n$ and n is sharp in $\mathcal{D}(N)$ if and only if n and N/n are relatively prime. Also, n is an atom in $\mathcal{D}(N)$ if and only if n is prime. Observe that the effect algebra structure of $\mathcal{D}(N)$ is an MV-effect algebra.

Theorem 4.4. *An effect algebra $\mathcal{E}$ is a divisor effect algebra if and only if $\mathcal{E} \approx [\widehat{0}, X]$ where $X \in \mathcal{M}_F(C_n)$ for some $n \in \mathbb{N}$.*

Proof. Suppose that $\mathcal{E} = (\mathcal{D}(N), 1, N, \oplus)$ is a divisor effect algebra and $N = p_1^{k_1} \ldots p_j^{k_j}$. Let $n = \sum k_i$, $\{\lambda_1, \ldots, \lambda_n\} \subseteq \mathbb{R} \smallsetminus \{0\}$ and define $X \in \mathcal{M}_F(C_n)$ by $X = \{(\lambda_i, k_i a)\}_{i=1}^{j}$ where a is the atom in the chain C_n. Then $X_1 \leq X$ if and only if X_1 has the form $X_1 = \{(\lambda_i, \ell_i a)\}$ where $0 \leq \ell_i \leq k_i$ and we delete the terms with $\ell_i = 0$ and define $X(0) = (n - \sum \ell_i)a$. Thus, the elements of $[\widehat{0}, X]$ are in a natural one-to-one correspondence with the elements of $\mathcal{D}(N)$. Moreover, if $X_2(\lambda_i) = r_i a$ and $X_2 \in [\widehat{0}, X]$ then $X_1 \perp X_2$ if and only if $\ell_i + r_i \leq k_i$ in which case

$$(X_1 \oplus X_2)(\lambda_i) = (\ell_i + r_i)a$$

Thus, if X_1 corresponds to $q = p_1^{\ell_1} \ldots p_j^{\ell_j}$ and X_2 corresponds to $p = p_1^{r_1} \ldots p_j^{r_j}$ then $X_1 \oplus X_2$ corresponds to qp. This shows that $\mathcal{E} \approx [\widehat{0}, X]$. Conversely, let $X \in \mathcal{M}_F(C_n)$ and let $\mathcal{E} = [\widehat{0}, X]$. Then X has the form $X(\lambda_i) = k_i a$, $i = 1, \ldots, j$, $\lambda_i \neq 0$. Just as before there is a one-to-one correspondence between elements of $[\widehat{0}, X]$ and elements of $\mathcal{D}(N)$ which preserves $\oplus$, where $N = p_1^{k_1} \ldots p_j^{k_j}$ □

It follows from Theorem 4.4 that $[\widehat{0}, X]$ for $X \in \mathcal{M}_F(C_n)$ is an MV-effect algebra. Also, since the sharp elements of an MV-effect algebra form a Boolean algebra this holds for $[\widehat{0}, X]_S$. If N has the form p^n then $\mathcal{D}(N)$ is the chain C_n and if N has the form $N = p_1 \dots p_n$ then $\mathcal{D}(N)$ is the Boolean algebra 2^n. We conclude that $[\widehat{0}, X]$ for $X \in \mathcal{M}_F(C_n)$ is either a completely sharp Boolean algebra, a completely unsharp chain or an MV-effect algebra between these extremes and if $n \geq 3$ all of these possibilities occur. The next result follows from Theorems 4.2 and 4.4.

Corollary 4.5. *If $N \in \mathbb{N}$ with prime factorization $N = p_1^{k_1} \dots p_j^{k_j}$, then $\mathcal{D}(N) \approx C_{k_1} \times \dots \times C_{k_j}$.*

Example 5. Let $X \in \mathcal{M}_F(C_4)$ be defined by

$$X = \{(\lambda_1, a), (\lambda_2, a), (\lambda_3, 2a)\}$$

where $\{\lambda_1, \lambda_2, \lambda_3\} \subseteq \mathbb{R} \setminus \{0\}$. Then $[\widehat{0}, X]$ contains $\widehat{0}$, X and the following ten elements.

$$\begin{aligned}
X_1 &= \{(\lambda_1, a), (0, 3a)\}, & X_1' &= \{(\lambda_2, a), (\lambda_3, 2a), (0, a)\}\\
X_2 &= \{(\lambda_2, a), (0, 3a)\}, & X_2' &= \{(\lambda_1, a), (\lambda_2, 2a), (0, a)\}\\
X_3 &= \{(\lambda_3, a), (0, 3a)\}, & X_3' &= \{(\lambda_1, a), (\lambda_2, a), (\lambda_3, a)(0, a)\}\\
X_4 &= \{(\lambda_3, 2a), (0, 2a)\}, & X_4' &= \{(\lambda_1, a), (\lambda_2, a), (0, 2a)\}\\
X_5 &= \{(\lambda_1, a), (\lambda_3, a)(0, 2a)\}, & X_5' &= \{(\lambda_2, a), (\lambda_3, 2a), (0, 2a)\}
\end{aligned}$$

Then $[\widehat{0}, X] \approx \mathcal{D}(60)$ under the isomorphism:

$$\begin{gathered}
\widehat{0} \to 1, \quad X \to 60, \quad X_1 \to 5, \quad X_1' \to 12, \quad X_2 \to 3, \quad X_2' \to 20\\
X_3 \to 2, \quad X_3' \to 30, \quad X_4 \to 4, \quad X_4' \to 15, \quad X_5 \to 10, \quad X_5' \to 6
\end{gathered}$$

5 Sequential Effect Algebras

We now consider various types of products of measurements on a σ-SEA $(\mathcal{E}, 0, 1, \oplus, \circ)$. Let $X \in \mathcal{M}_F(\mathcal{E})$ with $X = \{(\lambda_i, a_i)\}$ and let $Y \in \mathcal{M}(\mathcal{E})$. We define the **conditional measurement** $[Y \mid X] \colon \mathcal{B}(\mathbb{R}) \to \mathcal{E}$ by

$$[Y \mid X](\Delta) = \sum a_i \circ Y(\Delta)$$

It is easy to check that $[Y \mid X] \in \mathcal{M}(\mathcal{E})$. We say that $X, Y \in \mathcal{M}(\mathcal{E})$ are **compatible** if $X(\Delta_1) \circ Y(\Delta_2) = Y(\Delta_2) \circ X(\Delta_1)$ for all $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$. For $X \in \mathcal{M}_F(\mathcal{E})$ and $Y \in \mathcal{M}(\mathcal{E})$, if X and Y are compatible, then $[Y \mid X] = Y$ but the converse does not hold [1]. Since

$$[\widehat{b} \mid \widehat{a}]\,(\{1\}) = a \circ b \oplus a' \circ b$$

and

$$[\widehat{b} \mid \widehat{a}]\,(\{0\}) = a \circ b' \oplus a' \circ b'$$

we see that

$$[\widehat{b} \mid \widehat{a}] = (a \circ b' \oplus a' \circ b)^\wedge$$

For $a, b \in \mathcal{E}$ and $s \in \Omega(\mathcal{E})$ we define the **conditional probability of** b **given** a in the states s by $s(b \mid a) = s(a \circ b)/s(a)$ provided that $s(a) \neq 0$. For $Y \in \mathcal{M}(\mathcal{E})$ the **conditional expectation of** Y **given** $a \in \mathcal{E}$ in the state s is

$$E_s(Y \mid a) = \int \lambda s\,[X(d\lambda) \mid a]$$

For $a, b \in \mathcal{E}$ we have that

$$E_s\left([\widehat{b} \mid \widehat{a}]\right) = s(a \circ b) + s(a' \circ b) = s(a)s(b \mid a) + s(a')s(b \mid a')$$

and in general

$$\begin{aligned} E_s\left([Y \mid X]\right) &= \int \lambda \sum s\left(a_i \circ Y(d\lambda)\right) = \int \lambda \sum s(a_i) s\,[Y(d\lambda) \mid a_i] \\ &= \sum s(a_i) E_s(Y \mid a_i) \end{aligned}$$

Lemma 5.1. *If $X \in \mathcal{M}_F(\mathcal{E})$ and $Y, Z \in \mathcal{M}(\mathcal{E})$ with $Y \oplus Z$ defined then*

$$[(Y \oplus Z) \mid X] = [Y \mid X] \oplus [Z \mid X]$$

Proof. Since $Y \oplus Z$ is defined we have that

$$\begin{aligned} [Y \mid X]\left(\{0\}'\right) \oplus [Z \mid X]\left(\{0\}'\right) &= \sum a_i \circ Y\left(\{0\}'\right) \oplus \sum a_i \circ Z\left(\{0\}'\right) \\ &= \sum a_i \circ \left[Y\left(\{0\}'\right) \oplus Z\left(\{0\}'\right)\right] \end{aligned}$$

so $[Y \mid X] \perp [Z \mid X]$. For $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$ we have that

$$\begin{aligned}[(Y \oplus Z) \mid X](\Delta) &= \sum a_i \circ [(Y \oplus Z)(\Delta)] = \sum a_i \circ [Y(\Delta) \oplus Z(\Delta)] \\ &= \sum a_i \circ Y(\Delta) \oplus \sum a_i \circ Z(\Delta) \\ &= [Y \mid Z](\Delta) \oplus [Z \mid X](\Delta)\end{aligned}$$

Hence, $[(Y \oplus Z) \mid X] = [Y \mid X] \oplus [Z \mid X]$. $\square$

We now define a type of sequential product for finite measurements. For $X, Y \in \mathcal{M}_F(\mathcal{E})$ with $X = \{(\lambda_i, a_i)\}$, $Y = \{(\mu_j, b_j)\}$ we define $X\&Y\colon \mathcal{B}(\mathbb{R}^2) \to \mathcal{E}$ by

$$(X\&Y)(\Delta) = \bigoplus_{i,j} \{a_i \circ b_j \colon (\lambda_i, \mu_j) \in \Delta\}$$

Then $X\&Y$ is a normalized effect-valued measure on $\mathcal{B}(\mathbb{R}^2)$ because

$$\underset{i,j}{\oplus} a_i \circ b_j = \underset{i}{\oplus} a_i \circ \oplus b_j = \oplus a_i \circ 1 = \oplus a_i = 1$$

We call $X\&Y$ X **and then** Y and think of $X\&Y$ as a measurement based on $\mathcal{B}(\mathbb{R}^2)$ in which X is performed first and Y second. We also write

$$X\&Y = \{((\lambda_i, \mu_j), a_i \circ b_j)\}$$

where it is assumed that terms with $a_i \circ b_j = 0$ are omitted. Notice that for $a, b \in \mathcal{E}$ we have that

$$\widehat{a}\&\widehat{b} = \{((1,1), a \circ b), ((0,0), a' \circ b'), ((1,0), a \circ b'), ((0,1), a' \circ b)\}$$

It follows that $\widehat{a}\&\widehat{b} = \widehat{b}\&\widehat{a}$ if and only if $a \circ b = b \circ a$; that is, a and b are compatible. Of course $\widehat{a}\&\widehat{b} \neq (a \circ b)^\wedge$.

The **marginal measurements** $(X\&Y)(\Delta \times \mathbb{R})$ and $(X\&Y)(\mathbb{R} \times \Delta)$ as functions of Δ are both elements of $\mathcal{M}_F(\mathcal{E})$ and satisfy

$$(X\&Y)(\Delta \times \mathbb{R}) = \underset{i,j}{\oplus} \{a_i \circ b_j \colon \lambda_i \in \Delta\} = \oplus \{a_i \colon \lambda_i \in \Delta\} = X(\Delta) \tag{5.1}$$

$$(X\&Y)(\mathbb{R} \times \Delta) = \underset{i,j}{\oplus} \{a_i \circ b_j \colon \mu_j \in \Delta\} = \oplus a_i \circ Y(\Delta) = [Y \mid X](\Delta) \tag{5.2}$$

Equation (5.1) says that the second measurement does not affect the first, while Equation (5.2) says that the first measurement affects the second.

We define the **expectation** $E_s(X\&Y)$ **in the state** s by

$$E_s(X\&Y)=\sum_{i,j}s(a_i\circ b_j)(\lambda_i,\mu_j)\in\mathbb{R}^2$$

We then have

$$\begin{aligned}E_s(X\&Y)&=\left(\sum_{i,j}s(a_i\circ b_j)\lambda_i,\sum_{i,j}s(a_i\circ b_j)\mu_j\right)\\&=\left(\sum\lambda_i s(a_i),E_s\left([Y\mid X]\right)\right)\\&=\left(E_s(X),E_s\left([Y\mid X]\right)\right)\end{aligned}$$

We now briefly compare $X\&Y$ to a different candidate for a sequential product of measurements. For $X,Y\in\mathcal{M}(\mathcal{E})$ define $X\alpha Y)\colon\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R})\to\mathcal{E}$ by

$$(X\alpha Y)(\Delta_1\times\Delta_2)=X(\Delta_1)\circ Y(\Delta_2)$$

In this case the marginal measurements satisfy $(X\alpha Y)(\Delta\times\mathbb{R})=X(\Delta)$ and $(X\alpha Y)(\mathbb{R}\times\Delta)=Y(\Delta)$. It follows that $X\&Y\neq X\alpha Y$. To directly see that $X\&Y\neq X\alpha Y$ we compare them on the set $\{\lambda_1,\lambda_2\}\times\{\mu\}$. We then have that

$$\begin{aligned}(X\&Y)\left(\{\lambda_1,\lambda_2\}\times\{\mu\}\right)&=(X\&Y)\left(\{(\lambda_1,\mu),(\lambda_2,\mu)\}\right)\\&=X\left(\{\lambda_1\}\right)\circ Y\left(\{\mu\}\right)\oplus X\left(\{\lambda_2\}\right)\circ Y\left(\{\mu\}\right)\end{aligned}\tag{5.3}$$

and

$$\begin{aligned}(X\alpha Y)\left(\{\lambda_1,\lambda_2\}\times\{\mu\}\right)&=X\left(\{\lambda_1,\lambda_2\}\right)\circ Y\left(\{\mu\}\right)\\&=\left[X\left(\{\lambda_1\}\right)\oplus X\left(\{\lambda_2\}\right)\right]\circ Y\left(\{\mu\}\right)\end{aligned}\tag{5.4}$$

In general, the right hand sides of (5.3) and (5.4) do not agree.

Unfortunately, $X\alpha Y$ does not necessarily extend to an effect-valued measure on $\mathcal{B}(\mathbb{R}^2)$. To show this, suppose that $X\alpha Y$ has an extension to $\mathcal{B}(\mathbb{R}^2)$. For $\Delta_1,\Delta_2\in\mathcal{B}(\mathbb{R}^2)$ we then have that

$$\begin{aligned}Y(\Delta_2)&=(X\alpha Y)\left((\Delta_1\cup\Delta_1')\times\Delta_2\right)\\&=(X\alpha Y)\left[(\Delta_1\times\Delta_2)\cup(\Delta_1'\times\Delta_2)\right]\\&=(X\alpha Y)(\Delta_1\times\Delta_2)\oplus(X\alpha Y)(\Delta_1'\times\Delta_2)\\&=X(\Delta_1)\circ Y(\Delta_2)\oplus X(\Delta_1)'\circ Y(\Delta_2)\end{aligned}\tag{5.5}$$

Now (5.5) does not hold in general. For example, if (5.5) holds in $\mathcal{E}(\mathcal{H})$ then $X(\Delta_1)$ and $Y(\Delta_2)$ commute [10].

Again, let $X, Y \in \mathcal{M}_F(\mathcal{E})$ with $X = \{(\lambda_i, a_i)\}$, $Y = \{(\mu_j, b_j)\}$. We define another type of sequential product $X \circ Y \in \mathcal{M}_F(\mathcal{E})$ as follows. The value set $\Lambda(X \circ Y) \subseteq \Lambda(X)\Lambda(Y)$ and

$$X \circ Y\left(\{\lambda_i\mu_j\}\right) = \bigoplus_{r,s} \{a_r \circ b_s \colon \lambda_r\mu_s = \lambda_i\mu_j\} \tag{5.6}$$

if the right hand side of (5.6) is not 0 and otherwise $\lambda_i\mu_j \notin \Lambda(X \circ Y)$. The next result shows that $X \circ Y$ has some of the important properties of the sequential product $a \circ b$.

Theorem 5.2. *For $X, Y \in \mathcal{M}_F(\mathcal{E})$, the product $X \circ Y$ has the following properties.*
(i) $\widehat{1} \circ X = X \circ \widehat{1} = X$ *and* $\widehat{0} \circ X = X \circ \widehat{0} = \widehat{0}$.
(ii) $X \circ Y = \widehat{0}$ *implies that* $Y \circ X = \widehat{0}$.
(iii) $(a \circ b)^\wedge = \widehat{a} \circ \widehat{b}$.
(iv) *If $Z \in \mathcal{M}_F(\mathcal{E})$ and $Y \perp Z$, then $X \circ Y \perp X \circ Z$ and*

$$X \circ (Y \oplus Z) = X \circ Y \oplus X \circ Z$$

Proof. The proofs of (i) and (ii) are straightforward. (iii) Notice that

$$(a \circ b)^\wedge = \{(1, a \circ b), (0, (a \circ b)')\}$$

Since $\widehat{a} \circ \widehat{b}\,(\{1\}) = a \circ b$ and

$$\widehat{a} \circ \widehat{b}\,(\{0\}) = a \circ b' \oplus a' \circ b \oplus a' \circ b' = (a \circ b)'$$

we have that $(a \circ b)^\wedge = \widehat{a} \circ \widehat{b}$. (iv) First,we have that

$$\begin{aligned}\widehat{a} \circ (\widehat{b} \oplus \widehat{c}) &= \{(1, a), (0, a')\} \circ \{(1, b \oplus c), (0, (b \oplus c)')\}\\ &= \left\{(1, a \circ (b \oplus c)), \left(0, [a \circ (b \oplus c)]'\right)\right\}\end{aligned}$$

Moreover,

$$\begin{aligned}(\widehat{a} \circ \widehat{b}) \oplus (\widehat{a} \circ \widehat{c}) &= \{(1, a \circ b), (0, (a \circ b)')\} \oplus \{(1, a \circ c), (0, (a \circ c)')\}\\ &= \{(1, a \circ b \oplus a \circ c), (0, (a \circ b \oplus a \circ c)')\}\\ &= \left\{(1, a \circ (b \oplus c)), \left(0, [a \circ (b \oplus c)]'\right)\right\}\end{aligned}$$

Hence,

$$\widehat{a} \circ (\widehat{b} \oplus \widehat{c}) = (\widehat{a} \circ \widehat{b}) \oplus (\widehat{a} \circ \widehat{c})$$

Applying Lemma 4.1 gives

$$\widehat{a} \circ (Y \oplus Z) = \widehat{a} \circ Y \oplus \widehat{a} \circ Z$$

A little more work extends this to the general result. □

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