THE MYSTERIOUS 2-CROWN

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Abstract. We show that the 2-crown is not coproductive, which is to say that the class of those bounded distributive lattices whose Priestley spaces lack any copy of the 2-crown is not productive. We do this by first exhibiting a general construction to handle questions of this sort. We then use a particular instance of this constrution, along with some of the combinatorial features of projective planes, to show that the 2-crown is not coproductive.

1. Introduction

In any context in which a class of algebraic objects is dual to a class of geometric objects whose structures carry a partial order, it naturally provokes interest when the presence or absence of a particular finite poset in the geometric objects can be characterized by the satisfaction of first-order conditions by the algebraic objects. Bergman's recent article [6], which treats (somewhat more than) finite posets in the spectra of rings, is a good example of this line of investigation, but other examples occur earlier in the literature. In [1], Adams and Beazer provided, for each positive integer n, a list of first-order sentences in the language of bounded distributive lattices, whose satisfaction was both necessary and sufficient to rule out the occurrence of an n-chain in the Priestley space of the lattice. (This directly generalized the classical result that a bounded distributive lattice is complemented iff its Priestley space is trivially ordered.) And much earlier still in [14], Monteiro showed that a topological space is relatively normal, a first-order condition on the lattice of open sets, iff the prime ideals of open sets formed a forest, a condition which is equivalent to the absence of a copy of the three-element poset $\{a, b < c\}$, a unrelated to b, in the Priestley space of the lattice of open sets.

The authors have undertaken a general investigation of this phenomenon in the setting of Priestley duality, and this article is a continuation of that investigation. We are specifically interested in the question of which connected finite posets P have the feature that the absence of a copy of P in the Priestley space of a bounded distributive lattice can be captured by the satisfaction of a first-order formula in the lattice. We say *the prohibition of P is first-order.* In [3] we showed that the

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prohibition of a tree is first-order, and in [4] that, among connected finite posets with top element, only trees had this property.

Our interest has therefore focused on general finite connected posets P, i.e., on the case in which more than one maximal element may be present. In [5] we showed that the prohibition of acyclic posets is first-order in general. However, the general case diverges fundamentally from the topped case inasmuch as generic cycles are not just diamonds, but may also be k-crowns. (These posets will be defined in Section 2.) Also in [5], the first steps in proving the converse were taken, by showing that the prohibition of the k-crown, $k \geq 3$, is not first-order. The two-crown, however, resisted our techniques, and it is the purpose of this article to fill this gap by showing that the prohibition of the 2-crown is not first-order. This brings us a significant step closer to our overall objective, which is to establish the conjecture in its most general form.

Conjecture 1.1. *The prohibition of a finite connected poset P is first-order iff P is acyclic.*

Another question connected with these phenomena is that of the order structure of the infinite coproduct of Priestley spaces. This is a compactification of the disjoint union of the summands, and its structure is by no means well understood. In particular, one asks which *finite* posets cannot occur in such a coproduct without occurring in one of the summands. We say that such a poset is *coproductive.* In the topped case, coproductivity coincides with first-order definability, i.e., with the property of being a tree. Although we strongly believe this to be true in general, we cannot yet claim it. Just the same, we showed in [5] that k-crowns, $k \geq 3$, are not coproductive. And here we do the same, and a little more, for the 2-crown.

2. Preliminaries

In this section and the next, we fix the basic notational conventions, and briefly review the background results, which will allow the reader to make sense of what follows. A *configuration* is a finite connected partially ordered set. An *embedding* of a poset P in a poset Q is a mapping $m : P \to Q$ such that $x \leq y$ in P iff $m(x) \leq m(y)$ in Q. If such an embedding exists we will say that Q *contains a copy of* P, and write $P \hookrightarrow Q$. We indicate the negative by writing $P \not\hookrightarrow Q$. In this section we clear the ground for what follows; we begin by fixing notation.

A subset M of a partially ordered set $(X, \leq)$ is termed a *downset* (*upset*) provided that

$$x \leq y \in M \Longrightarrow x \in M \ (x \geq y \in M \Longrightarrow x \in M),$$

and for any subset $M \subseteq X$ we denote the downset (upset) it generates by

$$\downarrow M \equiv \{x : \exists\, y \in M \ \ (x \leq y)\} \ (\uparrow M \equiv \{x : \exists\, y \in M \ \ (x \geq y)\}\,.$$

Recall that a *Priestley space* is a compact ordered space X such that whenever $x \not\leq y$ there is a clopen downset $U \subseteq X$ such that $x \notin U \ni y$. We will work in the context of the famous *Priestley duality* connecting the category **PSp** of Priestley spaces and order-preserving continuous functions with the category **DLat** of bounded distributive lattices and bounded lattice homomorphisms. This duality is usually described by these formulas.

$$\mathcal{P}(L) \equiv \{x : x \text{ is a proper prime ideal of } L\}\,,\ \mathcal{P}(h)(x) \equiv h^{-1}[x]\,,$$
$$\mathcal{D}(X) \equiv \{U : U \text{ is a clopen downset of } X\}\,,\ \mathcal{D}(f)(U) \equiv f^{-1}[U]\,.$$

$\mathcal{P}(L)$ is endowed with a suitable topology, and it plays a pivotal role because it is the topology which controls the approximation of elements of the remainder in a coproduct by elements in the summands. However, the topology plays no role in the calculations in this article, for the intricacies of approximation are captured entirely by the appropriate description of the coproduct.

Let $\{X_i : i \in J\}$ be a family of nonempty Priestley spaces. Since the coproduct $X \equiv \coprod_{i\in J} X_i$ must be compact, it cannot simply be the disjoint union of the summands X_i. Now X does contain (a copy of) the disjoint union of the summands, and the additional points constitute what we call the *remainder of the coproduct.* But it is not difficult to show (see [5]) that X can be organized as the disjoint union

$$X = \bigcup_{u\in\beta J} X_u,$$

indexed by the Čech-Stone compactification βJ of the index set J. (We adhere to the convention that the points of βJ are the ultrafilters on J.) The sets X_u are order-independent, meaning that no element of one is comparable with any element of another. This fact has the important consequence that *a configuration, which is after all connected, can embed in X only if it embeds in some X_u.* Furthermore, each summand X_i can be identified with the set indexed by the principal ultrafilter $\{M \subseteq J : i \in M\}$, and this identification provides the canonical coproduct insertions. Thus the remainder consists precisely of the points of the X_u's indexed by free ultrafilters u.

A configuration P is said to be *coproductive* if for any coproduct $X = \coprod_J X_i$ of Priestley spaces,

$$P \hookrightarrow X \Longrightarrow \exists\, i \in J \ \ (P \hookrightarrow X_i)\,.$$

(Thus P is coproductive precisely when the class of lattices whose Priestley spaces contain no copy of P is productive.) In light of the organization of the coproduct outlined in the previous paragraph we may reformulate this as follows. P is coproductive iff for any family $\{X_i : i \in J\}$ of Priestley spaces, if $P \hookrightarrow X_u$ for some free ultrafilter u then P must embed in some X_i with $i \in J$.

A result of Koubek and Sichler [11] shows that in general, X_u is the Priestley space of the ultraproduct $\prod_u \mathcal{D}(X_i)$. Since first-order theories are closed under ultraproducts by Łoś's Theorem [12], we are led to an important deduction. *If $P \not\hookrightarrow X$ determines a first-order class of distributive lattices then P is coproductive.*

3. What makes the 2-crown special

What makes the 2-crown special is precisely this. *In spite of the fact that no diamond can be embedded in a combinatorial tree and that no k-crown with $k \geq 3$ can be embedded in a combinatorial tree, it is nevertheless true that a 2-crown can be embedded in a combinatorial tree.* We elaborate briefly on this important fact in this subsection. We begin with the definitions.

A *diamond* is a configuration $\{a < b, c < d\}$ with b and c incomparable. A *k-crown* ($k \geq 2$) is a configuration $\{a_i, b_i : 0 \leq i \leq k-1\}$ such that the a_i's are incomparable to one another and the b_i's likewise, and such that $a_i \leq b_j$ iff $j = i$ or $j = i + 1 \bmod k$. Let f be an embedding of the 2-crown $C = \{a_0, a_1 < b_0, b_1\}$ into a configuration P. We say that f is *improper* if there is some $c \in P$ such that $a_i < c < b_j$ for all i, j, and *proper* otherwise. We speak of $f(C)$ as an *improper 2-crown*, or *proper 2-crown in P*, as the case may be.

The distinction between properly and improperly embedded 2-crowns divides the corresponding coproductivity issue into two parts. Therefore, in addition to the question of whether the 2-crown can be embedded in a coproduct when it embeds in no summand, we shall also discuss the same question with respect to properly embedded 2-crowns – that is, whether there can be a coproduct in which the 2-crown embeds *properly* even though it cannot be *properly* embedded in any summand.

For a configuration $(P, \leq)$, we denote immediate precedence by the symbol $\prec$, so that

$$a \prec b \iff b \succ a \iff (a < b \text{ and } \forall c \ \ (a \leq c \leq b \implies (c = a \text{ or } c = b)))\,.$$

We denote the corresponding symmetric relation by the symbol $\succ\!\!\prec$, so that

$$a \succ\!\!\prec b \iff (a \prec b \text{ or } a \succ b)\,.$$

Thus $(P, \succ\!\!\prec)$ is a graph which can be visualized as the Hasse diagram of P disregarding its up-down orientation.

A configuration P is a *combinatorial tree* if $(P, \succ\!\!\prec)$ is a tree in the sense of graph theory, i.e., if $(P, \succ\!\!\prec)$ is acyclic, and we sometimes refer to the configuration itself as being *acyclic*. We use the adjective 'combinatorial' to distinguish the trees we wish to consider from those studied in computer science, which have the additional feature of a top element. The following result characterizes the acyclicity of $(P, \succ\!\!\prec)$ in terms of the order. Both assertions are well-known; a proof of the second can be found in [4].

Proposition 3.1. *A configuration with top is acyclic iff it contains no diamond. A general configuration is acyclic iff it contains no diamond, no k-crown for $k \geq 3$, and no proper 2-crown.*

In the construction we will employ in the sequel, and in similar constructions we have used in previous work, we need bipartite graphs with the following features. Assume a labeling of the points of the form $1, 2, \ldots, n$ and $1', 2', \ldots, n'$, with no edges between i and j or i' and j' when $i \neq j$.

(1) There should be no edge between any i and i'.
(2) The graph should contain no copy of the (graph equivalent to) the configuration being treated.
(3) Subject to these qualifications, the graph should be as dense as possible, i.e., have as many edges as possible.

In the case of the diamond, it is sufficient to join each i with each j', $i \neq j$, and this also works for the k-crown, $k \geq 4$. Although the same construction does not suffice for the 3-crown, it is sufficient in that case to join each i and j', $i < j$, and one still has quadratically many edges in the graph. However, avoiding 2-crowns, for which the corresponding graph is a square, reduces the number of possible edges drastically to at most $\sim n^{3/2}$. Luckily, this just suffices. (See Section 5.)

4. Increasingly colorful sequences, and a construction

In this section we first introduce the notion of an increasingly colorful sequence of relations, and then use it to construct a coproduct of finite Priestley spaces which has the virtue of carrying a copy of the configuration under consideration in its remainder. We have used similar constructions in several previous articles, and we develop it here in general terms in hopes of eventually employing it to resolve the

Conjecture 1.1. We use it in Section 6 specifically to show that the 2-crown is not coproductive.

We use $\widetilde{k}$ to denote $\{i : 1 \leq i \leq k\}$. For a relation $R \subseteq \widetilde{k} \times \widetilde{k}$, we abbreviate $(i, j) \in R$ to iRj.

Definition 4.1. *Let R be a relation on $\widetilde{k}$ such that iRj implies $i \neq j$. A subset $J \subseteq \widetilde{k}$ is* independent *provided that $(J \times J) \cap R = \emptyset$. The* chromatic number *of a subset $A \subseteq \widetilde{k}$, denoted $\chi_R(A)$ or simply $\chi(A)$, is the least cardinality of a cover of A by independent sets.*

Obviously,

$$\chi(A \cup B) \leq \chi(A) + \chi(B). \tag{$\blacklozenge$}$$

Definition 4.2. *Let $k_1 < k_2 < \ldots < k_n < \ldots$ be an increasing sequence of natural numbers, and for each n let R_n be a relation on $\widetilde{k_n}$ such that*

$$\chi_n(\widetilde{k_n}) \equiv \chi_{R_n}(\widetilde{k_n}) \to \infty.$$

Such a system $(\widetilde{k_n}, R_n)$ will be referred to as an increasingly colorful sequence of relations*, or briefly, as an ICS.*

We have assembled all of the apparatus necessary for the basic construction. On the set

$$I \equiv \left\{(n, j) : n \in \mathbb{N},\ j \in \widetilde{k_n}\right\}$$

choose a free ultrafilter u. Set

$$F \equiv \{J \subseteq I : \exists m \ \ (\{n : \chi_n \{j : (n, j) \notin J\} \leq m\} \in u)\}.$$

This set is a filter by ($\blacklozenge$) and is proper because $\chi_n(\widetilde{k_n}) \to \infty$. Choose an ultrafilter

$$v \supseteq F.$$

For $J \subseteq I$ define

$$f(J) \equiv \{(n, j) : \exists i \ \ ((n, i) \in J \text{ and } iR_nj)\}.$$

Lemma 4.3. *If $J \in v$ then $f(J) \in v$.*

Proof. Suppose for the sake of argument that $J \in v$ but $f(J) \notin v$, so that $I \smallsetminus f(J) \in v$. Then by replacing J by $J \cap (I \smallsetminus f(J))$ if necessary, we may assume that $J \cap f(J) = \emptyset$. But this means that for any n, the set $\{j : (n, j) \in J\}$ is independent, i.e.,

$$\chi_n \{j : (n, j) \notin I \smallsetminus J\} = \chi_n \{j : (n, j) \in J\} = 1,$$

and hence $I \smallsetminus J \in F \subseteq v$ and $J \notin v$. □

Now we are able to present the fundamental definition of the summands X_n. Let $\left(\widetilde{k_n}, R_n\right)$ be an ICS, and let a and b represent fixed elements of P such that $a \prec b$. Let

$$X_n \equiv P \times \widetilde{k_n},$$

and write $(p, i) \in X_n$ as pi. Order X_n as follows.

(1) If $p \leq a < b \leq q$, and for every r such that $p < r < q$ we have either $r \leq a$ or $r \geq b$, then

$$pi < qj \Longleftrightarrow iR_nj.$$

(2) If $p \le a < b \le q$ and there is an r such that $p < r < q$ and such that r is incomparable to either a or b, then

$$pi < qj \Longleftrightarrow iR_n j \text{ or } i = j.$$

(3) Otherwise

$$pi < qj \Longleftrightarrow p < q \text{ and } i = j.$$

It is easy to check that this defines a partial order. Let $pi < qj < rk$; if $i \neq j$ then the first inequality is of type (1) or (2), hence the third is of type (3), $j = k$ and $pi < rj$ by (1) or (2); similarly for $j \neq k$. If $i = j = k$ then either the first or the second inequality comes under (2) and $pi < ri$ by the same rule, or both of them are from (3) and then $pi < ri$ either following (3) or (2) − in the second case with q playing the role of the incomparable element.

Now we must embed P in the coproduct X of the summands X_n just defined. This requires the following specific realization of X. Let $A_n \equiv \mathcal{D}(X_n)$ be the bounded distributive lattice formed by all downsets of X_n, and let $A \equiv \prod A_n$. Then the desired realization of X is as the Priestley space $\mathcal{P}(A)$. Thus we may achieve our objective by defining a map $m : P \to \mathcal{P}(A)$ by the rule

$$m(p) \equiv \{\alpha \in A : \{(n,j) : pj \notin \alpha_n\} \in v\}, \ p \in P.$$

Here we use α_n to denote the value of α at index n for elements $\alpha \in A$; to reiterate, α_n is a downset of X_n.

Lemma 4.4. *For each $p \in P$, $m(p)$ is a proper prime ideal of A.*

Proof. $m(p)$ is obviously a downset. And for $\alpha, \beta \in m(p)$,

$$\begin{aligned} &\{(n,j) : pj \notin (\alpha \vee \beta)_n = \alpha_n \cup \beta_n\} \\ &= \{(n,j) : pj \notin \alpha_n\} \cap \{(n,j) : pj \notin \beta_n\} \in v, \end{aligned}$$

thus verifying that $\alpha \vee \beta \in m(p)$. If $\alpha \wedge \beta \in m(p)$ then v contains

$$\begin{aligned} &\{(n,j) : pj \notin (\alpha \wedge \beta)_n = \alpha_n \cap \beta_n\} \\ &= \{(n,j) : pj \notin \alpha_n\} \cup \{(n,j) : pj \notin \beta_n\}. \end{aligned}$$

Then the primeness of v forces it to contain one of the sets displayed on the right, meaning that either α or β must lie in $m(p)$. □

Proposition 4.5. *m is an embedding.*

Proof. To show that m is order-preserving, consider first $p < q$ by virtue of rule (3). If $\alpha \in m(p)$ then $\{(n,j) : pj \notin \alpha_n\} \in v$, and since α_n is a downset,

$$\{(n,j) : pj \notin \alpha_n\} \subseteq \{(n,j) : qj \notin \alpha_n\} \in v,$$

meaning $\alpha \in m(q)$. Next consider $p \le a < b \le q$. Since we have established that $m(p) \subseteq m(a)$ and $m(b) \subseteq m(q)$ in the preceding two sentences, it suffices to show that $m(a) \subseteq m(b)$. For that purpose consider $\alpha \in m(a)$, so that $\{(n,i) : ai \notin \alpha_n\} \in v$. Since α_n is a downset, if $ai \notin \alpha_n$ and $iR_n j$ then $bj \notin \alpha_n$. Therefore

$$f(\{(n,i) : ai \notin \alpha_n\}) \subseteq \{(n,j) : bj \notin \alpha_n\}.$$

Since the set displayed on the left lies in v by Lemma 4.3, the set on the right does also.

Now suppose that $p \nleq q$, and let α^p be defined by the rule $\alpha_n^p \equiv \{rj : r \ngeq p\}$ for all n. Then

$$\{(n,j) : pj \notin \alpha_n^p\} = \{(n,j) : p \geq p\} = I \in v,$$
$$\{(n,j) : qj \notin \alpha_n^p\} = \{(n,j) : q \geq p\} = \emptyset \notin v.$$

That is, $\alpha^p \in m(p) \smallsetminus m(q)$. □

5. The ICS of projective planes

In Section 4 we gave a general construction of a family of summands X_n, $n \in \mathbb{N}$, and then located a copy of the given configuration P in the coproduct. But showing P to be non-coproductive also requires showing that no copy of P can be found in any of the X_n's, and this usually requires a subtle argument. In the case of the 2-crown, the subtleties involve projective planes.

We begin by outlining all the facts about projective planes that we will need. The reader wishing more background may consult, for example, [8]. A *finite projective plane* consists of disjoint finite sets X and Y, and a relation $E \subseteq X \times Y$. The customary terminology is to refer to the elements of X as *points* and to those of Y as *lines*. E is called the *incidence relation*, so that to say that x *is incident on* y is to say that xEy. The defining qualities of E are

$$\forall x_i \in X \ \ (x_1 \neq x_2 \Longrightarrow \exists! \ y \in Y \ \ (x_1Ey \text{ and } x_2Ey)),$$
$$\forall y_i \in Y \ \ (y_1 \neq y_2 \Longrightarrow \exists! \ x \in X \ \ (xEy_1 \text{ and } xEy_2)).$$

In addition, to prevent trivialities one assumes that there exist four points, no three incident on the same line. From these axioms it follows that every line has the same number $n+1$ of points, that every point is incident on the same number $n+1$ of lines, and that

$$|X| = |Y| = n^2 + n + 1.$$

Therefore $|E| = (n+1)|X| = (n+1)\left(n^2+n+1\right) \approx n^3$, and

$$(*) \qquad |E| \approx |X|^{\frac{3}{2}}.$$

The number n is referred to as the *order* of the projective plane. For prime order, there is the standard construction of the projective plane $\mathbb{P}(p)$ of order p, which takes place in the context of the vector space of dimension 3 over the Galois field $G(p)$ of order p. The points x are taken to be the 1-dimensional subspaces and the lines y to be the 2-dimensional subspaces, with xEy iff $x \subseteq y$. There is at least one projective plane of each prime power order, and every known finite projective plane has prime power order. But the question of the possible orders of finite projective planes is open.

In Definition 5.2 we generate a relation from a projective plane. Because the sequence R_n of relations so generated will be used in the construction of Section 4, they must satisfy Definition 4.1, and in particular iR_nj must imply $i \neq j$. This is not difficult to arrange, but it requires a lemma.

Lemma 5.1. *Let $\mathbb{P} \equiv (X,Y,E)$ be a finite projective plane. Then there is a bijection $\phi : X \to Y$ such that no $x \in X$ is incident on $\phi(x)$.*

Proof. In the graph $G \equiv (X,Y,N)$, where $N \equiv (X \times Y) \smallsetminus E$, consider an arbitrary subset $M \subseteq X$. Then $MN = \{y \in Y : \exists x \in M \ \ (xNy)\}$. If the points of M are all incident on a single line y then $MN = Y \setminus \{y\}$, and if not then $MN = Y$. Thus in either case $|MN| \geq |M|$ and the result follows by Hall's Theorem. □

Definition 5.2. *Let $\mathbb{P}(p) \equiv (X, Y, E)$, let $\phi : X \to Y$ be a bijection such that no $x \in X$ is incident on $\phi(x)$, let $k \equiv |X| = |Y| = p^2 + p + 1$, and let $\psi : \widetilde{k} \to X$ be a bijection. Define the relation R on $\widetilde{k}$ by the rule*

$$iRj \iff \psi(i)\, E\phi(\psi(j)).$$

In particular, in case when $p = p_n$ is the n^{th} prime we label the corresponding entities

$$\widetilde{k}_n \text{ and } R_n.$$

The fundamental requirement imposed by Definition 4.2 is that the chromatic numbers grow without bound. In order to establish this fact, we introduce a notion of independence for sets of points in a finite projective plane. The relationship between this notion and that of Definition 4.1 will be spelled out in Lemma 5.4.

Definition 5.3. *A set A of points in a projective plane is said to be* independent *if there is a set B of lines such that no point of A is incident on any line of B and $|A| \leq |B|$.*

Lemma 5.4. *If A is an independent set of points of $\mathbb{P}(p)$ then A is also independent in $\left(\widetilde{k}, R\right)$, i.e., in the sense of Definition 4.1.*

Proof. Set $B \equiv A$. The independence of A under Definition 4.1 means that

$$(A \times A) \cap R_n = \emptyset,$$

which is to say that no point of A incident on any line of B. □

To complete the argument that the chromatic numbers associated with the relations R_n grow, we need to estimate the size of independent subsets of $\mathbb{P}(p_n)$. The relevant estimate follows immediately from a beautiful and deep inequality of N. Alon [2].

Proposition 5.5. *Let A be an independent set of points in $\mathbb{P}(p)$. Then*

$$|A| \leq p\sqrt{p}.$$

Proof. Let $x \equiv |A|$. As Alon himself remarks on page 216, taking $d = 2$ in [2, Theorem 2.3] shows that the number of lines on which points in A are incident is at least $(p+1)^2 x / (p + x)$, so that the number of lines which are incident on no points of A is at most

$$p^2 + p + 1 - \frac{(p+1)^2 x}{p+x} = (p+1)^2 - p - \frac{(p+1)^2 x}{p+x} = \frac{(p+1)^2 p}{p+x} - p.$$

If A is to be independent then we must have x bounded above by the figure displayed on the right, and upon rearranging this inequality we see that $(p+x)^2 \leq (p+1)^2 p$, and taking square roots yields $x \leq (p+1)\sqrt{p} - p$. The result follows. □

Corollary 5.6. $\chi_{R_n}(\widetilde{k}_n) \geq \sqrt{p_n} \to \infty$.

Up to this point we have established that $\left(\widetilde{k}_n, R_n\right)$ above is an ICS.

We refer to $\left(\widetilde{k}_n, R_n\right)$ as *the ICS of projective planes*. However, we must address one more issue before applying the construction of Section 4 to the ICS of projective planes. This issue arises because the n^{th} summand X_n used in the construction contains a copy of a poset associated with $\mathbb{P}(p_n)$, so we must take care that this poset is free of any copy of the 2-crown. It is most convenient to translate this consideration into the language of graph theory. With a projective plane $\mathbb{P}$ we associate the bipartite graph

$$B(\mathbb{P}) \equiv (X \cup Y, E).$$

For $\mathbb{P} = \mathbb{P}(p_n)$, where p_n is the n^{th} prime, we abbreviate $B(\mathbb{P}(p_n))$ to B_n. We must make sure that B_n contains no square, i.e., no pairs of distinct points $x_i \in X$ and $y_j \in Y$ such that $x_i E y_j$ for all i and j.

Lemma 5.7. *B_n contains no square, but if $(x_0, y_0) \notin E$ then*

$$(X \cup Y, E \cup \{x_0, y_0\})$$

does contain a square.

Proof. B_n contains no square because each pair of distinct points $x_i \in X$ have a unique line $y \in Y$ on which both are incident. On the other hand, if a point $x_0 \in X$ is not incident on a line $y_0 \in Y$, then by choosing any $y_1 \in Y$ on which x_0 is incident we get a unique $x_1 \in X$ incident on both y_0 and y_1. But then $x_0 \neq x_1$ and $\{x_0, x_1, y_0, y_1\}$ forms a square. □

B_n is, of course, a very special bipartite graph, and there are others which are square-free and even maximal with respect to this property. But it is a well-known fact that the number of edges in a square free bipartite graph is at most $n^{\frac{3}{2}} + n$ (Erdös 1938, [10], for a short proof see [13]).

6. The 2-crown is not coproductive

Theorem 6.1. *The 2-crown is not coproductive.*

Proof. For P take the 2-crown $C_2 \equiv \{1, 2 < 1', 2'\}$, and let $(\widetilde{k}_n, R_n)$ be the n-th member of the ICS of projective planes from Section 5. Construct the X_n's as in Section 4, setting $(a, b) \equiv (1, 1')$. Now by Proposition 4.5,

$$C_2 \hookrightarrow X \equiv \coprod X_n$$

However, no single X_n contains the crown. The part $\{1, 1'\} \times \widetilde{k}_n$ has B_n as its Hasse diagram, so that in light of Lemma 5.7, any representation of the crown has to contain $2i$ or $2'i$. Such a point is connected only with $1'i$ and $2'i$ (resp. $1i$ and $2i$). From $1'i$ we can now proceed further only to a $1j$ with $i \neq j$ and this is not connected to $2'i$. □

We immediately obtain the following corollary.

Corollary 6.2. *Prohibiting 2-crowns does not define a first-order class of distributive lattices. Consequently, prohibiting properly embedded 2-crowns does not define a first-order class of distributive lattices, either.*

Proof. The first sentence follows immediately from Theorem 6.1 by virtue of the fact we pointed out at the end of Section 2, namely that any configuration whose prohibition determines a first-order class of lattices is coproductive. On the other hand, an arbitrary 2-crown is either proper or improper, and the prohibition of proper 2-crowns is simply the prohibition of the combinatorial tree $\{1, 2 < c < 1', 2'\}$, which is known to define a first-order class of lattices. If the prohibition of proper 2-crowns defined a first-order class of lattices, the prohibition of the general 2-crown would define a class of lattices by the logical conjunction of the first-order conditions in each case. □

In fact, our construction yields a stronger result.

Theorem 6.3. *Prohibiting properly embedded* 2*-crowns is not coproductive.*

Proof. It suffices to prove that the embedding $C_2 \hookrightarrow X$ of Proposition 4.1 (as in the proof of Theorem 6.1) is proper, that is, that there is no prime ideal x with

$$m(1), m(2) \subseteq x \subseteq m(1'), m(2').$$

Let α^p be as in the final part of the proof of Proposition 4.1. Explicitly we have

$$\alpha_n^{1'} = \{rj : r = 1, 2, 2';\ j \le k_n\},\ \alpha_n^{2'} = \{rj : r = 1, 2, 1';\ j \le k_n\},$$
$$\alpha_n^1 = \{2j : j \le k_n\},\ \alpha_n^2 = \{1j : j \le k_n\},$$

so that

$$\left(\alpha^{1'} \wedge \alpha^{2'}\right)_n = \{rj : r = 1, 2\} = \alpha_n^1 \cup \alpha_n^2 \in m(1) \vee m(2) \subseteq x.$$

Since x is prime, either $\alpha^{1'}$ or $\alpha^{2'}$ is in $x \subseteq m(1') \cap m(2')$, while neither $\alpha^{1\prime}$ nor $\alpha^{2\prime}$ lies in both $m(1')$ and $m(2')$. □

Our X_n's do not contain *any* 2-crown, proper or not. In fact, to construct a family of summands which contain only proper 2-crowns while their coproduct contains a proper one is simpler than the construction presented here and does not require projective planes.

The construction in Section 4 provides non-coproductivity proofs for many more configurations than we have considered up to this point. Let us sketch a simple example. A cycle

$$(\bigstar) \qquad C = a_0 < a_1 < \ldots < a_{r_1} > \ldots > a_{r_2} < \ldots > a_{r_{2t}} = a_0$$

is *simple* if the only order relationships between listed points follow from those indicated, and the *number of turns* of such a cycle is $\tau(C) \equiv t$. Thus the diamond has one turn and the k-crown has k turns. A simple cycle is *replete* if the relations indicated in ($\bigstar$) are actually $\prec$ and $\succ$. It is easy to see that each simple cycle can be replaced by a replete one with the same number of turns.

Proposition 6.4. *Suppose all the replete cycles C of P having the minimum number m of turns share a couple $a \prec b$ of consecutive elements. Then P is not coproductive.*

Proof. Let $g : X_n \to P$ be the projection $pi \longmapsto p$, and consider a replete cycle C of X_n. Now g preserves $\prec$, so $g(C)$ is a replete cycle unless C contains consecutive elements of the form $ai \prec bj \succ ak$ or $bi \succ aj \prec bk$. In particular, if C contains neither $ai \prec bj$ nor $bi \succ aj$ as consecutive elements then $g(C)$ is a replete cycle of P which avoids the $a \prec b$ edge, and so

$$\tau(C) = \tau(g(C)) > m.$$

If C contains either $ai \prec bj$ or $bi \succ aj$ as a consecutive pair, then by arguing as in the proof of Theorem 6.1 we see that C must contain more than one such pair. It follows that $g(C)$ cannot be simple, and that $\tau(C) > m$ in this case also. Finally the case in which all points of C lie in $\{a, b\} \times \widetilde{k}_n$ can be dealt with by suitable choice of R_n: those from Section 5 if $m = 2$, $R_n \equiv \{(i,j) : i < j\}$ otherwise. □

This was, of course, a very primitive case. By finer reasoning we can considerably extend the class of known non-coproductive configurations. We do not know, however, if one can prove in this way that no cyclic configuration is coproductive, i.e., Conjecture 1.1. It may be necessary to further generalize the construction, for example by twisting more than one edge of P.

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