COMPRESSION BASES IN EFFECT ALGEBRAS

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Abstract

We generalize David Foulis's concept of a compression base on a unital group to effect algebras. We first show that the compressions of a compressible effect algebra form a compression basis and that a sequential effect algebra possesses a natural maximal compression basis. It is then shown that many of the results concerning compressible effect algebras hold for arbitrary effect algebras by focusing on a specific compression base. For example, the foci (or projections) of a compression base form an orthomodular poset. Moreover, one can give a natural definition for the commutant of a projection in a compression base and results concerning order and compatibility of projections can be generalized. Finally it is shown that if a compression base has the projection-cover property, then the projections of the base form an orthomodular lattice.

1 Introduction

An effect algebra is a mathematical structure that has recently become important in foundational studies of quantum mechanics and quantum measurement theory [1, 2, 3, 4, 10, 11, 16]. An effect algebra is a set E of effects together with a partial binary operation $\oplus$ on E. The effects correspond to yes-no (or one-zero) quantum measurements that may be unsharp. Alternatively, we may think of effects as fuzzy quantum events. The orthosum $a \oplus b$

of two effects $a, b \in E$ may be roughly interpreted as a "parallel combination" of the measurements a and b or as a "mutually exclusive union" of the fuzzy events a and b.

In a previous article the author has considered a special type of effect algebra called a compressible effect algebra [13]. These compressible effect algebras were inspired by the pioneering work of David Foulis [6, 7, 8, 9] on compressible groups. Although the common effect algebras that have been studied turn out to be compressible, there is a fairly large class of effect algebras that do not have this property [5, 9]. This is unfortunate because compressible effect algebras possess a well-structured collection of compressions which appear to be important and useful operations on effect algebras. For example, compressions can be used to define conditional probabilities and Lüders operations [13]. However, as again pointed out by Foulis for the case of unital groups [9] we can work in arbitrary effect algebras by considering compression bases.

We first show that the compressions of a compressible effect algebra form a compression basis. It is then shown that many of the results in [13] hold for arbitrary effect algebras by focusing on a specific compression base. For example, the foci (or projections) of a compression base form an orthomodular poset. Also, one can give a natural definition for the commutant of a projection in a compression base and previous results concerning order and compatibility of projections can be generalized. Moreover, we show that a sequential effect algebra [14, 15] possesses a natural maximal compression base. This is the reason why the results in [13] for sequential effect algebras hold even when they are not compressible. It is demonstrated that the projection-cover property and the Richart projection property are equivalent for compression bases. Finally, it is proved that if a compression base has the projection-cover property, then its projections form an orthomodular lattice.

2 Effect Algebra Definitions

This section summarizes the basic definitions and notations concerning effect algebras and sequential effect algebras. If $\oplus$ is a partial binary operation, we write $a \perp b$ if $a \oplus b$ is defined. An **effect algebra** is a system $(E, 0, 1, \oplus)$ where $0, 1$ are distinct elements of E and $\oplus$ is a partial binary operation on E that satisfies the following conditions.

(E1) If $a \perp b$ then $b \perp a$ and $b \oplus a = a \oplus b$.

(E2) If $a \perp b$ and $c \perp (a \oplus b)$, then $b \perp c$ and $a \perp (b \oplus c)$ and

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c$$

(E3) For every $a \in E$ there exist a unique $a' \in E$ such that $a \perp a'$ and $a \oplus a' = 1$.

(E4) If $a \perp 1$, then $a = 0$.

In the sequel, whenever we write $a \oplus b$ we are implicitly assuming that $a \perp b$. We define $a \leq b$ if there exists a $c \in E$ such that $a \oplus c = b$. If such a $c \in E$ exists, then it is unique and we write $c = b \ominus a$. It can be shown that $a \perp b$ if and only if $a \leq b'$. Moreover, $(E, \leq, ')$ is a partially ordered set with $0 \leq a \leq 1$ for all $a \in E$, $a'' = a$ and $a \leq b$ implies that $b' \leq a'$. An element $a \in E$ is **sharp** if $a \wedge a' = 0$ and we denote the set of sharp elements in E by E_S. An element $a \in E$ is **principal** if $b, c \leq a$ with $b \perp c$ imply that $b \oplus c \leq a$. It is easy to show that principal elements are sharp. A subset F of an effect algebra E is a **sub-effect algebra** of E if $0, 1 \in F$, $a' \in F$ whenever $a \in F$ and $a \oplus b \in F$ whenever $a, b \in F$ with $a \perp b$.

Although there are many examples of effect algebras [1, 4, 10], the most important for quantum theory comes from the set $\mathcal{E}(H)$ of all self-adjoint operators A on a Hilbert space H satisfying $0 \leq A \leq I$ [2, 3, 15]. For $A, B \in \mathcal{E}(H)$ we define $A \perp B$ if $A + B \in \mathcal{E}(H)$ in which case $A \oplus B = A + B$. Then $(\mathcal{E}(H),), I, \oplus)$ is an effect algebra that we call a **Hilbert space effect algebra**. The **quantum effects** $A \in \mathcal{E}(H)$ correspond to yes-no measurements that may be unsharp. The set of projection operators $\mathcal{P}(H)$ on H form an orthomodular lattice which is a sub-effect algebra of $\mathcal{E}(H)$. It can be shown that $\mathcal{P}(H) = \mathcal{E}(H)_S$ so the elements of $\mathcal{P}(H)$ correspond to sharp quantum effects.

Let E be an effect algebra and let $a \in E$ with $a \neq 0$. Define the **interval**

$$F = [0, a] = \{b \in E \colon 0 \leq b \leq a\}$$

For $b, c \in F$ we say that $b \oplus_F c$ is **defined** if $b \perp c$ and $b \oplus c \leq a$ in which case $b \oplus_F c = b \oplus c$. Then $(F, 0, a, \oplus_F)$ becomes an effect algebra. Another simple way to obtain new effect algebras is the following. Suppose that p and p' are principal elements of E. Let

$$F = [0, p] \oplus [0, p'] = \{a \oplus b \colon a \leq p, b \leq p'\}$$

Letting $\oplus|F$ be the restriction of $\oplus$ to F, it is easy to show that $(F, 0, 1, \oplus|F)$ is an effect algebra and hence a sub-effect algebra of E.

If E and F are effect algebras, we say that $\phi\colon E \to F$ is **additive** if $a \perp b$ implies $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. If $\phi\colon E \to F$ is additive and $\phi(1) = 1$, then ϕ is a **morphism**. If $\phi\colon E \to F$ is a morphism and $\phi(a) \perp \phi(b)$ implies $a \perp b$, then ϕ is a **monomorphism**. A surjective monomorphism is an **isomorphism**. It is easy to see that a morphism ϕ is an isomorphism if and only if ϕ is bijective and ϕ^{-1} is a morphism. An additive map $J\colon E \to E$ is a **retraction** if $a \le J(1)$ implies that $J(a) = a$. The converse, that $J(a) = a$ implies that $a \le J(1)$ automatically holds for any additive map J. We call $J(1)$ the **focus** of the contraction J. We denote the kernel of J by

$$\mathrm{Ker}(J) = \{a \in E\colon J(a) = 0\}$$

and the image of J by $J(E)$. The following result was proved in [13].

Lemma 2.1. *Let J be a retraction on E with focus p.* (i) $[0, p'] \subseteq \mathrm{Ker}(J)$. (ii) *$p$ is principal and hence sharp.* (iii) *If $p \le a$, then $J(a) = p$.* (iv)

$$J(E) = \{a \in E\colon J(a) = a\} = [0, p]$$

An element $p \in E$ is a **projection** if p is the focus $J(1)$ of a retraction J on E. The set of all projections on E is denoted by $P(E)$. It follows from Lemma 2.1(ii) that $P(E) \subseteq E_S$. In general, $P(E) \ne E_S$ [5, 13]. If J is a retraction, then by Lemma 2.1(i) we have that $a \le J(1)'$ implies that $J(a) = 0$. If the converse holds, then J is a **compression**. Thus, a retraction J with focus p is a compression if $\mathrm{Ker}(J) = [0, p']$. For retractions J and I on E, we say that I is a **supplement** of J if $\mathrm{Ker}(J) = I(E)$ and $\mathrm{Ker}(I) = J(E)$.

A **compressible effect algebra** is an effect algebra E such that every retraction on E is uniquely determined by its focus and every retraction on E has a supplement. It can be shown that if E is compressible, then every retraction on E is a compression and if a retraction J has focus p, then the unique supplement of J has focus p' [13].

We now briefly discuss sequential effect algebras. Besides the orthosum $\oplus$ of an effect algebra, it is also important to describe a series combination or sequential product of effects. We shall denote by $a \circ b$ the sequential measurement in which a is performed first and b second.

For a binary operation $\circ$, if $a \circ b = b \circ a$ we write $a \mid b$. A sequential effect algebra (SEA) is a system $(E, 0, 1, \oplus, \circ)$ where $(E, 0, 1, \oplus)$ is an effect algebra and $\circ\colon E \times E \to E$ is a binary operation that satisfies the following conditions.

(S1) $b \mapsto a \circ b$ is additive for every $a \in E$.

(S2) $1 \circ a = a$ for every $a \in E$.

(S3) If $a \circ b = 0$, then $a \mid b$.

(S4) If $a \mid b$, then $a \mid b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$ for every $c \in E$.

(S5) If $c \mid a$ and $c \mid b$, then $c \mid a \circ b$ and $c \mid (a \oplus b)$.

We call an operation that satisfies (S1)–(S5) a **sequential product** on E.

Again, there are many examples of SEA's [14, 15], but we shall only mention that a Hilbert space effect algebra $\mathcal{E}(H)$ is a SEA under the sequential product $A \circ B = A^{1/2}BA^{1/2}$, It is easy to show that if E is a SEA, then $a \in E_S$ if and only if $a \circ a = a$. Also if $a \in E$ and $b \in E_S$, then $a \leq b$ if and only if $a \circ b = b \circ a = a$ and $b \leq a$ if and only if $a \circ b = b \circ a = b$ [14]. Moreover, $P(E) = E_S$ and E is compressible if and only if every retraction J on E has the form $J(a) = p \circ a$ for some $p \in E_S$ [13].

3 Compression Bases

Let E be an effect algebra and let F be a sub-effect algebra of E. We say that F is a **normal sub-effect algebra** of E if for any $a, b, c \in E$, whenever $a \oplus b \oplus c$ exists in E and $a \oplus b, b \oplus c \in F$, then $b \in F$ [9]. We say that $a, b \in E$ **coexist** if there exist $r, s, t \in E$ such that $r \oplus s \oplus t$ exists in E and $a = r \oplus s$, $b = s \oplus t$.

Lemma 3.1. *Let F be a normal sub-effect algebra of E and let $a, b \in F$. If a and b coexist in E, then a and b coexist in F.*

Proof. Since a and b coexist in E, there exist $r, s, t \in E$ such that $r \oplus s \oplus t$ exists in E and $a = r \oplus s$, $b = s \oplus t$. Since F is normal and $r \oplus s, s \oplus t \in F$ we have that $s \in F$. But then $r = a \ominus s \in F$ and $t = b \ominus s \in F$ so that a and b coexist in F. □

Lemma 3.2. *Let E be an effect algebra. Suppose that J is a compression on E with focus p and J' is a retraction with focus p'. Then for every $a \in E$, $J(a) = 0$ if and only if $J'(a) = a$.*

Proof. If $a \in E$ we have that $J'(a) \leq J'(1) = p'$ so that $J(J'(a)) = 0$. Hence, if $J'(a) = a$ we have that $J(a) = J(J'(a)) = 0$. Conversely, suppose that $J(a) = 0$. Since J is a compression with focus p, we have that $a \leq p'$ so that $J'(a) = a$. □

Suppose that E is a compressible effect algebra. For $p \in P(E)$ we denote the unique compression on E with focus p by J_p.

Theorem 3.3. *Let E be a compressible effect algebra.* (i) *$P(E)$ is a normal sub-effect algebra of E.* (ii) *If $p, q, r \in P(E)$ with $p \oplus q \oplus r$ defined, then the composition*

$$J_{p \oplus r} \circ J_{r \oplus q} = J_r$$

Proof. (i) By [13, Corollary 4.5], $P(E)$ is a sub-effect algebra of E. Suppose that $a, b, c \in E$, $a \oplus b \oplus c$ exists in E and $a \oplus b, b \oplus c \in P(E)$. Define $J = J_{a \oplus b} \circ J_{b \oplus c}$. Then $J \colon E \to E$ is additive and

$$J(1) = J_{a \oplus b}\left(J_{b \oplus c}(1)\right) = J_{a \oplus b}(b \oplus c) = J_{a \oplus b}(b) \oplus J_{a \oplus b}(c)$$

Since $a \oplus b \oplus c$ exists, we have that $c \leq (a \oplus b)'$. Also, $b \leq a \oplus b$ so that $J(1) = b \oplus 0 = b$. Suppose that $d \in E$ with $d \leq b$. Then $d \leq a \oplus b, b \oplus c$ and it follows that

$$J(d) = J_{a \oplus b}\left(J_{b \oplus c}(d)\right) = J_{a \oplus b}(d) = d$$

Therefore, J is a retraction with focus b so that $b \in P(E)$. Hence, $P(E)$ is normal. (ii) It follows from the proof of (i) that

$$J_{a \oplus b} \circ J_{b \oplus c} = J_b$$

Replacing a, b, c by p, r, q, respectively, the result follows. □

Let E be an effect algebra. A family $(J_p)_{p \in P}$ of compressions on E, indexed by a normal sub-effect algebra P of E is called a **compression base** for E if the following conditions hold.

(C1) Each $p \in P$ is the focus of the corresponding compression J_p.

(C2) If $p, q, r \in P$ with $p \oplus q \oplus r$ defined in E, then

$$J_{p \oplus r} \circ J_{r \oplus q} = J_r$$

Of course, every effect algebra possesses a **trivial** compression base $\{J_0, J_1\}$. It follows from Theorem 3.3 that $(J_p)_{p\in P(E)}$ is a compression base for a compressible effect algebra E. However, there are noncompressible effect algebras that have nontrivial compression bases [9]. Notice that if $\mathcal{J}_1$ and $\mathcal{J}_2$ are compression bases for E, then $\mathcal{J}_1 \cap \mathcal{J}_2$ is a compression base for E and if $\mathcal{J}_\alpha$ is a chain of compression bases for E, then $\cup\mathcal{J}_\alpha$ is a compression base for E. A simple Zorn's lemma argument shows that any effect algebra possesss a maximal compression base. Also, if J_p and $J_{p'}$ are compressions, then J_p and $J_{p'}$ are contained in a maximal compression base. If E is a SEA and $p \in E_S = P(E)$, then J_p denotes the compression $J_p(a) = p \circ a$.

Theorem 3.4. *If E is a SEA, then $(J_p)_{p\in P(E)}$ is a maximal compression base for E. Moreover, if $F \subseteq P(E)$ is a sub-SEA, then $(J_p)_{p\in F}$ is a compression base for E.*

Proof. It is shown in [13] that $P(E)$ is a sub-effect algebra of E. Suppose that $p, q, r \in E$, $p \oplus q \oplus r$ exists in E and $p \oplus r, r \oplus q \in P(E)$. Since $r \le p \oplus r$ it follows that $r \mid p \oplus r$ and $(p \oplus r) \circ r = r$ [13]. Also, since $q \le (p \oplus r)'$ we have that $q \mid p \oplus r$ and $(p \oplus q) \circ q = 0$ [13]. Hence,

$$(p \oplus r) \circ (r \oplus q) = (p \oplus r) \circ r \oplus (p \oplus r) \circ q = r$$

Since $(p \oplus r) \mid (r \oplus q)$ we conclude that $r = (p \oplus r) \circ (r \oplus q) \in P(E)$. Hence, $P(E)$ is a normal sub-effect algebra of E. Certainly each $p \in P(E)$ is the focus of the corresponding compression J_p. Suppose that $p, q, r \in P(E)$ with $p \oplus q \oplus r$ defined in E. Then p, q and r are mutually orthogonal projections and $(p \oplus r) \mid (r \oplus q)$. Hence,

$$(p \oplus r) \circ (r \oplus q) = p \circ r \oplus p \circ q \oplus r \circ r \oplus r \circ q = r$$

It follows that

$$J_{p\oplus r} \circ J_{r\oplus q} = J_r$$

Hence, $(J_p)_{p\in P(E)}$ is a compression base for E. Suppose that $\mathcal{J}$ is a strictly larger compression base for E. Then $\mathcal{J}$ has the form $\mathcal{J} = (J_q)_{q\in Q}$ where Q strictly contains $P(E)$. But the elements of Q must be projections so $Q \subseteq P(E)$ which is a contradiction. Hence, $(J_p)_{p\in P(E)}$ is maximal. The proof of the last statement of the theorem is similar. □

Lemma 3.5. *Let $(J_p)_{p\in P}$ be a compression base for E. Then P is an orthomodular poset and if $p \in P$, then $J_{p'}$ is a supplement of J_p.*

Proof. By [13, Lemma 3.1(iii)] every element of P is principal. Hence, $P \subseteq E_S$ so P is an orthoalgebra. Let $p, q \in P$ with $p \perp q$. Then $p \oplus q \in P$ and $p, q \le p \oplus q$. If $r \in P$ with $p, q \le r$, then since r is principal, we have that $p \oplus q \le r$. Hence, $p \oplus q = p \vee q$. It follows that P is an orthomodular poset. If $p \in P$, then $p' \in P$ and by Lemma 3.2 we have that $J_p(a) = 0$ if and only if $J_{p'}(a) = a$ and $J_{p'}(a) = 0$ if and only if $J_p(a) = a$. Hence, $J_{p'}$ is a supplement of $J_{p'}$. □

Theorem 3.6. *Let $(J_p)_{p\in P}$ be a compression base for E. If $p, q \in P$, then the following statements are equivalent.* (i) $q \le p$. (ii) $J_p \circ J_q = J_q$. (iii) $J_p(q) = q$. (iv) $J_q \circ J_p = J_q$. (v) $J_q(p) = q$.

Proof. (i)⇒(ii) If $q \le p$, then $p \ominus q \in P$ and $(p \ominus q) \oplus 0 \oplus q = p$. Hence, by definition

$$J_q = J_{(p\ominus q)\oplus q} \circ J_{q\oplus 0} = J_p \circ J_q$$

(ii)⇒(iii) If (ii) holds, then

$$J_p(q) = J_p\left(J_q(1)\right) = J_q(1) = q$$

(ii)⇒(iv) If (iii) holds, then $q = J_p(a) \le p$. Hence, $p \ominus q \in P$ and as before we have that

$$J_q = J_{0\oplus q} \circ J_{q\oplus(p\ominus q)} = J_q \circ J_p$$

(iv)⇒(v) If (iv) holds, then

$$J_q(p) = J_q\left(J_p(1)\right) = J_q(1) = q$$

(v)⇒(i) If (v) holds, then

$$J_q(p') = J_q(1 \ominus p) = q \ominus q = 0$$

so that $p' \le q'$. Hence, $q \le p$. □

Theorem 3.7. *Let $(J_p)_{p\in P}$ be a compression base for E. If $p, q \in P$, then the following statements are equivalent.* (i) $p \circ q = 0$. (ii) $p \perp q$. (iii) $q \circ p = 0$. (iv) $p \perp q$ *and* $(p \oplus q)' = p' \circ q' = q' \circ p'$.

Proof. (i)⇒(ii) If $p \circ q = 0$, then $q \le p'$ so that $p \perp q$. (ii)⇒(iii) If $p \perp q$ then

$$q \circ p \oplus q = q \circ (p \oplus q) \le q$$

so by cancellation, $q \circ p = 0$. (iii)⇒(iv) If $q \circ p = 0$ then as before $p \perp q$. It follows that $p' \circ q = q$. Hence,

$$(p \oplus q)' = p' \ominus q = p' \ominus p' \circ q = p' \circ (1 - q) = p' \circ q'$$

and by symmetry, $(p \oplus q)' = q' \circ p'$. (iv)⇒(i) This is similar to (ii)⇒(iii). □

4 Commutants and Compatibility

In this section, P will denote a set of projections for which $(J_p)_{p \in P}$ is a compression base for E. For $p \in P$, we write $p \circ a = J_p(a)$ and define the **commutant** of p by

$$C(p) = \{a \in E \colon a = p \circ a \oplus p' \circ a\}$$

If $a \in C(p)$ we say that a is **compatible** with p.

Lemma 4.1. *If $p \in P$, $a \in E$, then the following statements are equivalent.* (i) $p \circ a \le a$. (ii) $a \in C(p)$. (iii) $a \in [0, p] \oplus [0, p']$.

Proof. (i)⇒(ii) Suppose that $p \circ a \le a$. Then

$$p \circ (a \ominus p \circ a) = p \circ a \ominus p \circ a = 0$$

so that

$$a \ominus p \circ a = p' \circ (a \ominus p \circ a) = p' \circ a$$

Hence, $a = p \circ a \oplus p' \circ a$ so that $a \in C(p)$. (ii)⇒(iii) If $a \in C(p)$, then $a = p \circ a \oplus p' \circ a$ where $p \circ a \le p$ and $p' \circ a \le p'$. (iii)⇒(i) Suppose that $a \in [0, p] \oplus [0, p']$. Then $a = b \oplus c$ where $b \le p$ and $c \le p'$. We then have that

$$p \circ a = p \circ b \oplus p \circ c = b \le a.$$ □

Theorem 4.2. *For $p, q \in P$ the following statements are equivalent.* (i) $J_p \circ J_q = J_q \circ J_p$. (ii) $p \circ q = q \circ p$. (iii) $p \circ q \leq q$. (iv) *p and q coexist.* (v) *There exists an $r \in P$ such that $J_p \circ J_q = J_r$.* (vi) $p \circ q \in P$. (vii) $p \in C(q)$.

Proof. (i)⇒(ii) If (i) holds, then

$$p \circ q = J_p\left(J_q(1)\right) = J_q\left(J_p(1)\right) = q \circ p$$

(ii)⇒(iii) If (ii) holds, then $p \circ q = q \circ p \leq q$. (iii)⇒(iv) Letting $r = p \circ q$ and assuming (iii) holds, we have that $r \leq p, q$. Then there exist $s, t \in E$ such that $s \oplus r = p$ and $r \oplus t = q$. Since

$$p \circ t = p \circ (q \ominus r) = r \ominus r = 0$$

we have that $t \leq p'$ so that $s \oplus r \oplus t$ is defined. Hence, p and q coexist. (iv)⇒(v) If (iv) holds, there exist $r, s, t \in E$ such that $p = s \oplus r$, $q = r \oplus t$ and $s \oplus r \oplus t$ is defined in E. Since P is normal, we conclude that $r, s, t \in P$ and since $(J_p)_{p \in P}$ is a compression base we have that

$$J_p \circ J_q = J_{s \oplus r} \circ J_{r \oplus t} = J_r$$

(v)⇒(vi) If (v) holds, then

$$p \circ q = J_p\left(J_q(1)\right) = J_r(1) = r \in P$$

(vi)⇒(vii) Assume that (vi) holds and let $r = p \circ q \in P$. Then $r \circ q \leq r \leq p$ so by Theorem 3.6 we have that

$$r \ominus r \circ q = r \ominus (J_r \circ J_p)(q) = r \ominus J_r\left(J_p(q)\right) = r \ominus (r \circ r) = 0$$

Hence, $r = r \circ q$ and it follows that $r \circ (q') = 0$ so that $q' \leq r'$. Therefore, $r \leq q$ so by Lemma 4.1, $q \in C(p)$. (vii)⇒(i) Assume that (i) holds. Then by Lemma 4.1, $p \circ q \leq q$ so (iii) holds. Since we have already shown that (iii) implies (iv),there exist $r, s, t \in P$ such that $s \oplus r \oplus t$ is defined and $p = s \oplus r$, $q = r \oplus t$. Therefore, by definition we have that

$$J_p \circ J_q = J_{s \oplus r} \circ J_{r \oplus t} = J_r = J_{t \oplus r} \circ J_{r \oplus s} = J_q \circ J_p$$ □

By symmetry, it follows from Theorem 4.2 that $p \in C(q)$ if and only if $q \in C(p)$. It follows that compatibility is a symmetric relation on P.

Corollary 4.3. *Let $p, q \in P$ with $p \in C(q)$. Then $q \circ p = p \circ q = p \wedge q$ is the greatest lower bound of p and q in both E and P. Moreover, we have that*

$$J_p \circ J_q = J_q \circ J_p = J_{p \wedge q}$$

Proof. By Theorem 4.2 there exists an $r \in P$ with $J_p \circ J_q = J_q \circ J_p = J_r$. Thus,

$$r = J_p\left(J_q(1)\right) = p \circ q = q \circ p \leq p, q$$

If $a \in E$ with $a \leq p, q$, then

$$a = J_p\left(J_q(a)\right) = J_r(a) \leq r$$

so r is the greatest lower bound of p and q in E and hence also in P. □

Theorem 4.4. *Let $p \in P$, define $H = J_p(E)$, $P_H = \{q \in P\colon q \leq p\}$ and for every $q \in P_H$, let J_q^H be the restriction of J_q to H. Then the following statements hold.* (i) *H is an effect algebra with unit p and*

$$H = \{a \in E\colon J_p(a) = a\} = [0, p]$$

(ii) *If $q \in P_H$, then J_q^H is a compression on H.* (iii) *$(J_p^H)_{q \in P_H}$ is a compression base for H.*

Proof. (i) Since J_p is idempotent, $H = \{a \in E\colon J_p(a) = a\}$ and since J_p is a contraction, $H = [0, p]$ (see Lemma 2.1). As mentioned in Section 2, $[0, p]$ is an effect algebra with unit p where $a \oplus_H b$ is defined in H whenever $a \oplus b$ is defined in E and in this case $a \oplus_H b = a \oplus b$.
(ii) If $q \in P_H$, then $J_q(a) \leq q \leq p$ so $J_q^H\colon H \to H$. Clearly, J_q^H is additive on H and $J_q^H(p) = q$. If $a \leq q$, then $a \leq p$ and $J_q^H(a) = q$. Hence, J_q^H is a contraction on H with focus q. If $a \in H$ and $J_q^H(a) = 0$, then $a \leq q'$. Since p is principal in E and since $a, q \leq p$ we have that $a \oplus q \leq p$. Hence, $a \leq p \ominus q$ so J_q^H is a compression on H.
(iii) We must first show that P_H is a normal sub-effect algebra of H. It is clear that P_H is a sub-effect algebra of H. To show that P_H is normal, suppose that $a, b, c \in H$, $a \oplus b \oplus c$ exists in H and $a \oplus b, b \oplus c \in P_H$. Then $a \oplus b \oplus c$ exists in E and $a \oplus b, b \oplus c \in P$. Since P is a normal sub-effect algebra of E we have that $b \in P$. But $b \leq p$ so $b \in P_H$. To show that $(J_q^H)_{q \in P_H}$ is a compression base for H, suppose that $q, r, s \in P_H$ with $q \oplus r \oplus s \leq p$. Then $q, r, s \in P$ and $q \oplus r \oplus s$ exists in E. Hence, $J_{q \oplus r} \circ J_{r \oplus s} = J_r$ and it follows that

$$J_{q \oplus r}^H \circ J_{r \oplus s}^H = J_r^H$$ □

Theorem 4.5. *Let $p \in P$ and let $C = C(p)$. For each $q \in C \cap P$, let J_q^C be the restriction of J_q to C.* (i) *$C = [0,p] \oplus [0,p']$ is a sub-effect algebra of E.* (ii) *If $q \in C \cap P$, then J_q^C is a compression on C.* (iii) *$(U_q^C)_{q \in C \cap P}$ is a compression base for C.*

Proof. (i) This result was mentioned in Section 2. (ii) If $a \in C$ and $q \in C \cap P$, we have by Theorem 4.2 that

$$\begin{aligned} J_q^C(a) &= J_q^C\left(J_p(a) \oplus J_{p'}(a)\right) = J_q\left(J_p(a)\right) \oplus J_q\left(J_{p'}(a)\right) \\ &= J_p\left(J_q(a)\right) \oplus J_{p'}\left(J_q(a)\right) \in C \end{aligned}$$

It now follows that J_q^C is a compression on C. (iii) This proof is similar to the proof of Theorem 4.4(iii). □

5 Projection-Cover Property

A compression base $(J_p)_{p \in P}$ on E has the **projection-cover property** [5, 12] if for every $a \in E$ there exists a smallest projection $\widehat{a} \in P$ such that $a \leq \widehat{a}$.

Lemma 5.1. *Let $(J_p)_{p \in P}$ be a compression base on E that has the projection-cover property and let $p, q, r \in P$ and $a \in E$.* (i) *$p \ominus (p \circ a)^\wedge \leq a'$.* (ii) *$r \perp p \circ q$ if and only if $q \perp p \circ r$.* (iii) *$p \circ q \leq r$ if and only if $q \perp p \circ r'$.*

Proof. (i) Notice that $p \circ a \leq p$ so $(p \circ a)^\wedge \leq p$ and $p \ominus (p \circ a)^\wedge$ is defined. Let $t = [(p \circ a)^\wedge]'$ and $s = (p \circ t)^\wedge$. Then $t, s \in P$, $s \leq p$ and since $t' \leq p$ we have that

$$p \circ t = p \ominus t' = p \ominus (p \circ a)^\wedge \in P$$

Hence,

$$s = p \circ t = p \ominus (p \circ a)^\wedge$$

Now $p \circ a \leq (p \circ a)^\wedge = t'$ so that $t \circ (p \circ a) = 0$. Since $(p \circ a)^\wedge \leq p$ we have that $(p \circ a)^\wedge \in C(p)$ so $t \in C(p)$. Thus

$$s \circ a = (p \circ t) \circ a = (t \circ p) \circ a = t \circ (p \circ a) = 0$$

Therefore, $s \leq a'$ so that $p \ominus (p \circ a)^\wedge \leq a'$. (ii) Assume that $r \perp p \circ q$. Then $p \circ q \leq r'$ which implies that $(p \circ q)^\wedge \leq r'$ and hence $r \leq ((p \circ q)^\wedge)'$. Applying (i) gives

$$p \circ r \leq p \circ ((p \circ q)^\wedge)' \leq q'$$

Thus, $q \perp p \circ r$ and the converse follows by symmetry. (iii) In (ii) replace r by r' to get $p \circ q \leq r$ if and only if $p \circ r' \leq q'$ or $q \perp p \circ r'$. $\square$

Theorem 5.2. *Let $(J_p)_{p\in P}$ be a compression base for E that has the projection-cover property. Then P is an orthomodular lattice in which $p \wedge q = p \ominus (p \circ q')^\wedge$ and $(p \circ q)^\wedge = p \wedge (q \vee p')$.*

Proof. By Lemma 5.1(i) $p \ominus (p \circ q')^\wedge \leq p, q$. Suppose that $r \in P$ satisfies $r \leq p, q$. Then $p \circ r = r \leq q$, so by Lemma 5.1(iii) we have that $p \circ q' \leq r'$. Hence, $(p \circ q')^\wedge \leq r'$ so that $r \leq [(p \circ q')^\wedge]'$. Therefore,

$$r = p \circ r \leq p \ominus (p \circ q')^\wedge$$

Hence, $p \wedge q = p \ominus (p \circ q')^\wedge$. To prove the last equation, we have that $(p \circ q')^\wedge = p \ominus p \wedge q'$. Since $q \vee p' \leq p'$, $q \vee p' \in C(p')$, so that $q \vee p \in C(p)$. Hence, by Corollary 4.3 we have that

$$\begin{aligned}(p \circ q)^\wedge &= p \ominus p \wedge q' = p \circ [(p \wedge q')'] = p \circ (q \vee p') \\ &= p \wedge (q \vee p')\end{aligned}$$

$\square$

A compression base $(J_p)_{p\in P}$ on E has the **Richart projection property** if there exists a map $\sim : E \to P$ such that for every $p \in P$ we have $p \leq \widetilde{a}$ if and only if $p \circ a = 0$ [7, 13].

Theorem 5.3. *A compression base $(J_p)_{p\in P}$ on E has the projection-cover property if and only if it has the Richart projection property.*

Proof. Suppose that $(J_p)_{p\in P}$ has the projection-cover property. Define the map $\sim : E \to P$ by $\widetilde{a} = (\widehat{a})'$. If $p \in P$ satisfies $p \leq \widetilde{a} = (\widehat{a})'$ then $a \leq \widehat{a} \leq p'$. It follows that $p \circ a = 0$. Conversely, if $p \circ a = 0$, then $a \leq p'$. Hence, $\widehat{a} \leq p'$ so that $p \leq (\widehat{a})' = \widetilde{a}$. Thus, $(J_p)_{p\in P}$ has the Richart projection property. Now suppose that $(J_p)_{p\in P}$ has the Richart projection property. Define the map $^\wedge : E \to P$ by $\widehat{a} = (\widetilde{a})'$. Now $\widetilde{a} \leq \widetilde{a}$ implies that $\widetilde{a} \circ a = 0$. Hence,

$$a = (\widetilde{a})' \circ a \leq (\widetilde{a})' = \widehat{a}$$

If $p \in P$ with $a \leq p$, then $p' \circ a = 0$ so that $p' \leq \widetilde{a}$. Hence, $\widehat{a} = (\widetilde{a})' \leq p$. We conclude that $(J_p)_{p\in P}$ has the projection-cover property. $\square$

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