Between a parabola and a nowhere differentiable place

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Let $f_0(x)$ be the distance from x to the nearest integer. Formally, f_0 is the periodic extension of the function $x \mapsto \begin{cases} x & if \quad 0 \leq x \leq 1/2 \\ 1-x & if \quad 1/2 \leq x \leq 1 \end{cases}$ to $\mathbb{R}$.

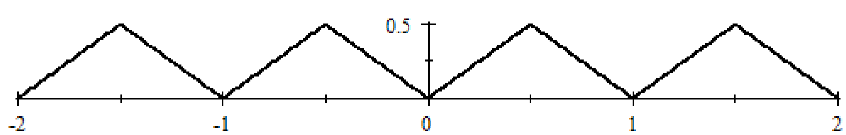

Figure 1: The graph of f_0 on $[-2, 2]$

For $k \geq 1$, let $f_k(x) = f_0(2^k x)$. For $|\lambda| \geq 2$, define F_λ by $F_\lambda(x) = \sum_{k=0}^{\infty} \frac{f_k(x)}{\lambda^k}$. Since $|f_0(x)| \leq 1/2$ for all x, this series converges uniformly on $\mathbb{R}$ as long as $|\lambda| > 1$, so each F_λ is continuous.

In 1903, Takagi [8] showed that the function (called here) F_2 is nowhere differentiable. The starting point of the present work occurred when the first author was studying the Takagi function. He plotted the graphs of F_2 and F_4 on $[0, 1]$ (see Figures 2 and 3) on his computer, walked into the office of the second author and said, "*The graph of F_4 looks like part of a parabola. Is it?*" We prove that it is in Theorem 2. Once we had done this, we began to wonder about the points of differentiability (if any) of other F_λ's and this paper is the result of this investigation. In addition to our result on F_4, we show that if $|\lambda| > 2$, $\lambda \neq 4$, left and right derivatives of F_λ exist at every point and F_λ is differentiable except at dyadic rationals.

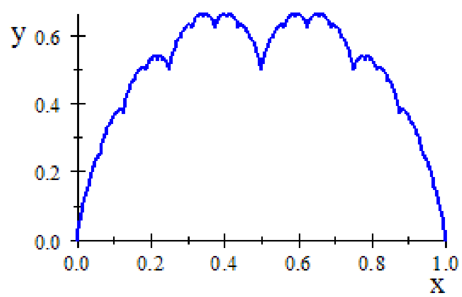

Figure 2: F_2

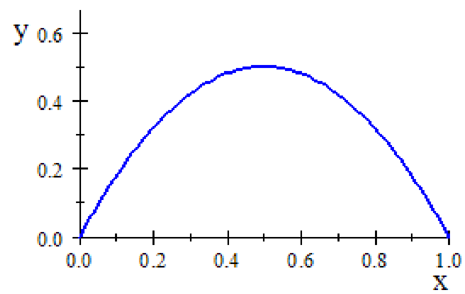

Figure 3: F_4

There is a considerable literature on the family of functions defined by $G(x) = \sum_{k=0}^{\infty} a^k f_0(b^k x)$ for various parameters a and b. Many of the results concerning these functions appear to have been discovered and rediscovered a number of times, at least when $|ab| \geq 1$. When $a = 1/2$ and $b = 2$, we get Takagi's function F_2. This function was rediscovered by Hildebrandt [5, 1933] who based his construction on yet another example of a continuous nowhere differentiable function discovered by van der Waerden [9] some 27 years later in 1930. (In van der Waerden's construction, $b = 10$ and $a = 1/10$.) Baouche and Dubuc [1, 1994] proved that G is continuous and nowhere differentiable if $a \in (0, 1)$ and $ab > 1$. They generalize the result of Knopp [6, 1918], who considered the case where $a \in (0, 1)$, b is an even integer and $ab > 4$. Much of the recent work we found concerns the computation of the fractal dimension of the graphs of various G's. It is worth noting that for us, $b = 2$ and $a = 1/\lambda$ with $|\lambda| \geq 2$ so $|ab| \leq 1$.

The original continuous nowhere differentiable function is due to Weierstrass in 1872 (see, for example, Hardy [4, 1916], for a long article on the Weierstrass function and its properties). Stromberg [7, p. 174] writes "$\cdots$ *that there seems to be good evidence to believe that examples were known to Bolzano as early as 1830.*"

Here is an informal summary of some of the things we discovered and some reasons we think these results might be of interest to students and instructors of real analysis classes.

1. We provide a complete description (in terms of λ, $|\lambda| > 2$) of when F_λ is differentiable and we provide explicit formulas for the left and right derivatives of F_λ at every number $a \in \mathbb{R}$. These left and right derivatives **always** exist. Further analysis leads to an explicit formula for right and left derivatives of $F_\lambda(a)$ in terms of the binary representation(s) of a. We also show that term by term right and left differentiability of the series for F_λ is justified even though term by term differentiability of a uniformly converging series is not generally permissible.

2. We show in two ways that $F_4(x) = 2x(1-x)$ on $[0,1]$. Our first (and original) method uses the contraction mapping theorem. Our second method is computational and uses equation ($\bigstar$) in the proof of Theorem 2(3). This computation clearly shows why the cases $\lambda = 4$ and $\lambda \neq 4$ differ.

3. For the sake of completeness, we include a proof that F_2 is not right or left differentiable at any point. This result is due to Cater [2, 1994]. (Cater proves much more. Our argument seems (to us) simpler because we can exploit the connections between the binary representation(s) of a number a and computation of $F_2(a)$. In fact, the argument we present is a simple extension of the usual parity argument used to prove the nowhere differentiability of F_2.)

4. All of the techniques we employ are elementary. The reader need only understand continuity, differentiability, uniform convergence of series of functions, summation of infinite (mostly geometric) series, and binary representations of real numbers. A slightly more advanced topic is the contraction mapping theorem. Almost any introductory real analysis should contain the necessary background. One such is the text by Goldberg [3].

The Main Results

Before stating our main results, we need some additional notation. For a function f and a number $a \in \mathbb{R}$, define

$$D^{+}f(a) = \lim_{x \to a^{+}} \frac{f(x) - f(a)}{x - a} \qquad \text{and} \qquad D^{-}f(a) = \lim_{x \to a^{-}} \frac{f(x) - f(a)}{x - a}$$

provided these limits exist. These are (respectively) the right and left derivatives of f at a. We would like to emphasize that the symbols $D^{+}f$ and $D^{-}f$ refer here to honest to goodness right and left derivatives of f, and not to Dini derivatives. We use this notation since $D^{+}f_k$ looks more natural and readable to us than, say, $f'_{k,+}$. Whenever a function f has a derivative at a, we will use the standard notation $f'(a)$. We state the following easy result without proof.

Lemma 1

1. *For $a \in [0,1)$, $D^{+}f_0(a) = 1$ if $0 \leq a < 1/2$ and $= -1$ if $1/2 \leq a < 1$. The right derivative $D^{+}f_0$ is periodic with period 1.*

2. *For $a \in (0,1]$, $D^{-}f_0(a) = 1$ if $0 < a \leq 1/2$ and $= -1$ if $1/2 < a \leq 1$. The left derivative $D^{-}f_0$ is periodic with period 1.*

3. *$D^{+}f_k(a) = 2^k D^{+}f_0\left(2^k a\right) = 2^k$ if $0 \leq a < 1/2^{k+1}$ and $= -2^k$ if $1/2^{k+1} \leq a < 1/2^k$. $D^{+}f_k$ is periodic with period $1/2^k$.*

4. *$D^{-}f_k(a) = D^{+}f_0\left(2^k a\right) = 2^k$ if $0 < a \leq 1/2^{k+1}$ and $= -2^k$ if $1/2^{k+1} < a \leq 1/2^k$. $D^{-}f_k$ is periodic with period $1/2^k$.*

For example, Figures 4 and 5 show the graphs of f_2 and $D^{+}f_2$ on the interval $[0,1]$.

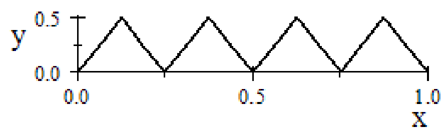

Figure 4: f_2

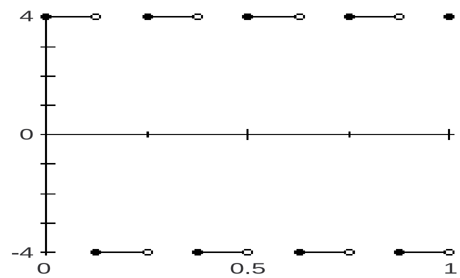

Figure 5: $D^+ f_2$

We now state our main result. The proof of this result is postponed until the final section of this paper.

Theorem 2

1. *For* $0 \le x \le 1$, $F_4(x) = 2x(1-x)$.

2. *Fix* $|\lambda| > 2$. *Then* F_λ *is right differentiable and left differentiable at every point. If* $a \in \mathbb{R}$, *then*

$$D^+ F_\lambda(a) = \sum_{k=0}^{\infty} \frac{D^+ f_k(a)}{\lambda^k} = \sum_{k=0}^{\infty} \varepsilon_k \left(\frac{2}{\lambda}\right)^k$$

where $\varepsilon_k = D^+ f_0\left(2^k a\right)$ *and*

$$D^- F_\lambda(a) = \sum_{k=0}^{\infty} \frac{D^- f_k(a)}{\lambda^k} = \sum_{k=0}^{\infty} \delta_k \left(\frac{2}{\lambda}\right)^k$$

where $\delta_k = D^- f_0\left(2^k a\right)$. *In particular, the right (respectively left) derivative of* F_λ *can be computed by term by term right (respectively left) differentiation of the series for* F_λ.

3. *Fix* $|\lambda| > 2$, $\lambda \ne 4$. *Then* F_λ *is differentiable except at dyadic rationals (i.e., numbers of the form* $k/2^m$, *where* $m = 0, 1, 2, \cdots$ *and* $k \in \mathbb{Z}$*).*

4. F_2 is not right or left differentiable at any point $a \in \mathbb{R}$.

To arrive at concrete formulas for $D^{\pm}F_{\lambda}(a)$ and to set the stage for some later work (see the proof of Theorem 2(4)) we use the important observation that $f_k(a)$ and $D^{\pm}f_k(a)$ can easily be determined from the binary representation(s) of a. To see this, let us suppose first that $a \in [0,1)$. (By periodicity, it is enough to show computations for $a \in [0,1)$ or $a \in (0,1]$.) Pick a binary representation $a = (.a_1a_2a_3\ldots)_2$ not ending in an infinite string of 1s. Then

$$(.a_1a_2a_3\ldots)_2 \in \begin{cases} [0,1/2) & \text{if} \quad a_1 = 0 \\ \\ [1/2,1) & \text{if} \quad a_1 = 1. \end{cases}$$

So, $f_0(a) = a$ if $a_1 = 0$ and $f_0(a) = 1 - a$ if $a_1 = 0$. Similarly, $2^k a = 2^k\left((.a_1a_2a_3\ldots a_k)_2\right) + (.a_{k+1}a_{k+2}a_{k+3}\ldots)_2$ with $2^k\left((.a_1a_2a_3\ldots a_k)_2\right) \in \mathbb{Z}$. Then

$$f_k(a) = \begin{cases} (.a_{k+1}a_{k+2}a_{k+3}\ldots)_2 & \text{if} \quad a_{k+1} = 0 \\ \\ 1 - (.a_{k+1}a_{k+2}a_{k+3}\ldots)_2 & \text{if} \quad a_{k+1} = 1. \end{cases}$$

Also, $\varepsilon_k = D^+f_0\left(2^k a\right) = D^+f_0\left((.a_{k+1}a_{k+2}a_{k+3}\ldots)_2\right)$, so $\varepsilon_k = 1$ if $a_{k+1} = 0$ and $= -1$ if $a_{k+1} = 1$. Said another way,

$$D^+f_0\left(2^k a\right) = 1 - 2a_{k+1}$$

for each k.

Similarly, if $b \in (0,1]$, consider a binary representation of b, say $b = (.b_1b_2b_3\ldots)_2$ where each $b_j = 0$ or 1 and the binary representation of b does not end in an infinite string of 0s, we find that $\delta_k = D^-f_0\left(2^k b\right) = 1$ if $b_{k+1} = 0$ and $= -1$ if $b_{k+1} = 1$. So,

$$D^-f_0\left(2^k b\right) = 1 - 2b_{k+1}$$

for each k. We have proven (assuming Theorem 2(2)) the following.

Corollary 3 *Fix $|\lambda| > 2$. Then F_λ is right differentiable and left differentiable at every point. If $a \in \mathbb{R}$, $z_1 \in \mathbb{Z}$ and $a = z_1 + (.a_1a_2a_3\ldots)_2$ where the binary representation of a does not end in an infinite string of 1's, then*

$$D^+F_\lambda(a) = \frac{\lambda}{\lambda-2} - 2\sum_{k=0}^{\infty} a_{k+1}\left(\frac{2}{\lambda}\right)^k.$$

If $a \in \mathbb{R}$, $z_2 \in \mathbb{Z}$ and $a = z_2 + (.b_1b_2b_3\ldots)_2$ where the binary representation of a does not end in an infinite string of 0's, then

$$D^-F_\lambda(a) = \frac{\lambda}{\lambda-2} - 2\sum_{k=0}^{\infty} b_{k+1}\left(\frac{2}{\lambda}\right)^k.$$

Examples and Additional Diagrams

This brief section contains some consequences of Theorem 2 and Corollary 3. Following this, we provide some additional diagrams.

Examples We first use Corollary 3 to compute $F_\lambda'(1/3)$ for $|\lambda| > 2$. Since $1/3 = \left(.\overline{01}\right)_2$ is not a dyadic rational, we get that $a_k = b_k = 0$ if k is odd and $= 1$ if k is even. So

$$F_\lambda'\left(\frac{1}{3}\right) = \frac{\lambda}{\lambda-2} - 2\sum_{j=0}^{\infty}\left(\frac{2}{\lambda}\right)^{2j+1} = \frac{\lambda}{\lambda-2} - 2\left(\frac{2\lambda}{\lambda^2-4}\right) = \frac{\lambda}{\lambda+2}.$$

When $\lambda = 4$, we get $F_4'(1/3) = 2/3$, which agrees with the direct computation

$$\left.\frac{d}{dx}\left(2x(1-x)\right)\right|_{x=1/3} = \left.(2-4x)\right|_{x=1/3} = \frac{2}{3}$$

Next, let us compute $D^\pm F_\lambda(1/2)$ for $|\lambda| > 2$ using Corollary 3 again. The two binary representations of $1/2$ are $(.1\bar{0})_2$ and $(.0\bar{1})_2$. In the first representation $a_1 = 1$ and $a_k = 0$ if $k \geq 2$. A simple computation yields

$$D^+F_\lambda\left(\frac{1}{2}\right) = \frac{\lambda}{\lambda-2} - 2a_1 = -\frac{\lambda-4}{\lambda-2}.$$

Similarly, in the second representation, $b_1 = 0$ and $b_k = 1$ if $k \geq 2$. So

$$D^- F_\lambda \left(\frac{1}{2}\right) = \frac{\lambda}{\lambda - 2} - 2\sum_{k=1}^{\infty} \left(\frac{2}{\lambda}\right)^k = \frac{\lambda - 4}{\lambda - 2}.$$

We see (as we would expect from visual evidence once we know that both $D^+F_\lambda(1/2)$ and $D^-F_\lambda(1/2)$ exist) that $D^+F_\lambda(1/2) = -D^-F_\lambda(1/2)$. We also see computationally that $D^+F_\lambda(1/2) \neq D^-F_\lambda(1/2)$ unless $\lambda = 4$.

Let us examine these derivatives when $\lambda = -4$. Then $F_{-4}(1/3) = 2$, $D^+F_{-4}(1/2) = -4/3$ and $D^-F_{-4}(1/2) = 4/3$. These results are supported by the graph of F_{-4} which is the first of our additional diagrams.

Additional Diagrams Routine computations using the definition of F_λ and the formula for $F'_\lambda(1/3)$ are summarized in the following table.

λ	$F_\lambda(1/3)$	$F'_\lambda(1/3)$
-6	$2/7$	$3/2$
-4	2	$4/15$
2.5	$5/9$	$5/9$
6	$2/5$	$3/4$

Figures 6, 8, 10 and 12 show the graph of F_λ for these λ on $[0, 1]$ and and the tangent line to the graph at $(1/3, F_\lambda(1/3))$, plotted as a dashed line. Figures 7, 9, 11 and 13 show the corresponding right derivative of F_λ on $[0, 1]$. (These graphs are not fine enough to distinguish between the right and left derivatives of F_λ, which differ only at dyadic rationals.)

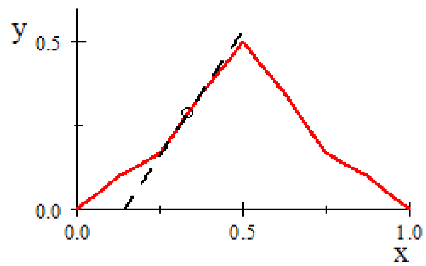

Figure 6: F_{-6}

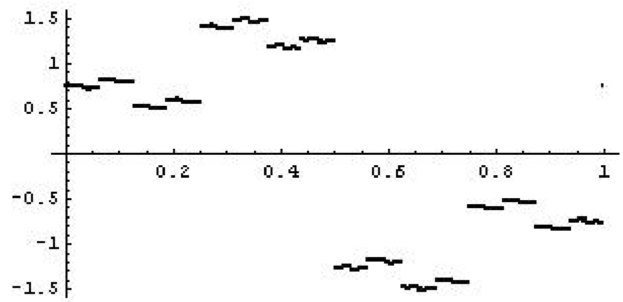

Figure 7: D^+F_{-6}

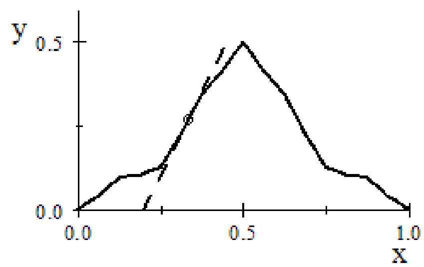

Figure 8: F_{-4}

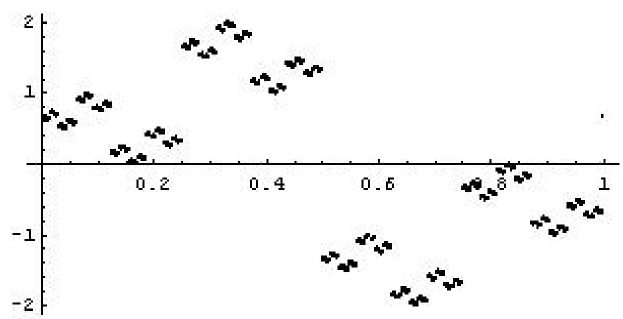

Figure 9: D^+F_{-4}

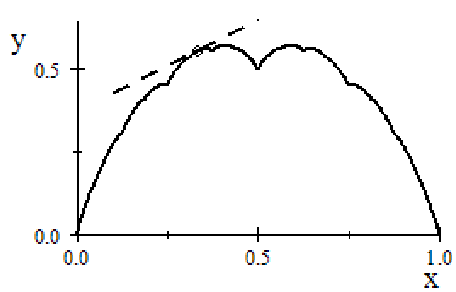

Figure 10: $F_{2.5}$

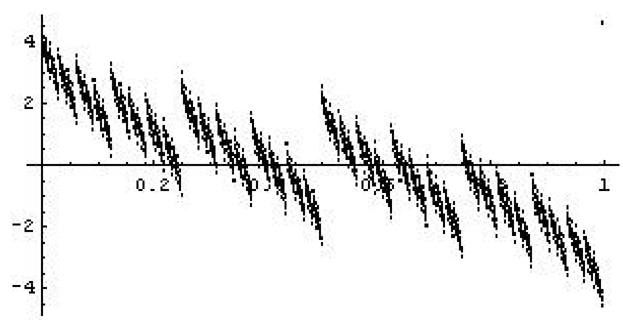

Figure 11: $D^+F_{2.5}$

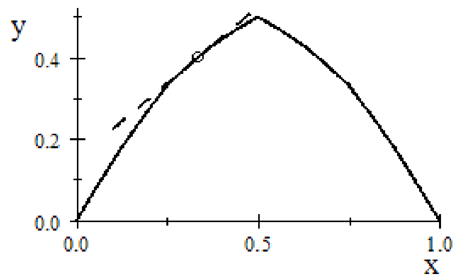

Figure 12: F_6

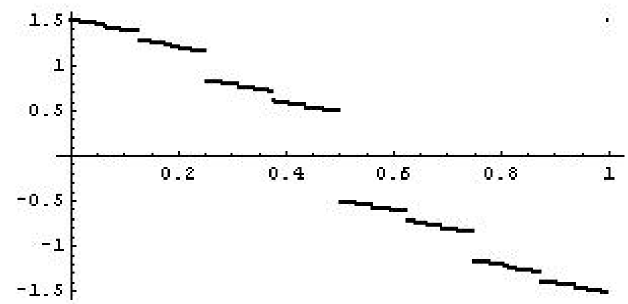

Figure 13: D^+F_6

These graphs suggest the following problem.

Problem: For $|\lambda| > 2$ it follows from Corollary 3 that $|D^+F_\lambda(a)| \leq \left|\frac{\lambda}{\lambda-2}\right|$. Investigate the possible values of $D^+F_\lambda(a)$. In particular, let

$$A = \left\{\lambda : D^+F_\lambda(\mathbb{R}) \supseteq \left(-\left|\frac{\lambda}{\lambda-2}\right|, \left|\frac{\lambda}{\lambda-2}\right|\right)\right\}$$

and $B = \left\{\lambda : \overline{D^+F_\lambda(\mathbb{R})} \text{ is a Cantor set}\right\}$. (In this context, a Cantor set K is an uncountable compact subset of $\mathbb{R}$ which is perfect, i.e. every point of K is a limit of a sequence of other points from K. The closure of the set O is denoted $\overline{O}$.)

1. For what values of λ is $\lambda \in A$? We know that $4 \in A$ and Figure 11 suggests that $2.5 \in A$. Is $-4 \in A$? Figure 9 is inconclusive.

2. For what values of λ is $\lambda \in B$? Figures 7 and 13 suggest that $-6, 6 \in B$.

3. If $\lambda \in A$, does D^+F_λ have the intermediate value property?

Finally, we show the graph of F_{-2} on $[0,1]$.

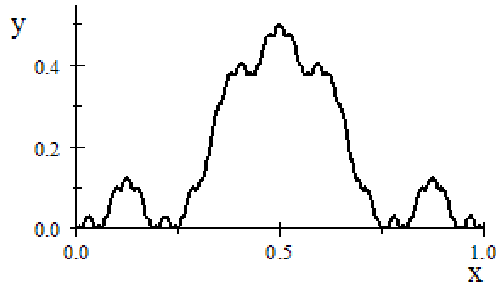

Figure 14: F_{-2}

Problem: Does F_{-2} have points of right or left differentiability? (We'd suggest consulting the proof of Theorem 2(4)) first.)

Proof of Theorem 2

This section is devoted to the proof of Theorem 2. We begin by using the contraction mapping theorem to show that $F_4(x) = 2x(1-x)$ on $[0,1]$. We will indicate a completely computational approach to this result following the proofs of parts (2) and (3) of Theorem 2.

Proof of Theorem 2(1). Let C denote the the set of continuous (real valued) periodic functions on $\mathbb{R}$ with period 1. It is well known that with the metric $d(f,g) = \max_{x\in\mathbb{R}} |f(x) - g(x)|$, C is a complete metric space. Also, a sequence (g_n) converges to g in the metric space (C,d) if and only if the sequence (g_n) converges to g uniformly on $\mathbb{R}$. Completeness of C is just a combination of the facts that a uniformly Cauchy sequence (g_n) in C converges uniformly to some period 1 function g, and $g \in C$ since a uniform limit of continuous functions is continuous (see, for example, Goldberg [**3, p. 260**]).

Let $\Phi : C \to C$ be defined by

$$\Phi F(x) = f_0(x) + F(2x)/4.$$

The following are easily checked.

1. $\Phi F \in C$ if $F \in C$.

2. If H is the periodic extension of the function $2x(1-x)$ on $[0,1]$, then $\Phi H = H$.

   Indeed, if $x \in [0, 1/2]$, then

   $$\Phi H(x) = f_0(x) + \frac{H(2x)}{4} = x + \frac{2(2x)(1-2x)}{4} = 2x(1-x) = H(x).$$

   If $x \in [1/2, 1)$, then $2x - 1 \in [0, 1)$ and $H(2x) = H(2x-1)$. So,

   $$\Phi H(x) = f_0(x) + \frac{H(2x)}{4} = (1-x) + \frac{2(2x-1)(1-(2x-1))}{4} = H(x).$$

3. The map Φ is a contraction on C. To see this observe that if $F, G \in C$, then

   $$d(\Phi F, \Phi G) = \max_{x\in\mathbb{R}} |\Phi F(x) - \Phi G(x)| = \max_{x\in\mathbb{R}} \frac{1}{4}|F(2x) - G(2x)| = \frac{1}{4} d(F, G).$$

4. Let us iterate the Φ function starting at f_0. Then, denoting the n^{th} iterate of Φ by Φ^n, an easy induction argument shows that

   $$\Phi^n f_0(x) = f_0(x) + \frac{f_0(2x)}{4} + \frac{f_0(4x)}{16} + \cdots + \frac{f_0(2^n x)}{4^n} = \sum_{k=0}^{n} \frac{f_k(x)}{4^k}$$

   which is the n^{th} partial sum of $F_4(x)$. So $\Phi^n f_0 \to F_4$ uniformly on $\mathbb{R}$.

By the contraction mapping theorem (see, for example, [3, pp 158-9]), $\Phi^n f_0 \to H$, where H is the unique fixed point of Φ determined in step 2. But by step 4, $\Phi^n f_0 \to F_4$ so $F_4 = H$, as claimed. ∎

We next turn to the proof of Theorem 2(2). Lemmas 4 and 5 are used to provide important bounds that we use to compute $\lim_{x\to a^{\pm}} \frac{F_\lambda(x) - F_\lambda(a)}{x - a}$. Here and in the rest of the paper, we concentrate on the arguments for right derivatives whenever possible.

Lemma 4 *Given any $n \in \mathbb{N}$ there is a $\delta_n > 0$ such that, if $0 < x - a < \delta_n$ then*

$$f_k(x) - f_k(a) = D^+ f_k(a) \cdot (x - a)$$

for all $k = 0, \cdots, n$.

Proof. Without loss of generality, assume that $0 \le a < 1$. Partition the nonnegative real numbers into half open intervals $[\beta, \beta + 1/2)$ where $\beta \in \{m/2 : m = 0, 1, 2, \cdots\}$. For each k, there is a unique β_k so that $2^k a \in [\beta_k, \beta_k + 1/2)$. For each k let

$$d_k = \frac{\beta_k + \frac{1}{2} - 2^k a}{2^k}$$

Notice that $d_k > 0$, since $\beta_k + 1/2 > 2^k a$. Let $\delta_n = \min_{k=0,\cdots,n}\{d_k\}$. Clearly, $\delta_n > 0$. Let $a < x < a + \delta_n$ and fix $k \in \{0, \cdots, n\}$. Then $0 < 2^k x - 2^k a < 2^k d_k = \beta_k + \dfrac{1}{2} - 2^k a$, so

$$2^k a < 2^k x < \beta_k + \frac{1}{2}.$$

So, $2^k x$ and $2^k a$ are in the same half intervals $[\beta_k, \beta_k + 1/2)$ and since $D^+ f_0$ is constantly 1 or -1 in these intervals, we get

$$f_k(x) - f_k(a) = f_0\left(2^k x\right) - f_0\left(2^k a\right) = \varepsilon \cdot 2^k \cdot (x - a)$$

where $\varepsilon = 1$ if β_k is an integer and -1 if $\beta_k = m + 1/2$ for some $m = 0, 1, 2, \cdots$. Similarly,

$$D^+ f_k(a) \cdot (x - a) = 2^k D^+ f_0(2^k a) \cdot (x - a) = \varepsilon \cdot 2^k \cdot (x - a)\,.$$

■

Problem: Supply a second proof of Lemma 4 using the binary representation of a which does not terminate in an infinite string of 1s.

Lemma 5 *For each $k = 0, 1, 2, \cdots$, for each $x, a \in \mathbb{R}$*

$$|f_k(x) - f_k(a)| \le 2^k |x - a|$$

Remark: This lemma has this simple geometrical interpretation: *The maximum absolute value of the slope of any chord joining two points on the graph of* f_k *is* 2^k. Shown in Figure 15 is the graph of $f_2(x)$ on $[0,1]$ and the chord joining $(.3, f_2(.3))$ and $(.6, f_2(.6))$.

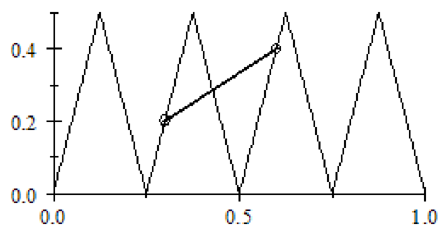

Figure 15:

Even though the lemma is geometrically clear, we provide a short proof.

Proof. First observe that the result for arbitrary k follows from the result for $k = 0$. For then, if $x, a \in \mathbb{R}$,

$$|f_k(x) - f_k(a)| = \left|f_0\left(2^k x\right) - f_0\left(2^k a\right)\right| \leq \left|2^k x - 2^k a\right| = 2^k\,|x - a|\,.$$

To establish the result for $k = 0$, assume $x > a$. If $x - a \geq 1$, then, since the range of f_0 is $[0, 1/2]$, we have that $|f_0(x) - f_0(a)| \leq 1 \leq x - a$. If $x - a \leq 1$, we consider cases. Because of the periodicity of f_0 we may assume (without loss of generality) that $a, x \in [0, 1/2)$, **or** $a \in [0, 1/2)$ and $x \in [1/2, 1)$, **or** $a \in [-1/2, 0)$ and $x \in [0, 1/2)$ **or** $a, x \in [1/2, 1)$. In the first case, $|f_0(x) - f_0(a)| = |x - a|$. In the second case, observe that $a \leq 1 - a$ and $1 - x \leq x$. So

$$f_0(x) - f_0(a) = (1 - x) - a$$

and

$$a - x \leq (1 - a) - x = (1 - x) - a \leq x - a$$

giving $|f_0(x) - f_0(a)| \leq |x - a|$ as desired. The last two cases are similar to the first two. ∎

We now turn to the proofs of parts 2 and 3 of Theorem 2. As mentioned above, we restrict our attention to right derivatives whenever possible.

Proof of Theorem 2(2). Fix $|\lambda| > 2$. Let $\varepsilon > 0$. Pick n so that

$$\frac{4}{|\lambda| - 2} \cdot \left(\frac{2}{|\lambda|}\right)^n < \varepsilon$$

and pick $\delta = \delta_n$ from Lemma 4. Routine computations and the use of the triangle inequality gives

$$\left|\frac{F_\lambda(x) - F_\lambda(a)}{x - a} - \textstyle\sum_{k=0}^{\infty} \frac{D^+ f_k\,(a)}{\lambda^k}\right| \le \left|\sum_{k=0}^{n} \frac{1}{\lambda^k} \cdot \left(\frac{f_k(x) - f_k(a)}{x - a} - D^+ f_k\,(a)\right)\right|$$

$$+ \sum_{k=n+1}^{\infty} \frac{1}{|\lambda|^k} \cdot \left|\frac{f_k(x) - f_k(a)}{x - a}\right|$$

$$+ \sum_{k=n+1}^{\infty} \frac{|D^+ f_k\,(a)|}{|\lambda|^k}.$$

Name these three terms as $\boldsymbol{I}$, $\boldsymbol{II}$ and $\boldsymbol{III}$ (respectively). Let $0 < x - a < \delta_n$. Then

- Lemma 4 gives that term $\boldsymbol{I} = 0$.
- Lemma 5 gives that term $\boldsymbol{II} \le \left(\frac{2}{|\lambda|}\right)^n \frac{2}{|\lambda| - 2}$.
- Lemma 1(3) gives that term $\boldsymbol{III} \le \left(\frac{2}{|\lambda|}\right)^n \frac{2}{|\lambda| - 2}$.

Combining these, we conclude that if $0 < x - a < \delta_n$ then

$$\left|\frac{F_\lambda(x) - F_\lambda(a)}{x - a} - \sum_{k=0}^{\infty} \frac{D^+ f_k\,(a)}{\lambda^k}\right| < \varepsilon.$$

■

We have shown that F_4 is differentiable on $(0, 1)$ and that for all $|\lambda| > 2$ that F_λ is left differentiable and right differentiable on $[0, 1]$. We now show that if $|\lambda| > 2, \lambda \neq 4,\ F_\lambda$ is differentiable except at the dyadic rationals. So, these F_λ's are both differentiable and non differentiable at dense sets of points.

Lemma 6 *Let $a = m/2^n$ for integers m, n where m is odd. Then*

1. $D^- f_k(a) = D^+ f_k(a)$ *for* $k = 0, \cdots, n-2$.
2. $D^- f_k(a) - D^+ f_k(a) = 2^{k+1}$ *for* $k = n-1$.
3. $D^- f_k(a) - D^+ f_k(a) = -2^{k+1}$ *for* $k \geq n$.

This works with $n = 0, 1$ as well, in which case properties (1) and possibly (2) are vacuous.

Proof. For $1 \leq k \leq n-2$, $f_k(a) = f_0(2^k a) = f_0(\frac{m}{2^{n-k}})$ and $\frac{m}{2^{n-k}} \neq \frac{j}{2}$ for any integer j. So, f_k is differentiable at a , which establishes (1). For (2),

$$2^{n-1}a = \frac{m}{2}$$

where m is odd, so $D^- f_k(a) = 2^k$ and $D^+ f_k(a) = -2^k$. For (3) observe that if $k \geq n$, then $2^k a$ is an integer, so $D^- f_k(a) = -2^k$ and $D^+ f_k(a) = 2^k$. ■

Proof of Theorem 2(3). Fix λ, $|\lambda| > 2$, $\lambda \neq 4$, and let $a = \frac{m}{2^n}$ be as in Lemma 6.

Assume $n \geq 2$. (The argument in case $n = 0$ or $n = 1$ requires obvious minor modifications.) Then

$$\begin{aligned} D^+ F_\lambda(a) - D^- F_\lambda(a) &= \sum_{k=0}^{\infty} \frac{D^+ f_k(a) - D^- f_k(a)}{\lambda^k} \qquad (\bigstar) \\ &= \left(D^+ f_{n-1}(a) - D^- f_{n-1}(a)\right) + \sum_{k=n}^{\infty} \frac{D^+ f_k(a) - D^- f_k(a)}{\lambda^k} \\ &= 2\left(\left(\frac{2}{\lambda}\right)^{n-1} - \left(\sum_{k=n}^{\infty}\left(\frac{2}{\lambda}\right)^k\right)\right) = 2\left(\frac{2}{\lambda}\right)^{n-1}\frac{\lambda-4}{\lambda-2} \neq 0 \end{aligned}$$

since $\lambda \neq 4$.

Now assume that a is not a dyadic rational. Then a has only one binary representation, which contains both infinitely many 0's and 1's. By Corollary 3, $D^{+}f_0\left(2^k a\right) = D^{-}f_0\left(2^k a\right)$ for every $k = 0, 1, 2, \cdots$ and so $D^{+}F_{\lambda}(a) = D^{-}F_{\lambda}(a)$. ■

Equation (★) also yields that if $\lambda = 4$ and a is a dyadic rational, then $D^{+}F(a) = D^{-}F(a)$. This allows us to give a second proof that $F_4(x) = 2x(1-x)$ on $[0,1]$.

Second Proof of Theorem 2(1). Suppose that $a \in (0,1)$ and that the binary representation of a not terminating in infinitely many 1's is $a = (.a_1 a_2 \ldots a_n \ldots)_2$. Then by Corollary 3,

$$D^{+}F_4(a) = \frac{4}{2} - 2\sum_{k=0}^{\infty} a_{k+1}\left(\frac{2}{4}\right)^k = 2 - 2\sum_{k=0}^{\infty}\frac{a_{k+1}}{2^k}$$

$$= 2 - 4\sum_{k=0}^{\infty}\frac{a_{k+1}}{2^{k+1}} = 2 - 4\left((.a_1 a_2 \ldots a_n \ldots)_2\right) = 2 - 4a.$$

Now, if a is not a dyadic rational, we already know from the proof of Theorem 2(3) that F_4 is differentiable at a. If a is a dyadic rational, then equation (★) in that proof shows that

$$D^{+}F_4(a) - D^{-}F_4(a) = 0$$

so F_4 is differentiable at a as well. In all cases, then $F_4'(a) = D^{+}F_4(a) = 2-4a$. A simple integration (and the fact that $F_4(0) = 0$) shows that $F_4(x) = 2x(1-x)$ on $(0,1)$. ■

Proof of Theorem 2(4). We sketch the proof that F_2 is not right or left differentiable at any point. The argument is similar to the classical proofs that F_2 is nowhere differentiable, so we just provide a sketch of the details. It is clearly enough to present the argument for nowhere right differentiability.

To this end assume that $0 \leq a < 1$, and let us pick a binary representation for a containing infinitely many 0's. So for infinitely many n we may write

$$a = (.a_1 a_2 \dots a_n 0)_2 + \frac{d}{2^{n+1}} = (.a_1 a_2 \dots a_n)_2 + \frac{d}{2^{n+1}}$$

where $0 \leq d < 1$. Put

$$b = (.a_1 a_2 \dots a_n 1)_2 + \frac{1-d}{2^{n+1}} = (.a_1 a_2 \dots a_n)_2 + \frac{2-d}{2^{n+1}}$$

Notice that $b > a$ since $b - a = \dfrac{1-d}{2^n} > 0$. A routine computation shows that, for $k = 0, \cdots, n-1$

$$f_k(b) - f_k(a) = \varepsilon_k 2^k (b-a)$$

where $\varepsilon_k = 1$ if $a_{k+1} = 0$ and -1 if $a_{k+1} = 1$. For $k = n$, we get that $2^n a = \dfrac{d}{2}$ while $2^n b = \dfrac{2-d}{2} = 1 - \dfrac{d}{2}$, so $f_n(b) - f_n(a) = 0$. For $k > n$, $f_k(b) - f_k(a) = 0$ as well. It follows that

$$\frac{F_2(b) - F_2(a)}{b-a} = \sum_{k=0}^{n-1} \pm \varepsilon_k.$$

Notice that these difference quotients are even integers if n is even and odd if n is odd. If there happen to be 0's in infinitely many even *and* infinitely many odd positions in the binary representation of a, then we are already done. If not, then, past some position N, the bit 0 occurs only at even positions or at odd positions. If we label these positions as $n_1 < n_2 < \cdots$, then we are done unless these sums are all eventually the same, so we may assume, by relabeling the sequence (n_j) if necessary, that there is a fixed number A such that

$$\sum_{k=0}^{n_j - 1} \varepsilon_k = A$$

for all j. In this case, we must have (since the n_j's occur only in even positions or odd positions) that $n_{j+1} \geq n_j + 2$, that $a_{n_j+1} = 1$ and that

$$\sum_{k=n_j}^{n_{j+1}-1} \varepsilon_k = 0$$

for $j = 1, 2, \cdots$. It follows immediately that $n_{j+1} = n_j + 2$. (If there are two consecutive 1's between n_j and $n_{j+1} - 1$, then there are at least two 1's and just one 0 in this interval of integers.)

We have shown that F_2 is not right differentiable at a unless a has a representation of the form

$$a = \left(.a_1 a_2 \ldots a_n \overline{01}\right)_2 = (.a_1 a_2 \ldots a_n)_2 + \frac{1}{3 \cdot 2^n}.$$

We finish the argument by showing that F is not right differentiable at these a's. Take $m > n$ with $a_{m+1} = 0$, consider

$$c = a + \frac{1}{2^{m+1}} = (.a_1 a_2 \ldots a_m)_2 + \frac{1}{2^{m+1}} + \frac{1}{3 \cdot 2^m} = (.a_1 a_2 \ldots a_m)_2 + \frac{5}{6 \cdot 2^m}.$$

Note that $c > a$ and that $c - a = \dfrac{5}{6 \cdot 2^m}$. Computation shows that

$$\frac{F_2(c) - F_2(a)}{c - a} = \sum_{k=0}^{m-1} \pm\varepsilon_k - 1/3 = A - 1/3.$$

To summarize this final case, given $\delta > 0$, we can find two numbers $b, c > a$ with $b - a < \delta$, $c - a < \delta$ and

$$\left| \frac{F_2(c) - F_2(a)}{c - a} - \frac{F_2(b) - F_2(a)}{b - a} \right| = \frac{1}{3}.$$

This completes the proof that F_2 is not right differentiable at any a. ■

Acknowledgements: This work was done while the first author was an undergraduate at the University of Denver. He acknowledges with gratitude the support of the PINS (Partner's in Scholarship) undergraduate research program at D.U. The second author was the faculty partner of the first author.

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