PERIODS IN MISSING LENGTHS OF RAINBOW CYCLES

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Abstract. A cycle in an edge-colored graph is said to be rainbow if no two of its edges have the same color. For a complete, infinite, edge-colored graph G, define

$$\mathfrak{S}(G) = \{n \geq 2 \mid \text{no } n\text{-cycle of } G \text{ is rainbow}\}.$$

Then $\mathfrak{S}(G)$ is a monoid with respect to the operation $n \circ m = n + m - 2$, and thus there is a least positive integer $\pi(G)$, the period of $\mathfrak{S}(G)$, such that $\mathfrak{S}(G)$ contains the arithmetic progression $\{N + k\pi(G) \mid k \geq 0\}$ for some sufficiently large N.

Given that $n \in \mathfrak{S}(G)$, what can be said about $\pi(G)$? Alexeev showed that $\pi(G) = 1$ when $n \geq 3$ is odd, and conjectured that $\pi(G)$ always divides 4. We prove Alexeev's conjecture:

Let $p(n) = 1$ when n is odd, $p(n) = 2$ when n is divisible by four, and $p(n) = 4$ otherwise. If $2 < n \in \mathfrak{S}(G)$ then $\pi(G)$ is a divisor of $p(n)$. Moreover, $\mathfrak{S}(G)$ contains the arithmetic progression $\{N + kp(n) \mid k \geq 0\}$ for some $N = O(n^2)$. The key observations are: If $2 < n = 2k \in \mathfrak{S}(G)$ then $3n - 8 \in \mathfrak{S}(G)$. If $16 \neq n = 4k \in \mathfrak{S}(G)$ then $3n - 10 \in \mathfrak{S}(G)$.

The main result cannot be improved since for every $k > 0$ there are G, H such that $4k \in \mathfrak{S}(G)$, $\pi(G) = 2$, and $4k + 2 \in \mathfrak{S}(H)$, $\pi(H) = 4$.

1. Introduction

Let G be a complete, infinite, edge-colored graph. In [2], the *spectrum* of G was defined as

$$\mathfrak{S}(G) = \{n \geq 2 \mid \text{no } n\text{-cycle of } G \text{ is rainbow}\}.$$

It is easy to see (cf. [2, Proposition 3.1]) that $\mathfrak{S}(G)$ is a monoid with respect to the operation $n \circ m = n + m - 2$, and thus that $\mathfrak{S}(G) - 2$ can be regarded as a submonoid of $(\mathbb{N}_0, +)$.

It is therefore reasonable to ask if a given submonoid of $(\mathbb{N}_0, +)$ can be realized as $\mathfrak{S}(G) - 2$ for some G. If the submonoid contains 2, the answer is "yes," by [2, Propositions 3.2, 3.3]. In particular, there is G with $\mathfrak{S}(G) = \{2, 4, 6, 8, \dots\}$. However, Alexeev [1] noticed that not every submonoid of $(\mathbb{N}_0, +)$ can be so realized, making the situation much more interesting.

Consequently, Alexeev became interested in the related question *"What can be said about $\mathfrak{S}(G)$, provided $n \in \mathfrak{S}(G)$?"*, and proved:

(i) There is G with $\mathfrak{S}(G) = \{2, 6, 10, 14, \dots\}$.
(ii) If $n = 2k + 1 \in \mathfrak{S}(G)$ then $k(2k + 1) \in \mathfrak{S}(G)$.
(iii) If $n = 2k + 1 \in \mathfrak{S}(G)$ then $3n - 6 \in \mathfrak{S}(G)$.
(iv) If $n = 2k + 1 \in \mathfrak{S}(G)$ then $m \in \mathfrak{S}(G)$ for every $m \geq 2n^2 - 13n + 23$.

Moreover, he gave a table of computer generated results, among which one can find: $8 \in \mathfrak{S}(G) \Rightarrow 16 \in \mathfrak{S}(G)$, $10 \in \mathfrak{S}(G) \Rightarrow 22 \in \mathfrak{S}(G)$, and $12 \in \mathfrak{S}(G) \Rightarrow 26 \in \mathfrak{S}(G)$.

All results mentioned up to this point will be used below without reference.

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Let us now recall a simple fact about subsemigroups of natural numbers. Given a subsemigroup A of $(\mathbb{N}_0,+)$, let $\Pi(A)=\{m-n \mid m,\ n\in A,\ n<m\}$, and $\pi(A)=\min\Pi(A)$.

Proposition 1.1. *Let $A\neq\{0\}$ be a subsemigroup of $(\mathbb{N}_0,+)$, and let $p\in\Pi(A)$. Then there is N such that A contains the arithmetic progression $\{N+kp \mid k\geq 0\}$. Moreover, $\pi(A)=\gcd\Pi(A)$.*

Proof. Let $p\in\Pi(A)$, and fix m, $n\in A$ such that $p=m-n$. Pick $s\geq n$. Then A contains sn, $(s-1)n+m=sn+p$, $\dots$, $(s-n)n+nm=sn+np$. Using $s=n$, we see that A contains n^2, n^2+p, $\dots$, n^2+np. Using $s=n+p$, we see that A contains $(n+p)n=n^2+np$, $(n+p)n+p=n^2+(n+1)p$, $\dots$, $(n+p)n+np=n^2+2np$, and so on. Thus A contains the arithmetic progression $\{n^2+kp\,|\,k\geq 0\}$.

In particular, there are N_0, N_1 such that $\{N_0+kp;k\geq 0\}\subseteq A$, $\{N_1+k\pi(A);\ k\geq 0\}\subseteq A$. If p does not divide $\pi(A)$, there is $r\in\Pi(A)$ such that $r<\pi(A)$, a contradiction. □

Let $N(A)$ be the least positive integer such that

$$\{N(A)+k\pi(A) \mid k\geq 0\}\subseteq A.$$

The proof of Proposition 1.1 shows that $N(A)\leq n^2$ whenever there are m, $n\in A$ such that $m-n=\pi(A)$. Here is another way of estimating $N(A)$:

Theorem 1.2 (Frobenius coin-exchange problem)**.** *Let A be a subsemigroup of $(\mathbb{N}_0,+)$ containing relatively prime integers n, m. Then $\pi(A)=1$ and $N(A)\leq(n-1)(m-1)$.*

Corollary 1.3. *Let A be a subsemigroup of $(\mathbb{N}_0,+)$ containing pn, pm, where n, m are relatively prime. Then $\pi(A)$ is a divisor of p, and there is $N\leq p(n-1)(m-1)$ such that $\{N+kp \mid k\geq 0\}\subseteq A$.*

Proof. Let $B=\{x/p \mid x\in A \text{ is divisible by } p\}$. Then B is a subsemigroup of $(\mathbb{N}_0,+)$, $m\in B$, $n\in B$. By the Frobenius coin-exchange problem, $\pi(B)=1$ and $N(B)\leq(n-1)(m-1)$. Thus $\pi(pB)=p$ and $N(pB)\leq p(n-1)(m-1)$. Since pB is a subsemigroup of A, we are done. □

Using this terminology, (iv) can be restated roughly as follows: If $\mathfrak{S}(G)$ contains an odd integer n then $\pi(\mathfrak{S}(G))=1$, and $N(\mathfrak{S}(G))=O(n^2)$.

Alexeev conjectured in [1] that $\pi(\mathfrak{S}(G))$ is a divisor of 4 for every G. This would mean that the two constructions yielding $\mathfrak{S}(G)=\{2,\ 4,\ 6,\ 8,\ \dots\}$ and $\mathfrak{S}(G)=\{2,\ 6,\ 10,\ 14,\ \dots\}$ are exceptional. We establish his conjecture and more, as described in the abstract. The asymptotic behavior of $\mathfrak{S}(G)$ is therefore fully understood.

Finally, when $3\in\mathfrak{S}(G)$, let us call G a *Gallai graph*. Note that Gallai graphs have no rainbow cycles. All finite Gallai graphs can be built iteratively [3], and the iterative construction is very useful in proving results about Gallai graphs. Is there a similar iterative construction for G with $4\in\mathfrak{S}(G)$, $5\in\mathfrak{S}(G)$, etc? The results obtained here could shed some light into this question.

2. The notation and technique

The edge with vertices i, j will be denoted by (i,j), and its color by $\gamma(i,j)$.

Given $n\in\mathfrak{S}(G)$, how can one go about proving that $m\in\mathfrak{S}(G)$ for some specific $m>n$? In a typical scenario, we start with a complete graph G on m vertices 0, $\dots$, $m-1$ drawn in the usual way (the vertices form a regular m-gon) and color the edges on the perimeter cycle by $\gamma(i,i+1)=i$, with vertices and edge colors labeled modulo m. There is no a priori restriction on the possible colors of the inner edges of G, but by carefully selecting n-cycles in G, we might manage to restrict colors on the inner edges, until, ultimately, we might prove that some inner edge cannot be colored at all, hence reaching a contradiction. For instance, since the n-cycle $0 \to 1 \to 2 \to \cdots \to n-1 \to 0$ cannot be rainbow when $n \in \mathfrak{S}(G)$, we see that $\gamma(0, n-1) \in \{0, \ldots, n-2\} \subset \{0, \ldots, m-1\}$.

The difficult part in this strategy is the selection of "good" n-cycles that lead to a systematic restriction of colors on the inner edges of G. Once a suitable n-cycle is found, the argument becomes routine. In this sense, the drawings of n-cycles accompanying our proofs say (almost) everything. Some n-cycles will be used more than once, and that is the reason why we have labeled their vertices only by letters—the meaning of the letters will be clarified in every instance the cycle is used.

Most of our results are of the form "if $n \in \mathfrak{S}(G)$ and $n > c$ then ...". It is usually not difficult to verify the conclusion for large values of n, but it takes some effort to pin down the constant c. This suggests a strategy in which a proof is first read with a large enough n in mind, and once the structure of the proof is understood, the constant c can be carefully estimated during second pass. In every such proof we point out at least one step where $n > c$ is needed.

3. The even case.

Fix $n \geq 4$. In Lemmas 3.1, 3.2, 3.3, let G be a complete, edge-colored graph with $n \in \mathfrak{S}(G)$, containing a rainbow $3n-8$ cycle $0 \to 1 \to \cdots \to 3n-9 \to 0$ such that $\gamma(i, i+1) = i$ for every i.

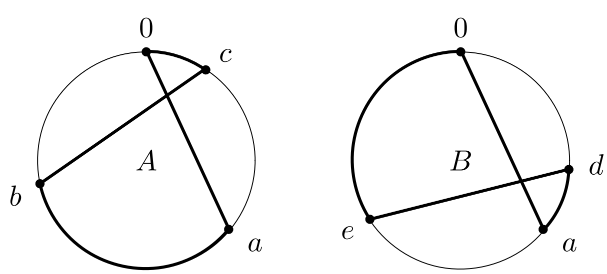

Figure 1. Proof of Lemmas 3.1 and 4.1.

Lemma 3.1. *Let $n > 4$. Then we can assume without loss of generality that*

$$\gamma(i, i+n-1) \in \{i, i+1\}$$

for every i.

Proof. Consider the n-cycle A of Figure 1, where $a = n-1$, $b = 2n-5$, $c = 2$. The edges of A have colors $\gamma(0, n-1)$, $n-1$, ..., $2n-6$, $\gamma(2, 2n-5)$, 1, 0. Since $0 \to 1 \to \cdots \to n-1 \to 0$ is an n-cycle and $n \in \mathfrak{S}(G)$, we have $\gamma(0, n-1) \in \{0, \ldots, n-2\}$. Similarly, $\gamma(2, 2n-5) \in \{2n-5, \ldots, 3n-9, 0, 1\}$. Since A is not rainbow, at least one of the colors $\gamma(0, n-1)$, $\gamma(2, 2n-5)$ occurs twice on A. Thus either $\gamma(0, n-1) \in \{0, 1\}$ or $\gamma(2, 2n-5) \in \{0, 1\}$. The cycle A is symmetrical with respect to the line passing through the center and the vertex 1. The edges $(0, n-1)$ and $(2, 2n-5)$ therefore play a symmetrical role in the construction. Hence, without loss of generality, $\gamma(0, n-1) \in \{0, 1\}$.

Consider the n-cycle B of Figure 1, where $d = n-3$, $e = 2n-4$. As above, we reach the conclusion that either $\gamma(n-3, 2n-4) \in \{n-3, n-2\}$, or $\gamma(0, n-1) \in \{n-3, n-2\}$. But $\gamma(0, n-1) \in \{0, 1\}$ and $\{0, 1\} \cap \{n-3, n-2\} = \emptyset$ (we need $n > 4$ here). Thus $\gamma(n-3, 2n-4) \in \{n-3, n-2\}$.

Just as we have rotated A clockwise by $n-3$ steps to obtain B, we can rotate B clockwise by $n-3$ steps, etc. Since the integers $3n-8$ and $n-3$ are relatively prime, it follows that $\gamma(i, i+n-1) \in \{i, i+1\}$ for every i. □

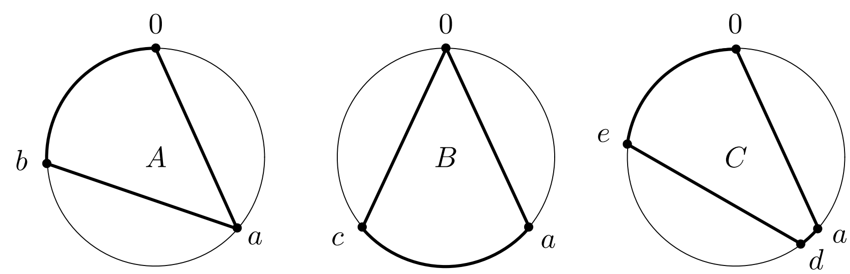

Figure 2. Proof of Lemmas 3.2 and 4.2.

Lemma 3.2. *Let $n > 5$. Then we can assume without loss of generality that*

$$\gamma(i, i+n-1) \in \{i, i+1\}, \quad \gamma(i, i+n-5) \in \{i+n-5, i+n-4\}$$

for every i.

Proof. By Lemma 3.1, we can assume that $\gamma(i, i+n-1) \in \{i, i+1\}$ for every i. Consider the n-cycle A of Figure 2, where $a = n-5$ (we need $n > 5$ here), $b = 2n-6$. Since $(2n-6)-(n-5) = n-1$, we conclude that $\gamma(0, n-5) \in \{n-5, n-4\} \cup \{2n-6, \dots, 3n-9\}$. Consider the n-cycle B of Figure 2, where $c = 2n-7$. Since $(2n-7)+(n-1) = 3n-8$, we conclude that $\gamma(0, n-5) \in \{n-5, \dots, 2n-8\} \cup \{2n-7, 2n-6\}$. Finally, consider the n-cycle C of Figure 2, where $d = n-4$ and $e = 2n-5$. Since $(2n-5)-(n-4) = n-1$, we conclude that $\gamma(0, n-5) \in \{n-5, n-4, n-3\} \cup \{2n-5, \dots, 3n-9\}$. Altogether, $\gamma(0, n-5) \in \{n-5, n-4\}$. As we could have started with any i, not necessarily with $i = 0$, we have $\gamma(i, i+n-5) \in \{i+n-5, i+n-4\}$ for every i. □

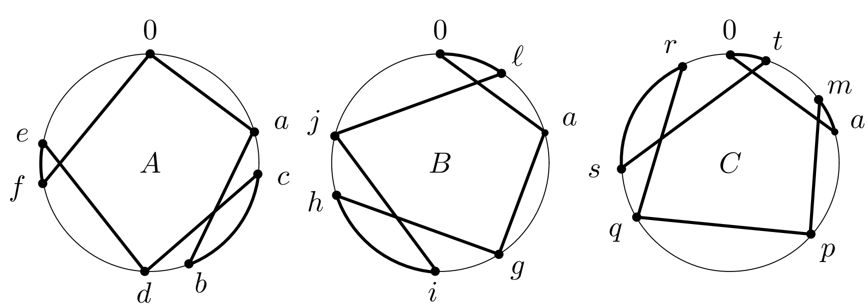

Figure 3. Proof of Lemmas 3.3 and 4.3.

Lemma 3.3. *Let $n > 10$. Then we can assume without loss of generality that*
$\gamma(i, i+n-1) \in \{i, i+1\}$, $\gamma(i, i+n-5) \in \{i+n-5, i+n-4\}$, $\gamma(i, i+7) \in \{i, i+1\}$
for every i.

Proof. By Lemma 3.2, we can assume that $\gamma(i, i+n-1) \in \{i, i+1\}$ and $\gamma(i, i+n-5) \in \{i+n-5, i+n-4\}$ for every i (in fact, we will only need the second assumption).

Consider an n-cycle starting at 0, containing 1 forward (clockwise) edge of length 7, 4 forward edges of length $n-5$, and $n-5$ backward (counterclockwise) edges of length 1. This is indeed an n-cycle, since $7 + 4(n-5) - (n-5) = 3n - 8 \equiv 0$. The three n-cycles of Figure 3 will be of this form.

Consider the n-cycle A of Figure 3 with $a = 7$, $b = n+2$, $c = 9$, $d = n+4$, $e = 2n-1$, $f = 2n-3$, and deduce $\gamma(0, 7) \in \{0, 1\} \cup \{9, \dots, n+5\} \cup \{2n-3, \dots, 2n\}$.

Consider the n-cycle B with $g = n+2$, $h = 2n-3$, $i = n+7$ (we need $n > 10$ to have $h < i$), $j = 2n+2$ and $\ell = 5$, and deduce $\gamma(0, 7) \in \{0, \dots, 6\} \cup \{n+2, n+3\} \cup \{n+7, \dots, 2n-2\} \cup \{2n+2, 2n+3\}$. Combined with the restrictions from cycle A, we have $\gamma(0, 7) \in \{0, 1\} \cup \{n+2, n+3\} \cup \{2n-3, 2n-2\}$.

Finally, consider the n-cycle C with $m = 5$, $p = n$, $q = 2n-5$, $r = 3n-10$, $s = 2n$, $t = 3$, and deduce $\gamma(0, 7) \in \{0, \dots, 6\} \cup \{n, n+1\} \cup \{2n-5, 2n-4\} \cup \{2n, \dots, 3n-9\}$. Combined with the previous restrictions, we obtain $\gamma(0, 7) \in \{0, 1\}$.

Starting at an arbitrary i instead of at $i = 0$, we see that $\gamma(i, i+7) \in \{i, i+1\}$ for every i. □

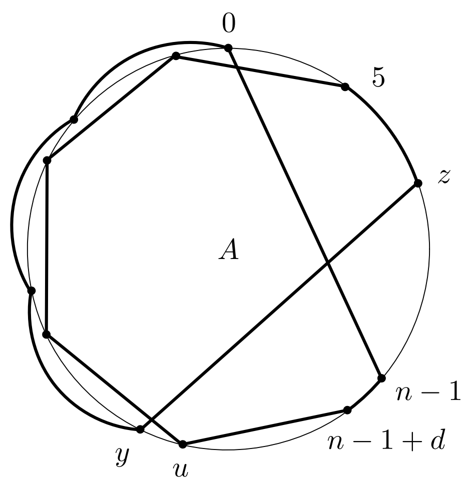

FIGURE 4. Proof of Theorem 3.4.

Theorem 3.4. *Let $n \geq 4$ be an even integer. Assume that G is a complete, edge-colored graph containing no rainbow n-cycles. Then G contains no rainbow $(3n-8)$-cycles.*

Proof. There is nothing to show for $n = 4$. When $n = 6$, we have $10 = 6 \circ 6 \in \mathfrak{S}(G)$. When $n = 8$ (resp. $n = 10$), Alexeev's computer calculations show that $16 \in \mathfrak{S}(G)$ (resp. $22 \in \mathfrak{S}(G)$). Therefore, we can take $n \geq 12$. Assume, for a contradiction, that G contains a rainbow $(3n-8)$-cycle $0 \to 1 \to \cdots \to 3n-9 \to 0$ colored $\gamma(i, i+1) = i$, and restrict all further considerations to the subgraph of G induced by this cycle. By Lemma 3.3, we can assume that $\gamma(i, i+7)$, $\gamma(i, i+n-1) \in \{i, i+1\}$ for every i.

Note that $2\cdot(n-1)+((n/2)-2)\cdot 7+(n/2)\cdot 1 = 6n-16 = 2(3n-8)\equiv 0 \pmod{3n-8}$. We will therefore consider an n-cycle A containing 2 edges of length $n-1$, $(n/2)-2$ edges of length 7, and $n/2$ edges of length 1, all forward. Such an n-cycle makes two clockwise revolutions modulo $3n-8$.

Assume that $n\ge 16$ (we will deal with $n=12$, 14 later), and construct A as follows (cf. Figure 4): Start with the edge $(0,n-1)$ followed by the least possible number d of edges of length 1 so that $(3n-8)-(n-1+d)\equiv 2 \pmod 7$. This is possible, since

$$0\le d\le 6\le n/2 \tag{3.1}$$

holds whenever $n\ge 12$. Continue the cycle by edges of length 7 until you reach 5. Then add the remaining $(n/2)-d$ edges of length 1, stopping at z. Note that

$$z = 5+(n/2)-d\le n-1-2 \tag{3.2}$$

thanks to $d\ge 0$ and $n\ge 16$. Add an edge of length $n-1$, ending at some $y=z+n-1$. Note that y is past $n-1+d$ if and only if

$$z = 5+(n/2)-d > d, \tag{3.3}$$

which holds because $2d\le 12<5+8\le 5+n/2$. Hence there is an already constructed edge $(u,u+7)$ of A such that $u<y\le u+7$. Since we have by now used all two edges of length $n-1$ and all $n/2$ edges of length 1, the cycle A closes itself upon adding the remaining edges of length 7. But this shows that $y-u=2$. The conditions $\gamma(i,i+7)$, $\gamma(i,i+n-1)\in\{i,i+1\}$ then imply that A is rainbow, a contradiction.

When $n=14$, the same construction of A goes through, because $d=5$, and the needed inequalities (3.1)–(3.3) hold. When $n=12$, we let A be the cycle $0\to 11\to 12\to 13\to 14\to 15\to 16\to 23\to 2\to 9\to 20\to 27\to 0$, for instance. □

4. The doubly even case

Fix $n=4k$. In Lemmas 4.1, 4.2, 4.3, let G be a complete, edge-colored graph with $n\in\mathfrak{S}(G)$, containing a rainbow $3n-10$ cycle $0\to 1\to\cdots\to 3n-11\to 0$ such that $\gamma(i,i+1)=i$ for every i.

Lemma 4.1. *Let $n=4k\ge 12$. Then we can assume without loss of generality that*

$$\gamma(i,i+n-1)\in\{i,i+1,i+2\}$$

for every i.

Proof. Consider the n-cycle A of Figure 1 with $a=n-1$, $b=2n-6$, $c=3$, and deduce that $\gamma(0,n-1)\in\{0,1,2\}$ or $\gamma(3,2n-6)\in\{0,1,2\}$. Without loss of generality, $\gamma(0,n-1)\in\{0,1,2\}$.

The rotated cycle B of Figure 1 with $d=n-4$ and $e=2n-5$ then shows that either $\gamma(0,n-1)\in\{n-4,n-3,n-2\}$ or $\gamma(n-4,2n-5)\in\{n-4,n-3,n-2\}$, and hence the latter must be true.

Since $\gcd\{3n-10,n-4\}=2$, we conclude that $\gamma(i,i+n-1)\in\{i,i+1,i+2\}$ for every even i. If $\gamma(i,i+n-1)\in\{i,i+1,i+2\}$ holds for at least one odd i, it then holds for every odd i, and we are done.

Let us therefore assume, for a contradiction, that $\gamma(1,1+n-1)\notin\{1,2,3\}$. Using the n-cycle A rotated clockwise by 1, we see that $\gamma(2n-5,4)\in\{1,2,3\}$. Proceeding as above but with counterclockwise rotations by $n-4$, we conclude that $\gamma(i,i+n-1)\in\{i+n-4,i+n-3,i+n-2\}$ for every odd i.

Note that $n/2$ is even, and consider the n-cycle E of Figure 5. (When $n=12$, the vertices $(5/2)n-4$ and 0 coincide.) Since 0 is even, n is even, and $(3/2)n-3$ is odd, the

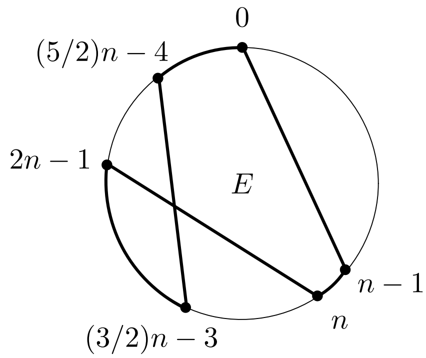

FIGURE 5. Proof of Lemma 4.1.

edges of E are colored by $\gamma(0,n-1)\in\{0,1,2\}$, $n-1$, $\gamma(n,2n-1)\in\{n,n+1,n+2\}$, $(3/2)n-3$, …, $2n-2$, $\gamma((3/2)n-3,(5/2)n-4)\in\{(5/2)n-7,\ (5/2)n-6,\ (5/2)n-5\}$, $(5/2)n-4$, …, $3n-11$. The condition $n\ge 12$ then guarantees that E is rainbow, a contradiction. ☐

Lemma 4.2. *Let $n=4k\ge 12$. Then we can assume without loss of generality that*

$$\gamma(i,i+n-1)\in\{i,i+1,i+2\},\quad \gamma(i,i+n-7)\in\{i+n-7,i+n-6,i+n-5\}$$

for every i.

Proof. We can assume that $\gamma(i,i+n-1)\in\{i,i+1,i+2\}$ for every i, by Lemma 4.1. Consider the n-cycles A, B, C of Figure 2 with $a=n-7$, $b=2n-8$, $c=2n-9$, $d=n-5$, and $e=2n-6$. We get $\gamma(0,n-7)\in\{n-7,\ n-6,\ n-5\}\cup\{2n-8,\ \dots,\ 3n-11\}$ from A, $\gamma(0,7)\in\{n-7,\ \dots,\ 2n-7\}$ from B, and $\gamma(0,n-7)\in\{n-7,\ \dots,\ n-3\}\cup\{2n-6,\ \dots,\ 3n-11\}$ from C. Altogether, $\gamma(0,n-7)\in\{n-7,\ n-6,\ n-5\}$, and thus $\gamma(i,i+n-7)\in\{i+n-7,\ i+n-6,\ i+n-5\}$. ☐

Lemma 4.3. *Let $n=4k\ge 20$. Then we can assume without loss of generality that*

$$\gamma(i,i+n-1)\in\{i,i+1,i+2\},\quad \gamma(i,i+n-7)\in\{i+n-7,i+n-6,i+n-5\},$$
$$\gamma(i,i+13)\in\{i,i+1,i+2\}$$

for every i.

Proof. Consider the cycles A, B, C of Figure 3 with $a=13$, $b=n+6$, $c=16$, $d=n+9$, $e=2n+2$, $f=2n-3$, $g=n+6$, $h=2n-1$, $i=n+12$, $j=2n+5$, $\ell=8$, $m=8$, $p=n+1$, $q=2n-6$, $r=3n-13$, $s=2n+2$, and $t=5$. Note that each of the cycles contains 1 forward edge of length 13, 4 forward edges of length $n-7$, and $n-5$ backward edges of length 1. The restrictions on $\gamma=\gamma(0,13)$ obtained from A, B, and C, respectively, are: $\gamma\in\{0,1,2\}\cup\{16,\dots,n+11\}\cup\{2n-3,\dots,2n+4\}$, $\gamma\in\{0,\dots,10\}\cup\{n+6,n+7,n+8\}\cup\{n+12,\dots,2n+1\}\cup\{2n+5,2n+6,2n+7\}$ (we need $n\ge 20$ to have $2n+7<3n-10$), and $\gamma\in\{0,\dots,12\}\cup\{n+1,n+2,n+3\}\cup\{2n-6,2n-5,2n-4\}\cup\{2n+2,\dots,3n-11\}$. It is then easy to see that $\gamma(0,13)\in\{0,1,2\}$, and thus $\gamma(i,i+13)\in\{i,i+1,i+2\}$ for every i. ☐

We will now imitate the proof of Theorem 3.4. But the situation is more delicate because we will need cycles with four revolutions, and because the uncertainty as to the color is larger (3 instead of 2 choices on many edges). On the other hand, the edges of length 13 give us a bit more wiggle room than the edges of length 7.

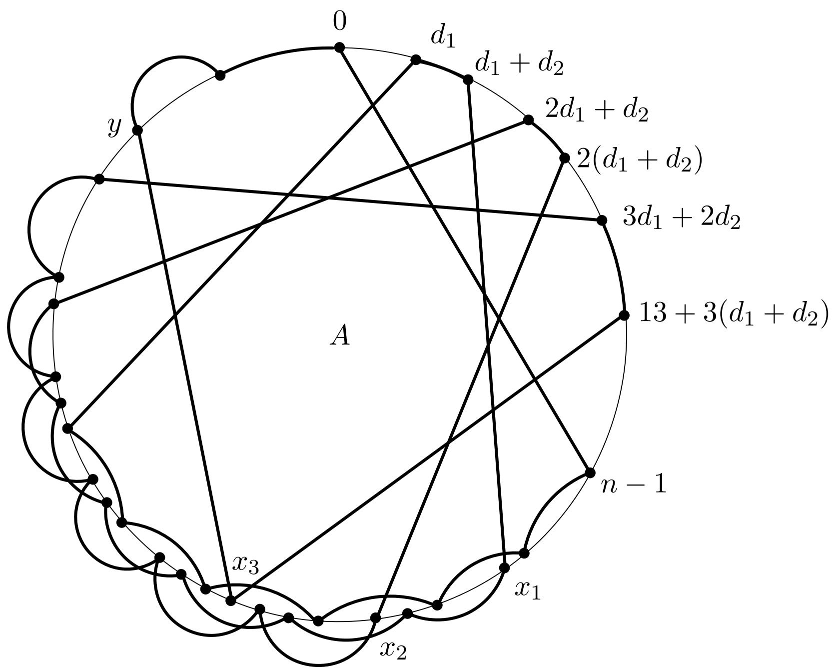

FIGURE 6. Proof of Theorem 4.4.

Theorem 4.4. *Let $n = 4k \neq 16$. Assume that G is a complete, edge-colored graph containing no rainbow n-cycles. Then G contains no rainbow $(3n - 10)$-cycles.*

Proof. When $n = 4$, there is nothing to prove since $2 \in \mathfrak{S}(G)$. When $n = 8$, $8 \circ 8 = 14 \in \mathfrak{S}(G)$. When $n = 12$, Alexeev's computer calculations show that $26 \in \mathfrak{S}(G)$. We can therefore assume that $n = 4k \geq 20$. Suppose that G contains a rainbow $(3n - 10)$-cycle $0 \to 1 \to \cdots \to 3n - 11 \to 0$ colored $\gamma(i, i + 1) = i$, and restrict all further considerations to the subgraph of G induced by this cycle. By Lemma 4.3, we can assume that $\gamma(i, i + n - 1)$, $\gamma(i, i + 13) \in \{i, i + 1, i + 2\}$ for every i.

Note that $8 \cdot (n - 1) + (k - 2) \cdot 13 + (3k - 6) \cdot 1 = 4(3n - 10) \equiv 0 \pmod{3n - 10}$. We will therefore consider an n-cycle A containing 8 edges of length $n - 1$, $k - 2$ edges of length 13, and $3k - 6$ edges of length 1, all forward. Such an n-cycle makes four clockwise revolutions modulo $3n - 10$.

Let r be the least integer such that

$$2(4k - 1) + 13r \geq 12k - 10 + 3,$$

i.e.,

$$r = \left\lceil \frac{4k - 5}{13} \right\rceil.$$

We will now describe the cycle (see Figure 6) and later check under which conditions the construction makes sense.

Let d_2 be the least nonegative integer such that $2(n - 1) + 13r - (3n - 10) + d_2 \equiv 3 \pmod{13}$, which is equivalent to $d_2 \equiv n + 8 \pmod{13}$. Let $d_1 = 2(n - 1) + 13r - (3n - 10)$.

Repeat three times: Add an edge of length $n-1$, r edges of length 13, an edge of length $n-1$, and d_2 edges of length 1. (Note the position of d_1 in the Figure.) After the three repetitions, add 13 edges of length 1, 2 edges of length $n-1$, all remaining edges of length 13, and all remaining edges of length 1.

[Enough edges of length 13?] In the construction, we need at least $3r$ edges of length 13. We have precisely $k-2$ such edges available. Thus we must have

$$3\left\lceil\frac{4k-5}{13}\right\rceil \leq k-2. \tag{4.1}$$

Careful analysis of this inequality shows that it holds for all $k \geq 35$.

[Estimating d_1 and d_2.] By our choice of r, $3 \leq d_1 < 3+13 = 16$. Clearly, $0 \leq d_2 \leq 12$.

[Enough edges of length 1?] In the construction, we need at least $13+3d_2$ edges of length 1. We have $3k-6$ such edges. Thus we must have

$$13+3d_2 \leq 3k-6. \tag{4.2}$$

Since $d_2 \leq 12$, this inequality holds as long as $k \geq 19$.

[Safely below $n-1$ after three rounds?] After the first round (1 edge of length $n-1$, r edges of length 13, 1 edge of length $n-1$, d_2 edges of length 1), we are in position d_1+d_2. The second round starts with an edge of length $n-1$, which brings us to some $x_1 = n-1+d_1+d_2$. By the definition of d_2, we have $x_1 > n-1$ and $x_1-(n-1) \equiv 3 \pmod{13}$. Moreover, upon completing the second round, we will be in position $2(d_1+d_2)$, and the next edge of length $n-1$ brings us to some $x_2 > x_1$ satisfying $x_2-(n-1) \equiv 6 \pmod{13}$. Finally, after the third round we are in position $3(d_1+d_2)$. We then add 13 edges of length 1. In order to be safely below $n-1$, we demand

$$13+3(d_1+d_2) \leq n-1-3 = 4k-4. \tag{4.3}$$

Since $d_1 \leq 15$, $d_2 \leq 12$, this inequality holds whenever $k \geq 25$.

[Far enough after all long edges have been used?] We now continue by adding two edges of length $n-1$. The first edge moves us to some $x_3 > x_2$ satisfying $x_3-(n-1) \equiv 9 \pmod{13}$. The second edge brings us to $y = 3(d_1+d_2)+13+2(n-1)$. Are we safely past all the edges of length 13 used so far? The farthest edge of length 13 used so far occurred as the last such edge in round 3, and terminated at $(n-1)+13r+2(d_1+d_2)$. We therefore demand

$$(n-1)+13r+2(d_1+d_2)+3 \leq 3(d_1+d_2)+13+2(n-1). \tag{4.4}$$

Since $d_1 \geq 3$ and $r < (4k+8)/13$, this inequality always holds.

[Finishing the cycle.] We have used all edges of length $n-1$, completed more than three revolutions, and there are no edges between our current position y and 0. The remaining edges of length 1 and 13 can therefore be adjoined in any order, and we are guaranteed to end exactly at 0.

When all inequalities (4.1)–(4.3) hold, the construction yields a rainbow n-cycle, a contradiction.

We are therefore done when $k \geq 35$. Let us have a closer look at k in the interval $5 \leq k \leq 34$. It is not hard to check that the inequalities (4.1)–(4.4) hold for every $k \in X = \{11, 20, 23, 24, 26, 27, 29, 30, 32, 33\}$.

For all remaining values $k \in \{5, \dots, 34\} \setminus X$, we ran a greedy algorithm that attempts to construct a $4k$-cycle with 8 edges of length $4k-1$, $k-2$ edges of length 13, and $3k-6$ edges of length 1, all forward, so that Lemma 4.3 guarantees that the cycle is rainbow. The depth-first backtrack algorithm first tries to extend paths by edges of length 1, then by edges of length 13, and finally by edges of length $n-1$. It succeeds for all $k \in \{5, \dots, 34\} \setminus X$, except for $k = 22$, $k = 25$ (when it was terminated after a few minutes). For instance, it finds $0 \to 1 \to 2 \to 3 \to 16 \to 17 \to 18 \to 37 \to 6 \to 25 \to 26 \to 27 \to 46 \to 9 \to 22 \to 41 \to 42 \to 43 \to 12 \to 31 \to 0$ as a valid cycle for $k = 5$.

The remaining two cases $k = 22$, $k = 25$ can be constructed by hand, by essentially following the general construction. To describe the two cycles, we use compact notation, in which $\stackrel{m}{\to}$ means that an edge of length m was used once, and $\stackrel{t.m}{\to}$ means that t edges of length m were used in succession.

Here is the cycle for $k = 22$: $0 \stackrel{87}{\to} 87 \stackrel{13.1}{\to} 100 \stackrel{6.13}{\to} 178 \stackrel{87}{\to} 11 \stackrel{5.1}{\to} 16 \stackrel{87}{\to} 103 \stackrel{7.13}{\to} 194 \stackrel{87}{\to} 27 \stackrel{5.1}{\to} 32 \stackrel{87}{\to} 119 \stackrel{7.13}{\to} 210 \stackrel{87}{\to} 43 \stackrel{5.1}{\to} 48 \stackrel{2.87}{\to} 222 \stackrel{32.1}{\to} 0$. And here is the cycle for $k = 25$: $0 \stackrel{99}{\to} 99 \stackrel{13.1}{\to} 112 \stackrel{7.13}{\to} 203 \stackrel{99}{\to} 12 \stackrel{4.1}{\to} 16 \stackrel{99}{\to} 115 \stackrel{8.13}{\to} 219 \stackrel{99}{\to} 28 \stackrel{4.1}{\to} 32 \stackrel{99}{\to} 131 \stackrel{8.13}{\to} 235 \stackrel{99}{\to} 44 \stackrel{4.1}{\to} 48 \stackrel{2.99}{\to} 246 \stackrel{44.1}{\to} 0$. □

5. Main result

Theorem 5.1. *Let G be an infinite, complete, edge-colored graph with $n \in \mathfrak{S}(G)$.*

(i) *If $n \geq 3$ is odd, there is $N \leq 2n^2 - 13n + 23$ such that $\{N + k;\ k \geq 0\} \subseteq \mathfrak{S}(G)$.*
(ii) *If $n = 4k+2 \geq 6$, there is $N \leq (9/4)n^2 - 18n + 37$ such that $\{N + 4k;\ k \geq 0\} \subseteq \mathfrak{S}(G)$.*
(iii) *If $n = 4k$, there is $N \leq (9/2)n^2 - 39n + 86$ such that $\{N + 2k;\ k \geq 0\} \subseteq \mathfrak{S}(G)$.*

Proof. Part (i) is proved in [1].

Assume that $n = 4k+2 \geq 6$. Then $\mathfrak{S}(G)$ contains $n \circ n \circ n = 3n-4$, and also $3n-8$, by Theorem 3.4. Hence $A = \mathfrak{S}(G)-2$ contains $3n-6 = 4(3k)$ and $3n-10 = 4(3k-1)$. Since $3k$, $3k-1$ are relatively prime, Corollary 1.3 implies that A contains $\{N + 4k;\ k \geq 0\}$ for some $N \leq 4(3k-2)(3k-1)$. Hence $\mathfrak{S}(G)$ contains $\{N + 4k;\ k \geq 0\}$ for some $N \leq 4(3k-2)(3k-1) + 2 = (9/4)n^2 - 18n + 37$. This proves (ii).

Assume that $n = 4k$. If $n = 4$ then $\mathfrak{S}(G)$ contains all positive even integers, and $9/2 \cdot n^2 - 39n + 86 = 2$, so (ii) holds. Assume that $4 < n = 4k \neq 16$. Then $\mathfrak{S}(G)$ contains $3n-8$ by Theorem 3.4, and also $3n-10$ by Theorem 4.4. Thus $A = \mathfrak{S}(G) - 2$ contains $3n-10 = 2(6k-5)$ and $3n-12 = 2(6k-6)$. Since $6k-5$, $6k-6$ are relatively prime (we need $n > 4$ here, else $6k-6 = 0$), Corollary 1.3 implies that A contains $\{N+2k;\ k \geq 0\}$ for some $N \leq 2(6k-7)(6k-6)$. Hence $\mathfrak{S}(G)$ contains $\{N+2k;\ k \geq 0\}$ for some $N \leq 2(6k-7)(6k-6) + 2 = (9/2)n^2 - 39n + 86$.

Finally, assume that $n = 16$. We have 16, $3 \cdot 16 - 8 = 40 \in \mathfrak{S}(G)$, by Theorem 3.4, and also $3 \cdot 40 - 10 = 110 \in \mathfrak{S}(G)$, by Theorem 4.4. Then $A = \mathfrak{S}(G) - 2$ contains 14, 38 and 108, and a check by computer shows that it contains $\{216 + 2k;\ k \geq 0\}$. Hence $\mathfrak{S}(G)$ contains $\{218 + 2k;\ k \geq 0\}$. Since $218 \leq 9/2 \cdot 16^2 - 39 \cdot 16 + 86 = 614$, we are through. □

6. Comments and suggestions for future research

6.1. **Comments.** 1) The notion of $\mathfrak{S}(G)$ makes sense for finite graphs, too, but then we have $\pi(\mathfrak{S}(G)) = 1$, since there are no n-cycles for sufficiently large n.

2) The results on $\pi(\mathfrak{S}(G))$ for $n = 4k$, $n = 4k+2$ cannot be improved in general, as witnessed by any graph G with $\mathfrak{S}(G) = \{2, 4, 6, \dots\}$ (see [2] for an example), and by any graph H with $\mathfrak{S}(H) = \{2, 6, 10, \dots\}$ (see [1] for an example).

3) Even if the bound in Lemma 4.3 could be improved to $n \geq 16$, Theorem 4.4 would not go through for $n = 16$ with the current proof, since there is no "valid" 16-cycle with 8 edges of length 15, 2 edges of length 13, and 6 edges of length 1, all forward (by a computer search).

4) The greedy algorithm used in the proof of Theorem 4.4 is available at

http://www.math.du.edu/˜petr.

6.2. **Suggestions for future research.** Let $n \in \mathfrak{S}(G)$.

1) Improve the bound on $N(\mathfrak{S}(G))$. Can you do better than $N(\mathfrak{S}(G)) = O(n^2)$? We are not aware of any examples in which $N(\mathfrak{S}(G))$ would get close to n^2.

2) For a more ambitious project, describe $\mathfrak{S}(G)$ between n and $N(\mathfrak{S}(G))$.

3) Extend Theorem 4.4 to cover the case $n = 16$, if possible.

4) Find an iterative construction for all finite, complete, edge-colored graphs with $4 \in \mathfrak{S}(G)$.

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References

[1] B. Alexeev, *On lengths of rainbow cycles*, Electron. J. Combin. **13** (2006), no. **1**, Research Paper 105, 14 pp.

[2] R. N. Ball, A. Pultr and P. Vojtěchovský, *Colored graphs without colorful cycles*, Combinatorica **27** (2007), no. **4**, 407–427.

[3] T. Gallai, *Transitiv oreintierbare Graphen*, Acta Math. Acad. Sci. Hungar. **18** (1967), 25–66. English translation by F. Maffray and M. Preissmann, in Perfect Graphs, edited by J. L. Ramirez-Alfonsin and B. A. Reed, John Wiley and Sons.

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