QUOTIENTS OF BING SPACES

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Dedicated to the memory of Yaki Sternfeld.

ABSTRACT. A Bing space is a compact Hausdorff space whose every component is a hereditarily indecomposable continuum. We investigate spaces which are quotients of a Bing space by means of a map which is injective on components. We show that the class of such spaces does not include every compact space, but does properly include the class of compact metric spaces.

Our entire development is based on Krasinkiewicz's penetrating notions of folding and double pairs. In section 1, we provide the definitions of double pairs and folding and show how these concepts are related to hereditarily indecomposable continua. Section 2 is the heart of this paper. We begin with a fixed compact space X. In Proposition 2.4, a simple two dimensional construction of a subspace Y of $X \times [-1,1]$ and the properties of Y are presented. In Lemma 2.8 the components of Y are characterized in terms of components of X and certain subspaces of X. In section 3, we use the machinery developed in section 2 to describe when a selection of components from Y, one for each component in X, can be made in such a way that their union Z is compact (see Theorem 3.5 and Corollary 3.7). This leads rather quickly to Theorem 3.17 and Corollary 3.18, which give classes of spaces (including compact metric spaces and connected spaces) which are guaranteed to be quotients of a Bing space by means of a map which is injective on components. We conclude by showing (Theorem 3.21) that there are spaces which are not such quotients of the above sort.

Throughout this paper, all topological spaces will be assumed to be Hausdorff without further mention. Since our development focuses on compact Hausdorff spaces, spaces not otherwise specified will be assumed to be compact.

1. Double pairs, folding and continua

We begin with a review of some of the basic notions. This section owes a particularly heavy debt to Krasinkiewicz [3] where the notion of a "double pair" is fundamental. The pairs used in the current paper are composed of open sets (rather than closed sets as in [3]), but we do not change the terminology and continue to call them double pairs.

Throughout this section, X and Y denote compact spaces.

Definition 1.1. A *double pair* is an ordered pair $((A_{-1}, B_{-1}), (A_1, B_1))$ of ordered pairs (A_i, B_i) of open subsets of X such that $A_i \subseteq B_i$ and $\overline{A_i} \cap \overline{B_j} = \emptyset$ for $i, j \in \{-1, 1\}$, $i \neq j$. *Throughout this paper, we adopt the convention that $i \in \{-1, 1\}$ and $j = -i$. For technical reasons (see Lemma 3.2 below) we allow the A_i's and B_i's to be empty. Of course, $\overline{A}$ denotes the closure of A.*

Date: August 25, 2006.

2000 *Mathematics Subject Classification.* Primary 54F15 ; Secondary 54D15,54B15.

Key words and phrases. Hereditarily indecomposable continuum, Bing space.

We recall that disjoint closed sets C_{-1} and C_1 in X are *completely separated*, i.e., for any real numbers r_i we may find some $f \in C(X)$ such that f has value r_i on C_i. See [1]. (We use $C(X)$ to designate the set of continuous real-valued functions on the compact space X.)

Proposition 1.2. *The following are equivalent for a double pair $((A_i, B_i))$.*

(1) *There exists $f \in C(X)$, $-1 \leq f \leq 1$, such that $f(A_i) \subseteq \{i\}$ and*
$$f^{-1}(-1,0) \cap B_{-1} = f^{-1}(0,1) \cap B_1 = \emptyset.$$
(2) *There exist disjoint clopen subsets F_i of $\overline{B_i}$ such that $A_i \subseteq F_i$.*

Proof. To show that (1) implies (2) take a function f satisfying (1) and set $F_1 \equiv f^{-1}[1/3, 1]$ and $F_{-1} \equiv f^{-1}[-1, -1/3]$. For the reverse implication, consider sets F_i satisfying (2). Then find $f_i \in C(X)$, $-1 \leq f_i \leq 1$, such that f_i is i on F_i and j on $\left(\overline{B_i} \smallsetminus F_i\right) \cup F_j$. Clearly $f \equiv \frac{1}{2}(f_{-1} + f_1)$ satisfies (1). □

Definition 1.3. We say that a double pair $((A_i, B_i))$ is *folded* if it satisfies either of the equivalent conditions in Proposition 1.2. In this case we say that subsets F_i satisfying (2) *fold* $((A_i, B_i))$, and that a function f satisfying (1) is a *folding function* for $((A_i, B_i))$.

Definition 1.4. A *continuum* is a compact connected space. *(To emphasize, a continuum is **not** assumed to be a metric space.)* A *subcontinuum* of a space is a subset which is a continuum. A *component* of a space is a maximal connected subset. Components are subcontinua. A space is *decomposable* if it is the union of two proper subcontinua. A space is *indecomposable* if it is not decomposable. A space is *hereditarily indecomposable* if every subcontinuum is indecomposable. A space is *Bing* if every component is hereditarily indecomposable.

We now begin to make the connections between continua and double pairs.

Definition 1.5. A double pair $((A_i, B_i))$ *detects* subcontinua C_i provided that $C_i \subseteq \overline{B_i}$ and $C_i \cap \overline{A_i} \neq \emptyset$.

Remark 1.6. *Observe the following.*

(1) *Two subcontinua detected by a double pair are nonempty and distinct. If they intersect then their union is a decomposable subcontinuum.*
(2) *Every decomposable subcontinuum is the union of two subcontinua detected by some double pair.*
(3) *X is Bing iff no double pair detects intersecting subcontinua.*

We remind the reader that if E_{-1} and E_1 are disjoint closed sets such that E_{-1} is a union of components, then there exists a clopen set F such that $E_{-1} \subseteq F$ and $F \cap E_1 = \emptyset$. The next result is a reformulation of Lemma 2.2 of [3].

Proposition 1.7. *A double pair is folded iff it detects no intersecting subcontinua.*

Proof. Let P be a double pair. If P detects intersecting subcontinua C_i then P certainly cannot be folded, for sets F_i folding P would have to satisfy $F_i \supseteq C_i$. Conversely, suppose $P = ((A_i, B_i))$ does not detect intersecting subcontinua, and let E_i be the union of the components of $\overline{B_i}$ which meet $\overline{A_i}$. Note that the E_i's are closed sets which are disjoint by hypothesis. Since E_{-1} is a union of components of $\overline{B_{-1}}$ disjoint from the closed set $\overline{B_{-1}} \cap E_1$, there exists a clopen subset F_{-1} of $\overline{B_{-1}}$ such that $E_{-1} \subseteq F_{-1}$ and $F_{-1} \cap E_1 = \emptyset$. Since E_1

is a union of components of $\overline{B_1}$ disjoint from the closed set $\overline{B_1} \cap F_{-1}$, there exists a clopen subset $F_1$of $\overline{B_1}$ such that $E_1 \subseteq F_1$ and $F_1 \cap F_{-1} = \emptyset$. That is, the F_i's fold P. □

The following corollary is a reformulation of Theorem 2.4 of [3].

Corollary 1.8. *A space is Bing iff every double pair is folded.*

We close this section with a collection some standard facts about components of compact spaces.

Definition 1.9. Let $b_X : X \to bX$ designate the quotient map which collapses each component of X to a point of bX. (If no confusion will arise, we will often denote b_X by b.) We refer to bX as *the Boolean reflection of* X. For any compact Y and X and continuous map $\theta : Y \to X$ there is a unique continuous map $b\theta : bY \to bX$ such that $(b\theta)\, b_Y = b_X\theta$. In fact, $b\theta$ simply maps each component C of Y to the unique component of X which contains $\theta(C)$. *This terminology is motivated by categorical considerations. Formally, the category* **bK** *of Boolean spaces with continuous maps constitutes a reflective subcategory of the category* **K** *of compact Hausdorff spaces and continuous maps. We shall not use categorical notions in the sequel.*

Lemma 1.10. *Let X be a compact space.*

(1) *Let D be a closed subset of X. The union of components of X which intersect D is a closed set.*
(2) *Let C be a component of the space X which is a subset of an open set U. Then there is a clopen set A such that $C \subseteq A \subseteq U$.*
(3) *Let X be a connected space and let $A \subseteq X$ be open and nonempty. Then every component of the set $X \smallsetminus A$ intersects $\overline{A}$. (This is a "boundary bumping theorem." See Nadler* [5, Ch. 20] *for a more complete discussion.)*
(4) *Suppose that $\tau : Y \to X$ is a continuous onto map and that X is connected. If for all $x \in X$, $\tau^{-1}\{x\}$ is a subset of a component of Y, then Y is connected.*

Proof. To prove (1), let K be the union of components C of X which intersect D. Let $b_X : X \to bX$ be the Boolean reflection of X. Then $K = b_X^{-1}(b_X D)$ is closed since X is compact, D is closed and b_X is continuous.

To establish (2), recall ([4, p. 169]) that in a compact space, every component is a quasi-component. That is, every component is the intersection of the clopen sets that contain it. An easy compactness argument yields a finite number of clopen sets, each containing C, whose intersection A is a subset of U.

For (3), let A, X be as in the hypothesis. Let us assume that C is a component of $X \smallsetminus A$ which does not intersect $\overline{A}$. By (2), there is a clopen set U in the space $X \smallsetminus A$ containing C and disjoint from $\overline{A}$. Since U is a closed subset of the closed in X set $X \smallsetminus A$, U is closed in X. Since U is an open subset of the open in X set $X \smallsetminus \overline{A}$, U is open in X. But X is connected, so it cannot contain a proper clopen set.

Finally we establish (4). Suppose that Y is not connected. Then there are nonempty disjoint clopen sets A and B whose union is Y. By the assumption, $\tau(A)$ and $\tau(B)$ are disjoint closed sets whose union is X. Therefore $\tau(A)$ and $\tau(B)$ are both clopen and so X is not connected, a contradiction. □

2. Folding preimages

Regarding non-folded double pairs as defects in X, we propose to remove such defects by passage to a preimage. But we require that the preimage should have no more components than X, i.e., we hope to fold double pairs in a "conservative" preimage.

Recall that a space Y is called *Boolean* or *totally disconnected* if the Boolean algebra of its clopen sets, $\operatorname{Clop} Y$, serves as a base for the open sets, and in the presence of the assumption that Y is compact, this is equivalent to Y being homeomorphic to the Stone space of $\operatorname{Clop} Y$.

The proof of the following proposition is routine.

Proposition 2.1. *The following are equivalent for a continuous surjection $\tau : Y \to X$.*

(1) *Every clopen subset of Y is a union of τ fibers. That is, $y \in A \in \operatorname{Clop} Y$ implies $\tau^{-1}\tau(y) \subseteq A$.*
(2) *The map $A \mapsto \tau^{-1}(A)$ is a Boolean isomorphism from $\operatorname{Clop} X$ onto $\operatorname{Clop} Y$.*
(3) *The map $b\tau : bY \to bX$ is a homeomorphism.*

Definition 2.2. A *conservative map* is a function which satisfies the conditions of Proposition 2.1. A *preimage of X* is a space Y for which there exists a continuous surjection $\tau : Y \to X$, called the *quotient map.* If τ is conservative we refer to Y as *a conservative preimage of X*, and we refer to X as a *conservative quotient* of Y.

Definition 2.3. We say that a double pair $P \equiv ((A_i, B_i))$ *is folded in a preimage Y of X* if $\tau^{-1}(P) \equiv ((\tau^{-1}(A_i), \tau^{-1}(B_i)))$ is folded in Y, where τ is the quotient map. In this case we refer to Y as a *folding preimage of X for P*. A *universal folding preimage for P* is a pair (Y, f), where Y is a preimage of X with surjection $\tau : Y \to X$, $f \in C(Y)$ is a folding function for $\tau^{-1}(P)$, and the following universal property holds. For any preimage Z with quotient map ψ, and for any folding function $g \in C(Z)$ for $\psi^{-1}(P)$, there is a unique continuous function $\theta : Z \to Y$ such that $\tau\theta = \psi$ and $f\theta = g$.

$$\begin{tikzcd}
{[-1,1]} & Z \arrow[l, "g"'] \arrow[d, "\psi"] \arrow[dl, "\theta"'] \\
Y \arrow[u, "f"] \arrow[r, "\tau"'] & X
\end{tikzcd}$$

A universal folding preimage for P must be unique when it exists. That is, if (Y_j, f_j), $j = 1, 2$, are universal folding preimages for P, with surjections $\tau_j : Y_j \to X$, then there is a homeomorphism $\theta : Y_1 \to Y_2$ such that $\tau_2\theta = \tau_1$ and $f_2\theta = f_1$.The existence of θ follows from the usual abstract nonsense.

Proposition 2.4. *For every double pair P there is a unique universal folding preimage for P.*

Proof. For $P = ((A_i, B_i))$ let Y consist of those points $(x, r) \in X \times [-1, 1]$ with the following properties.

(1) If $x \in A_i$ then $r = i$, $i = \pm 1$.
(2) If $r \in (-1, 0)$ then $x \notin B_{-1}$, and if $r \in (0, 1)$ then $x \notin B_1$.

Now Y is a closed subset of $X \times [-1, 1]$ and is therefore compact. Let $\tau : Y \to X$ be the projection map on the first coordinate, and let $f \in C(Y)$ be the projection map on the second coordinate. Then f folds P by construction.

Consider a folding preimage Z for P, say $\psi : Z \to X$ has $g \in C(Z)$ folding $\psi^{-1}(P)$. Define $\theta : Z \to Y$ by the rule

$$\theta(z) \equiv (\psi(z), g(z)), \ z \in Z.$$

Clearly $\tau\theta = \psi$ and $f\theta = g$, and θ is unique with respect to these properties. $\square$

Example 2.5. *Throughout the paper, we will provide diagrams that we hope will motivate and explain the concepts. It is almost a theorem to say that "the pictures tell the truth." In fact, many/most of the results of this paper arose from looking at pictures, then verifying that what appeared to be true in two dimensions was true in general.*

Let $X = [0,1] \cup \left[1\frac{1}{4}, 1\frac{3}{4}\right] \cup \left[2, 2\frac{1}{2}\right]$,

$$A_{-1} = \left(\tfrac{1}{5}, \tfrac{2}{5}\right) \qquad B_{-1} = \left[0, \tfrac{3}{5}\right) \cup \left[1\tfrac{1}{4}, 1\tfrac{3}{4}\right] \cup \left[2, 2\tfrac{1}{2}\right]$$

$$A_{1} = \left(\tfrac{3}{4}, 1\right] \qquad B_{1} = \left(\tfrac{1}{2}, 1\right] \cup \left(1\tfrac{1}{2}, 1\tfrac{3}{4}\right] \cup \left[2, 2\tfrac{1}{2}\right]$$

Figure 1 illustrates the construction of Y *is from* X *and the double pair. Later, after stating Proposition 2.8, we will classify the components of* Y. *Here,* $X \times [-1,1]$ *is shown as the union of three large gray rectangles. Representations of* $A_{\pm 1}$ *and* $B_{\pm 1}$ *are shown above and below* $X \times [-1,1]$ *as an aid to understanding the construction. (The* A_i*'s are light gray rectangles, the* B_i*'s are darker gray.) The subspace* Y *of* $X \times [-1,1]$ *is shown in solid black.*

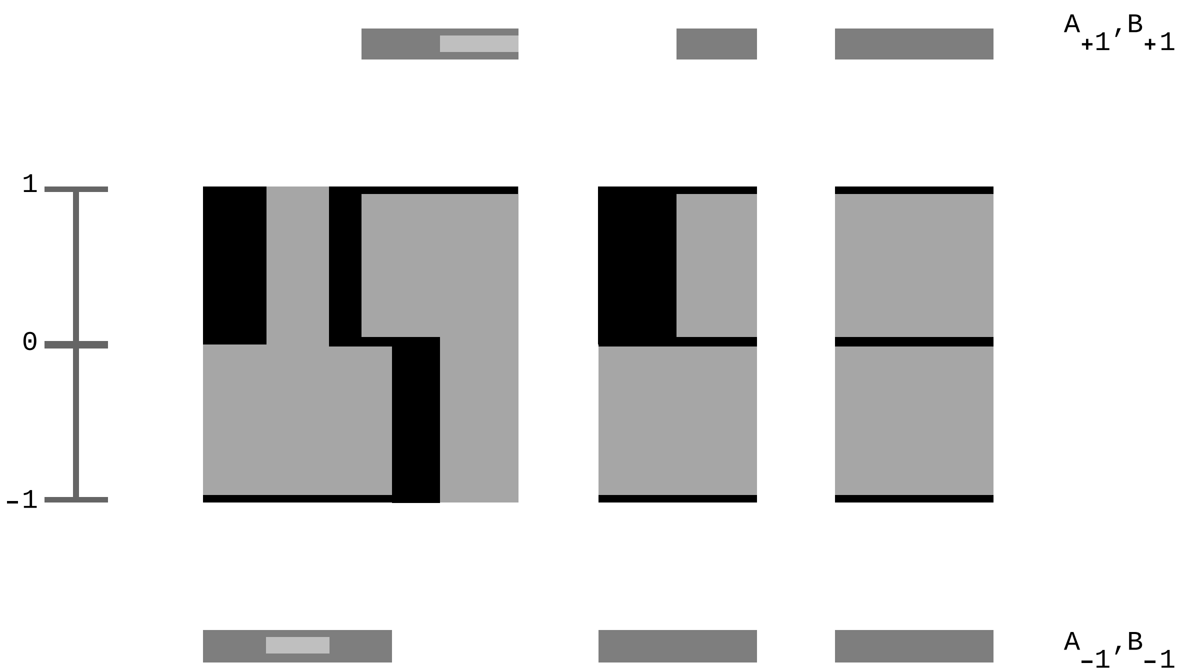

Figure 1

In future diagrams, we will leave out this level of detail and visualize this same example as shown in Figure 2. (Here, we show X *below* Y*, and not representations of the* A_i*'s and* B_i*'s.)*

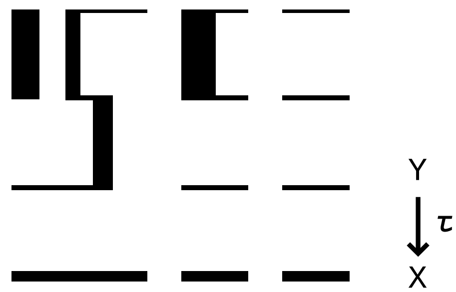

Figure 2

Corollary 2.6. *A double pair P is folded if and only if X is a retract of the universal folding preimage Y for P. That is, P is folded iff there is a continuous injection $\theta : X \to Y$ such that $\tau\theta$ is the identity function on X, where Y is the universal folding preimage for P from Proposition 2.4 and τ is the quotient map.*

Proof. If a function θ exists as above then one readily checks that $f\theta$ folds P. Now suppose that P is folded, choose a folding function $g \in C(X)$, and let $\theta : X \to Y$ be the map given by the universal property of Y. $\square$

In most instances the map $\tau : Y \to X$ given in Proposition 2.4 is not conservative. (This is the case in Example 2.5 above.) This raises a natural question. When does a double pair have a conservative folding preimage? We answer this question in Theorem 3.5. A key to this result is Proposition 2.8, which gives a description of the components of the universal folding preimage of a double pair.

Let us establish a small amount of notation before characterizing components of Y. Put $I = [-1, 1]$, $I_{-1} = [-1, 0]$ and $I_1 = [0, 1]$. For a set $B \subseteq X$, B^c denotes the complement of B in X. The following lemma is crucial in Proposition 2.8.

Lemma 2.7. *Let $P = ((A_i, B_i))$ be a double pair in X and let C be a component of X contained in neither B_{-1} nor B_1. For $i = \pm 1$ let UC_i be the union of components of $C \smallsetminus A_j$ which intersect B_j^c. Let UC_0 be the union of components of $C \smallsetminus (A_{-1} \cup A_1)$ which intersect both B_{-1}^c and B_1^c.*

Then

(1) $UC_0 = UC_{-1} \cap UC_1$.
(2) $C = UC_{-1} \cup UC_1$.

Proof. (1) Let $x \in UC_0$. Let K_0 denote the component of x in $X \smallsetminus (A_{-1} \cup A_1)$, and let K_i be the component of x in $X \smallsetminus A_j$. Clearly, $K_0 \subseteq K_i$. Since K_0 meets B_j^c, so does K_i, showing that $x \in UC_i$.

Next suppose that $x \in UC_{-1} \cap UC_1$. Then $x \notin A_{-1} \cup A_1$ and we define K_k, $k = -1, 0, 1$ as above. For $i = \pm 1$, we will show that $K_0 \cap B_j^c \neq \emptyset$ so $x \in UC_0$. For concreteness, let $i = -1$ and $j = 1$. If $K_{-1} \cap A_{-1} = \emptyset$ then $K_0 = K_{-1}$ and, since $x \in UC_{-1}$, we have $K_0 \cap B_1^c = K_{-1} \cap B_1^c \neq \emptyset$. If $K_{-1} \cap A_{-1} \neq \emptyset$, we apply Lemma 1.10(3) to the space K_{-1} and the nonempty, open in K_{-1} set $A_{-1} \cap K_{-1}$. We conclude from the lemma that K_0 contains a point of $\overline{A_{-1}} \subseteq B_1^c$. So in either case, K_0 intersects B_1^c. Similarly, using the fact that $x \in UC_1$, we conclude that K_0 intersects B_{-1}^c.

(2) follows easily from boundary bumping. Let $x \in C$. If $x \in A_i$ then the component of x in $X \smallsetminus A_j$ meets $\overline{A_j}$ so $x \in UC_i$. If $x \notin A_{-1} \cup A_1$ then the component K of x in

$C \setminus (A_{-1} \cup A_1)$ meets at least one of $\overline{A_{\pm 1}}$. If K meets $\overline{A_i}$ then the component of x in $C \setminus A_j$ contains K and meets $\overline{A_i} \subset B_j^c$ so $x \in UC_i$. $\square$

Proposition 2.8. *Let $P = ((A_i, B_i))$ be a double pair, and let Y be the universal folding preimage for P.*

(1) *Every component D of Y satisfies exactly one of the following descriptions.* (The components are named so that a type n component intersects exactly n of the "horizontal slices" $X \times \{-1, 0, 1\}$ in Y.)
Type $(1, i)$: Let C be a component of X contained in B_i. Then $D = C \times \{i\}$.
Type $(1, 0)$: $D = C \times \{0\}$, where C is a component of X contained in $B_{-1} \cap B_1$.
Type 2: Let C be a component of $X \setminus A_j$ contained in B_j but not B_i. Then
$$D = C \times \{i, 0\} \cup (C \setminus B_i) \times I_i.$$
(We say that D is a type 2 component spanning 0 and i.)
Type 3: Let C be a component of X contained in neither B_{-1} nor B_1. Then
$$\begin{aligned} D = &(UC_{-1} \times \{-1\}) \cup (UC_0 \times \{0\}) \cup (UC_1 \times \{1\}) \\ &\cup ((UC_{-1} \setminus B_{-1}) \times I_{-1}) \cup ((UC_1 \setminus B_1) \times I_1) \end{aligned}$$

(2) *In all cases τ maps D onto C, which is a component of X when D is type 1 or 3.*

Definition 2.9. We will say that a connected subset C of X *generates a type n component* $(n = 1, 2, 3)$ if $C = \tau(D)$ where D is a type n component. Note that the same C can generate more than one type of component in Y.

Example 2.5 (continued)**.** *Before proving this proposition, we can now identify the components of Y.*

- *The type (1,i) components are $[1\frac{1}{4}, 1\frac{3}{4}] \times \{-1\}$, $\left[2, 2\frac{1}{2}\right] \times \{-1\}$ and $\left[2, 2\frac{1}{2}\right] \times \{1\}$.*
- *The only type $(1, 0)$ component is $\left[2, 2\frac{1}{2}\right] \times \{0\}$.*
- *The type 2 components are $[0, \frac{1}{5}] \times I_1$ and $\left([1\frac{1}{4}, 1\frac{1}{2}] \times I_1\right) \cup \left([1\frac{1}{4}, 1\frac{3}{4}] \times \{0, 1\}\right)$. Note that the first of these components is not mapped by τ onto a component of X while the second is.*
- *The only type 3 component is the "S"–shaped component over the interval $[0, 1]$.*

We now turn to the proof of Proposition 2.8.

Proof. It is routine, but tedious, to check that Y is a disjoint union of sets of the above types. (To do this, assume $(y, r) \in Y$ and then consider cases depending on the value of r.) We continue by showing that all sets described above are components of Y.

Type $(1, i)$: Let $i = \pm 1$ and $x \in C$. Clearly, D is a closed, connected subset of Y containing (x, i). Now let $(y, r) \in Y \setminus D$. If $y \notin C$, then the points (y, r) and (x, i) cannot be in the same component of Y since τ is continuous and the points $\tau(y, r)$ and $\tau(x, i)$ are in different components of X. If $y \in C$, and $(y, r) \in Y \setminus D$ then $|i - r| \geq 1$ since $C \subseteq B_i$. Let $U \subseteq B_i$ be any clopen subset of X containing C. We note that since $U \subseteq B_i$, $U \times \{i\} = \left(U \times \left(i - \frac{1}{2}, i + \frac{1}{2}\right)\right) \cap Y$, so $U \times \{i\}$ is a clopen subset of Y. This set contains (x, i) but not (y, r) and we conclude that D is a component.

Type $(1, 0)$: Note that the set D is a subset of Y since $C \subseteq B_{-1} \cap B_1$ and therefore cannot intersect $A_{-1} \cup A_1$. The above proof works with $i = 0$ and $B_0 = B_{-1} \cap B_1$.

Type 2: Let $i = \pm 1$, $j = -i$. Then C is a component of $X \setminus A_j$ contained in B_j but not B_i. The sets $C \times \{0\}$, $C \times \{i\}$ are both connected subsets of Y. The set $C \setminus B_i$ is nonempty and for every $x \in C \setminus B_i$, the set $\{x\} \times I_i$ is a connected subset of Y which intersects both $C \times \{0\}$ and $C \times \{i\}$. Therefore D is connected.

To show that D is a component, let $(y, r) \in Y \smallsetminus D$. The point y is either in C, in another component of $X \setminus A_j$ or in A_j. If $y \notin C$ let U be a clopen set in $X \setminus A_j$ such that $U \subset B_j$, and U contains C but not y. Otherwise let U be any clopen subset of $X \setminus A_j$ inside B_j. Note that if $y \in C$ or $y \in A_j$ we have $r = j$ and $|r - i| = 2$. So in any case the set $(U \times I_i) \cap Y = \left(U \times \left(i - \frac{3}{2}, i + \frac{3}{2}\right)\right) \cap Y$ is a clopen subset of Y containing D but not (y, r).

Type 3: The set D is closed since (by Lemma 1.10(1)) it is a finite union of closed sets. It follows immediately from Lemma 2.7(2) that $\tau(D) = C$. We now use Lemma 1.10(4) to show that D is connected. To do this, we need only show that for any $x \in C$, $\tau^{-1}\{x\} \cap D$ is a subset of a connected component of D. If $|\tau^{-1}\{x\}| = 1$ then we are done. Suppose $|\tau^{-1}\{x\}| > 1$. It follows (Lemma 2.7(1)) that $x \in UC_0$. Since $x \in UC_0$, there are points $a_i \in B_i^c$ for $i = \pm 1$ which are both in the same component of $X \setminus (A_{-1} \cup A_1)$ as x. Again let K denote this component. The sets $K \times \{-1\}$, $\{a_{-1}\} \times I_{-1}$, $K \times \{0\}$, $\{a_1\} \times I_1$, $K \times \{1\}$ are all connected, nonempty subsets of D, with each intersecting the next on the list. This connected set contains $\tau^{-1}\{x\} \cap D$. Therefore, by Lemma 1.10(4), D is connected.

To show that D is a component, take $(y, s) \in Y \smallsetminus D$. If $y \notin C$ then (y, r) is not in the same component as any point in D since the point $\tau(y, r)$ is in a component of X different from $C = \tau(D)$. Now assume $y \in C$, but $(y, r) \notin D$. Note that $A_i \cap C \subset UC_i$ so $y \in X \smallsetminus (A_{-1} \cup A_1)$. Also note that $(y, 0) \notin D$ since this forces $(y, r) \in D$ for any $r \in [0, 1]$ such that $(y, r) \in Y$. Let L denote the component of y in $X \smallsetminus (A_{-1} \cup A_1)$, we know that L intersects B_j^c for exactly one $j \in \{\pm 1\}$. Since $L \cap B_j^c \neq \emptyset$, the component of y in $X \setminus A_j$, a superset of L, intersects B_j^c as well. This puts $y \in UC_i$, and $(y, i) \in D$. Therefore, $r \neq i$. Since $L \cap B_i^c = \emptyset$, $L \times \{s\} \cap Y = \emptyset$ for all s strictly between 0 and i which means $s \in I_j$. Further, $L \subset B_i$ implies L does not intersect $\overline{A_j}$ which means that L is the component of y in $X \setminus A_i$. Thus we have y in a component of $X \setminus A_j$ which is contained in B_j but not B_i and $s \in I_j$. That is, (y, s) is in a type 2 component of the space Y. Since D is disjoint from this component, we are done. $\square$

3. Existence and nonexistence of Conservative Bing Preimages

At last, we turn to existence and nonexistence of conservative preimages which fold specified double pairs. To this point, given X and a double pair P, we have constructed the universal folding preimage Y of X in which P is folded. This issue is to select from Y, if possible, a conservative preimage Z of X. The example given above (Example 2.5) has only a finite set of components in Y. The following slightly more complicated example shows what issues must be confronted if there is an infinite set of components in Y.

Example 3.1. *Take $X = [0, 1] \cup \{x_n : n = 1, 2, \cdots\}$ where $x_n \to 0$ as shown. Let $A_{-1} = \left(\frac{1}{5}, \frac{2}{5}\right)$, $A_1 = \left(\frac{3}{4}, 1\right]$, $B_{-1} = \left[0, \frac{3}{5}\right) \cup \{x_n\}$ and $B_1 = \left(\frac{1}{2}, 1\right] \cup \{x_n\}$. Since $x_n \in B_{-1} \cap B_1$ for all n, each of these points generates three type 1 components in Y, as pictured in Figure 3. It is clear that to select a conservative preimage of X, we must select the type 3 component over $[0, 1]$. To end up with a compact space Z we must eventually select all of the type $(1, -1)$ components $\{(x_n, -1)\}$ in Y. So some care must be taken in selecting components from Y. With the possible exception of a finite number of points above the sequence (x_n), we end up with a conservative preimage Z as in Figure 4. We will show formally how to build Z*

following the proof of Theorem 3.5 and Corollary 3.7.

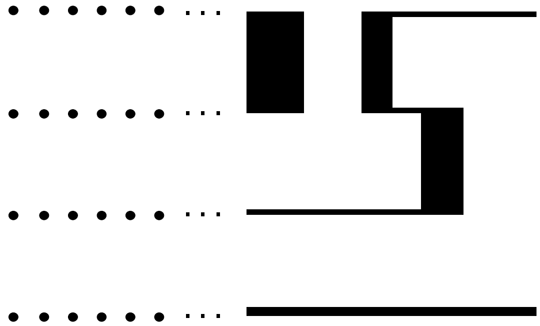

Figure 3

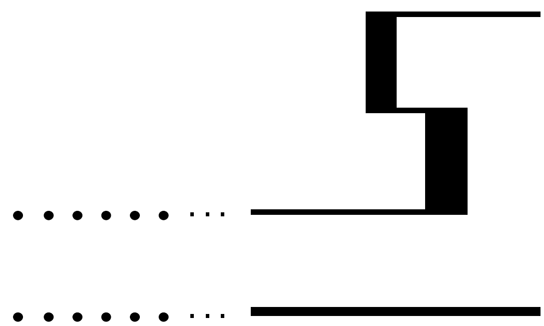

Figure 4

Our main theorem of this section (Theorem 3.5) requires some additional discussion, motivation and notation. Let $P = ((A_i, B_i))$ be a double pair in X. For an open subset U of X let $X' = X \diagdown U$ and $P' = ((A_i', B_i'))$, where $A_i' = A_i \cap X'$ and $B_i' = B_i \cap X'$. It is clear that P' is a double pair in X'. It will always be clear from context what X, P and U are, so no confusion should arise. Consistent with this notation, we let Y' be the universal folding preimage for P', $\tau' : Y' \to X'$ the quotient map and $f' : Y' \to I$ the folding function for Y'. We will denote a conservative preimage of X' which folds P' (if one exists) by Z'.

We collect relevant information into the following lemma.

Lemma 3.2. *Let $P = ((A_i, B_i))$ be a double pair in X. Let U be an open subset of X and let X', P' and Y' be as defined above. Then*

(1) $Y' = Y \setminus \tau^{-1}(U) = Y \cap \tau^{-1}(X')$, $\tau' = \tau|_{Y'}$ *and* $f' = f|_{Y'}$.
(2) *Every component D' of Y' is contained in a component D of Y.*
(3) *If D is a component of Y which is contained in Y', then D is a component of Y'.*
(4) *If D' is a component of Y' not meeting $\tau^{-1}\left(\bar{U}\right)$ then D' is a component of Y.*

Proof. We first consider (1). Let $x \in X$, $r \in [-1, 1]$. The proof easily from the definitions of Y and Y' by considering cases depending on r. We only give one case as the others are similar.

$$(x, -1) \in Y \cap \tau^{-1}(U) \Leftrightarrow x \notin A_1 \text{ and } x \notin U \Leftrightarrow x \in X' \text{ and } x \notin A_1' \Leftrightarrow (x, -1) \in Y'.$$

Property (2) follows from the fact that if V is an open subset of X and C' is a component of $X' \setminus V$, then there is a unique component C of $X \setminus V$ containing C'. We apply this when $V = \emptyset$, $V = A_i$ (in which case $X' \setminus V = X' \setminus A_i'$) and $V = A_{-1} \cup A_1$. The argument again proceeds by considering the type of a component D' of X'. The argument parallels the proof of Proposition 2.8, by considering an exhaustive set of cases. Since most of these cases are similar, we give only two of them (when D' is a type 1 or type 2 component) and leave the other cases to the reader.

First suppose that D' is a type 1 component of the form $C' \times \{i\}$ where C' is a component of X' contained in B_i'. Note that C' is disjoint from both A_j' and U so (regarded as a subset of X) C' is disjoint from A_j. Let C be the component in $X \setminus A_j$ which contains C'. Then $C \times \{i\}$ is a subset of a component D of Y which contains D'. Note that $(C \setminus A_j) \times \{i\}$. If D' is a type $(1, 0)$ component of X', the argument is similar.

Next suppose that $D' = C' \times \{i, 0\} \cup (C' \setminus B_i') \times I_i$ is a type 2 component of Y'. Then C' is a component of $X' \setminus A_j'$ contained in B_j' but not in B_i' and $C' \cap A_i' = \emptyset$. Let C be the component in $X \diagdown A_j$ containing C'. Then C is not contained in B_i. If C is contained in B_j then

$$D = C \times \{i, 0\} \cup (C \smallsetminus B_i) \times I_i$$

is a type 2 component of X containing the type 2 component D'. If C is not contained in B_j then the component K of $X \smallsetminus (A_{-1} \cup A_1)$ containing C' generates a type 3 component D of Y. Then (using the notation in the proof of Proposition 2.8) $C' \subseteq UC_i$ and $C' \subseteq UC_0$ so $C \times \{i, 0\} \subseteq D$. Also $C' \smallsetminus B_i' = C' \smallsetminus B_i \subseteq (K \smallsetminus B_i)$ so

$$(C' \smallsetminus B_i') \times I_i \subseteq D.$$

Property (3) is true generally. Property (4) is a direct consequence of boundary bumping (Lemma 1.10(3)). $\square$

Corollary 3.3. *Suppose that $P = ((A_i, B_i))$ is a double pair in X and $Z \subseteq Y$ is a conservative preimage of X which folds P. Put $A = A_{-1} \cup A_1$ and $Z' = Z \diagdown \tau^{-1}(A)$. Then*

$$b\tau' : bZ' \to bX'$$

is a homeomorphism. (That is, Z' is a conservative preimage of X' folding P'.)

Example 3.4. *Easy examples show that Corollary 3.3 fails for an arbitrary open subset U of X. Put $X = [0,1]$, $A_{-1} = \left(\frac{1}{5}, \frac{2}{5}\right)$, $A_1 = \left(\frac{3}{4}, 1\right]$, $B_{-1} = \left[0, \frac{3}{5}\right)$ and $B_1 = \left(\frac{1}{2}, 1\right]$. This is just Example 2.5 restricted to the subspace $[0,1]$. Figure 5 shows a diagram of X and Z, where we make the only possible choice for Z, the type 3 component mapping onto $[0,1]$. Since each of bX and bZ are singletons, $b\tau : bZ \to bX$ is a homeomorphism. Put $U_1 = \left(\frac{1}{2}, \frac{3}{5}\right)$. Then, as illustrated in the Figure 6, $b(X \diagdown U)$ has two points, each of whose preimages in $b(Z \diagdown \tau^{-1}(U))$ has two points. So $b\tau : b(Z \diagdown \tau^{-1}(U)) \to b(X \diagdown U)$ is not a homeomorphism. On the other hand Figure 7 shows $X \diagdown A$ and $Z \diagdown \tau^{-1}(A)$ where $A = A_{-1} \cup A_1$. In this case, $b\tau : b(Z \diagdown \tau^{-1}(A)) \to b(X \diagdown A)$ is a homeomorphism. Corollary 3.3 asserts that this is the case for any $A = A_{-1} \cup A_1$.*

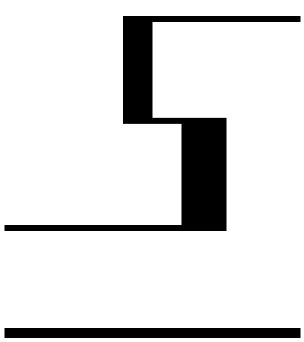

Figure 5

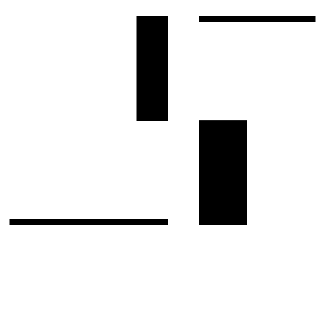

Figure 6

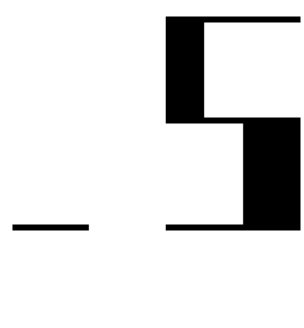

Figure 7

Proof of Corollary 3.3. The issue here is to show that the map $b\tau' = b\tau|_{Z \diagdown \tau^{-1}(A)}$ is injective. Let C' be a component of X', and let D be the component in Z satisfying $\tau(D) \supset C'$. We will be done if we can show that $\tau^{-1}(C') \cap D$ is connected. The argument splits into cases depending upon the type of the component D.

The statement is clear if D is type 1 since $\tau^{-1}(C') \cap D = C' \times \{k\}$ for some $k \in \{0, \pm 1\}$. If D is type 2 then, since $D \subset Z'$, $\tau(D)$ is a subset of X', so $\tau^{-1}(C') \cap D = D$.

Finally suppose that D is a type 3 component and $\tau(D) = C \supset C'$. From Lemma 2.7, $C = UC_{-1} \cup UC_1$, $UC_0 = UC_{-1} \cap UC_1$ and, since each UC_k is a union of components and

$C' \subset X \backslash (A_{-1} \cup A_1)$, it follows that if C' meets UC_k then $C' \subset UC_k$ $(k = 0, \pm 1)$. If $C' \subset UC_0$ then $\tau^{-1}(C')$ is a union of the sets $C' \times \{-1\}$, $(C' \setminus B_{-1}) \times I_{-1}$, $C' \times \{0\}$, $(C' \setminus B_1) \times I_1$, and $C' \times \{1\}$. Each $C' \times \{k\}$ is connected for $k = 0, \pm 1$, and for each $x \in (C' \setminus B_i)$, $\{x\} \times I_i$ $(i = \pm 1)$ is a connected set intersecting both $C' \times \{i\}$ and $C' \times \{0\}$. (That is $\tau^{-1}(C') \cap D$ is a type 3 component of Y'.) If C' is not a subset of UC_0, then for one i, $C' \subset UC_i$ and $C' \cap UC_j = \emptyset$. So by the definition of D in Proposition 2.8, $\tau^{-1}(C') \cap D = C' \times \{i\}$, which is connected. □

Recall that if X is a compact space $b_X : X \to bX$ is the map which collapses components of X into singletons in its Boolean reflection bX. For clarity of exposition in the next result, components in a space will be denoted by capital letters (say C or C'), and their image in the Boolean reflection will be denoted by the corresponding lower case letter (say c or c'). Formally, for $c, c' \in bX$, we have $C = b_X^{-1}(c)$ and $C' = b_X^{-1}(c')$.

Theorem 3.5. *Suppose that $P = ((A_i, B_i))$ is a double pair in X, $A = A_{-1} \cup A_1$, and $X' = X \diagdown A$. Then the following are equivalent:*

(1) *There is a conservative preimage $Z \subseteq Y$ of X folding P.*
(2) *In $b_{X'}X'$, there are disjoint open sets U_{-1}, U_1 and open sets V_{-1}, V_1 satisfying the following conditions:*
   (a) *$U_i \subseteq V_i \setminus V_j$,*
   (b) *For every $c' \in b_{X'}X'$, if $C' \subseteq B_i$ then $c' \in U_i$ or $c' \in V_j$;* (Recall that $C' = b_{X'}^{-1}(c')$.)
   (c) *For every $c' \in b_{X'}X'$, if $C' \subseteq B_i$ and $C' \cap \overline{A_i} \neq \emptyset$ then $c' \in U_i$.*

Proof. $(1 \Rightarrow 2)$ First assume that $Z \subseteq Y$ is a conservative preimage for X folding P. Then the following diagram commutes and by Corollary 3.3, $b\tau' : bZ' \to bX'$ is a homeomorphism.

$$\begin{tikzcd}
Y \arrow[dr, "\tau"'] & \supset \; Z \arrow[d, "\tau"] & \supset & Z' \arrow[r, "b_{Z'}"] \arrow[d, "\tau'"] & bZ' \arrow[d, "b\tau'"] \\
& X & \supset & X' \arrow[r, "b_{X'}"'] & bX'
\end{tikzcd}$$

Recall (Definition 1.3) that $f : Y \to I$ is the folding function for $\tau^{-1}(P)$. For $i = \pm 1$ put $\widetilde{U_i} = \{d' \in bZ' : D' \subseteq f^{-1}(i)\}$, put $\widetilde{V_{-1}} = \{d' \in bZ' : D' \subseteq f^{-1}(-3/2, 1/2)\}$ and $\widetilde{V_1} = \{d' \in bZ' : D' \subseteq f^{-1}(-1/2, 3/2)\}$. Since $b_{Z'}$ is an open map, each of the $\widetilde{U_i}$'s and $\widetilde{V_i}$'s is open in bZ'. It is clear that $\widetilde{V_{-1}} \cup \widetilde{V_1}$ is the set of all components of Z' with the exception of type 3 components. For $i = \pm 1$ put $U_i = (b\tau')\widetilde{U_i}$ and $V_i = (b\tau')\widetilde{V_i}$. Since $b\tau'$ is a homeomorphism, each of these sets is open in bX'.

We now show that the U_i's and V_i's satisfy (a), (b) and (c) by showing that the $\widetilde{U_i}$'s and $\widetilde{V_i}$'s satisfy analogous conditions in bZ' and then applying the homeomorphism $b\tau'$. It is clear that $\widetilde{U_i} \subseteq \widetilde{V_i} \setminus \widetilde{V_j}$ which proves (a).

For (b) assume that C' is a component of X' contained in B_i. Then by the construction in Proposition 2.8, C' generates a type $(1, i)$ component in Y', and may also generate a type $(1, 0)$ and a type $(1, j)$ component, or a type 2 component spanning 0 and j. So, any choice of a component D' in Y' with $\tau'(D') = C'$ puts d' in $\widetilde{U_i}$ or $\widetilde{V_j}$.

For property (c), assume that C' is a component of X' with $C' \subseteq B_i$ and $C' \cap \overline{A_i} \neq \emptyset$. Then D' is either the type $(1, i)$ component $C' \times \{i\}$ or a type 2 component over C' spanning 0 and j. But since $\tau^{-1}(A_i) = A_i \times \{i\}$, it follows that for any point $x \in \overline{A_i}$, $\tau^{-1}\{x\} \cap Z = (x, i)$. This implies that $D' \cap (Z \times \{i\}) \neq \emptyset$ so $D' = C' \times \{i\}$. That is, $d' \in \widetilde{U}_i$.

$(2 \Rightarrow 1)$ For the converse, we assume that conditions (a)-(c) hold and use them to construct Z. We think of this construction as selecting, for each $x \in X$, a subset of $\tau^{-1}\{x\}$ so that the union Z of the selected points is a conservative preimage.

For each $a \in A$ the set $\tau^{-1}\{a\}$ is a singleton. Clearly, we put all such points $\{\tau^{-1}\{a\} : a \in A\}$ into Z.

Now let $c' \in bX'$. We will select a component D' of Y' which maps onto C'.

Case 1: If $C' \nsubseteq B_{-1}$ and $C' \nsubseteq B_1$, then there is only one choice, since there is only one component (a type 3 component) of Y' which maps onto C'.

Case 2: If $C' \subseteq B_i$ and $C' \nsubseteq B_j$, then there are two possibilities - the type $(1, i)$ component $C' \times \{i\}$ and a type 2 component spanning levels 0 and j. Choose the type $(1, i)$ component if $c' \in U_i$ and the type 2 component spanning 0 and j if $c' \in V_j$.

Case 3: If $C' \subseteq B_i$ and $C' \subseteq B_j$ there are three possibilities, the type 1 components $C' \times \{k\}, k = 0, \pm 1$. Choose $C' \times \{i\}$ if $c' \in U_i$ for some i and select $C' \times \{0\}$ otherwise, i.e., if $c' \in V_{-1} \cap V_1$.

This completes the process of constructing Z. It remains to show that $\tau^{-1}(C) \cap Z$ is connected for every component C of X, and that Z is compact.

For the first, let C be a component of X. If $C \cap A = \emptyset$ then C is a component of X' and we know that $\tau^{-1}(C') \cap Z$ is connected for every component C' of X'. So let us suppose $C \cap A \neq \emptyset$. Observe that there is a unique component D of Y which contains $\{\tau^{-1}(x) : x \in C \cap A\}$ and therefore D is the unique component of Y such that $\tau(D) = C$. We will show that for any component C' of $C \setminus A$, $\tau^{-1}(C') \cap Z \subseteq D$. Each set of the form $\tau^{-1}(C') \cap Z$ is connected, so we will be done if we can show $\tau^{-1}(C') \cap Z \cap D \neq \emptyset$. But now, using the boundary bumping property, $C \cap A \neq \emptyset$ implies $C' \cap \overline{A_i} \neq \emptyset$ for some $i = \pm 1$. Let $x \in C' \cap \overline{A_i}$. The component D is a closed set which contains all of $(C \cap A_i) \times \{i\}$. Therefore, D contains (x, i). If $C' \cap B_i^c \neq \emptyset$ then C' generates a type 3 component of Y' and $(x, i) \in \tau^{-1}(C') \cap Z$. If not, $c' \in U_i$ and by our construction procedure for Z, $\tau^{-1}(C') \cap Z = C' \times \{i\}$, so $(x, i) \in \tau^{-1}(C') \cap Z$. Therefore, $(x, i) \in \tau^{-1}(C') \cap Z \cap D$.

To show that Z is compact, we show that Z^c is open in Y. Let D be a component of Z^c. Since every type 3 component is automatically in Z, D must be a component of type 1 or 2. Further, note that $\tau(D) \cap A = \emptyset$, so $\tau(D)$ is a component of X'. We now consider cases.

Suppose first that D is a type $(1, i)$ component for $i = \pm 1$. Then $\tau(D) \subseteq B_i$ and, by the construction procedure for Z, $D \subseteq Z^c$ if and only if $b_{X'}\tau(D) \in V_j$. The set

$$W = \{D : \tau(D) \subseteq B_i\} \cap b_{X'}\tau^{-1}(V_j)$$

is open in Y and $D \subset W \subset Z^c$.

Next, suppose that D is a type $(1, 0)$ component. Then $\tau(D) \subseteq B_{-1} \cap B_1$ and $D \subseteq Z^c$ if and only if $b_{X'}\tau(D) \in U_{-1} \cup U_1$. The set

$$W = \{D : \tau(D) \subseteq B_{-1} \cap B_1\} \cap b_{X'}\tau^{-1}(U_{-1} \cup U_1)$$

is open in Y and $D \subset W \subset Z^c$.

Finally, suppose that D is a type 2 component spanning levels 0 and j. Then $\tau(D) \subseteq B_i$ and $D \subseteq Z^c$ if and only if $b_{X'}\tau(D) \in U_i$. The set

$$W = \{D : \tau(D) \subseteq B_i\} \cap b_{X'}\tau^{-1}(U_i)$$

is open in Y and $D \subset W \subset Z^c$.

Therefore Z^c is open and Z is compact. □

We will see in Theorems 3.20 and 3.21 that conservative preimages of X folding P containing type 2 components of Y can create an obstacle to building a conservative Bing preimage of X. So it is crucial to know when type 2 components can be avoided. We would also like to avoid type $(1,0)$ components as well. The following lemma shows that if type 2 components can be avoided, then so can type type $(1,0)$ components.

Lemma 3.6. *Suppose that $P = ((A_i, B_i))$ is a double pair in X. Suppose also there is a conservative preimage $Z \subseteq Y$ of X folding P consisting of type 1 and type 3 components of Y. Then there is a conservative preimage $W \subseteq Y$ of X folding P consisting of type $(1,-1)$, type $(1,1)$ and type 3 components of Y.*

Proof. Suppose that $\hat{Z}$ is a conservative preimage of X, folding P, and containing only type 1 and type 3 components. Let T be the union of the type 3 and the types $(1,-1)$ and $(1,1)$ components of $\hat{Z}$. It is clear that T is closed in Y. Let S_0 be the union of the type $(1,0)$ components in $\hat{Z}$ and let S_1 be the set obtained by replacing each point $(x,0) \in S_0$ by $(x,1)$. Let $Z = T \cup S_1$. That is, Z is the space obtained from $\hat{Z}$ by replacing each type $(1,0)$ component in $\hat{Z}$ by the corresponding type $(1,1)$ component in Y.

We claim that Z is a conservative preimage of X. To prove that Z is closed, it suffices to prove that any cluster point of S_1 is in Z. To this end, let $(x,1)$ be such a point. But then $(x,0)$ is a cluster point of S_0, hence is in $\hat{Z}$. If $(x,0) \in S_0$, then $(x,1) \in S_1 \subset Z$. If $(x,0) \notin S_0$, then $(x,0) \in D$ where D is a type 3 component of $\hat{Z}$, hence of Z. By the characterization of type 3 components in the proof of Proposition 2.8 $(x,1) \in D \subset W$ as well. So, Z is closed. It is now easy to see that Z is conservative preimage of X. □

Corollary 3.7. *Suppose that $P = ((A_i, B_i))$ is a double pair in X, $A = A_{-1} \cup A_1$, and $X' = X \diagdown A$. Then the following are equivalent:*

1. *There is a conservative preimage $Z \subseteq Y$ of X folding P consisting of type 1 and type 3 components of Y.*
2. *There is a conservative preimage $Z \subseteq Y$ of X folding P consisting of type $(1,-1)$, type $(1,1)$ and type 3 components of Y. (That is, type $(1,0)$ components can be avoided.)*
3. *In $b_{X'}X'$, there are disjoint open sets U_{-1} and U_1 satisfying the following conditions:*
   (a) *$U_{-1} \cup U_1 = \{c' \in b_{X'}X' : C' \subseteq B_{-1}$ or $C' \subseteq B_1\}$.*
   (b) *For every $c' \in b_{X'}X'$, if $C' \subseteq B_i$ and $C' \cap \overline{A_i} \neq \emptyset$ then $c' \in U_i$.*

Proof. We have already (Lemma 3.6) established the equivalence of (1) and (2). We now show that $(2 \Rightarrow 3)$. To this end let Z be as in (2). (We continue with the notation introduced in the argument of Theorem 3.5.) For $i = \pm 1$, put $\widetilde{U_i} = \{d' \in bZ' : D' \subseteq f^{-1}(i)\}$ and $U_i = (b\tau')\,\widetilde{U_i}$. Each of these sets is open. Since Z contains no type 2 components, neither does Z'. It is now straightforward to check that conditions 3(a) and 3(b) hold.

For $(3 \Rightarrow 2)$ we first apply the construction in the proof of Theorem 3.5 with $V_i = U_i$ to form a conservative preimage $\hat{Z}$ of X. The space $\hat{Z}$ may contain type 2 components (from case 2) but it contains no type $(1,0)$ components, since these are built only in case 3 and only when $V_{-1} \cap V_1 \neq \emptyset$.

Let Z be the space formed by by exchanging each type 2 component spanning 0 and j in $\hat{Z}'$ (or $\hat{Z}$) with the corresponding type $(1,i)$ component in Y'. Specifically, let C' be a component of X' and D' a type two component over C' spanning 0 and j. Then remove D' from $\hat{Z}$ and add $C' \times \{i\}$ to Z. Note that $C' \cap \overline{A_i} = \emptyset$. So the components of Z are precisely the types 3 and 1 components of $\hat{Z}$ and these new exchanged type $(1,i)$ components. We are done once we show that the space Z formed in this way is closed. (The argument is similar to that in Lemma 3.6.) Let T denote the union of the type 2 components in $\hat{Z}$ and S the union of the corresponding $(1,i)$ components (the replacements) in Z. If (x,i) is a cluster point of S, then some (x,r) (where r is between 0 and j) is a cluster point of T. Now, since the union of the points in type 2 and type 3 components is closed, either $(x,r) \in T$ or $(x,r) \in D$, where D is a type 3 component in Z. In the former case, $(x,i) \in S \subset Z$. In the latter case, $(x,i) \in D$, again by the characterization of type 3 components in the proof of Proposition 2.8. In either case, $(x,i) \in Z$ and Z is closed. $\square$

Example 3.1 *(continued). Let us show how Corollary 3.7 leads to the construction of the conservative preimage Z as discussed in the first part of this example. The space $bX' = \{x_n : n = 1, 2, \cdots\} \cup \{c_1, c_2\}$ is shown in Figure 8.*

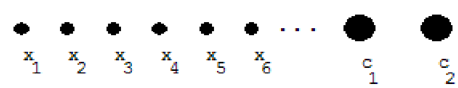

Figure 8

In bX', $x_n \to c_1$ and C_1 is the component of X' meeting $\overline{A_{-1}}$ and contained in B_{-1}, C_2 meets both $\overline{A_{-1}}$ and *$\overline{A_1}$. Let us build $U_{\pm 1}$. By condition 2(b), $c_1 \in U_{-1}$ and since U_{-1} must be open, there is an N so that $\{x_n : n \geq N\} \subset U_{-1}$. Our construction process gives $\{x_n\} \times \{-1\} \subset Z$ for $n \geq N$. The component C_2 is a type 3 component, so it must be a subset of Z as well. The remaining points $\{x_n\}$ for $n < N$ are also isolated, so they may distributed into $U_{\pm 1}$ in any way whatsoever.*

We arrive at the following critical result, which gives a sufficient condition for folding every double pair in X. The condition states that every open subspace W of bX satisfies the following strong normality property.

Definition 3.8. A totally disconnected compact space K is *strongly hereditarily normal* if it satisfies the following property:

Let W be an open set in K and let E_{-1} and E_1 be two disjoint closed in W subsets. Then there are two disjoint clopen in W sets U_{-1} and U_1 such that $E_i \subset U_i$ and $W = U_{-1} \cup U_1$.

Theorem 3.9. *Suppose that P is a double pair in X. Suppose also that bX' is strongly hereditarily normal. Then there is a conservative preimage $Z \subseteq Y$ of X folding P consisting of type $(1,-1)$ type $(1,1)$ and type 3 components of Y.*

Proof. Let P be a double pair in X. Form X' and Y' from P. Let $\hat{Z} = \{c' \in b_{X'}X' : C' \subseteq B_{-1}$ or $C' \subseteq B_1\}$. For $i = \pm 1$ let $E_i = \left\{c' \in b_{X'}X' : C' \cap \overline{A_i} \neq \emptyset\right\}$. By 1.10(1), each E_i is closed in $\hat{Z}$ and the E_i's are disjoint, since if $C' \cap \overline{A_i} \neq \emptyset$ for $i = \pm 1$, then $c' \notin \hat{Z}$. Select $U_i \supset E_i$ so that the U_i's are clopen in $\hat{Z}$ and partition $\hat{Z}$. A direct application of Corollary 3.7 gives a conservative preimage Z of X folding P in the stated form. $\square$

To show that this class of spaces is interesting and to set the stage for showing that every metric space has a conservative Bing preimage, we show here that totally disconnected compact metric spaces K are strongly hereditarily normal. In addition, we show that the ordinal space $[1, \Omega]$ with the order topology is strongly hereditarily normal. (Ω denotes the first uncountable ordinal.)

Lemma 3.10. *Suppose that K is a totally disconnected compact space in which every open set is an F_σ set. Then K is strongly hereditarily normal. In particular, every totally disconnected compact metric space is strongly hereditarily normal.*

Proof. Let W be an open subset of M and E_{-1}, E_1 disjoint, closed in W sets. Write $W = \cup_{n=1}^{\infty} W_n$ where each W_n is clopen in M and $W_n \cap W_m = \emptyset$ if $n \neq m$. For $n = 1, 2, \ldots$ and $i = \pm 1$ put $F_{n,i} = W_n \cap E_i$. Since $F_{n,-1}$ and $F_{n,1}$ are disjoint closed sets in M_n, there are disjoint clopen sets $U_{n,-1}$ and $U_{n,1}$ in M_n so that $F_{n,i} \subseteq U_{n,i}$ and $U_{n,-1} \cup U_{n,1} = W_n$.

Now for $i = \pm 1$ let $U_i = \cup_{n=1}^{\infty} U_{n,i}$. It is immediate that U_{-1} and U_1 have the desired properties. $\square$

Lemma 3.11. *Let $K = [1, \Omega]$ where Ω is the first uncountable ordinal. Then K is strongly hereditarily normal.*

Proof. The proof is a standard exercise, essentially the same as proving that the space $[1, \Omega)$ is normal so we provide only a sketch. Assume first that $W = [1, \Omega)$. The key is the fact that if E_{-1} and E_1 are two disjoint closed in W sets, then at most one of them can have Ω as a cluster point. Taking the closures of these sets in $[1, \Omega]$, it follows that $\overline{E_{-1}}$ and $\overline{E_1}$ are disjoint. So, $\overline{E_{-1}}$ and $\overline{E_1}$ can be separated in $[1, \Omega]$ by disjoint clopen sets V_{-1} and V_1. Letting $U_i = V_i \cap W$ gives the desired clopen partition of W. The case for arbitrary open W is now similar. The case where $\Omega \notin W$ is similar to the above. The case where $\Omega \in W$ is essentially the same as $[1, \Omega]$. $\square$

Definition 3.12. Suppose that $Q = ((A_{-1}, B_{-1}), (A_1, B_1))$ and $P = ((U_{-1}, V_{-1}), (U_1, V_1))$ are double pairs in the space X. We say that P *covers* Q if $A_i \subset U_i$ and $B_i \subset V_i$ for $i = \pm 1$. We call a family $\mathcal{P}$ of double pairs of X a *covering family* if for every double pair Q in X, there is a $P \in \mathcal{P}$ which covers Q.

The importance of covering families is indicated in the next result.

Lemma 3.13. *Suppose that X is a compact space and $\mathcal{P}$ is a covering family of double pairs. Suppose that every double pair $P \in \mathcal{P}$ is folded in X. Then every double pair in X is folded.*

Proof. Given a double pair $Q = ((A_{-1}, B_{-1}), (A_1, B_1))$ in X, find $P = ((U_{-1}, V_{-1}), (U_1, V_1)) \in \mathcal{P}$ as assumed. Let $f : X \to [-1, 1]$ fold P. Then f folds Q as well. $\square$

Lemma 3.14. *Suppose that X is a compact space and $\mathcal{U}$ a base for the topology of X which is closed under finite unions. Then $\{((A_{-1}, B_{-1}), (A_1, B_1)) : A_i, B_i \in \mathcal{U}$ for $i = \pm 1\}$ is a covering family.*

Proof. This is a standard compactness exercise. $\square$

We now begin a construction to show that if X is any compact metric space, then X has a conservative Bing preimage. The first step is Lemma 3.15, which produces a conservative metric preimage Z of X in which every double pair in X (but perhaps not in Z) is folded. This lemma provides the central step in an inductive process that we use in the proof of

Theorem 3.17 to produce a Bing preimage of X. The machinery we have developed makes the arguments fairly straightforward. We use repeatedly the fact that if X is a metric space, then so is any subspace X' of X. In particular, bX' is metric, so it is strongly hereditarily normal and Lemma 3.10 may be applied.

Theorem 3.15. *Let K be a compact metric space. Then X has a metric conservative preimage Z in which every double pair in X is folded.*

Proof. The proof proceeds via an inductive construction. Observe that X has a countable covering family $\{P_n : n = 1, 2, \dots\}$ of double pairs. (Such a family exists since each of the sets in a double pair may be covered by a finite union of sets from a countable base.)

Put $Z_0 = X$. Let $\tau_{1,0} : Z_1 \to Z_0$ be the quotient map where Z_1 is a conservative preimage of Z_0 folding P_1. (The space Z_1 exists because bZ_0' is strongly hereditarily normal.)

Now suppose that the sets $Z_0, \dots, Z_n$, bonding maps $\tau_{k,m} : Z_k \to Z_l$ for all $0 \le m \le k \le n$ such that

(1) $\tau_{k,m}$ maps Z_k conservatively onto Z_m. (In particular, $bZ_k = bZ_0 = bX$ for all $0 \le k \le n$.)
(2) If $0 \le m \le p \le k \le n$ then $\tau_{k,m} = \tau_{p,m} \circ \tau_{k,p}$.
(3) For all $k = 0, \dots n$, P_k is folded in Z_n.

We now construct Z_{n+1} and $\tau_{n+1,k} : Z_{n+1} \to Z_k$ so that 1–3 are satisfied.

Let Z_{n+1} be a conservative preimage of Z_n folding P_{n+1}. Let $\tau_{n+1,n} : Z_{n+1} \to Z_n$ be the quotient map. For $k \le n$ put $\tau_{n+1,k} = \tau_{n,k} \circ \tau_{n+1,n}$. This completes the inductive construction.

Put $Z = \varprojlim \{Z_n : n = 0, 1, 2, \dots\}$ and for $n = 0, 1, 2, \dots$ let $\tau_n : Z \to Z_n$ be the natural projection onto Z_n. It is clear that every double pair in X is folded in Z.

To show that Z is a conservative preimage of each Z_n, we need only show that $b\tau_n : bZ \to bZ_n$ is a homeomorphism. But this is easy, since each bonding map $b\tau_{m,n} : bZ_m \to bZ_n$ is a homeomorphism for each $m \ge n$.

Since $Z_n \subset Z_{n-1} \times [-1, 1]^{\aleph_0}$ for all $n \ge 1$, each Z_n is also metric. Since the inverse limit $Z = \varprojlim \{Z_n : n = 0, 1, 2, \dots\}$ of compact metric spaces is again compact metric, we obtain a metric conservative preimage Z of X folding each double pair in X. $\square$

Remark 3.16. *Suppose we know the following: X is a compact space and $\mathcal{P}$ is a covering family of open double pairs in X such that for* **every** *$P \in \mathcal{P}$, bX' is strongly hereditarily normal. Then a transfinite induction similar to the induction process above gives a conservative preimage Z of X which folds every double pair in X. We omit the details.*

Theorem 3.17. *Let X be a compact metric space. Then X has a metric conservative Bing preimage Z.*

Proof. The argument uses Lemma 3.15 and a standard induction. Put $Z_0 = X$ and inductively construct a sequence Z_n of spaces and quotient maps $\tau_n : Z_n \to Z_{n-1}$ so that

(1) Every double pair in Z_{n-1} is folded in Z_n.
(2) Z_n is a conservative preimage of Z_{n-1} with quotient map $\tau_n : Z_n \to Z_{n-1}$.

Let $Z = \varprojlim \{Z_n : n = 0, 1, 2, \dots\}$ and $\tau : Z \to X$ be the natural projection onto $X = Z_0$. As in the proof of Lemma 3.15, it is easy to see that τ is conservative. To show that every double pair in Z is folded, consider the family of sets

$$\left(O_0 \times O_1 \times O_2 \times \cdots O_n \times \prod_{m>n} Z_m \right) \bigcap Z$$

where each O_j is open in Z_j. Let $\mathcal{U}$ be the family of finite unions of these open sets. The construction shows that if $P \in \mathcal{U}$, then P is folded in Z. Also, $\mathcal{U}$ is a base of open sets satisfying Lemma 3.14. By Lemma 3.13, such a family is covering and so every double pair in Z is folded. Since $Z_n \subset Z_{n-1} \times [-1,1]^{\aleph_0}$ for all $n \geq 1$, each Z_n is also metric. Since the inverse limit $Z = \varprojlim\{Z_n : n = 0, 1, 2, \ldots\}$ of compact metric spaces is again compact metric, we obtain a metric conservative Bing preimage Z of X. $\square$

Theorem 3.18. *Let X be a connected compact space. Then X has a connected conservative Bing preimage Z.*

Proof. If X is connected, then for each double pair P in X, there is a type 1 or type 3 component D in Y such that $\tau(D) = X$. Put $Z = D$. In particular, Z is connected, the transfinite induction argument discussed in Remark 3.16 gives a hereditarily indecomposable continuum mapping onto X. $\square$

Open Problem. *If bX is metric or strongly hereditarily normal, does X have a conservative Bing preimage Z? The distinction between this and the results of Theorem 3.15 is that for subspaces X' of X, bX' need not be metric and so the construction discussed in Remark 3.16 does not work.*

We now show (Theorem 3.20) the existence of a space X and double pair P in X for which a conservative preimage Z folding P exists, but where any conservative preimage must have type 2 components. In Theorem 3.21 we exhibit a space X_1 and double pair Q in X_1 for which no conservative preimage Z folding Q exists. The first step is showing that there is a totally disconnected compact space which is not strongly hereditarily normal.

Lemma 3.19. *Let $X = [1,\Omega] \times [1,\Omega]$ where Ω is the first uncountable ordinal and each factor has the order topology. Then X is not strongly hereditarily normal.*

Proof. Put $R = \{\Omega\} \times [1,\Omega]$ and $W = X \setminus R$. Let $E_{-1} = \{(\alpha,\Omega\} : \alpha < \Omega\}$ and $E_1 = \{(\alpha,\alpha\} : \alpha < \Omega\}$. Observe that W is open in X, and E_{-1} and E_1 are disjoint closed in W subsets of W. It is well known and easy to establish ([2, problem E, p131]) that E_1 and E_{-1} cannot be separated by disjoint open in W sets. (This is a standard example that shows that the product $[1,\Omega] \times [1,\Omega)$ of two normal spaces need not be normal.) $\square$

Lemma 3.19 shows that the next two results are not vacuously true.

Theorem 3.20. *Let X be any totally disconnected compact space which is not strongly hereditarily normal. Then there is a double pair P in X so that if $Z \subseteq Y$ is any conservative preimage of X folding P, then Z has type two components.*

Proof. Let W be an open subset of X and E_{-1} and E_1 disjoint closed in W subsets of W which cannot be separated by disjoint clopen sets. Let $A_{-1} = A_1 = \emptyset$, $B_{-1} = W \setminus E_1$, $B_1 = W \setminus E_{-1}$ and $P = ((\emptyset, B_i))$. Since X is totally disconnected, components are singletons. Since $A = A_{-1} \cup A_1 = \emptyset$, $X = X' = bX = bX'$ and we do not distinguish among them. Also, B_{-1} and B_1 are open both in W and in X and $B_{-1} \cup B_1 = W$.

Now form Y and observe that

- If $x \notin W$, then $x \notin B_{-1} \cup B_1$, so x generates a type 3 component $\{x\} \times I$.
- If $x \in B_{-1} \cap B_1$, then x generates three type 1 components $\{(x,-1)\}$, $\{(x,0)\}$ and $\{(x,1)\}$.

- If $x \in E_{-1}$ then $x \in B_{-1} \diagdown B_1$, giving the type 1 component $\{(x,-1)\}$ and the type 2 component $\{x\} \times I_1$.
- If $x \in E_1$ then $x \in B_1 \diagdown B_{-1}$, giving the type 1 component $\{(x,1)\}$ and the type 2 component $\{x\} \times I_{-1}$.

We now produce a conservative preimage $X_1 \subset Y$ which folds P. (We call it X_1 because we will show in Theorem 3.21 that there is a double pair Q in X_1 which cannot be folded in any conservative preimage.) To do this, use the type 3 components $\{\{x\} \times I : x \in R\}$, the type 2 components $\{\{x\} \times I_1 : x \in E_{-1}\}$ and $\{\{x\} \times I_{-1} : x \in E_1\}$. Finally, use the type $(1,0)$ components $\{\{(x,0)\} : x \in B_{-1} \cap B_1\}$. It is routine to verify that X_1 is a closed, conservative preimage of X.

In the construction of X_1 in Theorem 3.5, this corresponds to letting $U_{-1} = U_1 = \emptyset$, $V_{-1} = B_1$ and $V_1 = B_{-1}$. It is immediate that conditions (a)-(c) are satisfied. Indeed, (a) is clear, (c) is vacuous, and if $c \in B_i$, then $c \in V_j$, establishing (b).

We showed above that if X is metric, we can always construct Z using only types 1 and 3 components. We now show that any conservative preimage Z of X folding P must have type 2 components.

So let us suppose by way of contradiction that another conservative preimage Z of X exists with no type 2 components. Then included in Z must be the type 3 components $\{\{x\} \times I : x \notin W\}$ and the sets of type 1 components $E_1 \times \{1\}$ and and $E_{-1} \times \{-1\}$. For $i = \pm 1$, let

$$U_i = \{x \in X : \{(x,i)\} \text{ is a type } (1,i) \text{ component in } Z\}.$$

The U_i's are disjoint clopen sets in W, partition W, and $U_i \supseteq E_i$ for $i = \pm 1$. This contradicts the fact that E_{-1} and E_1 cannot be separated by disjoint clopen sets in W. We have shown that condition 3 in Theorem 3.7 doesn't hold, so no conservative preimage without type 2 components can exist. □

Theorem 3.21. *Let X be any totally disconnected compact space which is not strongly hereditarily normal. Then there is a space X_1 with $bX_1 = X$ and a double pair Q in X_1 that cannot be folded in any conservative preimage of X_1. In particular, X_1 has no conservative Bing preimage.*

Proof. Let X, P, X_1, and $\tau : X_1 \to X$ be as in Theorem 3.20. We will show that there is a double pair Q in X_1 which cannot be folded in any conservative preimage of X_1. Let

$$A^1_{-1} = \left\{(x,r) \in X_1 : x \in W \text{ and } r > \frac{1}{2}\right\}$$

and

$$A^1_1 = \left\{(x,r) \in X_1 : x \in W \text{ and } r < -\frac{1}{2}\right\}.$$

Let $B^1_i = \tau^{-1}(B_i) = \{(x,r) \in X_1 : x \in B_i\}$. It is readily checked that $Q = ((A^1_i, B^1_i))$ is a double pair in X_1.

Form Y_1 from X_1 and Q as usual. For purposes of notational clarity, let us denote the natural projection from Y_1 onto X_1 by σ. We will show that there is no closed Z_1 in Y_1 so that σ maps Z_1 conservatively onto X_1 and folds Q. This follows from a number of observations.

Since X_1 may be thought of as arising from X by replacing certain point components (points in $E_1 \cup E_{-1}$) by intervals, it is clear that $bX_1 = X$ and that the map $b : X_1 \to bX_1$ coincides with the quotient map $\tau : X_1 \to X$ given by $\tau(x,r) = x$ for $(x,r) \in X_1$. If

$x \in E_{-1}$ then the interval $\{x\} \times I_1$ is a component in X_1 and its preimage in Y_1 under σ consists of a type $(1, -1)$ component and a type 2 component, as shown in Figure 11. (A similar statement/picture holds for components mapping into $\{x\} \times I_{-1}$ for $x \in E_1$.)

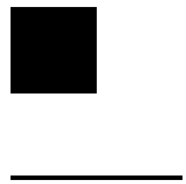

Figure 11

Note that no type 2 component in Y_1 maps under σ **onto** a component of X_1 so it cannot be selected as part of any conservative preimage of X_1. (Precisely, the only type 2 components in Y_1 are those of the form $\{x\} \times [0, 1/2] \times I_1$ for $x \in E_{-1}$ or $\{x\} \times [-1/2, 0] \times I_{-1}$ for $x \in E_1$.) So by Lemma 3.6, we may assume that Z_1 consists of type $(1, -1)$, type $(1, 1)$ and type 3 components of Y_1.

In particular there exist disjoint open sets U_1 and U_{-1} in $b_{X_1'} X_1'$ satisfying conditions (3a) and (3b) in Corollary 3.7. The argument is completed by interpreting these conditions in the current setting.

The components of X_1' are as follows:

(1) $\{x\} \times I$ for $x \notin W$.
(2) $\{x\} \times [0, 1/2]$ for $x \in E_{-1}$ and $\{x\} \times [-1/2, 0]$ for $x \in E_1$.
(3) $(x, 0)$ for $x \in W \diagdown (E_{-1} \cup E_1)$.

So $bX_1' = X$ and the Boolean reflection $b : X_1' \to bX_1' = X$ is just $\tau|_{X_1'}$.

Condition (3a) is $U_{-1} \cup U_1 = W$. For condition (3b), observe that the components of X_1' which are contained in B^1_{-1} and meet $\overline{A^1_{-1}}$ are precisely those of the form $\{x\} \times [0, 1/2]$ for $x \in E_{-1}$. So $E_{-1} \subset U_{-1}$. Similarly, $E_1 \subset U_1$.

The assumption that a conservative preimage of X_1 folding Q gives the following: Two disjoint open in W sets U_{-1} and U_1 with $U_i \supseteq E_i$. This is again a contradiction. $\square$

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