THE C* AXIOMS AND THE PHASE SPACE FORMULATION OF QUANTUM MECHANICS

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Dedicated to G. G. Emch.

Abstract. It is proven that the phase space localization operators on the Hilbert spaces of ordinary quantum mechanics provide a set of operators that are physically motivated and form a C* algebra. Then, it is proven that the set of localization operators, when extended, are informationally complete in the original Hilbert space.

1. Introduction

In 1972, "Algebraic Methods in Statistical Mechanics and Quantum Field Theory" was published by Gerard Emch [8], giving the axioms for a physical system in an algebraic setting using the language of Irving Segal [19], and then going on to obtain the C*-algebraic formalism for a physical system. Of these axioms, only the fifth contained an assumption that was questionable in its physical content. Bearing in mind that for each observable A and state ϕ, one obtains a distribution of values for the observed results of measurement, we have:

Axiom 5: For any element A in the set of observables $\mathfrak{A}$ and any non-negative integer n, there is at least one element, denoted A^n, in $\mathfrak{A}$ such that (i) the set of dispersion-free states for A^n is contained in the set of dispersion-free states for A, (ii) $< \phi; A^n > = < \phi; A >^n$ for all ϕ in the set of dispersion-free states for A. (Here $< \phi; B >$ is the expectation of observable B in state ϕ.)

This axiom is nessessary in order to define the product of two or more observables as a member of $\mathfrak{A}$. Our objective here is to justify that claim (as well as all the other axioms of the C* approach) for a set of observables **that are physically motivated** as well as **informationally complete** in the Hilbert spaces of ordinary quantum mechanics. In this way, we will obtain a completely physically motivated basis for the C* formalism for quantum mechanics. Then, the C* formalism may be used for more general physical situations, with an additional restriction that we will place on the algebra of observables in the phase space formalism.

It is first incumbent upon us to justify why this is a problem at all. We take the position operator, Q, or the momentum operator, P, as examples. They are unbounded self-adjoint operators with a purely continuous spectrum on any of the non-relativistic Hilbert spaces in quantum mechanics. We first treat them as being generated by their spectral projections onto compact sets of their spectrum, each being a bounded self-adjoint operator. But you have a problem; each of these

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has a piecewise continuous spectrum, and thus has no purely discrete spectrum (eigenvalues) associated with it, much less a complete basis of eigenvectors. The same problem appears for any operator with a purely continuous spectrum in any Hilbert space. We, next, may use a theorem [22] that says that if T is a self-adjoint, bounded operator on a separable Hilbert space, then there exists a self-adjoint compact operator K such that $T+K$ has eigenvectors that span the space. Moreover, by a theorem of von Neumann [21], $\| K \|$ may be made arbitrarily small and T doesn't have to be bounded. But there is no physical interpretation for what K is!

A way to circumvent this came from considering the phase space formalism for quantum mechanics. [16] In this formalism, the Hilbert spaces in which quantum mechanics is done are the usual ones and on each one we may define a phase space localization operator; we shall investigate the properties of such a phase space localization operator. It will have all the properties we want.

In Section 2, we define the phase space representations for any locally compact Lie group and obtain the phase space localization operators on them. In the next section, we define the usual quantum mechanical representation spaces for the Galilei and Poincaré groups, obtain an embedding of these spaces into the appropriate phase space representation spaces, and then pull back the phase space localization operators. We also discuss the previously known informational completeness of such phase space localization operators. In Section 4, we investigate in detail the spectral properties of these pulled back phase space localization operators. We find that for any function that is L^p in the phase space, these operators have a purely discrete spectrum. In Section 5, we show that they comprise a set of operators that are physically observable and form a C* algebra. Finally, in Section 6, we show that we may construct a Hilbert space in which the localization operators form an informationally complete set.

2. Phase space representations of a locally compact group

The majority of this section has been discussed in [6].

From *classical* experiments, one learns that classical (Newtonian) equations of motion are invariant under translations, boosts (relative velocity transformations between inertial [Galileian] reference frames), and rotations. Prior to 1887, these were viewed to generate the group of Galilean transformations on spacetime. Since the Michelson-Morley[14] experiment, and the subsequent analysis of numerous luminaries, these spacetime translations, boosts and rotations were interpreted as the generators of the group of Lorentz (or Poincaré) transformations on either energy-momentum space or on spacetime. These transformations generate the entire group from those transformations acting on an arbitrarily small neighborhood of any point. Transformations *infinitesimally* near the identity transformation form a vector space (the Lie algebra of the group) on which a non-associative operation (the Lie bracket) is defined. Thus, classical experiments reveal the kinematical groups of relevance.

The lesson learned through the efforts of mathematicians over the last 250 years is that we may use a space with a Poisson bracket, a *phase space,* to describe a classical conservative mechanics. A phase space is mathematically a (symplectic) manifold which possesses a closed, non-degenerate 2-form on it. Furthermore, the relevant Galilei or Poincaré group acts on this space in such a way as to preserve the Poisson bracket (acts "symplectically"). The phase space is thus a "G space", G the kinematical group. As a consequence of this set-up, "conjugate variables" are coordinates on the phase space which realize the canonical skew-symmetric form of the Poisson bracket, etc. With the experience of the Galilei and Poincaré groups, one may abstract this formulation to the setting of the action of a Lie group G on *any* phase space on which G acts symplectically.

A Lie group G generates, as above, a Lie algebra $\mathfrak{g}$; we may think of $\mathfrak{g}$ as the collection of all left-invariant vector fields on G. This process is invertable by e*xponentiation* that associates an element of the group (near the identity) to any element of the Lie algebra sufficiently near the origin (zero). One may thus go from the Lie group to the Lie algebra, and *vice versa*.

In the following, it is essential that $\mathfrak{g}$ is a finite-dimensional vector space. If $\wedge$ designates the anti-symmetric tensor product on $\mathfrak{g}$ then one may form the skew-symmetric tensor algebra $*(\mathfrak{g})$ over $\mathfrak{g}$ consisting of $\mathbb{R}$, $\mathfrak{g}$, $\mathfrak{g}\wedge\mathfrak{g}$, $\mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g}$, etc. Let their duals be denoted by $\mathfrak{g}^*$, etc. and note that $\mathfrak{g}^*$ may be thought of as the collection of all left-invariant 1-forms on G, $(\mathfrak{g}\wedge\mathfrak{g})^*$ as the left-invariant 2-forms on G, and so on. We then define the coboundary operator δ

$$\mathbb{R} \longrightarrow^{\delta_0} \mathfrak{g}^* \longrightarrow^{\delta_1} (\mathfrak{g}\wedge\mathfrak{g})^* \longrightarrow \cdots$$

as follows: Let $\{A_i\}$ be a basis of $\mathfrak{g}$ and let $\{\omega^i\}$ be the associated dual basis of $\mathfrak{g}^*$ so that $\omega^i(A_j) = \delta^i_j$. The structure constants $C^k_{ij} \in \mathbb{R}$ of $\mathfrak{g}$, defined relative to the basis $\{A_i\}$, are determined by the Lie bracket relations: $[A_i, A_j] = \sum_k C^k_{ij} A_k$. The $\mathbb{R}$ in the sequence above can be considered to be the collection of left-invariant functions on the group G, which is assumed to be connected, so that the $\mathbb{R}$ may be thought of as the left-invariant 0-forms f on the group. We define $\delta_0 f = 0$ as an element of $\mathfrak{g}^*$. Next, thinking of the ω^i as left-invariant 1-forms, we find that the Maurer-Cartan equations hold: $\mathrm{d}\omega^k = -\frac{1}{2}\sum_{i,j} C^k_{ij}\omega^i\wedge\omega^j$. We then define

$$\delta_1\omega^k = -\frac{1}{2}\sum_{i,j} C^k_{ij}\omega^i\wedge\omega^j$$

recognizing that this 2-form is actually in $(\mathfrak{g}\wedge\mathfrak{g})^*$. One extends this expression for δ_1 linearly, obtaining $\mathfrak{g}^* \longrightarrow^{\delta_1} (\mathfrak{g}\wedge\mathfrak{g})^*$. Making use of the skew-derivation property for δ_2

$$\delta_2(\lambda\wedge\mu) \equiv (\delta_1\lambda)\wedge\mu - \lambda\wedge(\delta_1\mu),$$

for $\lambda, \mu \in \mathfrak{g}^*$, one defines δ_2 and proceeds inductively to define δ.

Next, let

$$Z^2(\mathfrak{g}) \equiv \left\{\omega \in (\mathfrak{g}\wedge\mathfrak{g})^* \mid \delta_2(\omega) = 0\right\}$$

denote the space of closed, left-invariant 2-forms on G, and, for $\omega \in Z^2(\mathfrak{g})$, define

$$h_\omega \equiv \{\xi \in \mathfrak{g} \mid \omega(\xi,\cdot) = 0\}.$$

It turns out that h_ω is a Lie sub-algebra of $\mathfrak{g}$; so by exponentiation, h_ω determines a subgroup H_ω of G. We must assume that H_ω is a *closed* subgroup of G to obtain a manifold:

$$\Gamma \equiv G/H_\omega.$$

It is a symplectic manifold as the 2-form ω, when factored by its kernel, is the pull-back of a non-degenerate closed 2-form on G/H_ω. That Γ is a symplectic G space follows because G acts on G/H_ω by left multiplication on left cosets: $gx = g(g_1 H_\omega) = (gg_1)H_\omega$, where $x = g_1 H_\omega$ for some g_1 in G. Since $\Gamma \equiv G/H_\omega$ is a symplectic manifold, it naturally possesses a left-invariant Liouville measure μ equal to the m-th exterior power of ω, where the dimension of Γ is equal to 2m for some integer m.

The following result (Theorem 25.1 of [10]) captures the essence of the need for the construction outlined above and is sufficient for our purposes, but only in the context of *single-particle* kinematics.

Theorem 1. *Any symplectic action of a connected Lie group G on a symplectic manifold M defines a G morphism, $\Psi : M \to Z^2(\mathfrak{g})$. Since the map Ψ is a G morphism, $\Psi(M)$ is a union of G orbits in $Z^2(\mathfrak{g})$. In particular, if the action of G on M is transitive, then the image of Ψ consists of a single G orbit in $Z^2(\mathfrak{g})$.*

For the Galilei group and the Poincaré group, elements of $Z^2(\mathfrak{g})$ are fixed by a choice of mass and spin. Consequently, one obtains all the *single-particle* symplectic spaces on which G acts symplectically and transitively, and one has a unified mathematical picture of kinematics in the two (Galileian and Lorentzian) cases of relevance to one-particle physics. Using the "méthode de fusion" [20], we describe multi-particle kinematics by a phase space that is a Cartesian product of the single-particle phase spaces with symplectic form equal to the "sum" of the symplectic forms on each of the single-particle factors. Thus we may start from the symplectic action of a group on a classical single-particle phase space, and obtain all the phase spaces (single- or multi-particle) on which G acts symplectically, in a physically meaningful way.

One may form $L^2_\mu(\Gamma)$, a Hilbert space on which one may represent G by unitary operators $V^\alpha(g)$

$$[V^\alpha(g)\Psi](x) \equiv \alpha(h(g^{-1}, x))\Psi(g^{-1}x)$$

for $\Psi \in L^2_\mu(\Gamma)$, $x \in \Gamma$, $g \in G$, h a generalized cocycle, and α a one-dimensional representation of H; i.e., incorporate a phase factor in the left-regular representation. Then, define an operator $A(f)$, for all μ-measurable f, by

$$[A(f)\Psi](x) \equiv f(x)\Psi(x).$$

When the f are characteristic functions $\chi(\Delta)$, these operators on $L^2_\mu(\Gamma)$ have a clear classically-motivated interpretation of localization observables in the phase space region Δ. The $A(f)$ form a commuting set, reflecting the classical property that the operators of position, momentum, etc. are all obtainable with precision simultaneously. For this reason and others, it will become evident that $L^2_\mu(\Gamma)$ is not a Hilbert space of fundamental importance to the description of quantum mechanical models of elementary (i.e., irreducible) single-particle systems. It will turn out that it is reducible into a direct sum (or integral) of such irreducible spaces.

3. Quantum Mechanical Representation Spaces

This section is also largely taken from [6].

In the case where G is one of the inhomogeneous Galilei and Lorentz groups, we know that all continuous, irreducible, unitary Hilbert space representations are obtained through the "Mackey Machine" [13] and the earlier Wigner classification [23] and that these representations are characterized by the Casimir invariants in the universal enveloping algebra of the Lie algebra. These Casimir elements are identifiable as the physical quantities of rest mass and spin (or helicity in the mass-zero case) in the case $G =$ the inhomogeneous Lorentz group. For the inhomogeneous Galilei group, the analysis of Lévy-Leblond [12] achieved a similar picture physically characterized by mass and spin.

In what follows, U will denote an irreducible unitary representation of G on an irreducible representation space, denoted $\mathcal{H}$, with inner product denoted $< \cdot, \cdot >$.

We wish to encode the entire content of the state vector $\varphi \in \mathcal{H}$ into a complex-valued function on the phase space Γ in a manner that is reversible. The goal is to be able to reconstruct the state from the complex numbers $[W^{\eta}(\varphi)](x)$ which encode it. Hence, to intertwine $\mathcal{H}$ with $L^2_{\mu}(\Gamma)$, we perform the following: we define a linear transformation W^{η} from $\mathcal{H}$ to $L^2_{\mu}(\Gamma)$ by

$$[W^{\eta}(\varphi)](x) \equiv \;\; < U(\sigma(x))\eta, \varphi >$$

for $x \in \Gamma = G/H_{\omega}$, for all $g \in G$, and for all $\varphi \in \mathcal{H}$, where η is a vector in $\mathcal{H}$ and where σ is a (Borel measurable) section

$$\sigma : G/H_{\omega} \longrightarrow G.$$

To ensure that the image of W^{η} actually lies in $L^2_{\mu}(\Gamma)$ we rely on our choice of η. One first selects and fixes, once and for all, a (Borel measurable) section $\sigma : G/H_{\omega} \longrightarrow G$. Now, one says that η is *admissible* with respect to the section σ if

$$\int_{\Gamma} |< U(\sigma(x))\eta, \eta >|^2 \, d\mu(x) < \infty.$$

Assuming that η is admissible with respect to σ, one says that η is *α-admissible with respect to σ* if in addition to admissibility of η one also has

$$U(h)\eta = \alpha(h)\eta$$

for all h in H_{ω}, where α is a one-dimensional representation of H_{ω}. If η is *α-admissible with respect to σ* then we may define the mapping W^{η} from $\mathcal{H}$ to $L^2_{\mu}(\Gamma)$.[16] As we shall see, this is also enough to describe states $\varphi \in \mathcal{H}$ by their images $W^{\eta}(\varphi)$ in $L^2_{\mu}(\Gamma)$.

To illustrate that these conditions are achievable for all representations indexed by mass and spin of the Galilei and Poincaré groups, consider, for example:

(1) the case of a massive, spinless, relativistic particle ($G =$ Poincaré group) in which one finds [1] that η must be rotationally-invariant under $H_{\omega} = SU(2)$, and square-integrable over $\Gamma \equiv G/H_{\omega} \cong \mathbb{R}^6 \cong \mathbb{R}^3_{position} \times \mathbb{R}^3_{momentum}$, the classical phase space of a massive, relativistic spinless particle.

(2) the case of a massive, relativistic particle with non-zero spin ($G =$ Poincaré group) in which one finds [3, 5] that η must be rotationally invariant about the "spin axis" (but not necessarily invariant under all rotations in $SU(2)$), i.e., invariant under $H_{\omega} =$ double covering of $O(2) \cong$ stabilizer in $SU(2)$ of the spin axis, and square-integrable over $\Gamma \equiv G/H_{\omega} \cong \mathbb{R}^3_{position} \times \mathbb{R}^3_{momentum} \times S^2_{spin}$, the classical phase space of a massive, relativistic, spinning particle.

Orthogonality relations are present insuring that the images of the W^{η} are orthogonal in $L^2_{\mu}(\Gamma)$. We have the

Theorem 2. [11] *Let G be a locally compact group, H a closed subgroup, $\sigma_k : G/H \to G$ any Borel sections, and U_k any unitarily inequivalent representations of G square integrable with respect to $\sigma_k(G/H)$ on Hilbert space $\mathcal{H}^k$. ($k \in \{1,2\}$.) Also let $\mathcal{H}^{k\alpha_k}$ denote the non-trivial closed subspaces of $\mathcal{H}^k$ generated by the set of α_k-admissible vectors in $\mathcal{H}^k$. Assume that the vectors $\eta_k, \xi_k \in \mathcal{H}^{k\alpha_k}$, and $\varphi_k, \psi_k \in \mathcal{H}^k$. Then there exists a unique, positive, invertible operator C on $\mathcal{H}^{k\alpha_k}$ such that*

$$\int_{G/H} < \varphi_1, U(\sigma_1(x))\eta_1 >_{\mathcal{H}^1} < U(\sigma_2(x))\eta_2, \varphi_2 >_{\mathcal{H}^2} d\mu(x) = 0;$$

$$\begin{aligned}\int_{G/H} < \quad & \varphi_1, U(\sigma_1(x))\eta_1 >_{\mathcal{H}^1} < U(\sigma_1(x))\xi_1, \psi_1 >_{\mathcal{H}^1} d\mu(x) \\ = \quad & < C\xi_1, C\eta_1 >_{\mathcal{H}^1} < \varphi_1, \psi_1 >_{\mathcal{H}^1} .\end{aligned}$$

Therefore, we have a prescription for when the representations are orthogonal in $L^2_\mu(\Gamma)$: when $< C\xi_1, C\eta_1 >_{\mathcal{H}^1} = 0$.

Note that in the case of a compact group G, the positive operator C is just a positive *constant*. In general, this does not hold on all locally compact groups, for example for the Poincaré group in the representations with non-zero spin. See [16, pp. 328 - 329] for a condition which guarantees that C is a constant.

We work, now, with a single choice of G, H, σ, α, and η. For the sake of simplicity we also denote the closure of the image of W^η by $W^\eta(\mathcal{H}) \subset L^2_\mu(\Gamma)$. Let P^η denote the canonical projection

$$P^\eta : L^2_\mu(\Gamma) \longrightarrow W^\eta(\mathcal{H})$$

and denote by $A^\eta(f)$ the pulled back mapping [16]

$$A^\eta(f) \equiv [W^\eta]^{-1} P^\eta A(f) W^\eta : \mathcal{H} \longrightarrow \mathcal{H}.$$

This is a plausible candidate for the quantum mechanical operator that corresponds to the classical observable f. For example, for the Heisenberg group and for $\eta =$ the ground state wave function of the harmonic oscillator, then $A^\eta(q) = Q$ = the position operator, and $A^\eta(p) = P =$ the momentum operator. Note that we have gone from a commuting set of $A(f)$s to a non-commuting set, the $A^\eta(f)$s.

One can prove [16] that $A^\eta(f)$ has an operator density $T^\eta(\cdot)$:

$$\begin{aligned} A^\eta(f) &= \int_\Gamma f(x) T^\eta(x) d\mu(x), \\ T^\eta(x) &\equiv |\, U(\sigma(x))\eta >< U(\sigma(x))\eta \,|, \end{aligned}$$

and that, up to a finite renormalization of μ if necessary,

$$A^\eta(1) = 1.$$

With this set-up one can make a number of remarks:

1) We may restate the orthogonality relation in this case by replacing $|\, U(\sigma(x))\eta >< U(\sigma(x))\eta \,|$ with $T^\eta(x)$, yielding just a multiple of $< \varphi_1, \psi_1 >_{\mathcal{H}^1}$ on the right-hand side of the second relation of Theorem 2.

2) Let ρ denote any quantum density operator; i.e., ρ is non-negative and has trace one. Then one may write $\rho = \sum \rho_i P_{\psi_i}$, the ψ_i forming an orthonormal set and P_{ψ_i} denoting the corresponding projection. Now, using the interpretation of

$|< U(\sigma(x))\eta, \psi_i >|^2$ as the transition probability from ψ_i to $U(\sigma(x))\eta$, one has the quantum expectation value given by

$$Tr(\rho A^{\eta}(f)) = \sum_i \rho_i \int_{\Gamma} f(x) \, |< U(\sigma(x))\eta, \psi_i >|^2 \, d\mu(x);$$

i.e., the sum over the transition probabilities. [16]

For example, when using a "screen" to detect a particle in a vector state given by ψ, one idealizes the detector (the screen) as a multi-particle quantum system consisting of identical sub-detectors. In a fixed laboratory frame of reference a sub-detector is represented by a state vector η whose phase space counterpart $W^{\eta}\eta$ is peaked about a reference phase space point which may be referred to as "the origin". For a fixed space-time reference frame, one may "position" a detector at all "points" of space-time (space-time events) exactly as Einstein located rods and clocks. Of course, one must now position mass spectrometers (devices that measure rest-mass in their own rest frames) and Stern-Gerlach devices at all space-time events in addition to rods and clocks. As Einstein imagined that the rods and clocks were also equipped (at all space-time coordinate events) in all inertially-related space-time reference frames, so must we imagine that our inertially-related space-time reference frames carry identical mass spectrometers and Stern-Gerlach devices in addition to rods and clocks (boosted relative to the rest "laboratory" frame). So, instead of rods and clocks situated at each space-time event and at rest in inertially-related (uniformly moving) rest frames, we must add to that *imagery* a more elaborate set of apparati. For a fixed value of momentum p there are infinitely many pairs (m, u) such that $p = mu$; of course the momentum does not alone characterize the uniform relative velocity (boost) represented by p - one requires also the rest-mass m. The totality of all such "placements" of detectors constitutes the phase-space distribution of detectors - the classical phase space frame analogous to the classical space-time (Lorentz) frame (of rods and clocks). Thus the complete detector is composed of sub-detectors each located at different "positions" (points of Γ). The sub-detector located at "position" $x \in \Gamma$, obtained from η by a kinematical placement procedure (with the same intent as Einstein's placement of identical rods and clocks at all points of spacetime), is $U(\sigma(x))\eta$. Since the probability that ψ is captured in the state given by $U(\sigma(x))\eta$ is $|< U(\sigma(x))\eta, \psi >|^2$, the formula for the expectation is justified. *One cannot improve upon this procedure when measuring, by quantum mechanical means, the distribution of the particle.*

3) Since $T^{\eta}(x) \geq 0$ and $A^{\eta}(1) = 1$, then $\rho_{class}(x) \equiv Tr(\rho T^{\eta}(x))$ is a classical (Kolmogorov) probability function [16]. Consequently,

$$\begin{aligned} \text{quantum expectation} &= Tr(\rho A^{\eta}(f)) \\ &= \int_{\Gamma} f(x) Tr(\rho T^{\eta}(x)) d\mu(x) \\ &= \int_{\Gamma} f(x) \rho_{class}(x) d\mu(x) \\ &= \text{classical expectation.} \end{aligned}$$

4) Since the operators $A^{\eta}(f)$ enjoy the feature of the same expectation as the "classical" observables f, one might ask whether these operators are sufficient to distinguish states of the quantum system.

Definition 1. [15]*: A set of bounded self-adjoint operators* $\{A_\beta \mid \beta \in I,\ I$ *some index set*$\}$ *is* ***informationally complete*** *iff for all states* ρ, ρ' *such that* $Tr(\rho A_\beta) = Tr(\rho' A_\beta)$ *for all* $\beta \in I$ *then* $\rho = \rho'$.

Example [15]: In spinless quantum mechanics, the set of all spectral projections for position is not informationally complete. Neither is the set of all spectral projections for momentum, nor even the union of them.

The $\{A^\eta(f) \mid f \text{ is measurable}\}$ (or, equivalently $\{T^\eta(x) \mid x \in \Gamma\}$) is known to be informationally complete in a number of cases and under a single condition on η, that $< U(g)\eta, \eta > \neq 0$ for $a.e. g \in G$:

a) spin-zero massive representations of the Poincaré group [1]
b) mass-zero, arbitrary helicity representations [4] of the Poincaré group
c) the affine group [11]
d) the Heisenberg group [11]
e) massive representations [2, 16] of the inhomogeneous Galilei group.

This leaves the case of massive, non-zero spin representations [5] of the Poincaré group. We will have more to say on this in Section 6.

5) If $\{A_\beta \mid \beta \in I\}$ is informationally complete then any bounded operator on $\mathcal{H}$ may be written as (a closure of) integrals over the set I. [7]

6) When we specialize $A^\eta(f)$ to $f = \chi(\Delta)$, $\chi(\Delta)$ the characteristic function for the Borel set $\Delta \subset \Gamma$, then

$$\begin{aligned} \chi(\Delta) &= \text{classical localization in } \Delta \subset \Gamma, \\ A(\chi(\Delta)) &= \text{operator on } L^2(\Gamma) \text{ localizing in } \Delta \subset \Gamma, \\ A^\eta(\chi(\Delta)) &= \text{operator on } \mathcal{H} \text{ localizing in } \Delta \subset \Gamma. \end{aligned}$$

4. Spectral Properties of the $A^\eta(f)$

These $A^\eta(f)$ have several properties [16] of relevance to us here. We first provide the

Definition 2. *Let* $\mathcal{H}$ *be a Hilbert space and let* A *be a compact operator on* $\mathcal{H}$*. Let* $\{\alpha_k\}$ *denote the set of singular values (eigenvalues) of* A*. The* n***th trace class****,* $\mathcal{B}_n$*, is defined to be the set of all compact operators such that* $\sum_k \mid \alpha_k \mid^n < \infty$*. We denote the corresponding norm by* $\parallel A \parallel_{B_n} \equiv [\sum_k \mid \alpha_k \mid^n]^{1/n}$.

Then we have the

Theorem 3. *Let* (X, Σ, μ) *be a measure space, let* $\mathcal{H}$ *be a Hilbert space, and let* $A : \Sigma \to B(\mathcal{H})$ *be a positive operator valued measure. Suppose* A *has an operator density* T *such that* $\parallel T_x \parallel \leq c$ *for all* $x \in X$*, and* $Tr(T_x) \leq k$ *for all* $x \in X$*,* c *and* k *constants. Let* $f \in L^p_\mu(X)$*. Then* $A(f) \equiv \int_X f(x) T_x d\mu(x)$ *is a bounded operator that is compact with* $\parallel A(f) \parallel \leq c^{1/p} \parallel f \parallel_p$ *and* $\parallel A(f) \parallel_{B_p} \leq r(p) \parallel f \parallel_p$ *for some constant* $r(p)$*. In the case* $p = 1$*,* $r(p) = k$.

The proof is an excursion in interpolation theory. [17]

In the case at hand, we have $c = k = 1$ and $X = \Gamma$. Thus, for $A^\eta(f) \neq 0$:

a) $A^\eta(f)$ is compact for all $f \in L^p_\mu(\Gamma)$, and thus for all $f \in L^1_\mu(\Gamma)$. In particular, we have that it is compact for all f equal to a characteristic function on a compact, measurable set in Γ.

b) Suppose $A^\eta(f)\varphi = \lambda\varphi$, for f in $L^1_\mu(\Gamma)$, $\lambda \sim 1$, $\| \varphi \| = 1$. Suppose also that $g \in L^1_\mu(\Gamma)$, and $\| f - g \|_1 << 1$. Then $\| \lambda\varphi - A^\eta(g)\varphi \| = \| A^\eta(f-g)\varphi \| \leq \| f - g \|_1 << 1$. Thus, φ is nearly an eigenfunction of $A^\eta(g)$ with the same eigenvalue as $A^\eta(f)$. The eigenvalues are close. In particular, this holds if we take g to be a characteristic function and f to be a fuzzy set function.

Now we have the

Definition 3. *Let A and B be two effects; i.e., self-adjoint, positive operators in $\mathcal{H}$ that have spectrum in [0,1]. Then A and B are **comeasurable** iff we can write $A = A_1 + C$, $B = B_1 + C$, for A_1, B_1 and C effects, and $A_1 + B_1 + C$ is an effect.*

In particular, for $A = A^\eta(f)$, and $B = A^\eta(g)$, then $C = A^\eta(\min\{f, g\})$. Couple that with the fact that in $\mathcal{H}$, two projections are comeasurable iff they commute and you obtain the

Theorem 4. [18] *The set $\{A^\eta(f) \mid 0 \leq f \leq 1,\ f\ \mu\text{-measurable}\}$ does not contain any two non-trivial projections.*

But the $T^\eta(x)$ are covariant as an easy proof will show. Thus the $A^\eta(f)$ are covariant:

$$\begin{aligned} U(g)A^\eta(f)U^{-1}(g) &= U(g)\int_\Gamma f(x)T^\eta(x)d\mu(x)U^{-1}(g) = A^\eta(g^{-1}.f), \\ [g.f](x) &= f(gx) \text{ for all } g \in G. \end{aligned}$$

Hence, if you have one non-trivial $A^\eta(f)$ that is a projection, you have many. Consequently,

c) The set $\{A^\eta(f) \mid 0 \leq f \leq 1,\ f\ \mu\text{-measurable}\}$ does not contain any non-trivial projection. [18] Hence, any non-trivial operator in the set has spectrum in (0,1). This is in spite of the informational completeness of the set $\{A^\eta(f) \mid f$ is μ-measurable$\}$.

Now, one may prove such things as

d) If Δ is a compact subset of Γ with a piecewise differentiable boundary, we have shown that $\sum(\lambda_i - \lambda_i^2)$, $\{\lambda_i\}$ the eigenvalues of $A^\eta(\chi(\Delta))$, is small. Then $A^\eta(\chi(\Delta))$ has a decreasing spectrum which starts out just below 1, remains just below 1 until it suddenly drops to values just above zero [16, pp. 281-283]. Notice that $\chi(\Delta) \in L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma)$, the additional restriction to which we referred in the introduction.

e) For all $\Delta \subset \Gamma$, $\| A^\eta(\chi(\Delta)) \| \leq \mu(\Delta)$.

5. The other C* axioms

For completeness, we discuss cursorily the other axioms of the C* approach to physical systems. See [8] for a complete discussion. They are

Axiom 1) For each physical system Σ we can associate the triple $(\mathfrak{A}, \mathfrak{S}, <;>)$ formed by the set $\mathfrak{A}$ of all its observables, the set $\mathfrak{S}$ of all its states, and a mapping $<;>: (\mathfrak{S}, \mathfrak{A}) \to \mathbb{R}$ which associates with each pair (ϕ, A) in $(\mathfrak{S}, \mathfrak{A})$ a real number $< \phi; A >$ that we interpret as the expectation value of the observable A when the system is in the state ϕ.

Definition 4. *For fixed $A \in \mathfrak{A}$, we have $< \cdot; A >: \mathfrak{S} \to \mathbb{R}$. If $\mathfrak{T} \subseteq \mathfrak{S}$, denote by $A|_\mathfrak{T}$ the restriction $< \cdot; A >: \mathfrak{T} \to \mathbb{R}$. Declare $A\ |_\mathfrak{T} \leq B\ |_\mathfrak{T}$ whenever $< \phi; A > \leq$*

$< \phi; B > \forall \phi \in \mathfrak{T}$. *If* $\mathfrak{T}$=$\mathfrak{S}$, *then we write simply* $A \leq B$. *A subset* $\mathfrak{T}$ *is said to be* ***full*** *with respect to a subset* $\mathfrak{B} \subseteq \mathfrak{A}$ *iff* A *and* B *in* $\mathfrak{B}$, *and* $A\,|_{\mathfrak{T}} \leq B\,|_{\mathfrak{T}} \Rightarrow A \leq B$.

Example: Let $\mathcal{H}$ be a separable Hilbert space. Let $\{\psi_i\}$ be an orthonormal basis for $\mathcal{H}$. For $\psi \in \mathcal{H}$ and $\parallel \psi \parallel = 1$, let $P_\psi : \mathcal{H} \to \mathcal{H}$, $P_\psi \varphi = < \psi, \varphi > \psi$. Let $\mathfrak{B}_\Psi = \left\{ \sum_i \alpha_i P_{\psi_i}, \alpha_i \in \mathbb{C}, \sum_i |\alpha_i| < \infty \right\}$. Then $\mathfrak{T}_\Psi = \{P_{\psi_i}\}$ is full with respect to $\mathfrak{B}_\Psi$.

Axiom 2) The relation $\leq$ is a partial ordering relation on $\mathfrak{A}$.

Axiom 3) (i) There exist in $\mathfrak{A}$ two elements 0 and 1 such that, for all $\phi \in \mathfrak{S}$ we have $< \phi; 0 > = 0$ and $< \phi; 1 > = 1$.

(ii) For each observable $A \in \mathfrak{A}$ and any $\lambda \in \mathbb{R}$ there exists $(\lambda A) \in \mathfrak{A}$ such that $< \phi; \lambda A > = \lambda < \phi; A >$ for all $\phi \in \mathfrak{S}$.

(iii) For any pair of observables A and B in $\mathfrak{A}$ there exists an element $(A + B)$ in $\mathfrak{A}$ such that $< \phi; A + B > = < \phi; A > + < \phi; B >$ for all $\phi \in \mathfrak{S}$.

Definition 5. *Denote the set of all dispersion-free states for the observable* A *by* $\mathfrak{S}_A$.

Definition 6. *A subset* $\mathfrak{T} \subseteq \mathfrak{S}$ *is said to be* ***complete*** *if it is full with respect to the subset* $\mathfrak{A}_\mathfrak{T} \subseteq \mathfrak{A}$ *defined by* $\mathfrak{A}_\mathfrak{T} \equiv \{A \in \mathfrak{A} \mid \mathfrak{S}_A \supseteq \mathfrak{T}\}$. *A complete subset* $\mathfrak{T} \subseteq \mathfrak{S}$ *is said to be* ***deterministic*** *for a subset* $\mathfrak{B} \subseteq \mathfrak{A}$ *whenever* $\mathfrak{B} \subseteq \mathfrak{A}_\mathfrak{T}$. *A subset* $\mathfrak{B} \subseteq \mathfrak{A}$ *is said to be* ***compatible*** *if the set* $\mathfrak{S}_\mathfrak{B} \equiv \cap_{B \in \mathfrak{B}} \mathfrak{S}_B$ *is complete.*

Example: $\mathfrak{T}_\Psi$ is complete because it is full with respect to $\mathfrak{A}_\mathfrak{T} \equiv \mathfrak{B}_\Psi$. It is moreover deterministic for any subset $\mathfrak{C} \subseteq \mathfrak{A}_\mathfrak{T} \equiv \mathfrak{B}_\Psi$.

Example: Compatibility of $\mathfrak{B}$ in $\mathcal{H}$ is known to be given by $AB = BA \forall A, B \in \mathfrak{B}$.

Axiom 4) The set $\mathfrak{S}_A$ is deterministic for the one-dimensional subspace of $\mathfrak{A}$ generated by A; for any two observables A and B we have $\mathfrak{S}_{A+B} \supseteq \mathfrak{S}_A \cap \mathfrak{S}_B$, and $\mathfrak{S}_1 = \mathfrak{S}$.

Example: If in $\mathcal{H}$ we take a bounded observable A that has no eigenvalues, then $\mathfrak{S}_A$ is empty and axiom 4 is inachievable.

Axiom 5) The axiom discussed in Section 1.

Definition 7. *Let* A *and* $B \in \mathfrak{A}$. $A \circ B \equiv \frac{1}{2}([A + B]^2 - A^2 - B^2)$.

Axiom 6) For any three observables A, B, and C in which A and C are compatible, $(A \circ B) \circ C - A \circ (B \circ C)$ vanishes.

Axiom 7) The norm of $A \in \mathfrak{A}, \parallel A \parallel \equiv \sup_{\phi \in \mathfrak{S}} |< \phi; A >|$, is finite and $\mathfrak{A}$ is topologically complete when regarded as a metric space with the distance between any two elements A and B of $\mathfrak{A}$ defined by $\parallel A - B \parallel$. $\mathfrak{S}$ is then identified with the set of all continuous positive linear functionals ϕ on $\mathfrak{A}$ satisfying $< \phi; 1 > = 1$.

Axiom 8) A sufficient condition for a set $\mathfrak{B}$ of observables to be compatible is that $\overline{\mathfrak{P}(\mathfrak{B})}$ is associative. Here $\mathfrak{P}(\mathfrak{B})$ is the set of polynomials in $\mathfrak{B}$.

Axiom 9) $\mathfrak{A}$ can be identified with the set of all self-adjoint elements of a real or complex, associative, and involutive algebra $\mathfrak{R}$ satisfying

(i) For each $R \in \mathfrak{R}$ there exists an element A in $\mathfrak{A}$ such that $R^*R = A^2$;

(ii) $R^*R = 0$ implies $R = 0$.

We mention Axiom 10 for completeness only. It is not necessary to obtain a C* algebra.

Axiom 10) To each pair of observables A and B in $\mathfrak{A}$ corresponds an observable C in $\mathfrak{A}$ in the sense that for all $\phi \in \mathfrak{S}$, we have

$$< \phi; (A- < \phi; A > 1)^2 >< \phi; (B- < \phi; B > 1)^2 > \quad \geq \quad < \phi; C >^2 .$$

All axioms except axioms 4 and 5 hold in any Hilbert space construction with $< \phi; A > = Tr(\phi A)$. Now consider any Hilbert space for which the phase space formalism applies. Form

$$\mathfrak{A}^+ \equiv \{A^\eta(f) \mid f \in L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma),\ 0 \leq f \ \}.$$

Then, any $A^\eta(f) \in \mathfrak{A}^+$ is of the form $A^\eta(f) = \sum_i \lambda_i P_{\psi_i}$ for some orthonormal basis $\{\psi_i\}$. Consequently, axioms 4 and 5 are satisfied. (One must be aware that, in $A^\eta(f) + A^\eta(h) = A^\eta(f+h)$, we may have all three of $A^\eta(f), A^\eta(h), A^\eta(f+h)$ possessing distinctly different eigenvectors!) Take $\mathfrak{A} \equiv \mathfrak{A}^+ - \mathfrak{A}^+$, and we have a set on which all the axioms hold. Or take $\mathfrak{R} = \mathfrak{A} + i\mathfrak{A}$. Thus we have arrived at the result

Theorem 5. *Any Hilbert space in which the phase space formalism applies, including all the single particle Hilbert spaces of quantum mechanics, satisfies the axioms of the C* algebra formalism for a physical system.*

6. On the Informational Completeness of the Representation

Having a C* algebra in the phase space framework of quantum mechanics, we may now employ the G.N.S. construction [9, 19] which we assume is familiar to the reader:

Choose any state ϕ in the original Hilbert Space $\mathcal{H}$ in which the localization operators, $A^\eta(f)$, were defined and form $< \phi; A >, A \in \mathfrak{A}$. (Note: this includes $\phi = P_\psi$ for any $\psi \in \mathcal{H}$.) Let $\mathfrak{K}_\phi = \{K \in \mathfrak{A} \mid < \phi; R^* K >= 0 \forall R \in \mathfrak{A}\}$, which by the Cauchy-Schwarz-Buniakowski inequality is equal to $\{K \in \mathfrak{A} \mid < \phi; K^* K >= 0\}$. Since $\mathcal{H}$ is irreducible, any vector in $\mathcal{H}$ is cyclic; so, we will take $\phi = P_\psi$, with ψ of the form $U(g^{-1})\eta$, $g \in G$.

Then, abusing the notation, we have for our particular situation,

$$\begin{aligned} K_{U(g^{-1})\eta} &= \{A^\eta(f) \in \mathfrak{A} \mid \| A^\eta(f) U(g^{-1})\eta \| = 0\} \\ &= \{A^\eta(f) \in \mathfrak{A} \mid \| A^\eta(g^{-1}.f)\eta \| = 0\}. \end{aligned}$$

Since the set of fs is invariant under the group G, it suffices to consider just

$$\begin{aligned} K_\eta &= \{A^\eta(f) \in \mathfrak{A} \mid \| A^\eta(f)\eta \| = 0\} \\ &= \{A^\eta(f) \in \mathfrak{A} \mid A^\eta(f)\eta = 0\}. \end{aligned}$$

Now $A^\eta(f)\eta = \int f(x) < U(\sigma(x))\eta, \eta > U(\sigma(x))\eta d\mu(x)$. If $< U(\sigma(x))\eta, \eta >= 0$ a.e. x for $\sigma(x)$ in some compact set $\mathcal{O}$ with non-empty interior, then for all f with support in $\mathcal{O}$, $A^\eta(f)\eta = 0$. Thus $K_\eta \neq \{0\}$ in a way that is invariant under all infinitesimal transformations. If $< U(\sigma(x))\eta, \eta > \neq 0$ *a.e.* $\sigma(x)$ with $x \in \Gamma$, there may be some fs such that $A^\eta(f)\eta = 0$, but $A^\eta(g.f)\eta = 0$ does not hold for all g infinitesimally in all directions. Thus, $A^\eta(f)\eta = 0$ holds only for a thin set of fs.

Remarks: 1) If f is always positive or always negative, then since $spec(A^\eta(f)) \subseteq (0, 1)$, resp. $\subseteq (-1, 0)$, $A^\eta(f)\eta \neq 0$.

2) The α-admissibility of η implies

$$\begin{aligned} &< U(\sigma(x))\eta, \eta > \neq 0 \ a.e.\ \sigma(x) \text{ with } x \in \Gamma \\ \Leftrightarrow & \quad < U(g)\eta, \eta > \neq 0 \ a.e.\ g \in G; \end{aligned}$$

i.e., the same condition for obtaining informational completeness in the known cases of the Galilei and Poincaré groups.

3) By using the modular function we obtain: if η is admissible, then $U(g)\eta$ is admissible for all $g \in G$. Moreover, we have: if η is α-admissible, then $U(h)\eta$ is α-admissible for all $h \in H$. Thus, coupled with the results above, we have that any vector of the form $U(h)\eta$ will be suitable for obtaining the results below.

We will now obtain a representation of $\mathfrak{A}/K_\eta, \eta$ satisfying $< U(g)\eta, \eta > \neq 0$: For R, S in $\mathfrak{A}/K_\eta$, define $(R, S) \equiv < P_\eta; R^*S >$. This turns out to be a sesquilinear form and generates a norm on $\mathfrak{A}/K_\eta$. Hence $\mathfrak{A}/K_\eta$ is a pre-Hilbert space which has the Hilbert space $\mathcal{H}_\eta$ as its completion. Note that we are taking the completion in the topology dual to the strong or weak sense, by the cyclicity of η. This is the same topology as the topology for which informational completeness is discussed.

The representation π_η of $\mathfrak{A}$ is defined by $\pi_\eta(R) : \mathfrak{A}/K_\eta \to \mathfrak{A}/K_\eta$, $\pi_\eta(R)S = RS$. The G.N.S. theorem then goes on to show that $\pi_\eta(R)$ can be extended to a bounded operator on $\mathcal{H}_\eta$. Taking $\mathfrak{A}/K_\eta$ instead of $\mathfrak{A}$ is moot when we operate on $\mathcal{H}_\eta$. The set $\{A^\eta(f)\}$ is informationally complete in this representation. But for all practical purposes, we have that $\mathcal{H}_\eta \subseteq \mathcal{H}$. Define $\mathcal{U}(g) : A^\eta(f) \to A^\eta(g^{-1}.f)$, $g \in G$. $\mathcal{U}$ is an anti-representation of G on $\mathfrak{A}$. But using the covariance property of the $A^\eta(f)$, we see that this representation of the symmetry group is given by $\mathcal{U}(g)A^\eta(f) = U(g)A^\eta(f)U^{-1}(g)$; i.e., by the same U we had before. But that U is irreducible, and hence the Hilbert space obtained through the G.N.S. construction is the same as the original Hilbert space and $\pi_\eta(\mathfrak{A}/K_\eta)^\sim = B(\mathcal{H})$. Consequently,

Theorem 6. *The set $\{A^\eta(f) \mid f \in L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma)$, f real-valued, η α-admissible in $\mathcal{H}$ and $< U(g)\eta, \eta > \neq 0$ for almost every $g \in G\}$ is informationally complete in the G-irreducible representation space $\mathcal{H}$, for any G that is a Lie group.*

7. Conclusion

We have exhibited a set $\left\{A^\eta(f) \mid f \text{ real valued and } L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma)\right\}$ of operators that have a physical meaning in any experiment in which one measures by quantum mechanical means. These $A^\eta(f)$ each have a full set of eigenvectors. They form a C* algebra, and hence form a basis for the C*-algebraic formalism for physics in the free case. We may use the G.N.S. construction to obtain the informational completeness of $\{A^\eta(f) \mid f \text{ real valued and } L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma)\}$. Generalizing to any physical system that has the phase space localization operators defined on it, we obtain a C* algebra and the informational completeness of these $A^\eta(f)$.

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References

[1] Ali, S. T., Brooke, J. A., Busch, P., Gagnon, R., Schroeck, Jr., F. E., Can. J. Physics **66** (1988), 238 - 244.

[2] Ali, S. T. and Progovecki, E., Acta Appl. Math. **6** (1986), 19 - 45; **6** (1986) 47 - 62.

[3] Brooke, J. A., Schroeck, Jr., F. E., Nuclear Physics B (Proc. Suppl.) **6** (1989), 104 - 106.

[4] Brooke, J. A., Schroeck, Jr., F. E., J. Math. Phys.**37** (1996), 5958 - 5986.

[5] Brooke, J. A., Schroeck, Jr., F. E., *Phase Space Representations of Relativistic Massive Spinning Particles*, in preparation.

[6] Brooke, J. A., Schroeck, Jr., F. E., Int. J. Theor. Phys. **44** (2005), 1889 - 1904.

[7] Busch, P., Int. J. Theor. Phys. **30** (1991), 1217 - 1227.

[8] Emch, G. G., *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*, John Wiley and Sons, Inc., 1972.

[9] Gelfand, I., and Naimark, M. A., Mt. Sborn., N. S. **12** **[54]** (1943), 197 - 217.

[10] Guillemin, V., and Sternberg, S., *Symplectic Techniques in Physics,* Cambridge Univ. Press, 1991.

[11] Healy, Jr., D. M. and Schroeck, Jr., F. E., J. Math. Phys. **36** (1995), 433 - 507.

[12] Lévy-Leblond, J.-M., J. Math. Phys. **4** (1963), 776 - 788.

[13] Mackey, G. W., Ann. Math. **55** (1952), 101 - 139; **58** (1953), 193 - 221.

[14] Michaelson, A. A., and Morley, E. W., American J. of Science **XXXI** (May, 1886), 377 - 386; **XXXIV** (1887), 338 - 339.

[15] Prugovecki, E., Int. J. Theor. Physics **16** (1977), 321 - 331.

[16] Schroeck, F.E., Jr., *Quantum Mechanics on Phase Space*, Kluwer Academic Publishers, 1996.

[17] Schroeck, F.E., Jr., Int. J. Theor. Phys. **28** (1989), 247 - 262.

[18] Schroeck, F.E., Jr., Int. J. Theor. Phys. **44** (2005), 2091 - 2100.

[19] Segal, I. E., Ann. Math. **48** (1947), 930 - 948.

[20] Souriau, J.-M., *Structure des systèmes dynamiques*, Dunod, Paris; translation: *Structure of Dynamical Systems*, Prog. Math. **149**, Birkhäuser, Boston, 1997.

[21] von Neumann, J., *Chrakterisierung des Spektrums eines Integraloperators*, Act. Sci. et Ind., Paris, 229 (1935), 11.

[22] Weyl, H., "Überbeschränkte quadratische Formen, deren Differenz Vollstetig ist", Rend. Circ. Mat. Palermo **27** (1909), 373 - 392.

[23] Wigner, E. P., Ann. Math. **40** (1939), 149 - 204.

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