AN INTRINSIC CHARACTERIZATION OF MONOMORPHISMS IN REGULAR LINDELÖF LOCALES

RICHARD N. BALL, ANTHONY W. HAGER, AND JOANNE WALTERS-WAYLAND

Dedicated to Bernhard Banaschewski on the occasion of his eightieth birthday

Abstract. We characterize monomorphisms in **rLLoc**, the category of regular Lindelöf locales. Though somewhat complicated, the characterization is intrinsic in the sense that it refers only to the properties of the morphism itself, rather than to properties of some lifting of it to a distant category.

Monomorphisms in **rLLoc** cannot be said to be well understood, despite their characterizations in [2] and [7]. The problem with these characterizations is that, although in each case the condition is simple enough (injectivity of a functorially associated spatial map in the case of [2], and surjectivity of a functorially associated frame map in the case of [7]), it is applied in a category sufficiently remote from **rLLoc** to make verification troublesome.

On the other hand, there are two categories closely related to **rLLoc** in which (the morphisms corresponding to) monomorphisms are well understood. Epimorphisms in **W**, the category of archimedean lattice-ordered groups with weak order unit, have a rich theory, and **W** is related to **Loc**, the category of locales with locale maps, by the contravariant adjunction

$$\mathbf{W} \underset{C}{\overset{Y}{\rightleftarrows}} \mathbf{Loc}.$$

Here C associates with a given locale L the **W**-object CL of locale morphisms from L into the (localic) reals, and Y associates with a given **W**-object A its Yosida locale YA ([8], [1]). Restricting the adjunction to the ranges of the functors provides a contravariant equivalence between **CL**, the full subcategory of **W** consisting of the objects isomorphic to CL for $L \in \mathbf{Loc}$, and **rLLoc**.

Given this equivalence, one might wonder why the rich theory of **W**-epimorphisms is not immediately available for analysis of **rLLoc**-monomorphisms. The reason is that this theory is framed in terms of the classical (pointed) Yosida representation of **W**-objects, and does not translate across the localic Yosida adjunction. What has been lacking is a characterization of **W**-epimorphisms in purely algebraic terms, without reference to the classical (pointed) Yosida representation, so that one could hope to translate it into localic terms. Such a characterization has recently become available ([3]), and this article performs the translation. The translation itself makes heavy use of the machinery developed in [5].

The resulting condition has the advantage of being intrinsic, in the sense that it refers only to the properties of the underlying frame morphism, as opposed to the

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extrinsic conditions mentioned above. The condition has the disadvantage of being complicated. The authors hope it proves to be as useful for the study of locales as its progenitor has proven to be for the study of archimedean ℓ-groups.

The second category, closely related to **rLLoc** and having tractable monomorphisms, is **LSpFi**, the category of compact Hausdorff spaces endowed with filters of dense open subsets such that the filters are generated by their Lindelöf members. From the point of view of calculation, this category provides perhaps the most perspicuous context for the analysis of monomorphisms, and this analysis applies immediately to both **W** and **rLLoc**. (A full treatment may be found in [4].) In fact, it was by means of α**SpFi**, the generalization of **LSpFi** to arbitrary regular cardinals α, that monomorphisms in **Loc** were characterized in [2].

We work mainly in the category $\mathbf{rFrm}_\sigma$, of regular σ-frames and σ-frame homomorphisms, and in **W**. We begin by identifying a property of σ-frame morphisms, here termed uplifting, which implies (Proposition 1.1), but is not implied by (Example 1.2), their surjectivity. Although this property has been analyzed in connection with C- and C^*- frame quotients (Theorem 1.3), our purpose in introducing this notion is only to motivate a generalization, here termed weakly uplifting, which characterizes σ-frame epimorphisms (Theorem 5.2). The argument that weakly uplifting σ-frame morphisms are epimorphisms is relatively straightforward and direct (Proposition 2.1); the converse requires an excursion into **W** (Section 3).

Because we took considerable pains to provide motivation, background, and examples in the articles upon which this one is based, namely [5] and [4], we chose not to do so here. Thus this article may best be understood as an addendum to these papers. The exception is Section 4. This section contains a few technical computations in CL which, though firmly rooted in the notation and technique of [5], do not appear there. For that reason they are worked out in full here. Thus this article completes a detailed proof of the result it announces, but the details are distributed over three papers. Nevertheless, the reader who is willing to buy the (very few) results cited from [5] and [4] should be able to understand this article completely.

1. Uplifting σ-frame morphisms

A σ-frame morphism $m : L \to M$ is said to be *uplifting* if for all $a_i \in M$ such that $a_1 \vee a_2 = \top$ there exist $c_i \in L$ such that $c_1 \vee c_2 = \top$ and $m(c_i) \leq a_i$. We conform to the standard notation $a \prec b$ for a being well below b, and $a \prec\prec b$ for a being completely below b.

Proposition 1.1. *An uplifting σ-frame morphism into a regular σ-frame is surjective.*

Proof. Suppose $m : L \to M$ is uplifting, and consider $a \in M$. Express a in the form $a = \bigvee_{\mathbb{N}} a_i$, where $a_i \prec a$ for all $i \in I$. For each i choose b_i satisfying $a \wedge a_i = \bot$ and $a \vee b_i = \top$. Then use the uplifting property to find $c_i, d_i \in L$ such that $c_i \vee d_i = \top$, $m(c_i) \leq b_i$, and $m(d_i) \leq a_i$. Then since $m(c_i) \vee m(d_i) = \top$,

$$\begin{aligned} a_i &= a_i \wedge \top = a_i \wedge (m(c_i) \vee m(d_i)) \\ &= (a_i \wedge m(c_i)) \vee (a_i \wedge m(d_i)) \\ &\leq (a_i \wedge b_i) \vee (a_i \wedge m(d_i)) \\ &= m(d_i) \leq a, \end{aligned}$$

so that

$$a \geq m\left(\bigvee_{a\geq m(d)} d\right) \geq \bigvee_{\mathbb{N}} a_i = a,$$

i.e., $a \in m(L)$. □

The converse of Proposition 1.1 does not hold.

Example 1.2. *Take m to be the frame counterpart of the inclusion*

$$\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right] \subseteq [0,1].$$

Since the spaces are metric, every open set is a cozero, and so the frames are σ-frames, the map m is a σ-frame morphism which is evidently onto, and the frames are regular. But elements

$$a_1 \equiv \left[0, \frac{1}{2}\right),\ a_2 \equiv \left(\frac{1}{2}, 1\right]$$

violate the uplifting condition.

Frame morphisms $f : L \to M$ whose cozero part $\operatorname{coz} f : \operatorname{coz} L \to \operatorname{coz} M$ is uplifting have received some attention in the literature. We quote Theorem 7.1.1 of [5] by way of example, referring the interested reader to that paper for undefined terminology.

Theorem 1.3. *The following are equivalent for a frame surjection $m : L \to M$.*

(1) $\operatorname{coz} m : \operatorname{coz} L \to \operatorname{coz} M$ *is uplifting.*
(2) m *is a C^*-quotient, i.e., every element of CM extends over m. This means that for every frame map $h : \mathcal{O}[0,1] \to M$ there is a frame map $k : \mathcal{O}[0,1] \to L$ such that $mk = h$.*
(3) *In M, $b_0 \prec\prec b_1$ iff $b_0 \prec\prec_m b_1$.*
(4) *Two completely separated elements of M are m-completely separated.*
(5) *Two completely separated quotients of M are m-completely separated.*
(6) *Every binary cozero cover of M is refined by the image of a binary cozero cover of L.*
(7) *Every finite cozero cover of M is refined by the image of a finite cozero cover of L.*
(8) *Every binary cozero cover of M is the image of a binary cozero cover of L.*
(9) $(C^*m)(C^*L)$ *is uniformly dense in C^*L.*
(10) $C^*m : C^*L \to C^*M$ *is surjective.*

When L and M are completely regular and endowed with their Čech-Stone uniformities, these conditions are equivalent to the following.

(11) m *is a uniform surjection.*

2. Weakly uplifting σ-frame morphisms

For a σ-frame M, an *indicator* is a function $r : \mathbb{N}^2 \to M$ such that $r(i,j) \prec r(i,j+1)$ for all i and j, and such that $\bigvee_j r(i,j) = \top$ for all i. A *choice function* is a map $t \in \mathbb{N}^{\mathbb{N}}$. For an indicator r, choice function t, and positive integer k, we let

$$r(k,t) \equiv \bigwedge_{1\leq i\leq k} r(i, t(i)).$$

A σ-morphism $m : L \to M$ is said to be *weakly uplifting* if for every pair of elements $x_i \in M$ with $x_0 \vee x_1 = \top$ there is an indicator r such that for each choice function t there exists an integer $k \in \mathbb{N}$ and elements $y_i \in L$ satisfying $y_0 \vee y_1 = \top$ and

$$(*) \qquad m(y_i) \wedge r(k,t) \leq x_i.$$

Proposition 2.1. *A weakly uplifting σ-morphism is an epimorphism.*

Proof. For a weakly uplifting σ-morphism $m : L \to M$, consider σ-morphisms $n_i : M \to N$ such that $n_1 m = n_2 m$. Suppose for the sake of argument that there is some $x_0 \in M$ for which $n_1(x_0) \nleq n_2(x_0)$. Since x_0 is the join of elements well below it, there must exist $a \prec x_0$ such that $n_1(a) \nleq n_2(x_0)$. Find $x_1 \in M$ such that $x_0 \vee x_1 = \top$ and $a \wedge x_1 = \bot$, then find an indicator r which witnesses the weakly uplifting property of m as applied to x_0 and x_1.

Inductively define a choice function t satisfying

$$(\blacklozenge) \qquad n_1(r(n,t)) \wedge n_2(r(n,t)) \wedge n_1(a) \nleq n_2(x_0)$$

for all $n \in \mathbb{N}$, and let k and y_i be the entities gotten from t satisfying $(*)$. Then

$$\begin{aligned} a \wedge r(k,t) &= a \wedge r(k,t) \wedge \top \\ &= a \wedge r(k,t) \wedge m(\top) \\ &= a \wedge r(k,t) \wedge m(y_0 \vee y_1) \\ &= (a \wedge r(k,t) \wedge m(y_0)) \vee (a \wedge r(k,t) \wedge m(y_1)) \\ &\leq (a \wedge r(k,t) \wedge m(y_0)) \vee (a \wedge x_1) \\ &= a \wedge r(k,t) \wedge m(y_0) \leq r(k,t) \wedge m(y_0). \end{aligned}$$

Therefore

$$\begin{aligned} n_2(x_0) &\geq n_2(r(k,t) \wedge m(y_0)) \\ &\geq n_2(r(k,t)) \wedge n_1(r(k,t)) \wedge n_2 m(y_0) \\ &= n_2(r(k,t)) \wedge n_1(r(k,t)) \wedge n_1 m(y_0) \\ &= n_2(r(k,t)) \wedge n_1(r(k,t) \wedge m(y_0)) \\ &\geq n_2(r(k,t)) \wedge n_1(a \wedge r(k,t) \wedge m(y_0)) \\ &= n_2(r(k,t)) \wedge n_1(a \wedge r(k,t)) \\ &= n_1(r(k,t)) \wedge n_2(r(k,t)) \wedge n_1(a). \end{aligned}$$

This contradicts $(\blacklozenge)$. We conclude that no such element x_0 can exist in M, i.e., that $n_1(x_0) \leq n_2(x_0)$ for all $x_0 \in M$. A parallel argument establishes the opposite inequality, thus proving that m is an epimorphism. □

3. Epimorphisms in W

In this section we succinctly outline the (pointed Yosida) representation-free characterization of **W**-epimorphisms from [3], and in subsequent sections we translate it into the language of σ-frames. The characterization is in terms of density with respect to a particular convergence, called epi-convergence. In a **W**-object B, let

$$B \vee 1 \equiv \{b \in B : b \geq 1\}.$$

For a countable subset $R = \{r_n : n \in \mathbb{N}\} \subseteq B \vee 1$, choice function $t \in \mathbb{N}^{\mathbb{N}}$, $\varepsilon > 0$, and positive integer k, let $B(t|k, \varepsilon, b)$ designate the set of those elements $a \in B$ which satisfy

$$\bigwedge_{1 \le i \le k} (t(i) - r_i) \wedge |b - a| \le \varepsilon.$$

Then set

$$B(t, \varepsilon, b) \equiv \bigcup_{k \ge 1} B(t|k, \varepsilon, b).$$

The filter $\mathcal{F}(R, b)$ has base sets of the form $B(t, \varepsilon, b)$, $t \in \mathbb{N}^{\mathbb{N}}$, $\varepsilon > 0$. We say of an arbitrary filter $\mathcal{F}$ on B that $\mathcal{F}$ *epi-converges to* b *with indicators* R, and write $\mathcal{F} \xrightarrow{R} b$, provided that $\mathcal{F} \supseteq \mathcal{F}(R, b)$. We say that $\mathcal{F}$ *epi-converges to* b, and write $\mathcal{F} \to b$, provided that $\mathcal{F} \xrightarrow{R} b$ for some countable subset $R \subseteq B \vee 1$. A subset $A \subseteq B$ is said to be *dense* if for every $b \in B$ there is a filter $\mathcal{F}$ on B such that $A \in \mathcal{F}$ and $\mathcal{F} \to b$.

Here is the representation-free characterization of **W**-epimorphisms which we need. Its proof can be assembled from results in [3]; we offer brief instructions for doing so here.

Theorem 3.1. *A* **W***-subobject* $A \le B$ *is epically embedded iff it is dense, provided* A *is divisible.*

Proof. Together with the introductory comments about dominion at the beginning of Section 2, Corollary 2.6 establishes that A is epically embedded in B iff A is dense in B with respect to the closure operator cl^B. The fact that this closure operator coincides with the definition presented above is the content of Theorem 4.7. □

4. Some calculations in CL

We assume the results and notation of [5] in order to prove several technical lemmas. In particular, $\mathbb{R}_i \equiv \mathbb{R} \smallsetminus \{i\}$ for $i \in \{0, 1\}$. In the following lemmas L designates a regular σ-frame, f, g and h designate elements of CL, ε designates a positive real number and ${}_\varepsilon\mathbb{R}$ designates the open set $\mathbb{R} \smallsetminus [-\varepsilon, \varepsilon]$, and symbols U, U_i and V designate open subsets of $\mathbb{R}$. The first lemma generalizes Lemma 4.1.1 of [5].

Lemma 4.1. *The following are equivalent.*

(1) $g \wedge |f| \le \varepsilon$.
(2) $f({}_\varepsilon\mathbb{R}) \wedge g(\varepsilon, \infty) = \bot$.

Proof. Keeping in mind that

$$\varepsilon(U) = \begin{cases} \top & \text{if} \quad \varepsilon \in U \\ \bot & \text{if} \quad \varepsilon \notin U \end{cases},$$

Lemma 3.2.4 of [5] says that (1) is equivalent to

$$\begin{aligned} \bot &= (g \wedge |f|)(\varepsilon, \infty) = (g \wedge (f \vee -f))(\varepsilon, \infty) \\ &= \bigvee_{U_1 \wedge (U_2 \vee (-U_2)) \subseteq (\varepsilon, \infty)} g(U_1) \wedge f(U_2). \end{aligned}$$

But $U_1 \wedge (U_2 \vee (-U_2)) \subseteq (\varepsilon, \infty)$ iff $U_1 \subseteq (\varepsilon, \infty)$ and $U_2 \subseteq {}_\varepsilon\mathbb{R}$. Thus the condition displayed above is equivalent to (2). □

The next lemma generalizes Lemma 4.1.2 of [5].

Lemma 4.2. *The following are equivalent.*

1. $g \wedge |f-h| \leq \varepsilon$.
2. $f(U_1) \wedge h(U_2) \wedge g(\varepsilon,\infty) = \bot$ *for all pairs* U_i *which are at least* ε *units apart.*

Proof. By Lemma 4.1, (1) is equivalent to

$$\begin{aligned}\bot &= (f-h)\left({}_{\varepsilon}\mathbb{R}\right) \wedge g(\varepsilon,\infty) \\ &= \left(\bigvee_{(U_1-U_2)\subseteq_{\varepsilon}\mathbb{R}} f(U_1) \wedge h(U_2)\right) \wedge g(\varepsilon,\infty) \\ &= \bigvee_{(U_1-U_2)\subseteq_{\varepsilon}\mathbb{R}} f(U_1) \wedge h(U_2) \wedge g(\varepsilon,\infty).\end{aligned}$$

But the last supremum cannot be $\bot$ unless each of its terms is $\bot$, and this is (2). □

We remind the reader that, for open subsets $U, V \subseteq \mathbb{R}$, $U \subseteq_{\varepsilon} V$ means that

$$\{y : \exists\, x \in U \ \ (|x-y| < \varepsilon)\} \subseteq V.$$

Lemma 4.3. *If* $g \wedge |f-h| \leq \varepsilon$ *then*

$$h(U) \wedge g(\varepsilon,\infty) \leq f(V)$$

for all U *and* V *such that* $U \subseteq_{\varepsilon} V$.

Proof. For an open interval $U_2 \subseteq \mathbb{R}$ such that $U_2 \subseteq_{\delta} U$ for some $\delta > 0$, let U_1 designate the union of all open intervals $R \subseteq \mathbb{R}$ such that U_2 and R are at least ε units apart. Then by Lemma 4.2,

$$f(U_1) \wedge h(U_2) \wedge g(\varepsilon,\infty) = \bot,$$

and since

$$f(U_1) \cup f(V) = f(U_1 \cup V) = f(\top) = \top,$$

it follows that $h(U_2) \wedge g(\varepsilon,\infty) \leq f(V)$. Therefore

$$\begin{aligned}h(U) \wedge g(\varepsilon,\infty) &= \left(\bigvee_{\delta>0}\bigvee_{U_2\subseteq_{\delta}U} h(U_2)\right) \wedge g(\varepsilon,\infty) \\ &= \bigvee_{\delta>0}\bigvee_{U_2\subseteq_{\delta}U} h(U_2) \wedge g(\varepsilon,\infty) \\ &\leq f(V).\end{aligned}$$

This proves the lemma. □

Lemma 4.4. *For any* $n \in \mathbb{N}$,

$$(n-f)(\varepsilon,\infty) = f(-\infty, n-\varepsilon).$$

Proof. According to Proposition 3.1.1 of [5],

$$\begin{aligned}(n-f)(\varepsilon,\infty) &= \bigvee_{(U_1-U_2)\subseteq(\varepsilon,\infty)} n(U_1)\wedge f(U_2)\\ &= \bigvee_{\delta>0}\ \bigvee_{U_2\subseteq(-\infty,n-\varepsilon-\delta)} f(U_2)\\ &= f(-\infty,n-\varepsilon).\end{aligned}$$

A term contributes nontrivially to the first join only if

$$(n-\delta,n+\delta)\subseteq U_1$$

for some $\delta>0$, and in this circumstance the displayed condition forces U_2 to be contained in $(-\infty,n-\varepsilon-\delta)$. □

Lemma 4.5. $(f\wedge g)(\varepsilon,\infty)=f(\varepsilon,\infty)\wedge g(\varepsilon,\infty)$.

Proof. According to Proposition 3.1.1 of [5],

$$\begin{aligned}(f\wedge g)(\varepsilon,\infty) &= \bigvee_{(U_1\wedge U_2)\subseteq(\varepsilon,\infty)} f(U_1)\wedge g(U_2)\\ &= \bigvee_{U_i\subseteq(\varepsilon,\infty,)} f(U_1)\wedge g(U_2)\\ &= \bigvee_{U_1\subseteq(\varepsilon,\infty,)} f(U_1)\wedge \bigvee_{U_2\subseteq(\varepsilon,\infty,)} g(U_2)\\ &= f(\varepsilon,\infty)\wedge g(\varepsilon,\infty).\end{aligned}$$

The key observation is that $(U_1\wedge U_2)\subseteq(\varepsilon,\infty)$ iff each $U_i\subseteq(\varepsilon,\infty)$. □

5. σ-FRAME EPIMORPHISMS ARE WEAKLY UPLIFTING

Proposition 5.1. *Let $m:L\to M$ be a σ-frame morphism, and let $Cm:CL\to CM$ be the associated* **W***-morphism. If $(Cm)(CL)$ is dense in CM then m is weakly uplifting.*

Proof. Let $A\equiv(Cm)(CL)$ and $B\equiv CM$. Consider $x_i\in M$ such that $x_0\vee x_1=\top$. Use (the σ-frame version of) Proposition 5.1.2 in [5] to find $b\in B$ such that $b(\mathbb{R}\smallsetminus[i])=x_i$. By the density of A in B, there is a filter $\mathcal{F}$ such that $A\in\mathcal{F}\overset{R}{\to}b$ for some countable subset

$$R=\{r_n:n\in\mathbb{N}\}\subseteq B\vee 1.$$

Define indicator r by the rule

$$r(m,n)\equiv r_m(-\infty,n),\ m,n\in\mathbb{N}.$$

Consider now a choice function $t\in\mathbb{N}^{\mathbb{N}}$, and let s designate the choice function $n\longmapsto t(n)+1$. Since $A\in\mathcal{F}$ and

$$\bigcup_{k\geq1}B(s|k,1,b)\equiv B(s,1,b)\in\mathcal{F}(R,b)\subseteq\mathcal{F},$$

it follows that there is some $a\in A$ and integer k such that $a\in B(s|k,1,b)$, i.e.,

$$\bigwedge_{1\leq i\leq k}(s(i)-r_i)^{+}\wedge|b-a|\leq1.$$

We note that

$$\begin{aligned}\left(\bigwedge_{1\leq i\leq k}(s(i)-r_i)^+\right)(1,\infty) &= \bigwedge_{1\leq i\leq k}(s(i)-r_i)^+(1,\infty) \text{ by Lemma 4.5}\\ &= \bigwedge_{1\leq i\leq k} r_i(-\infty,s(i)-1) \text{ by Lemma 4.4}\\ &= \bigwedge_{1\leq i\leq k} r_i(-\infty,t(i)) = \bigwedge_{1\leq i\leq k} r(i,t(i)) = r(k,t)\end{aligned}$$

Now a is of the form $m(c)$ for some $c \in CL$. Fix a real number δ, $0<\delta<1$, and set

$$y_0 \equiv c({}_\delta\mathbb{R}),\ y_1 \equiv c(\mathbb{R}\setminus[1-\delta,1+\delta]).$$

Observe that $y_i \in L$ satisfy $y_0 \vee y_1 = \top$, and that

$$m(y_0) = a({}_\delta\mathbb{R}) \subseteq_\delta \mathbb{R}_0,$$

so that from Lemma 4.3 we get

$$\begin{aligned}m(y_0)\wedge r(k,t) &= a({}_\delta\mathbb{R})\wedge\left(\bigwedge_{1\leq i\leq k}(s(i)-r_i)^+\right)(1,\infty)\\ &\leq b(\mathbb{R}_0) = x_0.\end{aligned}$$

The argument that $m(y_1)\wedge r(k,t)\leq x_1$ is similar. □

We can now state and prove the main result.

Theorem 5.2. *The following are equivalent for a morphism $m : L \to M$ in* $\mathbf{rFrm}_\sigma$*.*

1. *m is an epimorphism.*
2. *Cm is an epimorphism in* **CL***.*
3. *Cm is an epimorphism in* **W***.*
4. *$(Cm)(CL)$ is dense in CM.*
5. *m is weakly uplifting.*

Proof. The fact that **CL** is categorically equivalent to $\mathbf{rFrm}_\sigma$ makes the equivalence of (1) and (2) obvious, and the fact that the functor C reflects and preserves epimorphisms makes the equivalence of (2) and (3) obvious. The equivalence of (3) and (4) is Theorem 3.1. The implication from (5) to (1) is Proposition 2.1, and the implication from (4) to (5) is Proposition 5.1. □

The final corollary follows immediately from the equivalence of $\mathbf{rFrm}_\sigma$ with **rLFrm**.

Corollary 5.3. *A morphism $m : L \to M$ in* **rLFrm** *is an epimorphism iff $\operatorname{coz} m : \operatorname{coz} L \to \operatorname{coz} M$ is weakly uplifting.*

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(Ball) Department of Mathematics, University of Denver, Denver, Colorado 80208, U.S.A.

E-mail address: `rick.ball@nsm.du.edu`

URL: `http://www.math.du.edu/~rball`

(Hager) Department of Mathematics, Wesleyan University, Middletown, Connecticut 06459, U.S.A.

E-mail address: `ahager@wesleyan.edu`

(Walters-Wayland) 21521 Rushford Drive, Lake Forest, California 92630, U.S.A..

E-mail address: `joanne@waylands.com`