FLOW-ORBIT EQUIVALENCE FOR MINIMAL CANTOR SYSTEMS

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ABSTRACT. This paper is about flow-orbit equivalence, a topological analogue of even Kakutani equivalence. In addition to establishing many basic facts about this relation, we characterize the conjugacies of induced systems that can be extended to a flow-orbit equivalence. We also describe the relationship between flow-orbit equivalence and a distortion function of an orbit equivalence. We show that if the distortion of an orbit equivalence is zero, then it is in fact a flow-orbit equivalence, and that the converse is true up to a conjugation by an element of the full group.

1. INTRODUCTION

In this paper, we examine flow-orbit equivalences, orbit equivalences which are simultaneously flow equivalences. For homeomorphisms of zero dimensional space, flow-orbit equivalence is the topological analog of even Kakutani equivalence in the measure-theoretic category. We view this relation as a restricted topological orbit equivalence in the spirit of restricted orbit equivalence in measure-theoretic dynamics as formulated in [29].

We focus on minimal $\mathbb{Z}$-actions on the Cantor set. In [17], Giordano, Putnam and Skau characterized of orbit equivalence for such actions. Their characterization is in terms of unital ordered groups. Specifically, for a minimal Cantor system (X,T), set $G(T) = C(X,\mathbb{Z})$ modulo the subgroup of infinitesimal functions, i.e., functions which integrate to 0 with any T-invariant Borel measure. Within the group $G(T)$, let $G(T)_+$ denote the semigroup of equivalence classes of nonnegative functions, and $[1_X]$ the equivalence class of the constant function 1 on X. Giordano, Putnam and Skau show that two minimal Cantor systems (X,T) and (Y,S) are orbit equivalent if and only if $G(T)$ is isomorphic to $G(S)$ by a isomorphism that takes $G(T)_+$ to $G(S)_+$ and takes $[1_X]$ to $[1_Y]$. This then gives a context in which to study more restrictive orbit equivalence classes.

Following another line of research in topological dynamics, flow equivalence is a topological version of measure-theoretic Kakutani equivalence in the sense that both are defined as being cross-sections of a common $\mathbb{R}$-flow. Flow equivalence is a well-studied concept which is strongly related to Morita equivalence in C^*-algebras [1, 3, 4, 5, 7, 8, 9, 10, 15, 19, 26, 27]. In both the topological and measure-theoretic situations, it turns out that being cross-sections of common $\mathbb{R}$-flows is equivalent to having conjugate discrete cross-sections, or induced maps [25, 27]. In the measure-theoretic category, a Kakutani equivalence is considered *even* if the sets upon which the induced maps act can be chosen to be of equal measure. To define

Date: March 23, 2007.
1991 *Mathematics Subject Classification.* Primary: 37B05, Secondary: 37A20, 37B10.
Key words and phrases. Topological Dynamics, Orbit Equivalence, Kakutani Equivalence.
* - Author supported by NSF grant DMS-0618030.

the topological analog of even Kakutani equivalence, we have to take into consideration the possibility that T and S have many invariant measures. The ordered groups used by Herman, Giordano, Putnam and Skau [17, 18] provide the ideal way to formulate the appropriate definition. There is a natural isomorphism from $G(T)$ to $G(T_A)$ for A a clopen subset of X. Now suppose (X,T) and (Y,S) are minimal Cantor systems and $A \subset X$, $B \subset Y$ are clopen subsets in the corresponding spaces such that T_A is conjugate to S_B. There is a natural isomorphism from $G(T_A)$ to $G(S_B)$. Then we may track $[1_X]$ as it passes through the isomorphisms $G(T) \to G(T_A) \to G(S_B) \to G(S)$. If the image of $[1_X]$ is equal to $[1_Y] \in G(S)$ then it follows from [17] that T and S are orbit equivalent in addition to being flow equivalent. We show (Theorem 2.16) that if the image is equal to $[1_Y] \in G(S)$ then the orbit equivalence from T to S can be chosen to be an extension of the conjugacy from T_A to S_B. We call the resulting map a flow-orbit equivalence, since it is at once a flow equivalence and an orbit equivalence.

In Section 3 we explore the relationship between flow-orbit equivalence and a distortion function D. The distortion function allows one to think of this relation (flow-orbit equivalence) as a restricted topological orbit equivalence analogous to the approach of [29]. In particular, a nearly identical notion is used in [11] for Kakutani equivalence of ergodic $\mathbb{Z}^d$ actions. Given an orbit equivalence φ between S and T, the function $D_\varphi(x)$ measures for each x, the asymptotic average of the cardinality of the symmetric difference of $\varphi\{T^n(x) : 0 \le n < N\}$ and $\{S^n\varphi(x) : 0 \le n < N\}$. We show (Theorem 3.5) that if $D_\varphi(x) < \frac{1}{4}$ then φ is a flow-orbit equivalence. Further, we show (Theorem 2.16) that although some flow-orbit equivalences have $D_\varphi \not\equiv 0$ (e.g. Example 3.1), for any two flow-orbit equivalent systems T and S and flow-orbit equivalence φ there is an element ψ of the full group of T such that $D_\varphi \equiv 0$ where φ is considered as a flow-orbit equivalence of $\psi^{-1}T\psi$ and S. We show (Theorem 3.11) that examples of flow-orbit equivalences with $D \not\equiv 0$ necessarily have cocycle functions with sets of discontinuities of positive measure for some invariant Borel measure.

Finally, we present an example (Example 4) which shows that the relation of flow-orbit equivalence is not the same as conjugacy. This example also addresses a question posed by Dartnell, Durand and Maass [10]. They show that given T and S where T is Sturmian, if S is both flow equivalent and orbit equivalent to T then S is conjugate to T. They go on to pose the question of whether the same statement is true when T is assumed to be a substitution system, or even for a general Cantor minimal system T. Our example involves substitution systems, and thereby resolves both of these questions in a negative way. However, in the example the two systems are not strongly orbit equivalent, so it leaves open the possibility that strong orbit equivalence and flow-equivalence could imply conjugacy for a more general class of systems.

2. Flow-Orbit Equivalence

We begin with some more specific definitions.

Definition 2.1. *A minimal Cantor system* (X,T) *is a Cantor set* X *along with a self-homeomorphism* $T : X \to X$ *such that for every* $x \in X$, *the set* $\{T^n x : n \in \mathbb{Z}\}$ *is dense in* X.

Definition 2.2. *Let (X,T), (Y,S) be minimal Cantor systems. Then T and S are* orbit equivalent *if they are conjugate to systems with the same orbits. I.e., there is a homeomorphism $\varphi : X \to Y$, and maps $\alpha, \beta : X \to \mathbf{Z}$ such that*

$$\varphi T(x) = S^{\alpha(x)} \varphi(x)$$
$$\varphi T^{\beta(x)}(x) = S \varphi(x)$$

If we have a minimal Cantor system (X,T) and a nonempty clopen set A, then we may define the first return time function $r_A : A \to \mathbb{N}$ as $r_A(x) = \min\{n > 0 : T^n(x) \in A\}$. Because (X,T) is a minimal Cantor system and A is clopen, r_A is a well-defined continuous function. With this, we can define the induced system $T_A : A \to A$ where $T_A(x) = T^{r_A(x)}(x)$.

Definition 2.3. *Let (X,T), (Y,S) be minimal Cantor systems. Then T and S are* flow equivalent *if they have conjugate induced systems. That is, there are clopen sets $A \subset X$ and $B \subset Y$ and a homeomorphism $\varphi : A \to B$ such that*

$$\varphi T_A(x) = S_B \varphi(x)$$

for all $x \in A$.

Remark 2.4. *As noted in the Introduction, the above is really a theorem not a definition. That is, flow equivalence was originally defined in terms of equivalence of the natural $\mathbb{R}$-flows on suspension spaces for T and S. It follows from results of Parry and Sullivan that the above definition coincides with the original one* [27].

We wish to examine orbit equivalences which are simultaneously flow equivalences.

Definition 2.5. *Let (X,T), (Y,S) be minimal Cantor systems. Then T and S are* flow-orbit equivalent *if they are orbit equivalent by a homeomorphism $\varphi : X \to Y$ which has the property that there is a clopen set $A \subset X$ such that for all $x \in A$,*

$$\varphi T_A(x) = S_{\varphi(A)} \varphi(x).$$

The following gives an alternate characterization of flow-orbit equivalence.

Proposition 2.6. *Let (X,T), (Y,S) be minimal Cantor systems which are orbit equivalent by a map φ. The maps T and S are flow-orbit equivalent if and only if there exists a nonempty clopen set A such that for all $x \in A$, and $r > 0$ such that $T^r(x) \in A$, we have $\varphi T^r(x) = S^q \varphi(x)$ for some $q > 0$.*

Proof. If T and S are flow-orbit equivalent then $\varphi T_A(x) = S_{\varphi(A)} \varphi(x)$ for some clopen set A. If $T^r(x) \in A$ for $r > 0$ then $T^r = (T_A)^n$ for some $n > 0$. Then

$$\varphi T^r(x) = \varphi (T_A)^n = \left(S_{\varphi(A)}\right)^n \varphi(x) = S^q \varphi(x)$$

for $q > 0$.

For the other direction, assume the set A exists as above. For $x \in A$, set $r(x)$ equal to the first return time of x to A. Fix $x \in A$ and let $y = \varphi(x)$. Since $r(x) > 0$, $\varphi T^{r(x)}(x) = S^q(y)$ for some $q > 0$. Since both y and $S^q y$ are in $\varphi(A)$, $S^q(y) = \left(S_{\varphi(A)}\right)^n (y)$ for some $n > 0$. Our goal is to show that $n = 1$.

Suppose $n > 1$. Then $\varphi^{-1} S_{\varphi(A)}(y) = (T_A)^m(x)$ for some $m \neq 0, 1$. If $m > 1$ then we have $\varphi (T_A)^{m-1} T_A(x) = \left(S_{\varphi(A)}\right)^{1-n} \varphi T_A(x)$, a contradiction. If $m < 0$ then we have $\varphi (T_A)^{-m} (T_A)^m(x) = \left(S_{\varphi(A)}\right)^{-1} \varphi (T_A)^m(x)$, another contradiction. □

Let $\mathcal{M}_T$ denote the space of T-invariant Borel probabilities on X.

Proposition 2.7. *Let φ be an orbit equivalence from (X, T) to (Y, S). Then φ induces an isomorphism between $\mathcal{M}_T$ and $\mathcal{M}_S$.*

Proof. Let μ be a T-invariant Borel probability on X. Then because φ is a homeomorphism, $\mu\varphi^{-1}$ is a Borel probability on Y. We just need to know that $\mu\varphi^{-1}$ is S-invariant. For this, we note that the set $E_n = \{y : T^n\varphi^{-1}(y) = \varphi^{-1}S(y)\} \subset Y$ is closed by the continuity of φ^{-1}, T and S^n and therefore μ-measurable. The sets E_n also form a countable partition of Y. For $B \subset Y$ a $\mu\varphi^{-1}$ measurable set, we have $B = \cup_n (B \cap E_n)$ and

$$\mu\varphi^{-1}SB = \mu\varphi^{-1} \cup_n S(B \cap E_n) = \mu \cup_n T^n\varphi^{-1}(B \cap E_n) = \mu\varphi^{-1} \cup_n (B \cap E_n) = \mu\varphi^{-1}B.$$

□

Definition 2.8. *A continuous function $f : X \to \mathbb{Z}$ is an* infinitesimal *for (X, T) if $\int f d\mu = 0$ for all $\mu \in \mathcal{M}_T$.*

Abusing notation slightly, extend the function r_A (the return time function for A) to all of X by setting it equal to 0 on $X \setminus A$.

Lemma 2.9. *Suppose φ is an orbit equivalence from (X, T) to (Y, S) and $A \subset X$, $B \subset Y$ are clopen sets such that $\varphi T_A = S_B\varphi$ then $(r_A) - (r_B \circ \varphi)$ is an infinitesimal.*

Proof. Let μ be any T-invariant Borel probability. By Kac's Theorem [20], we have $\int_A r_A d\mu = 1$. Then it follows

$$\int_A r_B \circ \varphi d\mu = \int_B r_B d\left(\mu\varphi^{-1}\right)$$

But since φ is an orbit equivalence, $\mu\varphi^{-1}$ is an S-invariant Borel probability on Y. Therefore, $\int_B r_B \circ \varphi d\mu = 1$ again by Kac's Theorem. Since every S-invariant Borel probability on Y is of the form $\mu\varphi^{-1}$, $(r_A) - (r_B \circ \varphi)$ is an infinitesimal. □

Conversely, we wish to know if that if φ is a conjugacy between induced systems T_A and S_B, and $(r_A) - (r_B \circ \varphi)$ is an infinitesimal for some clopen sets then T and S are flow-orbit equivalent. It follows from a theorem of Giordano, Putnam and Skau [17] that in this case, T and S are orbit equivalent. In fact, the results of [23] show that there is an orbit equivalence from T to S in which the cocycles are discontinuous at only 2 points, a point x_0 and its image $T(x_0)$. We give a proof of the stronger result below.

Theorem 2.10. *Suppose T_A is topologically conjugate to S_B by a map $\varphi : A \to B$. Furthermore, suppose that $(r_A) - (r_B \circ \varphi)$ is an infinitesimal. Then T and S are orbit equivalent by an orbit equivalence with at most 2 points of discontinuity for the associated cocycles.*

Proof. We set $K^0(T) = C(X,\mathbb{Z})$ modulo the subgroup of coboundaries, i.e., functions of the form $f - fT$. Let $[1_X]$ denote the equivalence class of the constant function 1. Note that there is a unital ordered group isomorphism

$$\left(K^0(T), K^0(T)_+, [1_X]\right) \cong \left(K^0(T_A), K^0(T_A)_+, [r_A]\right)$$

defined by the following. For $f \in C(X,\mathbb{Z})$, let

$$f_A = \begin{cases} \sum_{i=0}^{r_A(x)-1} f\left(T^i x\right) \text{ for } x \in A \\ 0 \text{ for } x \notin A \end{cases}.$$

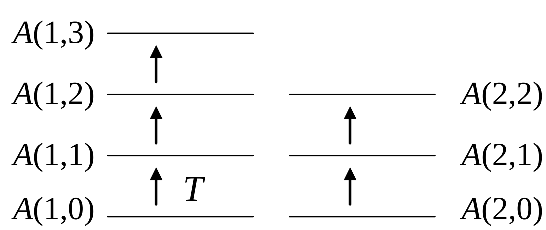
Further, the conjugacy φ gives an ordered group isomorphism between $(K^0(S_B), K^0(S_B)_+)$ and $\left(K^0(T_A), K^0(T_A)_+\right)$ defined by $[f] \mapsto [f\varphi]$. By composing these ordered group isomorphisms, we obtain an order isomorphism from $(K^0(S), K^0(S)_+)$ to $(K^0(T), K^0(T)_+)$ which maps $[1]_S \mapsto [r_B] \mapsto [r_B\varphi] = [r_A] + [g] \mapsto [1]_T + [g]$ where g is an infinitesimal. With such a set up, the results of [23] guarantee an orbit equivalence between T and S with at most 2 points of discontinuity for the cocycles. □

We aim to prove that the orbit equivalence above can be chosen to be an extension of the map φ. First we establish some notation.

2.1. **Tower Partitions.** Given a minimal Cantor system (X,T) we will be working with clopen *tower partitions* of X. A tower partition of X is a finite clopen partition of the form $\{A(i,j) : 0 \le j < h(i), 1 \le i \le I\}$ where $TA(i,j) = A(i,j+1)$ for $j+1 < h(i)$.

Given a clopen set $A \subset X$, we may construct a *tower partition over* A, i.e. a tower partition $\{A(i,j)\}$ with $A = \cup A(i,0)$, as follows. Set A_i equal to the clopen set $r_A^{-1}\{i\} \cap A$ and note that for some I, the sets $\{A_i : 1 \le i \le I\}$ form a finite clopen partition of A. Now set $A(i,j) = T^j A_i$ for $0 \le j < i$ and $1 \le i \le I$. Given any other partition $\mathcal{P}$ of X, we may refine this tower partition over A to another by partitioning the sets A_i according to $\bigvee_{j=0}^{i-1} T^{-j}\mathcal{P}$ to obtain a new tower partition $\{A'(i,j)\}$ over A which refines $\mathcal{P}$.

We will sometimes refer to a *tower partition over* x_0. By this we mean a tower partition over U where U is a clopen neighborhood of x_0. The tower partition then is visualized as a finite collection of columns with $A(i,0)$ at the base of the column, $A(i,h(i)-1)$ at the top, and the other *floors* $A(i,j)$ in between. The action of T then moves points up through the columns, and maps the union of the top floors to the union of the bottom floors.



The following are a few observations about tower partitions which will be useful for us.

Proposition 2.11. *Let $x_0 \in X$ and N be given. There is a $\delta > 0$ such that if $\tau = \{A(i,j) : 0 \le j < h(i), 1 \le i \le I\}$ is a tower partition over $A \ni x_0$ and the diameter of A is less than δ, then $h(i) > N$ for all i.*

Proposition 2.12. *Let $f : X \to \mathbb{Z}$ be a continuous coboundary. I.e., $f = g - gT$ where $g : X \to \mathbb{Z}$ is continuous. There is a $\delta > 0$ such that if A is a clopen set of diameter less than δ then for every $x \in A$,*

$$\sum_{j=0}^{r_A(x)-1} fT^j(x) = 0.$$

Notation 2.13. *Let $f, g : X \to \mathbb{Z}$ be continuous. We say $[f] > [g]$ if $[f] \neq [g]$ and $[f] - [g] \in G(T)_+$. For rational $\varepsilon = \frac{p}{q}$, we say that $\varepsilon[f] > [g]$ if $p[f] > q[g]$.*

Using the above and the Ergodic Theorem, we obtain the following.

Proposition 2.14. *Let $f, g : X \to \mathbb{Z}$ be continuous. Suppose $[f] > [g]$ in the ordered group $\left(G(T), G(T)_+\right)$. There is a $\delta > 0$ such that if A is a clopen set with diameter less than δ, then for all $x \in A$, then for every $x \in A$,*

$$\sum_{j=0}^{r_A(x)-1} fT^j(x) > \sum_{j=0}^{r_A(x)-1} gT^j(x).$$

Proposition 2.15. *Let $f : X \to \mathbb{Z}$ be an infinitesimal function, and let $\varepsilon > 0$ be given. There is an N such that if A is a clopen set with $r_A(x) > N$ for all $x \in A$, then for every $x \in A$,*

$$\frac{1}{r_A(x)} \sum_{j=0}^{r_A(x)-1} fT^j(x) < \varepsilon.$$

2.2. **Extension Theorem.**

Theorem 2.16. *Suppose T_A is topologically conjugate to S_B by a map $\varphi : A \to B$. The function $(r_A) - (r_B \circ \varphi)$ is an infinitesimal if and only if T and S are orbit equivalent by an orbit equivalence ψ with at most 2 points of discontinuity for each of the associated cocycles, and $\psi = \varphi$ on A.*

Proof. One direction is Lemma 2.9. For the other direction, note that there is a point $x_0 \in A$ such that $Tx_0 \notin A$ or else $A = X$ and we are done. Let $x_1 = Tx_0$. Let $y_0 = \varphi(x_0)$, and $y_1 = S^m(y_0)$ where $m = \min\{n > 0 : S^m(y_0) \notin B\}$. Fix a decreasing sequence of positive numbers $\{\delta_n\}$ with $\lim_{n\to\infty} \delta_n = 0$. Let $\psi = \varphi$ on A.

Define a clopen set U containing x_1 of diameter less than δ_1 such that

- $T^{-1}U \subset A$.

Now let V be a clopen set containing y_1 of diameter less than δ_1 with a diameter small enough to insure

- $S^{-m}V \subset B$
- $S^{-m}V \subsetneq \varphi T^{-1}U$.

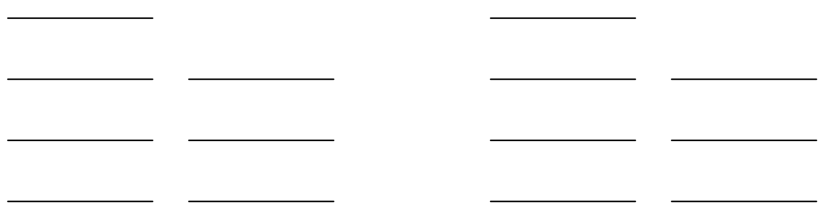
Since $S^{-m}V$ is a proper subset of $\varphi T^{-1}U$, nonempty clopen set, the function $1_U - 1_{\varphi^{-1}S^{-m}(V)}$ integrates to a positive number with any T-invariant Borel measure. Said another way, $[1_U] > \left[1_{\varphi^{-1}S^{-m}(V)}\right]$ in $G(T)$.

Now construct a T_A tower partition over x_0 where the functions $r_A, r_B\varphi, 1_U, 1_{\varphi^{-1}S^{-m}(V)}$ are constant on all floors of the tower partition and where every column has of sufficiently large height to insure that for each column and each x in the base of that column,

$$\sum_{j=0}^{H-1} (r_A - r_B\varphi)\, T^j x < \sum_{j=0}^{H-1} \left(1_U - 1_{\varphi^{-1}S^{-m}(V)}\right) T^j x \tag{1}$$

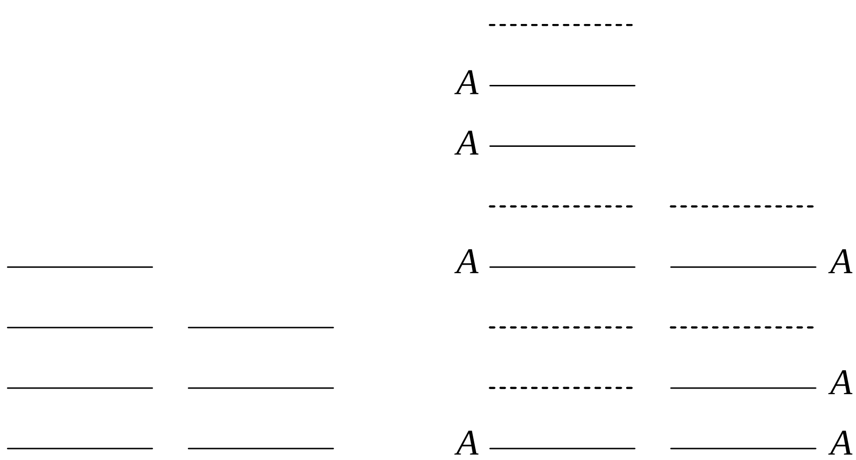
where H is the height of the column. By Proposition 2.14, we have that the above holds as long as all of the column heights are sufficiently large because $\left[1_U - 1_{\varphi^{-1}S^{-m}(V)}\right] > [r_A - r_B\varphi] = [0]$ in $G(T)$.

The tower partition is a tower partition for T_A, and via the conjugacy φ gives an identical tower partition for S_B. For example, we may have the two tower partitions pictured below.



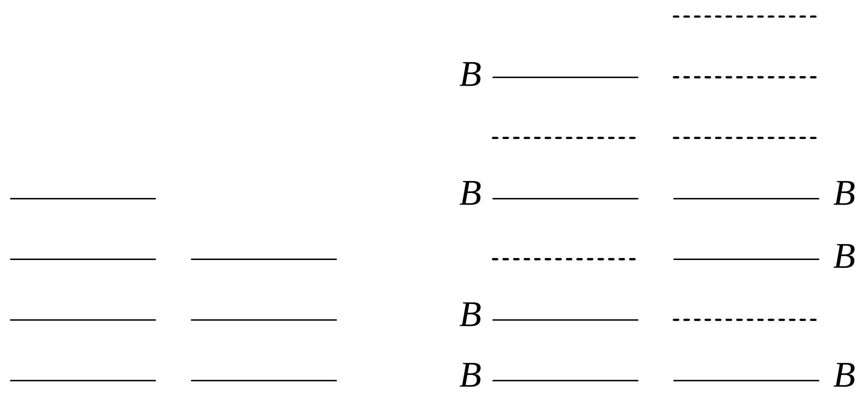
(A,T_A) (B,S_B)

Expand both tower partitions to tower partitions for T and S, respectively. That is, for $x \in A$, with $r_A(x) = r$, insert $r-1$ points $\{T(x), T^2(x), \ldots, T^{r-1}(x)\}$ between the floor containing x and the floor containing $T_A(x)$. This will result in a new clopen tower partition for (X,T). Our picture example is continued below, (dashed) floors from A^C are inserted between floors from A.



(A,T_A) (X,T)

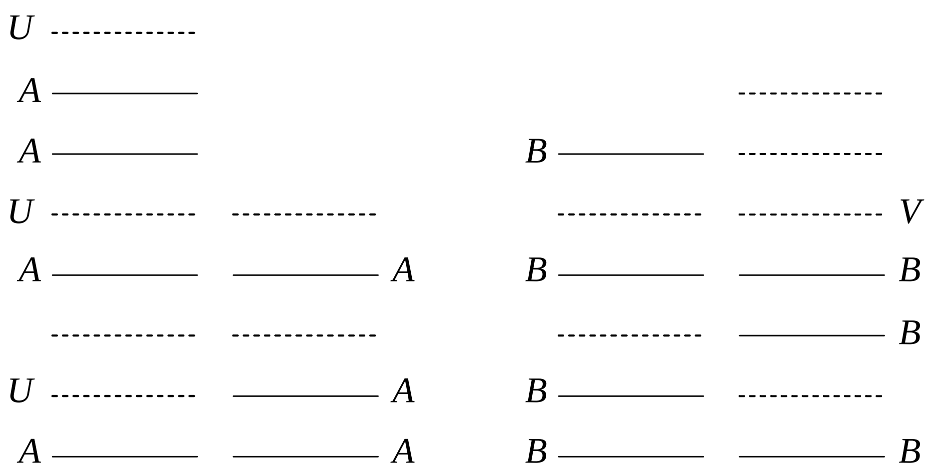
We can do the same for (Y,S).



(B,S_B) (Y,S)

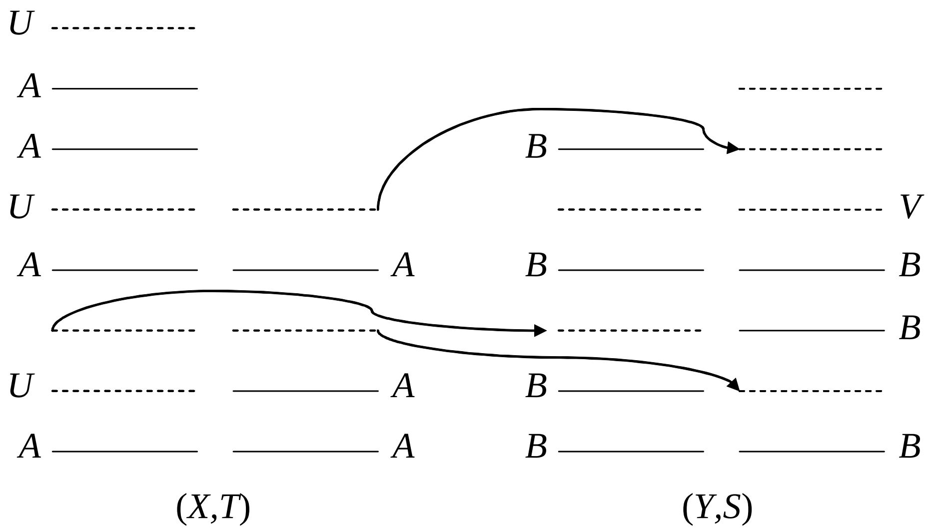
The results are tower partitions τ (for T) and σ (for S) with corresponding columns such that for each i, φ maps the base of the ith column of τ to the base of the ith column of σ. Further, the number of occurrences of A in the ith column of τ is equal to the number of occurrences of B in the ith column of σ.

Some of the floors from A^C in the tower partition τ are subsets of U and similarly some of the floors from B^C in σ are subsets of V. We add labels to these floors in our example.



(X,T) (Y,S)

By the inequality (1), we have that the number of floors in the ith column of τ which are subsets of $A^C \setminus U$ is less than the number of floors in the ith column of σ which are subsets of $B^C \setminus V$ as is the case in the example above. So far we have $\psi = \varphi$ on A, we continue to define the map ψ as follows. Fix i and work in the ith column of τ. Let F be the floor of minimal height in the ith column of τ which is not in U and on which ψ is not yet defined. Suppose F is at height h. For $x \in F$, let $\psi(x) = S^n \varphi T^{-h}(x)$, where n is the minimal height floor of the ith column of σ which is not in V and which is not yet the ψ image of a floor from τ. Continue in this way until ψ is defined on all floors of all columns of τ not in U. This is depicted for our example below.



The condition on numbers of occurrences of U and V guarantees that we can complete this. We now have a continuous, orbit-preserving extension of φ, $\psi : X \setminus U \to Y \setminus V$. Note that $\psi(X \setminus U)$ is a proper subset of $Y \setminus V$, and that $1_{\varphi T^{-1}U} - 1_{Y \setminus im(\psi)}$ is an S-infinitesimal.

This is the beginning half-step of a recursive construction. We call it a half-step because we next work with the roles of S and T reversed, and define ψ^{-1} on more of the space Y. Further the set A is now replaced by $X \setminus U$ as ψ is defined on all of $X \setminus U$ and B by $im\,(\psi)$, the set upon which ψ^{-1} is defined. The infinitesimal $r_A - r_B\varphi$, is replaced by the infinitesimal $1_{Y \setminus im(\psi)} - 1_{\varphi T^{-1}U}$.

To be specific, let us set $U_1 = U$, $V_1 = V$. Define a clopen set $V_2 \subset V_1$ containing y_1 with diameter less than δ_2. Similarly, let $U_2 \subset U_1$ be a clopen set containing x_1 of diameter less than δ_2 such that $\varphi T^{-1} U_2 \subsetneqq S^{-m} V_2$.

Construct σ_2, a tower partition for S over y_0 refining the first tower partition σ where 1_{V_2} and $1_{\varphi T^{-1}U_2}$ are constant on floors and where every column is of sufficiently large height to insure that the column sum of $1_{V_2} - 1_{\varphi T^{-1}U_2}$ is greater than $1_{Y \setminus im(\psi)} - 1_{\varphi T^{-1}U}$. Create the corresponding tower partition τ_2 for T by first inducing the tower partitions onto B, copying via the conjugacy φ to a T_A tower partition, then expanding to a T tower partition. The inequalities above guarantees that in the ith column of σ_2, the number of floors in $(Y \setminus im\,(\psi)) \cap (Y \setminus V_2)$ is less than the number of floors in the ith column of τ_2 which are subsets of $U_1 \setminus U_2$. Define ψ^{-1} as before, and continue.

This defines $\psi : X \setminus \{x_1\} \to Y \setminus \{y_1\}$ as a continuous, orbit-preserving surjection. Set $\psi\,(x_1) = y_1$. That ψ is continuous at x_1 is because given any $\varepsilon > 0$ there is a set V_n with diameter less than ε and a clopen set U_{n+1} around x_1 which guarantees that $\psi\,(x_1) \subset V_n$. That ψ is an orbit equivalence is by definition.

Let us consider which points are potential discontinuities for the cocycle function α where $\psi T\,(x) = S^{\alpha(x)} \psi\,(x)$. If $x \notin \{x_0, x_1\}$ then for some n, $\{x, Tx\} \cap U_n = \emptyset$. The way that ψ was defined, $\psi\,(x) = S^k \varphi T^{-h}\,(x)$ for some k, h. Further, $\psi = S^k \varphi T^{-h}$ on the floor of τ_n containing x. It is also true that $\psi = S^i \varphi T^{-j}$ for some i, j on the floor of τ_n containing Tx. Both $T^{-h}\,(x)$ and $T^{-j+1}\,(x)$ are in A. Therefore, $\varphi T^{-h}\,(x) = S^l \varphi T^{-j+1}\,(x)$ for some $l > 0$, and further that $\varphi T^{-h} = S^l \varphi T^{-j+1}$ on the floor of τ_n containing x. From this it follows that on the floor of τ_n containing x, we have

$$\begin{aligned}\psi T &= S^i \varphi T^{-j+1} \\ &= S^{i-l} \varphi T^{-h} \\ &= S^{i-l-k} \psi(x),\end{aligned}$$

which means that the cocycle α is continuous on $X \setminus \{x_0, x_1\}$. A similar argument shows that the cocycle β which satisfies

$$\psi T^{\beta(x)} = S\psi(x)$$

is continuous at all points in $Y \setminus \{\psi(x_1), S^{-1}\psi(x_1)\}$. □

3. Distortion Function

We now investigate the relationship of flow-orbit equivalence to the distortion function D as defined here. Let (X,T), (Y,S) be minimal Cantor systems. Let $\varphi : X \to Y$ be an orbit equivalence. Then we define

$$\begin{aligned}D_n(x,T,S,\varphi) &= \frac{|\varphi\{x,Tx,\dots,T^{n-1}x\} \triangle \{\varphi(x), S\varphi(x), \dots, S^{n-1}\varphi(x)\}|}{2n} \\ &= \frac{|\varphi\{x,Tx,\dots,T^{n-1}x\} \setminus \{\varphi(x), S\varphi(x), \dots, S^{n-1}\varphi(x)\}|}{n} \\ &= \frac{|\{\varphi(x), S\varphi(x), \dots, S^{n-1}\varphi(x)\} \setminus \varphi\{x,Tx,\dots,T^{n-1}x\}|}{n}.\end{aligned}$$

Define

$$D(x,T,S,\varphi) = \overline{\lim_{n\to\infty}} D_n(x,T,S,\varphi).$$

We establish some basic facts about the function D. Let us assume for the moment that φ, T and S are fixed, and suppress them in the notation.

Proposition 3.1. *$D(Tx) = D(x)$.*

Proof. There is an m such that $\varphi(Tx) = S^m\varphi(x)$. Fix $n > m$. Now compare

$$D_n(x) = \frac{|\varphi\{x,Tx,\dots,T^{n-1}x\} \triangle \{\varphi(x), S\varphi(x), \dots, S^{n-1}\varphi(x)\}|}{2n}$$

and

$$\begin{aligned}D_{n-m}(Tx) &= \frac{|\varphi\{Tx,\dots,T^{n-m}x\} \triangle \{\varphi(Tx), S\varphi(Tx), \dots, S^{n-m-1}\varphi(Tx)\}|}{2(n-m)} \\ &= \frac{|\varphi\{Tx,\dots,T^{n-m}x\} \triangle \{S^m\varphi(x), S^{m+1}\varphi(x), \dots, S^{n-1}\varphi(x)\}|}{2(n-m)}.\end{aligned}$$

So the numerators and denominators of the two terms differ by at most $2|m|$, which makes no difference to the limit as $n \to \infty$. □

Proposition 3.2. *D_n is lower semi-continuous.*

Proof. Fix $n > 1$. For $x \in X$, let $I(x)$ be the set of indices $i \in \{0, 1, \ldots, n-1\}$ such that $\varphi(T^i x) = S^j x$ for $j \in \{0, 1, \ldots, n-1\}$. Now $D_n(x) = 1 - |I(x)|/n$. There are only $n!$ possible distinct sets $I_1, I_2, \ldots, I_p$ where $I_k = I(x)$ for some point x. Furthermore, among all x with $I(x) = I_k$ there are only finitely many different functions $f_k^x : I_k \to \{0, 1, \ldots, n-1\}$ defined by $f_k^x(i) = j$ where j satisfies $S^j x = T^i x$.

Suppose $x_l \to z$, and $D_n(z) = d$. Choose a subsequence $\{x_{l(m)}\}$ of $\{x_l\}$ such that

(1) $\lim_{m\to\infty} D_n\left(x_{l(m)}\right) = \underline{\lim}_{l\to\infty} D_n(x_l)$
(2) there is a k such that $I\left(x_{l(m)}\right) = I_k$ for all l and
(3) all functions $f_k^{x_{l(m)}}$ are the same, say equal to f.

The set of x such that $\varphi(T^i x) = S^{f(i)}\varphi(x)$ for all $i \in I(x)$ is closed. Therefore, for every $i \in I_k$, $\varphi(T^i z) = S^{f(i)}\varphi(z)$. This implies that $|I(z)| \geq \left|I\left(x_{l(m)}\right)\right|$ for all m, and that $D_n(z) \leq \lim_{m\to\infty} D_n\left(x_{l(m)}\right) = \underline{\lim}_{l\to\infty} D_n(x_l)$. □

Proposition 3.3. *The set* $A_n^m = \left\{x : \forall N \geq n,\ D_N(x) \leq \frac{1}{m}\right\}$ *is closed.*

Proof. It follows from the lower semi-discontinuity of D_N that for all N, the set $\left\{x : D_N(x) \leq \frac{1}{m}\right\}$ is closed. Therefore, our set A_n^m is a countable intersection of closed sets. □

Proposition 3.4. *The set* $\{x : D(x) = 0\}$ *is an* $F_{\sigma\delta}$.

Proof. This follows from the previous proposition and the equation

$$\{x : D(x) = 0\} = \cap_{m\geq 1} \cup_{n\geq 1} A_n^m.$$

□

The following is one direction of our main result on the distortion function.

Theorem 3.5. *If T and S are orbit equivalent by φ with $D(\cdot, \varphi, T, S) \equiv 0$ then φ is a flow-orbit equivalence from T to S. In fact, if $D(x, \varphi, T, S) < \frac{1}{4}$ for all $x \in X$, then φ is a flow-orbit equivalence from T to S.*

Proof. Choose $m \geq 4$ and recall that $A_n^m = \left\{x : \forall N \geq n,\ D_N(x) \leq \frac{1}{m}\right\}$. Since $D(x) < \frac{1}{4}$ for all $x \in X$, we have that $X = \cup_{n\geq 1} A_n^m$, and each of the sets A_n^m is closed. Therefore, by the Baire Category Theorem, there is an $n \geq 1$ such that A_n^m contains a clopen set U. Let A be a clopen subset of U with the property that no point in A returns to A before time n. We wish to show that φ is a conjugacy between the induced maps T_A and $S_{\varphi(A)}$.

Fix $x \in A$, and set $r > 0$ as the smallest positive integer such that $T^r x \in A$. Let q be such that $\varphi(T^r x) = S^q \varphi(x)$. We will be done if we can show that $q > 0$. Since $r > n$, we know that

$$\frac{\left|\varphi\{x, Tx, \ldots, T^{r-1}x\} \vartriangle \{\varphi(x), S\varphi(x), \ldots, S^{r-1}\varphi(x)\}\right|}{2r} < \frac{1}{4}. \tag{2}$$

Notice that we also have

$$\frac{\left|\varphi\{T^r x, T^{r+1}x, \ldots, T^{2r-1}x\} \vartriangle \{S^q\varphi(x), S^{q+1}\varphi(x), \ldots, S^{q+r-1}\varphi(x)\}\right|}{2r} < \frac{1}{4} \tag{3}$$

and

$$\frac{\left|\varphi\left\{x, Tx, \ldots, T^{2r-1}x\right\} \vartriangle \left\{\varphi\left(x\right), S\varphi\left(x\right), \ldots, S^{2r-1}\varphi\left(x\right)\right\}\right|}{4r} < \frac{1}{4}. \tag{4}$$

It follows from the last inequality that

$$\frac{\left|\left\{\varphi\left(x\right), S\varphi\left(x\right), \ldots, S^{2r-1}\varphi\left(x\right)\right\} \setminus \varphi\left\{x, Tx, \ldots, T^{2r-1}x\right\}\right|}{2r} < \frac{1}{4}. \tag{5}$$

But if $q < 0$ then from inequalities 2 and 3 it follows that

$$\left|\left\{S^{r}\varphi\left(x\right), S^{r+1}\varphi\left(x\right), \ldots, S^{2r-1}\varphi\left(x\right)\right\} \setminus \varphi\left\{x, Tx, \ldots, T^{2r-1}x\right\}\right| > r - \frac{r}{2} = \frac{r}{2}$$

and therefore,

$$\left|\left\{\varphi\left(x\right), S\varphi\left(x\right), \ldots, S^{2r-1}\varphi\left(x\right)\right\} \setminus \varphi\left\{x, Tx, \ldots, T^{2r-1}x\right\}\right| > \frac{r}{2}.$$

But this contradicts inequality 5. □

We now observe in the example that follows that the converse of Theorem 3.5 does not hold in general.

3.1. **Example.** Here we give an example of two minimal actions T and S which are flow-orbit equivalent by the identity map, but where $D\left(\cdot, T, S, id\right)$ is not equivalently zero. We note that it follows from the ergodic theoretic results that D must be zero μ-almost everywhere for any invariant Borel probability measure $\mu \in \mathcal{M}_T$ [11].

Consider the dyadic odometer. Here, $X = \{0,1\}^{\mathbf{N}}$, and the transformation T is addition of 1 with carry to the right. I.e., for $x_1x_2x_3\ldots \neq 111...$, set $N = \max\left\{n : x_i = 1 \text{ for } i \leq n\right\}$. Define $T\left(x_1x_2x_3\ldots\right) = y_1y_2y_3\ldots$ where $y_i = 0$ for $i \leq N$, $y_{N+1} = 1$, $y_n = x_n$ for $n > N+1$. Define $T\left(111\ldots\right) = 000\ldots$. With the usual topology (product of discrete), this is a minimal Cantor system.

Create sequences of positive integers $n(i), m(i)$ recursively as follows. Set $n(1) = 3$ and $m(1) = 5$. Note that $m(1)2^{-n(1)} > \frac{1}{2}$. Given $n(i)$, $m(i)$, with $m\left(i\right)2^{-n(i)} > \frac{1}{2}$, define

- $n(i+1)$ such that $2m(i)2^{-n(i+1)} < m(i)2^{-n(i)} - \frac{1}{2}$
- $m\left(i+1\right) = m\left(i\right)2^{n(i+1)-n(i)} - 2m\left(i\right)$

Then it follows that $m\left(i+1\right)2^{-n(i+1)} = m\left(i\right)2^{-n(i)} - 2m\left(i\right)2^{-n(i+1)} > \frac{1}{2}$, and we can continue recursively.

For the construction of φ, begin with a tower of height $8 = 2^{n(1)}$. Select $m\left(1\right) = 5$ words of length $n\left(1\right)$: $w_1\left(1\right) = 001$, $w_1\left(2\right) = 010$, $w_1\left(3\right) = 100$, $w_1\left(4\right) = 101$, $w_1\left(5\right) = 110$. Let $A\left(i\right)$ be the cylinder set of $w_1\left(i\right)$.

Now consider words of length $n\left(2\right)$. For each word $w_1\left(i\right)$ of length $n\left(1\right)$, create two words of length $n\left(2\right)$, one by concatenating $w_1\left(i\right)$ with $\left(n\left(2\right) - n\left(1\right)\right)$ 0's to obtain $u_1\left(i\right) = w_1\left(i\right)00\ldots00$ and one by concatenating $w_1\left(i\right)$ with $\left(n\left(2\right) - n\left(1\right)\right)$ 0's and one 1 to obtain $v_1\left(i\right) = w_1\left(i\right)00\ldots01$. We define φ on the cylinder set of these two words as follows. If $x_1x_2\ldots x_{n(2)} = u_1\left(i\right)$ or $v_1\left(i\right)$, set $\varphi\left(x\right) = x_1x_2\ldots x_{n(2)-1}\left(x_{n(2)} + 1\right)x_{n(2)+1}x_{n(2)+2}\ldots$ where addition is mod 2. Now consider the collection of all words of length $n\left(2\right)$ which are extensions of the words $w_1\left(i\right)$ and are not equal to $u_1\left(i\right)$ or $v_1\left(i\right)$. There are exactly $m\left(2\right)$ such words. We can enumerate them as $w_2\left(1\right), w_2\left(2\right), \ldots w_2\left(m\left(2\right)\right)$.

Now work with words of length $n(3)$. Set $u_2(i) = w_2(i)\,00\ldots00$, and $v_2(i) = w_2(i)\,00\ldots01$, and if $x_1 \ldots x_{n(3)} = u_2(i)$ or $v_2(i)$, define $\varphi(x)$ by adding 1 to the $n(3)$ coordinate mod 2. Continue in this way, and set $S = \varphi T \varphi = \varphi T \varphi^{-1}$.

Let $x = 0000\ldots$. Consider the set $B = \left\{x, Tx, \ldots, T^{2^{n(j)}-1}x\right\}$. For all points in this set, the digits $n(j)+1$ through $n(j+1)$ are all equal to 0, and for each word w of length $n(j)$, there is precisely one element of this set $y \in B$ with $y_1 \ldots y_{n(j)} = w$. If $w = w_j(i)$ for some i, then $y \notin \left\{x, Sx, \ldots, S^{2^{n(j)}-1}x\right\}$. There are $m(j)$ such words, therefore,

$$\left|\left\{x, Tx, \ldots, T^{2^{n(j)}-1}x\right\} \triangle \left\{x, Sx, \ldots, S^{2^{n(j)}-1}x\right\}\right| > 2m(j) > 2^{n(j)}.$$

But this implies that $D(x) \geq \frac{1}{2}$.

Note that for the clopen set $C = \left(\cup_{i=1}^{5} A(i)\right)^c$, the set of all sequences which do not begin with any of the words $w_1(i)$. We have that $T_C = S_C$, which gives the flow-orbit equivalence of T and S.

Theorem 3.6. *Given any $\varepsilon > 0$, there is a flow-orbit equivalence with $D(x) < \varepsilon$ for all $x \in X$, but D not equivalently* 0.

Proof. To prove this, simply construct a tower of height greater than ε^{-1} over the dyadic odometer, and define φ as in the example, but equal to the identity map on any new sets in the tower. □

So although the converse of Theorem 3.5 does not hold, we note that the above differs from the kinds of flow-orbit equivalences we constructed previously. In particular, we showed that any flow-orbit equivalence can be achieved so that there are only two points of discontinuity for the associated cocycles. One can check that for the above example the set of discontinuities for the cocycle is a set of positive measure. We show in the next section, that this is no coincidence. In Theorem 3.11, we show that the only way to obtain a flow-orbit equivalence with $D \neq 0$ is to have a positive measure set of discontinuities for the cocycle.

3.2. **Distortion and discontinuities of the cocycle.**

Next we begin to relate properties of distortion function D to the measure(s) of the set of discontinuities of the cocycle function, or more specifically, if $\varphi T(x) = S^{\alpha(x)}\varphi(x)$, set

$$\operatorname{md}(\varphi, T, S) = \sup_{\mu \in \mathcal{M}_T} \mu\{x : \alpha \text{ is not continuous at } x\}$$

(md here stands for measure of discontinuities).

Remark 3.7. *Boyle showed that if α is continuous then T and S are flip conjugate (T is conjugate to S or S^{-1})* [2]*, see also* [6]*.*

We begin with some simple properties related to the quantity md.

Proposition 3.8. *Suppose T and S are minimal systems with the same orbits, so $T(x) = S^{\alpha(x)}(x)$ for all x. The set of $\{x : \alpha$ is not continuous at $x\}$ is a closed set with empty interior.*

Proof. Let $F_j = \{x : \alpha(x) = j\}$ and let E be the set of points at which n is not continuous. To say that α is continuous at x is to say that x is in the interior of an F_j. So E is equal to the complement of the union of the interiors of the F_j. In particular, E is closed. The Baire Category Theorem tells us that if E has interior, then $E \cap F_j$ has interior for some j, but this is a contradiction. □

Proposition 3.9. *Let E be a closed set and let $\mu \in \mathcal{M}_T$ with $\mu(E) = 0$. Suppose $E = \cap_{n\geq 1} K_n$ where K_n are clopen and $K_n \supset K_{n+1}$. Then given any $\varepsilon > 0$, there is an N such that $\mu(K_N) < \varepsilon$.*

Proof. Note that E^c is the countable union of disjoint clopen sets $C_n = (K_n)^c \setminus (K_{n-1})^c = (K_n)^c \cap K_{n-1}$. Therefore, $\mu(E^c) = 1 = \sum_{n=1}^{\infty} \mu(C_n)$, which means that there is an N such that $\sum_{n=N+1}^{\infty} \mu(C_n) < \varepsilon$ which implies that for $n \geq N$, $\mu(K_n) < \varepsilon$. □

Proposition 3.10. *Let E be a closed set with universal measure zero and suppose $E = \cap_{n\geq 1} K_n$ where K_n are clopen and $K_n \supset K_{n+1}$. Then given any $\varepsilon > 0$, there is an N such that $\mu(K_N) < \varepsilon$ for all $\mu \in \mathcal{M}_T$.*

Proof. Suppose not. Then there is an $\varepsilon > 0$ such that for all N, there is a μ_N with $\mu_N(K_N) \geq \varepsilon$. Since $\mathcal{M}_T$ is compact, we may assume that the μ_N converge to μ. Since $\mu(E) = 0$, there is an m such that $\mu(K_m) < \varepsilon/3$. Furthermore, since $\mu_N \to \mu$, there is an $N > m$ such that $|\mu_N(K_m) - \mu(K_m)| < \varepsilon/3$. Since $N > m$, $K_N \supset K_m$, so $\mu_N(K_m) > \mu_N(K_N) \geq \varepsilon$. But this puts $\mu(K_m) > 2\varepsilon/3$, a contradiction. □

Below is the main theorem relating md to D for a flow-orbit equivalence.

Theorem 3.11. *Let (X, T) and (Y, S) be minimal Cantor systems which are flow-orbit equivalent by a homeomorphism $\varphi : X \to Y$. If $\operatorname{md}(\varphi, T, S) = 0$ then $D(\varphi) \equiv 0$.*

Proof. First, let us replace S by $\varphi^{-1} S \varphi$ so that S and T are orbit equivalent by the identity map. Let α be the cocycle function associated with S and T, that is, $T(x) = S^{\alpha(x)}(x)$ for all x. First note that by the above propositions, given any $\varepsilon > 0$, there is a clopen set $U = U(\varepsilon)$ containing E, the set of discontinuities of the cocycle, with $\mu(U) < \varepsilon$ for all $\mu \in \mathcal{M}_T$. Note that the cocycle function is bounded off of this set U. This is because otherwise, there is an infinite sequence of points $x_k \in U^c$ with $|\alpha(x_k)| \to \infty$. But any such sequence of points has a convergent subsequence which converges to a point x, which is automatically a point of discontinuity for α. But U is open, so $x \notin U$, a contradiction.

Now suppose there is a point x such that $D(x) > 0$, i.e., $\overline{\lim} D_n(x) = d > 0$. The map φ is a conjugacy for some induced systems T_A and S_A. Since D is T-invariant, without loss of generality, $x \in A$. For $i \geq 0$, let $r(i)$ denote the time of the ith return of x to the set A under the map T and let $p(i)$ denote the time of the ith return of x to A under the map S. We know that given any $\varepsilon > 0$, for all k sufficiently large, $|r(k) - p(k)| < \varepsilon k$ (Lemma 2.9 and Proposition 2.15).

Let $R = \max_{x\in X} r_A(x)$. Choose $0 < \delta < \min\left\{\frac{d}{12}, \frac{d}{3R}\right\}$. Fix a clopen set U around the set of discontinuities with $\mu(U) < \delta$, and fix N such that N is an upper bound for $\{|\alpha(x)| : x \in U^c\}$. Pick m_0 such that $m > m_0$ guarantees that $|r(m) - p(m)| < \frac{\delta}{2} m$. Choose $n > \max\left\{\frac{2N}{R\delta}, \frac{m_0}{\delta}\right\}$ with the property that $D_n(x) > \frac{2}{3} d$.

Since $D_n(x) > \frac{2}{3}d$, we have

$$\left\{T^i x : 0 \le i \le n-1\right\} \setminus \left\{S^i x : 0 \le i \le n-1\right\} > \left(\frac{2}{3}d\right) n$$

and further,

$$\left\{T^i x : \delta n < i < (1-\delta) n\right\} \setminus \left\{S^i x : 0 \le i \le n-1\right\} > \left(\frac{2}{3}d\right) n - 2\delta n > \left(\frac{d}{2}\right) n.$$

Let $T^j x \in \{T^i x : \delta n < i < (1-\delta) n\} \setminus \{S^i x : 0 \le i \le n-1\}$. Then j is in one of the return intervals $r(i) < j < r(i+1)$. Since $T^{r(i)} x = S^{p(i)} x$, and $\frac{\delta}{2} n < p(i) < \left(1 - \frac{\delta}{2}\right) n$. This means that $\left|\sum_{k=r(i)}^{j-1} \alpha\left(T^k x\right)\right| > \frac{\delta}{2} n$, which means that one of the points $\left\{T^k x : r(i) \le k < j\right\}$ has $\left|\alpha\left(T^k x\right)\right| > N$.

Thus for each return interval $(r(i), r(i+1))$ which contains a j such that

$$T^j x \in \left\{T^i x : \delta n < i < (1-\delta) n\right\} \setminus \left\{S^i x : 0 \le i \le n-1\right\}$$

we obtain a point $T^k x$ such that $\left|\alpha\left(T^k x\right)\right| > N$, i.e. a point in U. There are at least $\left(\frac{d}{2}\right) n$ points in the set

$$\left\{T^i x : \delta n < i < (1-\delta) n\right\} \setminus \left\{S^i x : 0 \le i \le n-1\right\}.$$

Therefore, there are at least $\left(\frac{d}{2R}\right) n$ return intervals which contain such a point. This means that there are at least $\left(\frac{d}{2R}\right) n$ points in the set $\{T^i x : 0 \le i < n\}$ which are in the set U.

So whenever $n > \max\{2N/\delta, m_0/2\}$ with the property that $D_n(x) > \frac{2}{3}d$, we have

$$\#\left\{T^i x : 0 \le i < n\right\} \cap U \ge \left(\frac{d}{2R}\right) n > \left(\frac{3}{2}\delta\right) n.$$

Since $\overline{\lim_{n\to\infty}} D_n(x) = d$, there is an infinite sequence of such values of n. However, we know that $\mu U < \delta$ for all $\mu \in \mathcal{M}_T$ which means that

$$\overline{\lim_{n\to\infty}} \frac{\#\{T^i x : 0 \le i < n\} \cap U}{n} < \delta,$$

a contradiction. $\square$

The upshot of the above is that if we work within the category of orbit equivalences with md $= 0$, then the converse holds, that is, that if φ is a flow-orbit equivalence from S to T then $D = 0$. Since we know that flow-orbit equivalences can be achieved so that φ has only two points of discontinuity for the cocycle, clearly any two flow-orbit equivalent minimal Cantor systems are flow-orbit equivalent by a map with $D = 0$.

We take this one step further. We began this section with a definition of the functions D_n and D for two minimal Cantor maps T and S and an orbit equivalence φ from the T system to the S system. We have generally avoided the more cumbersome notation until now when we note

$$D(x, T, S, \varphi) = D\left(x, T, \varphi^{-1} S \varphi, id\right) = D\left(\varphi(x), \varphi T \varphi^{-1}, S, id\right).$$

Corollary 3.12. *Suppose T and S are minimal Cantor systems which are flow-orbit equivalent by a map φ. Then there is an element ψ of the full group of T such that $D\left(x, \psi T\psi^{-1}, S, \varphi\right) \equiv 0$.*

Proof. First let us consider $S' = \varphi^{-1}S\varphi$ and T which are flow-orbit equivalent by the identity map. Therefore, there is a clopen set A upon which $T_A = S'_A$. Apply Theorem 2.16 to obtain a flow-orbit equivalence ψ from T to S' which is equal to the identity map on A with at most two points of discontinuity for the cocycle. Since ψ is an orbit equivalence which is the identity on a nonempty clopen subset, ψ is in the full group of T. Therefore, we have

$$0 = D\left(x, T, \varphi^{-1}S\varphi, \psi\right) = D\left(\psi\left(x\right), \psi T\psi^{-1}, \varphi S\varphi^{-1}, id\right) = D\left(\psi\left(x\right), \psi T\psi^{-1}, S, \varphi\right),$$

but since this function is 0 everywhere, $D\left(x, \psi T\psi^{-1}, S, \varphi\right) = 0$. □

This shows that if we work "modulo the full group" any flow-orbit equivalence has $D = 0$, completing our version of a converse to Theorem 3.5.

We present the following theorem as a remark about strong flow-orbit equivalences, i.e. strong orbit equivalences which are simultaneously a flow equivalences. Recall that a strong orbit equivalence is one in which the cocycles relating both S to T and T to S have at most one point of discontinuity each.

Theorem 3.13. *Let (X, T) and (Y, S) be minimal Cantor systems. If T and S are flow-orbit equivalent by a strong orbit equivalence, then the cocycle function is continuous, and it follows that T and S are conjugate.*

Proof. Suppose that φ is a strong orbit equivalence from (X, T) to (Y, S) with cocycle α, and that A is a clopen subset of X such that $\varphi T_A = S_{\varphi(A)}\varphi$. Assume the function α has one point of discontinuity x_0. Let R be larger than the maximum T-return time to A for a point in A, and the maximum S-return time to $\varphi\left(A\right)$ for a point in $\varphi\left(A\right)$. Let $U \ni x_0$ be a clopen set with the property that the minimum return time to U for a point in U is $R+1$. In particular this means that if $x \in A$, and $j > 0$ is the smallest positive number such that $T^j\left(x\right) \in A$, then at most one of the points $\{x, Tx, \ldots T^{j-1}x\}$ is in U. Let $N = \max\left\{\left|\alpha\left(x\right)\right| : x \in U^c\right\} < \infty$.

Since x_0 is a point of discontinuity for α, there is a point $x \in U$ with $\left|\alpha\left(x\right)\right| > 2NR$. Let $k \geq 0$ be the minimal nonnegative number such that $T^{-k}\left(x\right) \in A$, and $j > 0$ the minimal positive number such that $T^j\left(x\right) \in A$. We know that $0 < k + j \leq R$ and that

$$0 < \sum_{i=-k}^{j-1} \alpha\left(T^i x\right) \leq NR$$

since the $k+j$ and $\sum_{i=-k}^{j-1} \alpha\left(T^i x\right)$ represent first return times to A and $\varphi(A)$, respectively. But now we have

$$\begin{aligned}\left|\sum_{i=-k}^{j-1} \alpha\left(T^i x\right)\right| &\geq \left|\alpha\left(T^j x\right)\right| - \left|\sum_{i=-k}^{-1} \alpha\left(T^i x\right) + \sum_{i=1}^{j-1} \alpha\left(T^i x\right)\right| \\ &\geq \left|\alpha\left(T^j x\right)\right| - \sum_{i=-k}^{-1} \left|\alpha\left(T^i x\right)\right| - \sum_{i=1}^{j-1} \left|\alpha\left(T^i x\right)\right| \\ &> 2NR - NR \\ &= NR,\end{aligned}$$

a contradiction. □

4. Example of Nonconjugate Flow-Orbit Equivalent Systems

To conclude, we present an example of two substitution systems which are flow-orbit equivalent, but not conjugate. In addition to showing that the above equivalence relation is not the same as conjugacy, this resolves in a negative way an open problem cited in [10]. There the authors show that two Sturmian systems which are both flow-equivalent and orbit equivalent are conjugate, and pose the question of whether this is true for general minimal Cantor systems, or for substitution systems.

Example 4.1. *Two substitution systems which are flow-orbit equivalent, but not strongly orbit equivalent.*

Consider the substitution rule σ below.

$$\begin{aligned}\sigma &: a \mapsto aabc \\ \sigma &: b \mapsto abc \\ \sigma &: c \mapsto bc\end{aligned}$$

One can check that the substitution above is primitive, proper and aperiodic and therefore defines a minimal Cantor system (Z,R) (see [28] for an introduction to substitution systems). The space Z is the subset of $\{a,b,c\}^{\mathbb{Z}}$ which consists of those sequences which can be uniquely written as concatenations of words $\sigma^k(a)$, $\sigma^k(b)$ and $\sigma^k(c)$ for any $k>0$, e.g.

$$z = \ldots abcbc.aabcaabcabcbc\ldots$$

The homeomorphism $R : Z \to Z$ is the shift map.

The substitution system (Z,R) is uniquely ergodic and the measure of a cylinder set corresponding to any of the three symbols can be read from the left Perron eigenvector of the adjacency matrix for the substitution

$$M_\sigma = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

One can check that $\left[\frac{1}{3},\frac{1}{3},\frac{1}{3}\right]$ is a left eigenvector for the Perron eigenvalue 3, and therefore, the cylinder sets corresponding to a, b, and c are all of equal measure.

We will create two systems (X,T) and (Y,S) for which (Z,R) is an induced system. Let X be the set of all sequences in the symbols $\{a,b,c,o\}$ which are obtained by taking a sequence in Z and inserting an o after every occurrence of a, e.g.

$$\ldots aobcbc.aoaobcaoaobcaobcbc\ldots$$

Similarly, let Y be the set of all sequences in the symbols $\{a,b,c,o\}$ which are obtained by taking a sequence in Z and inserting an o after every occurrence of b, e.g.

$$\ldots abocboc.aabocaabocabocboc\ldots$$

It is more or less clear that the system (Z,R) is an induced system of both S and T, conjugate to the return map on the clopen subset of points z with $z_0 \neq o$. In the system (X,T), the measure of the cylinder set corresponding to o is clearly equal to that of a. In the (Y,S) system, it is equal to that of b. But since both a and b have equal measure in Z, the cylinder set corresponding to o has equal measure in both X and Y. Then by Theorem 2.16, the systems are flow-orbit equivalent. That (X,T) and (Y,S) are substitution systems follows from [9, Corollary 2] which states that the class of substitution minimal Cantor systems is stable under flow equivalence.

Finally, we need to show that (X,T) and (Y,S) are not conjugate. This follows from the fact that the system (X,T) has a topological factor which is a periodic two point orbit and the system (Y,S) does not. That (X,T) has a period two factor is simple - replace the symbols a and b by 0 and c and o by 1 to obtain the factor. That (Y,S) has no such factor is a bit more complicated, but for example follows from the computation of the ordered group $\left(K^0(S), K^0(S)_+, [1_Y]\right)$. This computation can be carried out using the techniques of [9]. The result is that $K^0(S)$ is isomorphic to the subgroup of $\mathbb{Q}\times\mathbb{Z}$ which consists of elements of the form $\left(a3^{-k}, b\right)$ where $a,b\in\mathbb{Z}$, $a+b$ is even and $k\geq 0$. Via this isomorphism, the positive cone $K^0(S)_+$ corresponds to elements of the form $\left(a3^{-k}, b\right)$ where $a>0$ or where $a=b=0$. The order unit $[1_Y]$ corresponds to the element $(4,-2)$. As remarked in [17, Section 2], and proven explicitly in [24, Theorem 2.2], S has a periodic factor of period 2 if and only if there is an $[f]\in K^0(S)$ with $2[f]=[1_Y]$. There is no such element here since $(2,-1)$ is not in the group $K^0(S)$.

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