EXPLICIT CONSTRUCTIONS OF LOOPS WITH COMMUTING INNER MAPPINGS

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Abstract. In 2004, Csörgő constructed a loop of nilpotency class three with abelian group of inner mappings. Until now, no other examples were known. We construct many such loops from groups of nilpotency class two by replacing the product xy with xyh in certain positions, where h is a central involution. The location of the replacements is ultimately governed by a symmetric trilinear alternating form.

1. Introduction

As is well known, a group is of nilpotency class at most two if and only if its inner automorphism group is abelian. In 1946, Bruck published a long paper [1] that influenced the development of loop theory for decades, in which he observed that a loop of nilpotency class two possesses an abelian inner mapping group. This paper is concerned with the converse of Bruck's result.

While working on this problem in the early nineties, Kepka and Niemenmaa [11], [12] proved that a finite loop with abelian inner mapping group must be nilpotent. (Kepka later improved upon this result and showed that if the inner mapping group is abelian and finite, then the loop is nilpotent [7].) But they did not establish an upper bound on the nilpotency class of the loop, and, indeed, no such bound is presently known.

Many experts believed that the converse of Bruck's result holds, just as in the associative case. But in 2004 (the result was published in 2007), Csörgő [2] constructed a counterexample—a loop of nilpotency class three with an abelian group of inner mappings. Throughout the paper, we denote her loop by C.

Like Kepka and Niemenmaa, Csörgő has been using the technique of H-connected transversals of groups, and her counterexample is therefore fully embedded in group theory. She constructed a group G of order 8192 with a subgroup H of order 64 and a two-sided transversal A for H in G on which one can define a loop (of order 128) by $aH \cdot bH = cH$ if and only if $abH = cH$. As is acknowledged in [2], her goal was to prove the converse of Bruck's result, and she was gradually accumulating properties of a minimal counterexample so that its existence could be refuted. But, in a twist of events, she ended up constructing a counterexample.

Unfortunately, her approach did not lead to a general theory or construction method that would allow one to obtain additional examples, much less to decide how rare such examples are. Furthermore, the method of implicit construction by means of transversals makes even the task of explicitly formulating the loop operation somewhat nontrivial.

In this paper we deprive Csörgő's example of its solitary nature. We construct C in two different, explicit ways, and, more importantly, we show a general method that

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yields many other similar loops. The structure of the paper corresponds quite closely to the history of our investigation, and we shall now briefly describe both.

Using the GAP [6] package LOOPS [10], we have constructed the multiplication table of C, based on the description in [2], and determined the sizes of the nuclei and of the associator subloop. We were quite surprised that the latter consists of only two elements. This means that each nontrivial associator in C is equal to a central involution h. This led to an early conjecture that C can be obtained by the method of group table modifications, as used in our earlier work [3], [4], [5], [13]. More precisely, we conjectured that there exists a group—we shall denote it again by G—such that $C = (G, *)$, where $x * y \in \{xy, xyh\}$, for a fixed central involution $h \in G$.

We were now facing two tasks: to determine G, and to identify those pairs (x, y), where the group operation should be modified. It is to be expected that the modification is performed in a blockwise fashion, with (x', y') and (x, y) behaving in the same way whenever $x'K = xK$ and $y'K = yK$, for a (large) normal subgroup K of G.

Our first reconstruction of C has been obtained by different means, though. In Section 2, we develop the theory of nuclear extensions for loops, which allows us to give an explicit formula for C. Since we were not able to guess G from this formula, we applied a greedy algorithm (whose purpose was to maximize the number of associating triples) that resulted in another loop $\overline{C}$ with similar properties. The explicit formula for $\overline{C}$ is simple enough to connect it with a group G, which we demonstrate in Section 3.

By studying this single example $\overline{C}$, we developed a theory described in Section 4. The construction starts with a group G of nilpotency class at most three and produces a loop of nilpotency class three with abelian inner mapping group. At first we thought that the process cannot work if the starting group G is of nilpotency class two, and when we tried to refute this possibility we obtained some theorems that connect these groups to triadditive (or trilinear) mappings via the iterated commutator $[[-, -], -]$.

It turns out that the sought loops can be, in fact, obtained from groups G of nilpotency class two, cf. Section 5, and we were able to reconstruct the loop C in this way. In hindsight, our inability to do so in the first place got a natural explanation: the subgroup K that is used for blockwise modifications is of order 2.

The general construction (of obtaining loops from groups of nilpotency class two) has three steps. The first step is strictly governed by the associated group and the triadditive mapping. However, the second and the third steps depend on many free parameters, which results in a combinatorial explosion. Consequently, there are myriads (we do not know precisely how many) of loops of order 128, of nilpotency class three and with an abelian inner mapping group. The general construction is given in Section 5, and explicit examples are calculated in Section 6.

We conclude the paper with a list of open problems.

The paper relies heavily on machine computation, and all results not justified by theory have been checked computationally. The GAP code used here can be downloaded at `http://www.math.du.edu/~petr` in section Publications.

In the planned sequel, we shall explain why our method cannot work for orders less than 128, and why it cannot work for loops of odd order.

1.1. Background on loops.

A groupoid Q is a *loop* if the equations $ax = b$, $ya = b$ have unique solutions x, $y \in Q$ whenever a, $b \in Q$ are given, and if there is $1 \in Q$, the *neutral element* of Q, such that $a1 = a = 1a$ for every $a \in Q$. A nonempty subset S of a loop Q is a *subloop*, $S \leq Q$, if $1 \in S$ and S is a loop with respect to the multiplication inherited from Q. We say that $S \le Q$ is *normal* in Q, $S \trianglelefteq Q$, if $xS = Sx$, $x(yS) = (xy)S$, $S(xy) = (Sx)y$ for every x, $y \in Q$.

Every element x of a loop Q gives rise to two permutations of Q, the *left translation* $L_x : y \mapsto xy$, and the *right translation* $R_x : y \mapsto yx$. The *multiplication group* $\mathrm{Mlt}\, Q$ of Q is the group generated by $\{L_x,\ R_x;\ x \in Q\}$. The mappings

$$L(x, y) = L_{yx}^{-1} L_y L_x, \quad R(x, y) = R_{xy}^{-1} R_y R_x, \quad T(x) = R_x^{-1} L_x$$

are known as *left*, *right*, and *middle inner mappings*, respectively, and the *inner mapping group* $\mathrm{Inn}\, Q$ of Q is the group generated by all inner mappings of Q. In a complete analogy with groups, a subloop S of Q is normal in Q if and only if $\varphi(S) = S$ for every $\varphi \in \mathrm{Inn}\, Q$.

The *commutator* of x, $y \in Q$ is defined by $xy = yx \cdot [x, y]$, and the *associator* of x, y, $z \in Q$ is defined by $(xy)z = x(yz) \cdot [x, y, z]$. The *associator subloop* of Q is the smallest normal subloop $A(Q)$ of Q such that $Q/A(Q)$ is a group. In particular, $[x, y, z] \in A(Q)$ for every x, y, $z \in Q$.

The *left nucleus* $N_\lambda(Q)$ of a loop Q consists of all elements $x \in Q$ such that $[x, y, z] = 1$ for every y, $z \in Q$. Similarly, we have the *middle nucleus* $N_\mu(Q) = \{x \in Q;\ [y, x, z] = 1$ for every y, $z \in Q\}$, the *right nucleus* $N_\rho(Q) = \{x \in Q;\ [y, z, x] = 1$ for every y, $z \in Q\}$, and the *nucleus* $N(Q) = N_\lambda(Q) \cap N_\mu(Q) \cap N_\rho(Q)$. All nuclei are associative but not necessarily normal subloops of Q. The *center* $Z(Q)$ of Q consists of all elements $x \in N(Q)$ such that $[x, y] = 1$ for every $y \in Q$. It is then clear that $\varphi(Z(Q)) = Z(Q)$ for every $\varphi \in \mathrm{Inn}\, Q$, and hence $Z(Q) \trianglelefteq Q$.

When $S \trianglelefteq Q$, the *factor loop* Q/S is defined in the usual way. Given $Q = Q_0$, let $Q_{i+1} = Q_i/Z(Q_i)$. If there is $m \ge 0$ such that Q_m is trivial, we say that Q is *(centrally) nilpotent*, and if $m \ge 0$ is the least integer for which Q_m is trivial, we say that Q is of *nilpotency class* m.

2. Nuclear extensions

Let Q, K, F be loops. Then Q is an *extension* of K by F if $K \trianglelefteq Q$ and $Q/K \cong F$. Let us call an extension Q of K by F *nuclear* if K is an abelian group such that $K \le N(Q)$.

In this subsection we generalize group extensions by abelian groups to nuclear extensions of loops. We will need the following definitions:

A map $\theta : F \times F \to K$ is a *cocycle* if $\theta(x, 1) = \theta(1, x) = 1$ for every $x \in F$. Given a cocycle $\theta : F \times F \to K$ and a homomorphism $\varphi : F \to \mathrm{Aut}\, K$, $x \mapsto \varphi_x$, let $K \rtimes_\theta^\varphi F$ be the groupoid $(K \times F, \circ)$ defined by

$$(a, x) \circ (b, y) = (a\varphi_x(b)\theta(x, y),\ xy).$$

Here is the key observation (we write T_x for the inner mapping $T(x)$):

Lemma 2.1 (Leong, Theorem 3 of [9])**.** *Let Q be a loop with a normal subloop $K \le N(Q)$. For each $x \in Q$, define $\varphi_x = T_x|_K$. Then $\varphi_x \in \mathrm{Aut}\, K$, and the mapping $\varphi : Q \to \mathrm{Aut}\, K$, $x \mapsto \varphi_x$ is a homomorphism.*

Proof. First fix $a, b \in K$ and $x \in Q$. Since K is normal in Q, we have $\varphi_x(K) \le K \le N(Q)$. In particular, $T_x(ab) \cdot x = x \cdot ab = xa \cdot b = (T_x(a) \cdot x)b = T_x(a) \cdot xb = T_x(a)(T_x(b) \cdot x) = T_x(a)T_x(b) \cdot x$. Canceling x on the right then shows that φ_x is an automorphism of K.

Now fix $a \in K$ and $x, y \in Q$. Let $z = T_{xy}(a)$. By the first part, $z \in K$, and so $(zx)y = z(xy) = (xy)a = x(ya) = x(T_y(a) \cdot y) = xT_y(a) \cdot y$. Upon canceling y on the right, we get $zx = xT_y(a)$, i.e., $z = T_xT_y(a)$. Hence $\varphi_{xy} = \varphi_x\varphi_y$, as claimed. □

Theorem 2.2 (Nuclear extensions of loops)**.** *Let K be an abelian group and Q, F loops. Then the following conditions are equivalent:*

(i) *Q is an extension of K by F and $K \le N(Q)$,*
(ii) *$Q = K \rtimes_\theta^\varphi F$ for a cocycle $\theta : F \times F \to K$ and a homomorphism $\varphi : F \to \operatorname{Aut} K$.*

Proof. Let Q be an extension of K by F, $K \le N(Q)$. Let $\pi : Q \to Q/K = F$ be the natural projection, and let $\ell : F \to Q$ be such that $\ell(1) = 1$ and $\pi(\ell(x)) = x$ for every $x \in F$. Define $\varphi : F \to \operatorname{Aut} K$, $x \mapsto \varphi_x$, by $\varphi_x = T_{\ell(x)}|_K$. By Lemma 2.1, $\varphi_x \in \operatorname{Aut} K$ for every $x \in F$. We have $\pi(\ell(xy)) = xy = \pi(\ell(x))\pi(\ell(y)) = \pi(\ell(x)\ell(y))$, and thus for every x, $y \in F$ there is a unique $\theta(x, y) \in K$ such that $\ell(x)\ell(y) = \theta(x, y)\ell(xy)$.

Since $\ell(1) = 1$, we have $\theta(x, 1) = \theta(1, y) = 1$, and $\theta : F \times F \to K$ is a cocycle. Now, $\varphi_x\varphi_y = T_{\ell(x)}|_K \cdot T_{\ell(y)}|_K = T_{\ell(x)\ell(y)}|_K = T_{\theta(x,y)\ell(xy)}|_K = T_{\theta(x,y)}|_K \cdot \varphi_{xy}$, by Lemma 2.1. Since K is commutative, $T_k|_K = 1$ for every $k \in K$. Hence $\varphi_{xy} = \varphi_x\varphi_y$, and $\varphi : F \to \operatorname{Aut} K$ is a homomorphism.

By the definition of ℓ, for every $u \in Q$ there are uniquely determined $a \in K$, $x \in F$ such that $u = a\ell(x)$. Define $\psi : Q \to K \rtimes_\theta^\varphi F = (K \times F, \circ)$ by $\psi(u) = (a, x)$. It is clear that ψ is a bijection. Let $v = b\ell(y)$, with $b \in K$, $y \in F$. Then, on the one hand, $\psi(u) \circ \psi(v) = (a, x) \circ (b, y) = (a\varphi_x(b)\theta(x, y), xy)$. On the other hand, since $K \le N(Q)$, $uv = a\ell(x) \cdot b\ell(y) = a(\ell(x) \cdot b\ell(y)) = a(\ell(x)b \cdot \ell(y)) = a(T_{\ell(x)}(b)\ell(x) \cdot \ell(y)) = a(\varphi_x(b)\ell(x) \cdot \ell(y)) = a(\varphi_x(b) \cdot \ell(x)\ell(y)) = a\varphi_x(b) \cdot \ell(x)\ell(y) = a\varphi_x(b) \cdot \theta(x, y)\ell(xy) = a\varphi_x(b)\theta(x, y) \cdot \ell(xy)$, and, consequently, $\psi(uv) = (a\varphi_x(b)\theta(x, y), xy)$. Thus ψ is an isomorphism.

Conversely, assume that $\theta : F \times F \to K$ is a cocycle, $\varphi : F \to \operatorname{Aut} K$ is a homomorphism, and $Q = K \rtimes_\theta^\varphi F$. Then $(1, 1) \circ (b, y) = (\varphi_1(b)\theta(1, y), y) = (b, y)$, and, similarly, $(a, x)(1, 1) = (a, x)$, showing that $(1, 1)$ is the neutral element of Q. Note that $(a, x) \circ (b, y) = (c, z)$ holds if and only if $a\varphi_x(b)\theta(x, y) = c$ and $xy = z$. Hence, if (a, x), (c, z) are given, there is a unique (b, y) satisfying $(a, x) \circ (b, y) = (c, z)$, namely: y is the unique solution to $xy = z$, and $b = \varphi_x^{-1}(a^{-1}c\theta(x, y)^{-1})$. Similarly, there is a unique solution (a, x) when (b, y), (c, z) are given. Altogether, Q is a loop.

We now show in detail that $K = (K, 1)$ is a subloop of $N(Q)$. First, $(a, 1) \circ ((b, y) \circ (c, z)) = (a, 1) \circ (b\varphi_y(c)\theta(y, z), yz) = (ab\varphi_y(c)\theta(y, z), yz)$, and $((a, 1) \circ (b, y)) \circ (c, z) = (ab, y) \circ (c, z) = (ab\varphi_y(c)\theta(y, z), yz)$. Second, $(b, y) \circ ((a, 1) \circ (c, z)) = (b, y) \circ (ac, z) = (b\varphi_y(ac)\theta(y, z), yz)$, $((b, y) \circ (a, 1)) \circ (c, z) = (b\varphi_y(a), y) \circ (c, z) = (b\varphi_y(a)\varphi_y(c)\theta(y, z), yz)$. As φ_y is a homomorphism, the two expressions coincide. Finally, $(b, y) \circ ((c, z) \circ (a, 1)) = (b, y) \circ (c\varphi_z(a), z) = (b\varphi_y(c\varphi_z(a))\theta(y, z), yz)$, and $((b, y) \circ (c, z)) \circ (a, 1) = (b\varphi_y(c)\theta(y, z), yz) \circ (a, 1) = (b\varphi_y(c)\theta(y, z)\varphi_{yz}(a), yz)$. As φ_y and φ are homomorphisms, the two expressions coincide.

We proceed to show that $K \trianglelefteq Q$. Since $K \le N(Q)$, we get for free that K is closed under all left and right inner mappings of Q. It suffices to show that $T_{(a,x)}(K) \subseteq K$ for every $a \in K$, $x \in F$. Now, $T_{(a,x)}(b, 1)$ belongs to K if and only if there is $c \in K$ such that $(a, x) \circ (b, 1) = (c, 1) \circ (a, x)$. Since $(a, x) \circ (b, 1) = (a\varphi_x(b), x)$ and $(c, 1) \circ (a, x) = (ca, x)$, it suffices to take $c = \varphi_x(b)$.

Finally, we must establish $Q/K \cong F$. But this is clear, since $(a, x) \circ (b, y) = (a\varphi_x(b)\theta(x, y), xy)$ and $(1, x) \circ (1, y)$, $(1, xy)$ coincide modulo K. □

2.1. **The first example.** The loop C from the Introduction has a normal nucleus isomorphic to the elementary abelian group of order 16 and such that $C/N(C)$ is an elementary abelian group of order 8. It is therefore a nuclear extension of $N(C)$ by $C/N(C)$.

Note that once it is known that a loop Q is a nuclear extension of $K \le N(Q)$ by Q/K, the proof of Theorem 2.2 is constructive and provides the action φ and the cocycle θ, as soon as the section mapping ℓ is chosen.

Hence, starting with a multiplication table for C obtained from the original construction of Csörgő, we can easily (with a computer) reconstruct C as follows:

Let $K = \langle a_1, a_2, a_3, a_4\rangle$ be an elementary abelian group of order 16, and $F = \langle x_1, x_2, x_3\rangle$ an elementary abelian group of order 8. Set $a = a_1a_2a_3$ and $b = a_4$. Define a homomorphism $\varphi : F \to \operatorname{Aut} K$ by

$$x_i \mapsto (a \leftrightarrow b,\ a_{i+1} \mapsto a_{i+1},\ a_{i+2} \mapsto a_{i+2}),$$

where the addition in the subscripts is modulo $\{1, 2, 3\}$. Define a cocycle $\theta : F \times F \to K$ by

	1	x_1	x_2	x_1x_2	x_3	x_1x_3	x_2x_3	$x_1x_2x_3$
1	1	1	1	1	1	1	1	1
x_1	1	1	1	1	a_2	a_2	aba_2	aba_2
x_2	1	a_3	1	a_3	a_1	aa_2	a_1	aa_2
x_1x_2	1	a_3	1	a_3	aa_3	a	a_3b	b
x_3	1	1	1	1	1	1	1	1
x_1x_3	1	1	1	1	a_2	a_2	aba_2	aba_2
x_2x_3	1	aba_3	1	aba_3	a_1	a_2b	a_1	a_2b
$x_1x_2x_3$	1	aba_3	1	aba_3	aa_3	b	a_3b	a

The resulting loop $K \rtimes_\theta^\varphi F$ is isomorphic to C.

Here are some properties of C: $N(C) = N_\rho(C)$ is elementary abelian of order 16, $|N_\lambda(C)| = |N_\mu(C)| = 32$, $Z(C) = A(C)$, $|Z(C)| = 2$. One can interpret the fact that $|A(C)| = 2$ as an indication that C is very close to a group, indeed.

3. Extensions by crosshomomorphisms

It is not clear how to deduce a general construction from a specific nuclear extension, such as that of Subsection 2.1.

As far as nuclei are concerned, a more symmetric loop $\overline{C}$ is obtained from C by a simple greedy algorithm. We were able to develop the general theory of Section 4 only after we understood the loop $\overline{C}$, and we therefore devote considerable attention to it here.

Given a groupoid Q, let $\mu(Q) = |\{(a, b, c) \in Q \times Q \times Q;\ a(bc) \ne (ab)c\}|$. Hence $\mu(Q)$ is a crude measure of (non)associativity of Q.

Let T be a multiplication table of C split into blocks of size 16×16 according to the cosets modulo $N(C)$. Let h be the unique nontrivial central element of C.

(*) For $1 < i < j \le 8$, let T_{ij} be obtained from T by multiplying the (i, j)th block and the (j, i)th block of T by h on the right. Let (s, t) be such that $\mu(T_{st})$ is minimal among all $\mu(T_{ij})$. If $\mu(T_{st}) \ge \mu(T)$, stop and return T. If $\mu(T_{st}) < \mu(T)$, replace T by T_{st}, and repeat (*).

It turns out that the multiplication table T found by the above greedy algorithm yields another loop $\overline{C}$ of nilpotency class 3 whose inner mapping group is abelian. In addition, the following properties hold for $\overline{C}$: $N(\overline{C})$ is elementary abelian of order 16, $|N_\lambda(\overline{C})| = |N_\mu(\overline{C})| = |N_\rho(\overline{C})| = 64$, $Z(\overline{C}) = A(\overline{C})$, $|Z(\overline{C})| = 2$. In particular, $\overline{C}$ is not isomorphic to C.

We are now going to construct $\overline{C}$ anew. First we construct a certain group $\overline{G}$, using a cocycle θ based on a crosshomomorphism. The loop $\overline{C}$ can then be obtained in two ways: upon using a slight variation of θ, or, equivalently, by replacing xy in $\overline{G}$ with xyh for certain pairs $(x,y)\in\overline{G}\times\overline{G}$, where h is a nontrivial central element of $\overline{G}$.

3.1. **Crosshomomorphisms.** Recall that if $(A,\cdot)$, $(B,+)$ are groups and $\varphi:A\to\operatorname{Aut}B$ is an action, then a mapping $\gamma:A\to B$ is a *crosshomomorphism* if $\gamma(xy)=\gamma(x)+\varphi_x\gamma(y)$.

Let F_1, F_2 be multiplicative groups, and K an additive abelian group. Let $\psi:F_1\to(\operatorname{End}K,+)$, $x\mapsto\psi_x$, and $\varphi:F_2\to(\operatorname{Aut}K,\circ)$, $y\mapsto\varphi_y$ be homomorphisms. Assume further that ψ and φ commute, i.e., $\psi_x\varphi_y=\varphi_y\psi_x$. Extend the action φ to $F=F_1\times F_2$ by $\varphi_{(x_1,x_2)}=\varphi_{x_2}$.

Let $\gamma:F_2\to K$ be a map satisfying $\gamma(1)=0$, and let $\theta:F\times F\to K$ be defined by

$$\theta((x_1,x_2),(y_1,y_2))=\psi_{y_1}\gamma(x_2). \tag{3.1}$$

Lemma 3.1. *Let $F=F_1\times F_2$, K, ψ, φ, γ and θ be as above. Then $K\rtimes_\theta^\varphi F$ is a group if and only if γ is a crosshomomorphism.*

Proof. Direct computation shows that $K\rtimes_\theta^\varphi F$ is a group if and only if $\theta(x,y)+\theta(xy,z)=\varphi_x\theta(y,z)+\theta(x,yz)$. Now,

$$\theta((x_1,x_2),(y_1,y_2))+\theta((x_1y_1,x_2y_2),(z_1,z_2))=\psi_{y_1}\gamma(x_2)+\psi_{z_1}\gamma(x_2y_2),$$

while

$$\begin{aligned}&\varphi_{(x_1,x_2)}\theta((y_1,y_2),(z_1,z_2))+\theta((x_1,x_2),(y_1z_1,y_2z_2))=\varphi_{x_2}\psi_{z_1}\gamma(y_2)+\psi_{y_1z_1}\gamma(x_2)\\&=\psi_{z_1}\varphi_{x_2}\gamma(y_2)+\psi_{y_1}\gamma(x_2)+\psi_{z_1}\gamma(x_2)=\psi_{z_1}(\varphi_{x_2}\gamma(y_2)+\gamma(x_2))+\psi_{y_1}\gamma(x_2).\end{aligned}$$

The two expressions coincide if and only if

$$\psi_{z_1}(\varphi_{x_2}\gamma(y_2)+\gamma(x_2))=\psi_{z_1}\gamma(x_2y_2). \tag{3.2}$$

If γ is a crosshomomorphism then $\gamma(x_2y_2)=\gamma(x_2)+\varphi_{x_2}\gamma(y_2)$, and (3.2) holds. Conversely, if (3.2) holds, use $z_1=1$ and the fact that ψ_1 is the identity on K to conclude that γ is a crosshomomorphism. ☐

Lemma 3.2. *Let $K=\{(a_0,a_1,a_2);\ 0\le a_i\le 1\}$ be a three-dimensional vector space over the two-element field. Let $V_4=\{b_1^{c_1}b_2^{c_2};\ 0\le c_i\le 1\}$ be the Klein group. Then $\varphi:V_4\to\operatorname{Aut}K$ defined by*

$$\varphi_{b_1^{c_1}b_2^{c_2}}(a_0,a_1,a_2)=(a_0+c_2a_1+c_1a_2,a_1,a_2)$$

is a homomorphism, and $\gamma:V_4\to K$ defined by

$$\gamma(b_1^{c_1}b_2^{c_2})=(c_1+c_2+c_1c_2,c_1,c_2)$$

is a crosshomomorphism.

Proof. It is easy to see that V_4 acts on K via φ. It remains to check that $\gamma(xy)=\gamma(x)+\varphi_x\gamma(y)$ for every x, $y\in V_4$. We have

$$\gamma(b_1^{c_1}b_2^{c_2}\cdot b_1^{d_1}b_2^{d_2})=\gamma(b_1^{c_1+d_1}b_2^{c_2+d_2})=(c_1+d_1+c_2+d_2+(c_1+d_1)(c_2+d_2),c_1+d_1,c_2+d_2),$$

while

$$\begin{aligned}&\gamma(b_1^{c_1}b_2^{c_2})+\varphi_{b_1^{c_1}b_2^{c_2}}\gamma(b_1^{d_1}b_2^{d_2})\\&=(c_1+c_2+c_1c_2,c_1,c_2)+\varphi_{b_1^{c_1}b_2^{c_2}}(d_1+d_2+d_1d_2,d_1,d_2)\\&=(c_1+c_2+c_1c_2,c_1,c_2)+(d_1+d_2+d_1d_2+c_2d_1+c_1d_2,d_1,d_2)\\&=(c_1+c_2+c_1c_2+d_1+d_2+d_1d_2+c_2d_1+c_1d_2,c_1+d_1,c_2+d_2).\end{aligned}$$

☐

F	$(\ell',i')=(0,0)$	$(\ell',i')=(0,1)$	$(\ell',i')=(1,0)$	$(\ell',i')=(1,1)$
$(\ell,i)=(0,0)$	1	1	1	1
$(\ell,i)=(0,1)$	1	1	h	h
$(\ell,i)=(1,0)$	1	1	1	1
$(\ell,i)=(1,1)$	1	1	h	h

FIGURE 1. Modifying the group $\overline{G}$ by $h=(1,0,0)\in K$ to obtain $\overline{C}$

Let $F_2=\langle\rho,\ \sigma;\ \rho^4=\sigma^2=(\sigma\rho)^2=1\rangle$ be the dihedral group of order 8. With $f_1=\sigma$ and $f_2=\sigma\rho$, we have $F_2=\langle f_1,f_2\rangle$, and every element of F_2 can be written uniquely as $(f_1f_2)^{2i}f_1^jf_2^k$, where $0\le i,\ j,\ k\le 1$. We have $Z(F_2)=\{1,f_1f_2\}$, and the projection $\pi: F_2\to F_2/Z(F_2)\cong V_4=\langle b_1,b_2\rangle$ is determined by $f_i\mapsto b_i$. Hence the action φ of V_4 on K from Lemma 3.2 can be extended to an action φ of F_2 on K via $\varphi_x=\varphi_{\pi(x)}$, and the crosshomomorphism $\gamma: V_4\to K$ from Lemma 3.2 can be extended into a crosshomomorphism $\gamma: F_2\to K$ by $\gamma(x)=\gamma(\pi(x))$.

Let $F_1=\{0,1\}$ be the two element field, and let $\psi: F_1\to \operatorname{End}K$ be the scalar multiplication. Extend φ once again into an action of $F=F_1\times F_2$ on K by $\varphi_{(x_1,x_2)}=\varphi_{x_2}$. Then $\psi_x\varphi_y=\varphi_y\psi_x$ (since ψ_x is either the zero map or the identity on K). Let us calculate the explicit formula for the cocycle θ associated with γ via (3.1):

$$\begin{aligned}\theta((\ell,(f_1f_2)^{2i}f_1^jf_2^k),\,(\ell',(f_1f_2)^{2i'}f_1^{j'}f_2^{k'}))&=\psi_{\ell'}\gamma((f_1f_2)^{2i}f_1^jf_2^k)=\psi_{\ell'}\gamma(b_1^jb_2^k)\\&=\psi_{\ell'}(j+k+jk,j,k)=(\ell'(j+k+jk),\ell'j,\ell'k).\end{aligned}$$

By Lemma 3.1, $\overline{G}=K\rtimes_\theta^\varphi F$ is a group (of nilpotency class three).

3.2. **The loop $\overline{C}$.** Upon modifying the cocycle θ slightly, we obtain a copy of $\overline{C}$ and other loops of nilpotency class three with commuting inner mappings.

For instance, define $\theta': F\times F\to K$ by

$$\theta'((\ell,(f_1f_2)^{2i}f_1^jf_2^k),\,(\ell',(f_1f_2)^{2i'}f_1^{j'}f_2^{k'}))=(\ell'(i+j+k+jk),\ell'j,\ell'k).$$

Then $K\rtimes_{\theta'}^\varphi F$ is isomorphic to $\overline{C}$.

Let $h=(1,0,0)\in K$. Upon comparing the cocycles θ and θ', it is now easy to describe $\overline{C}$ as a modification of $\overline{G}$, where the product xy is replaced by xyh on certain blocks modulo K, as indicated in Figure 1. In the figure, elements of $F=\overline{G}/K$ are labeled as above, and each cell represents a 4×4 block in the multiplication table of F.

To better understand the relationship between $\overline{G}$ and $\overline{C}$ we have considered additional variations of θ. For $t_i\in\{0,1\}$, $0\le i\le 6$, and $t=\sum_{i=0}^6 t_i2^i$, let

$$\begin{aligned}&\theta_t((\ell,(f_1f_2)^{2i}f_1^jf_2^k),\,(\ell',(f_1f_2)^{2i'}f_1^{j'}f_2^{k'}))\\&\qquad=(\ell'(t_0i+t_1j+t_2ij+t_3k+t_4ik+t_5jk+t_6ijk),\,\ell'j,\,\ell'k),\end{aligned}$$

Then $K\rtimes_{\theta_t}^\varphi F$ is a group if and only if $t\in\{32,34,40,42\}$, and all groups obtained in this way are isomorphic to $\overline{G}$. More importantly, $K\rtimes_{\theta_t}^\varphi F$ is a nonassociative loop of nilpotency class three with commuting inner mappings if and only if $t\in\{1,\ 3,\ 9,\ 11,\ 33,\ 35,\ 41,\ 43\}$, and all these loops are isomorphic to the loop $\overline{C}$.

Since the cocycle θ_1 is especially easy to describe, we present the construction of $\overline{C}$ from scratch in Figure 2. This is presently the shortest description of a loop of nilpotency class three with an abelian group of inner mappings.

$\mathbb{F}_2 = \{0, 1\}$ ... two-element field
$K = (\mathbb{F}_2)^3$ ... normal subgroup of Q
$D_8 = \langle \sigma, \rho; \sigma^2 = \rho^4 = (\sigma\rho)^2 = 1 \rangle$ ... dihedral group of order 8
$F = \mathbb{F}_2 \times D_8$ ... factor group Q/K
$\varphi : F \to \mathrm{Aut}\, K$ ... action
$\varphi_{(\ell,\, \rho^{2i}\sigma^j(\sigma\rho)^k)}(a, b, c) = (a + kb + jc,\, b,\, c)$
$\theta : F \times F \to K$ ... cocycle
$\theta((\ell,\, \rho^{2i}\sigma^j(\sigma\rho)^k),\, (\ell',\, \rho^{2i'}\sigma^{j'}(\sigma\rho)^{k'})) = (\ell' i,\, \ell' j,\, \ell' k)$
$(Q, \circ) = K \rtimes_\theta^\varphi F$ ... loop of nilpotency class three with abelian $\mathrm{Inn}\, Q$
$(a, x) \circ (b, y) = (a + \varphi_x(b) + \theta(x, y),\, xy)$

FIGURE 2. Construction of the loop $Q = \overline{C}$ as an extension of K by F

.

Finally, the cocycle

$$\theta''((\ell, (f_1 f_2)^{2i} f_1^j f_2^k),\, (\ell', (f_1 f_2)^{2i'} f_1^{j'} f_2^{k'})) = (\ell'(i + (k - k')j), \ell' j, \ell' k)$$

produces a loop of nilpotency class three with commuting inner mappings that is isomorphic neither to C nor to $\overline{C}$. Since we will eventually be able to construct many such examples, we do not pursue extensions any further.

3.3. **A power-associative loop that is the union of its nuclei.** Allow us to digress in this subsection.

A loop is *power-associative* if each of its elements generates a subgroup. The loop $\overline{C}$ contains a nonassociative power-associative subloop that is a union of its nuclei. Since we are not aware of such a loop appearing in the literature, we construct it here:

Let $K = \langle a_1, a_2, a_3, a_4 \rangle$ be an elementary abelian group of order 16, and $F = \langle x_1, x_2 \rangle$ an elementary abelian group of order 4. As above, let $a = a_1 a_2 a_3$ and $b = a_4$. Define a homomorphism $\varphi : F \to \mathrm{Aut}\, K$ by

$$x_i \mapsto (a \mapsto a,\, b \mapsto b,\, a_i \mapsto a_i,\, a_3 \mapsto aba_3),$$

and a cocycle $\theta : F \times F \to K$ by

	1	x_1	x_2	x_1x_2
1	1	1	1	1
x_1	1	a_1	$a_2 b$	a_3
x_2	1	aba_2	a_2	1
x_1x_2	1	aa_3	a	a_3.

Then $Q = K \rtimes_\theta^\varphi F$ is a nonassociative power-associative loop of order 64 such that $N_\lambda(Q) \cup N_\mu(Q) \cup N_\rho(Q) = Q$, $|N(Q)| = 16$, $|N_\lambda(Q)| = |N_\rho(Q)| = |N_\mu(Q)| = 32$.

The subloop $Q \leq \overline{C}$ was first spotted by Michael K. Kinyon.

4. GROUP MODIFICATIONS

The properties of all examples constructed so far were verified by direct machine computation. To remedy the situation, we now develop a theory based on group modifications that also yields loops of nilpotency class three with commuting inner mappings, but which does not require any machine computation.

4.1. **Conditions that make** $\operatorname{Inn} Q$ **abelian.** Our point of departure is based on the structural properties of the loops C and $\overline{C}$.

For the rest of this section, let G be a group, $Z \le K \le N \trianglelefteq G$, where N is abelian, G/N is abelian, $Z \le Z(G)$, $K \trianglelefteq G$, and $N/K \le Z(G/K)$. Furthermore, let $\mu : G/K \times G/K \to Z$ be a mapping satisfying $\mu(xK, K) = 1 = \mu(K, xK)$ for every $x \in G$.

Write $\mu(x, y)$ instead of $\mu(xK, yK)$, and define a groupoid $Q = (G, *)$ by

$$x * y = xy\mu(x, y). \tag{4.1}$$

Lemma 4.1. *Q is a loop.*

Proof. We have $x * 1 = x = 1 * x$ since $1 \in K$. Assume that $x * y = x * z$ for some x, y, $z \in G$. Then $xy\mu(x, y) = xz\mu(x, z)$, hence $z^{-1}y = \mu(x, z)\mu(x, y)^{-1} \in K$, and so $z = yk$ for some $k \in K$. Thus $xy\mu(x, y) = x * y = x * z = xyk\mu(x, yk) = xyk\mu(x, y)$, so $k = 1$ and $y = z$. Similarly, if $y * x = z * x$ then $y = z$.

Note that $\mu(x, y\mu(u, v)) = \mu(x, y)$. Then for x, $z \in G$, we have $x * x^{-1}z\mu(x, x^{-1}z)^{-1} = z$, so $y = x^{-1}z\mu(x, x^{-1}z)^{-1}$ is the unique solution to $x * y = z$. Similarly, given y, $z \in Q$, $x = zy^{-1}\mu(zy^{-1}, y)^{-1}$ is the unique solution to $x * y = z$. □

Lemma 4.2. *$Z \le Z(G) \cap Z(Q)$, and $G/Z \cong Q/Z$ is a group.*

Proof. Let $z \in Z \le Z(G) \cap K$, x, $y \in G$. Then

$$\begin{aligned}
&z * x = zx = xz = x * z,\\
&z * (x * y) = z(x * y) = zxy\mu(x, y) = zx * y = (z * x) * y,\\
&x * (z * y) = x * zy = xzy\mu(x, y) = xz * y = (x * z) * y,\\
&x * (y * z) = x * (z * y) = (x * z) * y = (z * x) * y = z * (x * y) = (x * y) * z.
\end{aligned}$$

Thus $Z \le Z(Q)$, and $G/Z \cong Q/Z$ follows by the definition (4.1). □

Since $Q/Z \cong G/Z$ is a group, we have $A(Q) \le Z$, and thus $A(Q) \le Z(Q) \le N(Q) \trianglelefteq Q$. Such a situation has some well-known general consequences. For instance, the associator $[x, y, z]$ depends only upon classes modulo $N(Q)$, by [8, Lemma 4.2]. We will use this property freely.

Denote by L_x, R_x the translations by x in Q, rather than in G. For convenience, allow us to redefine inner mappings for Q by

$$L(x, y) = L_y^{-1}L_x^{-1}L_{x*y}, \quad R(x, y) = R_{x*y}^{-1}R_yR_x, \quad T(x) = L_x^{-1}R_x.$$

Lemma 4.3. *We have $L(x, y)z = z[x, y, z]$ and $R(x, y)z = z[z, x, y]$. The subgroup $\langle L(x, y), R(x, y);\ x,\ y \in G\rangle$ is abelian.*

Proof. $L(x, y)z = z * [x, y, z]$ is equivalent to $(x * y) * z = x * (y * (z * [x, y, z]))$, which holds since $[x, y, z] \in Z \le Z(Q)$, by Lemma 4.2. Similarly, $R(x, y)z = z * [z, x, y]$ is equivalent to $(z * x) * y = (z * [z, x, y]) * (x * y)$, which holds for the same reason.

Then

$$L(x, y)L(u, v)z = L(x, y)(z[u, v, z]) = z[u, v, z][x, y, z[u, v, z]] = z[u, v, z][x, y, z], \tag{4.2}$$

and $L(u, v)L(x, y)z = z[x, y, z][u, v, z]$. Also,

$$R(x, y)R(u, v)z = R(x, y)(z[z, u, v]) = z[z, u, v][z[z, u, v], x, y] = z[z, u, v][z, x, y], \tag{4.3}$$

and $R(u, v)R(x, y)z = z[z, x, y][z, u, v]$. Finally,

$$L(x, y)R(u, v)z = L(x, y)(z[z, u, v]) = z[z, u, v][x, y, z[z, u, v]] = z[z, u, v][x, y, z],$$

and

$$R(u,v)L(x,y)z = R(u,v)(z[x,y,z]) = z[x,y,z][z[x,y,z],u,v] = z[x,y,z][z,u,v].$$

□

Let $y^x = x^{-1}yx$, and $[y,x] = y^{-1}y^x$. The following lemma shows how conjugations differ in G and Q:

Lemma 4.4. $T(x)y = y^x\mu(y,x)\mu(x,y^x)^{-1}$.

Proof. Note that $L_a^{-1}(b)$ is the unique solution y to the equation $a*y=b$. Thus, as we have showed in the proof of Lemma 4.1, $L_a^{-1}(b) = a^{-1}b\mu(a,a^{-1}b)^{-1}$. Then

$$T(x)y = L_x^{-1}R_x(y) = x^{-1}R_x(y)\mu(x,x^{-1}R_x(y))^{-1},$$

and we are done by $x^{-1}R_x(y) = x^{-1}(y*x) = x^{-1}yx\mu(y,x) = y^x\mu(y,x)$. □

Define $\delta : G/K \times G/K \to Z$ by

$$\delta(x,y) = \mu(x,y)\mu(y,x)^{-1}.$$

Consider these conditions on μ and δ, that we have observed while attempting to construct C and $\overline{C}$ by extensions:

$$\mu(xy,z) = \mu(x,z)\mu(y,z) \text{ if } \{x,y,z\}\cap N \neq \emptyset, \tag{4.4}$$

$$\mu(x,yz) = \mu(x,y)\mu(x,z) \text{ if } \{x,y,z\}\cap N \neq \emptyset, \tag{4.5}$$

$$z^{yx}\delta([z,y],x) = z^{xy}\delta([z,x],y). \tag{4.6}$$

Lemma 4.5. *Suppose that* (4.4), (4.5) *hold. Then* $N \le N(Q)$, *and* $T(x)$ *commutes with* $L(u,v)$, $R(u,v)$.

Proof. It is easy to see that

$$[x,y,z] = \mu(x,y)\mu(xy,z)\mu(x,yz)^{-1}\mu(y,z)^{-1} \in Z. \tag{4.7}$$

Then $[x,y,z]=1$ whenever $\{x,y,z\}\cap N \neq \emptyset$, and so $N \le N(Q)$.

By Lemmas 4.3 and 4.4,

$$L(u,v)T(x)y = (T(x)y)[u,v,T(x)y] = y^x\mu(y,x)\mu(x,y^x)^{-1}[u,v,y^x],$$

and

$$\begin{aligned} T(x)L(u,v)y &= T(x)(y[u,v,y]) = (y[u,v,y])^x\mu(y[u,v,y],x)\mu(x,(y[u,v,y])^x)^{-1} \\ &= y^x[u,v,y]\mu(y,x)\mu(x,y^x)^{-1}. \end{aligned}$$

So the equality $L(u,v)T(x)y = T(x)L(u,v)y$ holds if and only if $[u,v,y^x] = [u,v,y]$. This is true since $y^x = y[y,x]$, and $[y,x] \in G' \le N \le N(Q)$.

Similarly

$$R(u,v)T(x)y = (T(x)y)[T(x)y,u,v] = y^x\mu(y,x)\mu(x,y^x)^{-1}[y^x,u,v],$$

and

$$\begin{aligned} T(x)R(u,v)y &= T(x)(y[y,u,v]) = (y[y,u,v])^x\mu(y[y,u,v],x)\mu(x,(y[y,u,v])^x)^{-1} \\ &= y^x[y,u,v]\mu(y,x)\mu(x,y^x)^{-1}. \end{aligned}$$

So the equality $R(u,v)T(x)y = T(x)R(u,v)y$ holds if and only if $[y^x,u,v] = [y,u,v]$, and we finish as before. □

Proposition 4.6. *Assume that* (4.4), (4.5) *hold. Then* $\operatorname{Inn} Q$ *is abelian if and only if* (4.6) *holds.*

Proof. In view of Lemmas 4.3 and 4.5, it suffices to show that $T(x)T(y) = T(y)T(x)$ for every x, y. Using Lemma 4.4, a straightforward calculation yields

$$T(x)T(y)z = z(yxz)^{-1}(zyx)\mu(z^y, x)\mu(x, z^{yx})^{-1}\mu(z, y)\mu(y, z^y)^{-1}. \tag{4.8}$$

Sine $G' \le N$, we have

$$\mu(z^y, x) = \mu(z[z, y], x) = \mu(z, x)\mu([z, y], x),$$
$$\mu(y, z^y) = \mu(y, z[z, y]) = \mu(y, z)\mu(y, [z, y]).$$

In any group, $[z, xy] = [z, y][z, x][[z, x], y]$, and since $[[z, x], y]$ belongs to K (as $N/K \le Z(G/K)$), we have

$$\mu(x, z^{yx}) = \mu(x, z[z, yx]) = \mu(x, z)\mu(x, [z, yx]) = \mu(x, z)\mu(x, [z, y])\mu(x, [z, x]).$$

Putting all these facts together, we can rewrite $T(x)T(y)z$ as

$$z\mu(z, x)\mu(x, z)^{-1}\mu(z, y)\mu(y, z)^{-1} (yxz)^{-1}(zyx)\mu([z, y], x)\mu(x, [z, y])^{-1} \mu(x, [z, x])^{-1}\mu(y, [z, y])^{-1}.$$

Upon interchanging x and y, we deduce that $T(y)T(x)z$ is equal to

$$z\mu(z, x)\mu(x, z)^{-1}\mu(z, y)\mu(y, z)^{-1} (xyz)^{-1}(zxy)\mu([z, x], y)\mu(y, [z, x])^{-1} \mu(x, [z, x])^{-1}\mu(y, [z, y])^{-1}.$$

The result then follows. □

4.2. **Consequences of the conditions.**

Proposition 4.7. *Assume that* (4.6) *holds. Then both* G *and* Q *are of nilpotency class at most three.*

Proof. By (4.6), $z^{-xy}z^{yx} = \delta([z, x], y)\delta([z, y], x)^{-1} \in Z \le Z(G)$. Since $z^{-xy}z^{yx} = (xy)^{-1}[z, [y^{-1}, x^{-1}]](xy)$ holds in any group,

$$[z, [y^{-1}, x^{-1}]] = \delta([z, x], y)\delta([z, y], x)^{-1} = z^{-xy}z^{yx} \tag{4.9}$$

follows.

Set $U = G'Z$ and consider the series $1 \le Z \le U \le G$. We have $Z \le Z(G) \cap Z(Q)$ by Lemma 4.2, and $U/Z \le Z(G/Z)$ by (4.9). The latter inclusion holds for both operations, as $G/Z \cong Q/Z$ by Lemma 4.2. □

Although we originally believed that Q of nilpotency class three cannot be obtained from G of nilpotency class two, it turns out that it can happen and that ample examples exist. We therefore focus on the (simpler) case when G is of nilpotency class two and Q is of nilpotency class three.

As an immediate consequence of (4.9), we have:

Corollary 4.8. *Assume that* (4.6) *holds. Then the following conditions are equivalent:*

(i) *G is of nilpotency class at most two,*
(ii) *$z^{xy} = z^{yx}$ for every x, y, $z \in G$,*
(iii) *$\delta([z, x], y) = \delta([z, y], x)$ for every x, y, $z \in G$.*

In view of Proposition 4.7, the following result is relevant. It is a consequence of the Hall-Witt identity for groups

$$[[x,y^{-1}],z]^y[[y,z^{-1}],x]^z[[z,x^{-1}],y]^x = 1.$$

Lemma 4.9. *Let H be a group of nilpotency class at most three. Then*

$$[x,[y,z]][y,[z,x]][z,[x,y]] = 1,$$

and $[x,[y,z]] = [x,[y^{-1},z^{-1}]]$ for every x, y, $z \in H$.

Lemma 4.10. *Assume that* (4.4), (4.5) *hold. Then $\delta([z,x],y) = \delta([x,z],y)^{-1}$.*

Proof. First note that $\mu(zx,y) = \mu(xz[z,x],y) = \mu(xz,y)\mu([z,x],y)$. This means that $\mu([z,x],y) = \mu(zx,y)\mu(xz,y)^{-1}$. Similarly, $\mu(y,[z,x]) = \mu(y,zx)\mu(y,xz)^{-1}$. Hence $\delta([z,x],y) = \mu([z,x],y)\mu(y,[z,x])^{-1} = \mu(zx,y)\mu(xz,y)^{-1}\mu(y,zx)^{-1}\mu(y,xz)$. The equality $\delta([z,x],y) = \delta([x,z],y)^{-1}$ follows. □

Proposition 4.11. *Assume that* (4.4)–(4.6) *hold. Then the following conditions are equivalent:*

(i) *Q is of nilpotency class at most two,*
(ii) *$G' \le Z(Q)$,*
(iii) *$\delta([x,y],z)\delta([y,z],x)\delta([z,x],y) = 1$.*

Proof. By Lemma 4.2, $Z \le Z(Q)$ and $Q/Z \cong G/Z$. This means that $Z(Q)$ is not only a subgroup of Q but also a subgroup of G, and $Q/Z(Q) \cong G/Z(Q)$. Now, Q is of nilpotency class at most two if and only if $Q/Z(Q)$ is abelian, which is the same as $G/Z(Q)$ being abelian, which is equivalent to $G' \le Z(Q)$.

By Lemma 4.5, $G' \le N \le N(Q)$. Thus the condition $G' \le Z(Q)$ holds if and only if $z * [y,x] = [y,x] * z$ for every x, y, $z \in G$. This is the same as $z[y,x]\mu(z,[y,x]) = [y,x]z\mu([y,x],z)$, or, equivalently,

$$[z,[y^{-1},x^{-1}]] = [z,[y,x]] = \delta([y,x],z) = \delta([x,y],z)^{-1},$$

by Lemmas 4.9 and 4.10. Then (4.9) and Lemma 4.10 yield

$$\delta([x,y],z)^{-1} = [z,[y^{-1},x^{-1}]] = \delta([z,x],y)\delta([z,y],x)^{-1} = \delta([z,x],y)\delta([y,z],x),$$

and we are done. □

In particular, if G is abelian then Q cannot be of nilpotency class three.

Corollary 4.12. *Assume that* (4.4)–(4.6) *hold. Then Q is of nilpotency class three and G is of nilpotency class two if and only if $\delta([x,y],z) = \delta([x,z],y)$ for every x, y, $z \in G$, and $\delta([x,y],z) \ne 1$ for some x, y, $z \in G$.*

Proof. By Corollary 4.8, $\delta([x,y],z) = \delta([x,z],y)$ holds for every x, y, $z \in G$ if and only if G is of nilpotency class two, in which case Lemma 4.10 yields

$$1 = \delta([y,z],x)\delta([y,z],x)^{-1} = \delta([y,z],x)\delta([z,y],x) = \delta([y,z],x)\delta([z,x],y). \tag{4.10}$$

Using (4.10) and Proposition 4.11, Q is then of nilpotency class three if and only if there are x, y, $z \in G$ such that $1 \ne \delta([x,y],z)\delta([y,z],x)\delta([z,x],y) = \delta([x,y],z)$. □

When A, B are groups then $f : A^3 \to B$ is said to be *symmetric triadditive* if $f(a_1,a_2,a_3) = f(a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)})$ for every permutation of σ of $\{1,2,3\}$ and every a_1, a_2, $a_3 \in A$, and $f(ab,c,d) = f(a,c,d)f(b,c,d)$ for every a, b, c, $d \in A$.

Proposition 4.13. *Assume that* (4.4)–(4.6) *holds, G is of nilpotency class two and Q is of nilpotency class three. Then there exists a subgroup $A \le Z$ of exponent two and a nontrivial symmetric triadditive mapping $f : (G/N)^3 \to A$ such that*

$$\delta([x, y], z) = f(xN, yN, zN)$$

for all x, y, $z \in G$.

Proof. Let $f : G^3 \to Z$ be defined by $f(x, y, z) = \delta([x, y], z)$. By Corollary 4.12, f is nontrivial, and $\delta([x, y], z) = \delta([x, z], y)$, so $f(x, y, z)$ is invariant under the permutation $(2, 3)$ of its arguments. By (4.10),

$$f(x, y, z) = \delta([x, y], z) = \delta([x, y], z)\delta([y, z], x)\delta([z, x], y),$$

which shows that $f(x, y, z)$ is invariant under the permutation $(1, 2, 3)$ of its arguments. Altogether, f is symmetric.

By Lemma 4.9, $1 = [x, [z^{-1}, y^{-1}]][z, [y^{-1}, x^{-1}]][y, [x^{-1}, z^{-1}]]$. By (4.9), we can rewrite this as

$$1 = \delta([x, y], z)\delta([x, z], y)^{-1}\delta([z, x], y)\delta([z, y], x)^{-1}\delta([y, z], x)\delta([y, x], z)^{-1},$$

and by Lemma 4.10 and (4.10) we can further simplify it to

$$1 = (\delta([x, y], z)\delta([y, z], x)\delta([z, x], y))^2 = \delta([x, y], z)^2.$$

This guarantees the existence of the subgroup $A \le Z$ of exponent two.

In any group, $[xy, z] = [x, z][[x, z], y][y, z]$. Since $[x, z]$, $[y, z] \in N$ and $[[x, z], y] \in K$, we have $\mu([xy, z], u) = \mu([x, z], u)\mu([y, z], u)$, and $\mu(u, [xy, z]) = \mu(u, [x, z])\mu(u, [y, z])$. Then $\delta([xy, z], u) = \delta([x, z], u)\delta([y, z], u)$, and f is triadditive.

Let $n \in N$. Then $[x, n] \in K$ and $\delta([x, n], z) = 1$. Thus, by additivity, $\delta([x, yn], z) = \delta([x, y], z)\delta([x, n], z) = \delta([x, y], z)$ and that is why the value $f(x, y, z)$ depends only upon classes modulo N. □

5. Constructing loops from symmetric trilinear alternating forms

As we have just shown, if Q is a loop of nilpotency class three with commuting inner mappings obtained as a modification of a group G of nilpotency class two by μ, then δ gives rise to a nontrivial symmetric triadditive form.

We now show a partial converse. Namely, that it is possible to construct G and μ (and hence Q) with the desired properties from certain groups of nilpotency class two.

Throughout this section, let H be a group of nilpotency class two such that $H' = Z(H)$, H/H' is an elementary abelian 2-group with basis $\{e_1H', \dots, e_dH'\}$, and H' is an elementary abelian 2-group with basis $\{[e_i, e_j];\ 1 \le i < j \le d\}$. In addition, let $A = \{1, -1\}$, and let $f : (H/H')^3 \to A$ be a symmetric trilinear alternating form (we can view H/H' as a vector space over A). For u, v, $w \in H$, we write $f(u, v, w)$ instead of the formally more precise $f(uH', vH', wH')$.

Starting with f, we are going to construct $\delta : H \times H \to A$ and $\mu : H \times H \to A$ so that $f(u, v, w) = \delta([u, v], w)$, $\delta(u, v) = \mu(u, v)\mu(v, u)^{-1}$, and such that (4.4)–(4.6) hold for μ and δ, with H' in place of N.

We can then set $G = A \times H$ (any extension of A by H would do) and use the mappings μ, $\delta : H \times H \to A$ to obtain the loop $Q = (G, *)$ according to (4.1). By Proposition 4.6 and Corollary 4.12, Q is then a loop of nilpotency class three with commuting inner mappings, *provided that f is nontrivial.*

Let $M = H'$. The construction of δ and μ is in three steps. First, the condition $f(u, v, w) = \delta([u, v], w)$ forces δ on $M \times H$. Second, the extension of δ from $M \times H$

to $H \times H$ depends on certain free parameters. Third, once $\delta : H \times H \to A$ is given, additional free parameters are needed to obtain μ.

5.1. **Constructing δ.** For $1 \le i, j, k \le d$ and $m \in M$ let

$$\delta([e_i, e_j], e_k m) = f(e_i, e_j, e_k), \tag{5.1}$$

and extend δ linearly into a mapping $\delta : M \times H \to A$. Then δ satisfies $\delta(m_1 m_2, h) = \delta(m_1, h)\delta(m_2, h)$ and $\delta(m, h_1 h_2) = \delta(m, h_1)\delta(m, h_2)$ for every m, m_1, $m_2 \in M$ and h, h_1, $h_2 \in H$. Also, $\delta(M, M) = 1$, because f vanishes whenever one of its arguments is trivial.

Our present task is to construct $\delta : H \times H \to A$ such that

$$\begin{aligned} &\delta([u, v], w) = f(u, v, w) \text{ for every } u,\ v,\ w \in H,\\ &\delta(u, v) = \delta(v, u)^{-1} \text{ for every } u,\ v \in H,\\ &\delta(u, vw) = \delta(u, v)\delta(u, w) \text{ if } \{u, v, w\} \cap M \ne \emptyset. \end{aligned} \tag{5.2}$$

(If $\{u, v, w\} \cap M \ne \emptyset$, we then also have $\delta(uv, w) = \delta(w, uv)^{-1} = \delta(w, u)^{-1}\delta(w, v)^{-1} = \delta(u, w)\delta(v, w)$.)

Lemma 5.1. *$\delta : M \times H \to A$ satisfies $\delta([u, v], w) = f(u, v, w)$ for every u, v, $w \in H$.*

Proof. Let $u \in e_1^{u_1} \cdots e_d^{u_d} M$, and $v \in e_1^{v_1} \cdots e_d^{v_d} M$, where $0 \le u_i$, $v_i \le 1$. Since H is of nilpotency class 2, we have $[xy, z] = [x, z][y, z]$ and $[x, yz] = [x, y][x, z]$ for every x, y, $z \in H$. Thus

$$\delta([u, v], w) = \prod_{i \ne j} \delta([e_i^{u_i}, e_j^{v_j}], w),$$

and, since f is trilinear and alternating,

$$f(u, v, w) = \prod_{i \ne j} f(e_i^{u_i}, e_j^{v_j}, w).$$

It therefore suffices to show that $\delta([e_i^{u_i}, e_j^{v_j}], w) = f(e_i^{u_i}, e_j^{v_j}, w)$. This is clearly true when $u_i = 0$ or $v_j = 0$, and when $u_i = v_j = 1$, it follows from the definition (5.1) of δ. □

We now extend $\delta : M \times H \to A$ to $\delta : H \times H \to A$. Let T be a transversal for M in H, $T = \{t_1, \dots, t_k\}$, $t_1 = 1$. For $1 \le i, j \le k$, choose $\delta(t_i, t_j) \in A$ as follows:

$$\begin{aligned} &\delta(t_1, t_j) = 1 \text{ for every } 1 \le j \le k,\\ &\delta(t_i, t_j) \text{ arbitrary when } 1 < i < j \le k,\\ &\delta(t_j, t_i) = \delta(t_i, t_j)^{-1} \text{ when } 1 < i < j \le k,\\ &\delta(t_i, t_i) = 1 \text{ for every } 1 \le i \le k. \end{aligned} \tag{5.3}$$

Every element $h \in H$ can be written uniquely as $h = mt$ for some $m \in M$, $t \in T$, and we define $\delta : H \times H \to A$ by

$$\delta(mt, m't') = \delta(m, t')\delta(m', t)^{-1}\delta(t, t'), \tag{5.4}$$

where $\delta(m, t')$, $\delta(m', t)$ have already been defined above.

Note that the new definition (5.4) gives $\delta(m, m't') = \delta(m, t')\delta(m', 1)\delta(t_1, t') = \delta(m, t')$, while the old definition (5.1) gives $\delta(m, m't') = \delta(m, m')\delta(m, t') = \delta(m, t')$. Hence $\delta : H \times H \to A$ extends the map $\delta : M \times H \to A$.

Lemma 5.2. *Let $\delta : H \times H \to A$ be defined as above. Then:*

(i) $\delta(u, v) = \delta(v, u)^{-1}$ *for every* u, $v \in H$,

(ii) $\delta(u, mv) = \delta(u, m)\delta(u, v)$ *for every* u, $v \in H$, $m \in M$,
(iii) $\delta(u, vm) = \delta(u, v)\delta(u, m)$ *for every* u, $v \in H$, $m \in M$,
(iv) $\delta(m, uv) = \delta(m, u)\delta(m, v)$ *for every* u, $v \in H$, $m \in M$.

Proof. (i) We have $\delta(mt, m't') = \delta(m, t')\delta(m', t)^{-1}\delta(t, t')$, and also $\delta(m't', mt)^{-1} = \delta(m', t)^{-1}\delta(m, t')\delta(t', t)^{-1}$. Hence we are done by $\delta(t, t') = \delta(t', t)^{-1}$ of (5.3).

(ii) Let $u = nt$, $v = n't'$, where n, $n' \in M$ and t, $t' \in T$. Then $\delta(u, mv) = \delta(nt, (mn')t') = \delta(n, t')\delta(mn', t)^{-1}\delta(t, t')$. On the other hand, $\delta(u, m)\delta(u, v)$ is equal to $\delta(nt, m)\delta(nt, n't') = \delta(m, t)^{-1}\delta(n, t')\delta(n', t)^{-1}\delta(t, t')$. Since $\delta(m, t)\delta(n', t) = \delta(mn', t)$, we are done.

Part (iii) is an immediate consequence of (ii), $M \subseteq Z(H)$, and the fact that A is abelian. We have already observed (iv). □

5.2. **Constructing μ.** We now need a map $\mu : H \times H \to A$ such that

$$\begin{aligned}
&\delta(u, v) = \mu(u, v)\mu(v, u)^{-1} \text{ for every } u,\ v \in H,\\
&\mu(uv, w) = \mu(u, w)\mu(v, w) \text{ if } \{u, v, w\} \cap M \neq \emptyset,\\
&\mu(u, vw) = \mu(u, v)\mu(u, w) \text{ if } \{u, v, w\} \cap M \neq \emptyset.
\end{aligned} \tag{5.5}$$

Let $T = \{t_1, \dots, t_k\}$ be the same transversal for M in H as above. Define $\mu : M \cup T \times M \cup T \to A$ as follows:

$$\begin{aligned}
&\mu(t_1, t_1) = 1,\\
&\mu(t_i, t_i) \text{ arbitrary, for } 1 < i \leq k,\\
&\mu(t_i, t_j) = \delta(t_i, t_j) \text{ if } 1 \leq i < j \leq k,\\
&\mu(t_j, t_i) = 1 \text{ if } 1 \leq i < j \leq k,\\
&\mu(m, n) = 1 \text{ for } m,\ n \in M,\\
&\mu(m, t) = \delta(m, t) \text{ for } m \in M,\ t \in T,\\
&\mu(t, m) = 1 \text{ for } m \in M,\ t \in T.
\end{aligned} \tag{5.6}$$

Since $\delta(M, 1) = 1$, $\mu : M \cup T \times M \cup T \to A$ is well-defined. Extend $\mu : M \cup T \times M \cup T \to A$ to $\mu : H \times H \to A$ by

$$\mu(mt, m't') = \mu(m, t')\mu(t, t'), \tag{5.7}$$

where m, $m' \in M$, t, $t' \in T$.

Lemma 5.3. $\delta(u, v) = \mu(u, v)\mu(v, u)^{-1}$ *for every* u, $v \in H$.

Proof. By the definitions (5.1), (5.4) and (5.7), $\delta(mt, m't') = \delta(m, t')\delta(m', t)^{-1}\delta(t, t')$, and

$$\begin{aligned}
&\mu(mt, m't')\mu(m't', mt)^{-1}\\
&\qquad = \mu(m, t')\mu(t, t')\mu(m', t)^{-1}\mu(t', t)^{-1} = \delta(m, t')\mu(t, t')\delta(m', t)^{-1}\mu(t', t)^{-1}.
\end{aligned}$$

Hence the desired equality holds if and only if $\mu(t, t')\mu(t', t)^{-1} = \delta(t, t')$.

Let $t = t_i$, $t' = t_j$. If $i = j$ then $\mu(t_i, t_i)\mu(t_i, t_i)^{-1} = 1 = \delta(t_i, t_i)$. If $i < j$ then $\mu(t_i, t_j)\mu(t_j, t_i)^{-1} = \delta(t_i, t_j)$. If $i > j$ then $\mu(t_i, t_j)\mu(t_j, t_i)^{-1} = \delta(t_j, t_i)^{-1} = \delta(t_i, t_j)$. □

Lemma 5.4. *The following properties hold for* $\mu : H \times H \to A$, u, $v \in H$, *and* $m \in M$:

(i) $\mu(mu, v) = \mu(m, v)\mu(u, v)$,
(ii) $\mu(um, v) = \mu(u, v)\mu(m, v)$,
(iii) $\mu(uv, m) = \mu(u, m)\mu(v, m)$,

(iv) $\mu(u, mv) = \mu(u, m)\mu(u, v)$,
(v) $\mu(u, vm) = \mu(u, v)\mu(u, m)$,
(vi) $\mu(m, uv) = \mu(m, u)\mu(m, v)$.

Proof. Since $\mu(m, 1) = \mu(t, 1) = 1$ for every $m \in M$, $t \in T$, we have $\mu(H, M) = 1$. Also, by definition (5.7), $\mu(mt, m't')$ does not depend on m'. We will use these properties and Lemma 5.2 without reference in this proof. For (i),

$$\begin{aligned}\mu(m \cdot m't', m''t'') &= \mu(mm't', t'') = \mu(mm', t'')\mu(t', t'') = \delta(mm', t'')\mu(t', t'')\\ &= \delta(m, t'')\delta(m', t'')\mu(t', t'') = \mu(m, t'')\mu(m', t'')\mu(t', t'')\\ &= \mu(m, t'')\mu(m't', t'') = \mu(m, m''t'')\mu(m't', m''t'').\end{aligned}$$

(ii) follows from (i) since $m \in Z(H)$ and A is commutative. (iii) follows from $\mu(H, M) = 1$. For (iv), $\mu(u, mv) = \mu(u, v)$ and $\mu(u, m) = 1$. (v) follows from (iv) since $m \in Z(H)$ and A is commutative.

For (vi), let $u = m't'$, $v = m''t''$. Then $\mu(m, uv) = \mu(m, t't'')$ and $\mu(m, u)\mu(m, v) = \mu(m, t')\mu(m, t'')$. With $t't'' = m^*t^*$, $\mu(m, t't'') = \mu(m, m^*t^*) = \mu(m, t^*) = \delta(m, t^*) = \delta(m, m^*)\delta(m, t^*) = \delta(m, m^*t^*) = \delta(m, t't'') = \delta(m, t')\delta(m, t'') = \mu(m, t')\mu(m, t'')$. □

6. Explicit examples

The minimal situation of Section 5 in which $f : (H/H')^3 \to A$ is nontrivial occurs when H/H' is a vector space of dimension three over $A = \{1, -1\}$, and f is the unique (up to equivalence) nontrivial symmetric trilinear alternating form, i.e., f is the determinant.

The commutator subgroup H' is then also of dimension three, with basis $\{[e_1, e_2], [e_1, e_3], [e_2, e_3]\}$, and so $|H| = 64$. One might wonder if there is any group H satisfying all these requirements. An inspection of the GAP libraries of small groups shows that there are precisely 10 such groups.

Furthermore, there are $21+7 = 28$ free parameters (5.3) and (5.6) used in constructing δ and μ from f. Altogether, when the direct product $G = A \times H$ is used, the procedure of Section 5 yields $10 \cdot 2^{28}$ loops (not necessarily pairwise nonisomorphic) of order 128 that are of nilpotency class three and have commuting inner mappings.

Throughout this section, let H be a group such that H/H' is an elementary abelian 2-group with basis $\{e_1H', e_2H', e_3H'\}$, and H' is an elementary abelian group with basis $\{[e_1, e_2], [e_1, e_3], [e_2, e_3]\}$. Furthermore, if $\mu : H \times H \to A$ is obtained from f by the procedure of Section 5, let $\mathcal{C}(H, \mu)$ denote the resulting loop Q defined on $G = A \times H$ via (4.1).

6.1. **The loop C.** The loop C is obtained as follows: Let H be the first suitable group in the GAP library of small groups, i.e., H is presented by

$$\begin{aligned}H = \langle g_1, g_2, g_3, g_4, g_5, g_6;\ & g_i^2 = 1 \text{ for every } 1 \le i \le 6,\\ & (g_ig_j)^2 = 1 \text{ for every } 1 \le j < i \le 6, \text{ except for}\\ & (g_2g_1)^2g_4 = (g_3g_1)^2g_5 = (g_3g_2)^2g_6 = 1\rangle,\end{aligned}$$

and let all the parameters for δ and μ be equal to 1. Then $\mathcal{C}(H, \mu)$ is isomorphic to C.

This shows: (i) the deep insight of Csörgő in constructing C, (ii) that the construction by group modifications is highly relevant to the problem at hand, (iii) that C is very natural among loops of nilpotency class three with commuting inner mappings.

6.2. **About the isomorphism problem.** Different choices of H and of the parameters for δ and μ produce generally nonisomorphic loops. We do not wish to pursue the general isomorphism problem here, but we offer some evidence that the number of loops $\mathcal{C}(H,\mu)$ is very large.

Lemma 6.1. *Let $Q=\mathcal{C}(H,\mu)$. Then $Z(Q)=A\times 1$.*

Proof. By Lemma 4.2, $A\le Z(Q)$. Now, $(a,h)\in Q$ commutes with $(b,k)\in Q$ if and only if $(ab\mu(h,k),hk)=(ba\mu(k,h),kh)$. Thus $(a,h)\in Z(Q)$ if and only if $h\in Z(H)=H'$ and $\delta(h,k)=1$ for every $k\in H$.

The form f is determined by its values $f(e_i,e_j,e_k)$, where $1\le i,\ j,\ k\le 3$, and we can assume without loss of generality that $f(e_i,e_j,e_k)=-1$ if and only $|\{i,j,k\}|=3$.

Suppose that $h\ne 1$. Then it is always possible to find k such that $\delta(h,k)=1$ leads to a contradiction. For instance, when $h=[e_1,e_2][e_1,e_3]$, we let $k=e_3$, and calculate $1=\delta(h,k)=\delta([e_1,e_2][e_1,e_3],e_3)=\delta([e_1,e_2],e_3)\delta([e_1,e_3],e_3)=f(e_1,e_2,e_3)f(e_1,e_3,e_3)=(-1)1=-1$. The remaining 6 cases are left to the reader. □

Lemma 6.2. *If H_1, H_2 are not isomorphic then $\mathcal{C}(H_1,\mu_1)$, $\mathcal{C}(H_2,\mu_2)$ are not isomorphic.*

Proof. Let $Q_i=\mathcal{C}(H_i,\mu_i)$. Assume, for a contradiction, that $Q_1\cong Q_2$. By Lemma 6.1, the two centers $Z(Q_1)$, $Z(Q_2)$ are equal to $A\times 1$, and thus $H_1\cong Q_1/Z(Q_1)\cong Q_2/Z(Q_2)\cong H_2$, by Lemma 4.2. □

In order to further demonstrate the multitude of nonisomorphic loops $\mathcal{C}(H,\mu)$, we conducted two experiments.

First, we let H be the group of Subsection 6.1, set all parameters (5.6) of μ to 1, and chose δ so that precisely one parameter of (5.3) was nontrivial. It turns out that the resulting 21 loops are pairwise nonisomorphic.

Second, we attempted to estimate the probability that $\mathcal{C}(H,\mu_1)$, $\mathcal{C}(H,\mu_2)$ are isomorphic, if the parameters for μ_1 and μ_2 are chosen at random. Let X be an n-element set partitioned into k nonempty blocks, and let p be the probability that two randomly chosen elements of X belong to the same block. When n and k are fixed, p is minimized when all blocks have the same size n/k, in which case $p=1/k$. None of the 2500 random pairs that we tested consisted of isomorphic loops. We can therefore conclude with some confidence that $p<1/2500$, and, consequently, that there are at least 2500 pairwise nonisomorphic loops with the desired properties. Of course, it is reasonable to expect that k is much larger. We did not check more random pairs since the test for isomorphism is time-consuming.

6.3. **Multiplication groups and inner mapping groups of loops $\mathcal{C}(H,\mu)$.** The original construction of Csörgő is based on multiplication groups, so we look at them more closely.

The multiplication groups of loops $\mathcal{C}(H,\mu)$ can have different orders. Let μ_0 denote the mapping obtained when all parameters (5.3) and (5.6) are trivial, and μ_1 the mapping obtained when all parameters are trivial except for $\mu_1(t_2,t_2)=-1$. Let H be any of the 10 suitable groups, and let $Q_i=\mathcal{C}(H,\mu_i)$. Then $|\operatorname{Mlt}Q_0|=2^{13}$ and $|\operatorname{Mlt}Q_1|=2^{17}$. We also came across a loop $\mathcal{C}(H,\mu)$ with multiplication group of order 2^{16}.

It appears that $|\operatorname{Mlt}\mathcal{C}(H,\mu)|\ge 2^{13}$, and that the equality holds if and only if μ is trivial.

Even if their orders agree, the multiplication groups need not be isomorphic. For instance, $\operatorname{Mlt}C$ is not isomorphic to any $\operatorname{Mlt}(H^*,\mu_0)$ when H^* is a group different from H of Subsection 6.1.

On the other hand, the structure of the inner mapping group of $\mathcal{C}(H,\mu)$ is clear:

Proposition 6.3. $\operatorname{Inn}\mathcal{C}(H,\mu)$ *is an elementary abelian* 2*-group.*

Proof. Let $Q=\mathcal{C}(H,\mu)=A\times H$. It suffices to show that the left, right, and middle inner mappings of Q are involutions, since we already know that $\operatorname{Inn}Q$ is abelian.

By (4.7), $[x,y,z]\in Z=A$ for every x, y, $z\in Q$. By (4.2) and (4.3), $L(x,y)^2z=z[x,y,z]^2=z$ and $R(x,y)^2z=z[z,x,y]^2=z$, since A is of exponent two.

By (4.8),

$$T(x)^2z=x^{-2}zx^2\mu(z^x,x)\mu(x,z^{x^2})^{-1}\mu(z,x)\mu(x,z^x)^{-1}.$$

Since $x^2\in Z(G)$, we can rewrite this as

$$\begin{aligned}T(x)^2z&=z\mu(z^x,x)\mu(x,z)^{-1}\mu(z,x)\mu(x,z^x)^{-1}=z\delta(z^x,x)\delta(z,x)\\&=z\delta(z[z,x],x)\delta(z,x)=z\delta(z,x)^2\delta([z,x],x)=z\delta([z,x],x).\end{aligned}$$

As $\delta([z,x],x)=f(z,x,x)=1$, we are done. ☐

7. Open problems

These are the main open problems of interest here:

Problem 7.1. *Let Q be a loop of nilpotency class at least three with abelian group of inner mappings.*

(i) *Can the nilpotency class of Q be bigger than three?*
(ii) *Can $|Q|$ be less than* 128*?*
(iii) *Can $|Q|$ be odd?*
(iv) *Can Q be constructed by the modifications of Section* 4*?*
(v) *Can Q be Moufang?*
(vi) *Can $|\operatorname{Mlt}Q|$ be less than* 8192*?*

While this paper was under review, G. P. Nagy and the second author constructed a Moufang loop of nilpotency class three and with commuting inner mappings, hence solving (v).

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