A DECOMPOSITION OF GALLAI MULTIGRAPHS

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Abstract. An edge-colored cycle is *rainbow* if its edges are colored with distinct colors. A *Gallai* (multi) graph is a simple, complete, edge-colored (multi) graph lacking rainbow triangles. As has been previously shown for Gallai graphs, we show that Gallai multigraphs admit a simple iterative construction. Moreover, we show that Gallai multigraphs give rise to a surprising and highly structured decomposition into directed trees.

1. Background

We assume throughout that all multigraphs are simple (no loops), complete (each pair of vertices is connected by at least one edge), finite, and edge-colored. We treat graphs as a special type of multigraph in which no pair of vertices is connected by more than one edge.

We say an edge-colored cycle is *rainbow* if its edges are colored distinctly. A multigraph that lacks rainbow triangles is called *Gallai*, and it follows from a simple inductive argument that such a multigraph lacks rainbow n-cycles for all n. Tibor Gallai [5] showed that Gallai graphs can be disconnected by the removal of two colors and as a corollary gave an elegant iterative construction of all such graphs. Gyárfás and Simonyi [6] consider certain monochromatic stars and spanning trees in Gallai graphs while Ball, Pultr, and Vojtěchovský [2] characterize those Gallai graphs each of whose triangles contains precisely two colors.

Some progress has been made toward understanding edge-colored graphs lacking rainbow n-cycles for a fixed n. Ball et al [2] give algebraic results about the sequence (n : G lacks rainbow n-cycles) as a monoid, and Vojtěchovský [7] extends the work of Alexeev

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[1] to find the densest arithmetic progression contained in this sequence. In the other direction, Frieze and Krivelevich [4] show that there is a constant c such that any edge-coloring of K_n in which no color appears more than cn times contains rainbow cycles of length k for all $3 \leq k \leq n$.

While searching for a general construction of graphs lacking rainbow n-cycles, we were confronted with the task of first understanding how the addition of multiedges affects the structure of Gallai multigraphs. While so far less studied that Gallai graphs, Mubayi and Diwan [3] make a conjecture about the possible color densities in Gallai multigraphs having at most three colors.

It is not difficult to show (as we do in §2) that Gallai multigraphs yield to essentially the same iterative construction as Gallai graphs. The key observation powering this construction is that, like Gallai graphs, any Gallai multigraph can be disconnected by the removal of at most two colors and thus can be decomposed into components connected by at most two colors.

After extending the previously known decomposition of Gallai graphs to the multigraph case, in §3 we explore an alternative and somewhat surprising decomposition of Gallai multigraphs into highly structured directed trees. We close with §4 by mentioning several related open problems.

2. Constructions of Gallai Graphs and Multigraphs

Implicit in his seminal work on transitively orientable graphs, Gallai [5] proved that every Gallai graph contains a set of at most two colors that, when removed, disconnects the graph.

Lemma 2.1. *If G is a Gallai graph having more than one vertex, then G can be disconnected by the removal of two colors.*

It follows easily from Lemma 2.1 that the following construction yields all Gallai graphs. Let $\mathcal{G}$ be the family of graphs defined inductively by:

(1) The single vertex graph is in $\mathcal{G}$.
(2) Fix colors A and B and graphs $\{G_i : 1 \le i \le t\} \subseteq \mathcal{G}$. For each $1 \le i \ne j \le t$, connect G_i and G_j by either A or B. The resulting graph is in $\mathcal{G}$.

Theorem 2.2. *$\mathcal{G}$ is the family of all Gallai graphs.*

Proof. It is clear that the construction does not introduce rainbow triangles. Suppose then that G is a Gallai graph having two or more vertices and let A and B be colors whose removal disconnects G into $\{G_i : 1 \le i \le t\}$ for $t \ge 2$.

For $1 \le i \ne j \le t$, we argue that G_i and G_j must be connected by A or B and not both. Let u_1u_2 be an edge in G_i colored neither A nor B. For each v in G_j, to avoid a rainbow triangle, the colors of u_1v and u_2v must agree. Since these edges span two distinct components, their colors must be A or B. By definition each pair of vertices in G_i is connected by a path whose edge-colors fall outside A and B. It thus follows that v is connected to all of G_i by either A or B. Repeating this argument for every v in G_j and then reversing the roles of G_i and G_j, we have that G_i and G_j must be connected by all A or all B.

Thus G will be constructed from smaller Gallai graphs in line (2) of the construction. □

As we have just seen, Lemma 2.1 is the key to Gallai's construction. Notice that the task of disconnecting a Gallai multigraph by the removal of colors becomes inherently more difficult with the addition of multiedges. Nonetheless, as we show in Lemma 2.3, the removal of two colors still suffices to disconnect any Gallai multigraph.

Lemma 2.3. *If M is a Gallai multigraph having more than two vertices, then M can be disconnected by the removal two colors.*

Once we have established Lemma 2.3, the following construction is easily seen to yield all Gallai multigraphs.

Let $\mathcal{M}$ be the family of multigraphs defined inductively by:

(1) Any multigraph with fewer than three vertices is in $\mathcal{M}$.
(2) Fix colors A and B and graphs $\{M_i : 1 \leq i \leq t\} \subseteq \mathcal{M}$. For each $1 \leq i \neq j \leq t$, connect M_i and M_j by either A or B. If $|M_i| = |M_j| = 1$, we may also connect M_i and M_j by both A and B. The resulting multigraph is in $\mathcal{M}$.

Theorem 2.4. *$\mathcal{M}$ is the family of all Gallai multigraphs.*

Proof. It is clear that the construction does not introduce rainbow triangles. Suppose then that M is a Gallai multigraph having three or more vertices and let A and B be colors whose removal disconnects M into $\{M_i : 1 \leq i \leq t\}$ for $t \geq 2$.

For $1 \leq i \neq j \leq t$, the argument given in the proof of Theorem 2.2 shows that M_i and M_j must be connected by a single color A or B except in the case when $|M_i| = |M_j| = 1$. Since we now allow multiedges, it is also possible that the single vertices of M_i and M_j are connected by both A and B.

Thus M can be constructed from smaller Gallai multigraphs in line (2) of the construction.

□

Proof of Lemma 2.3. Let G be a Gallai multigraph with at least three vertices and fix any vertex $v \in V(G)$. We argue that either

(1) v and G are connected by at most two colors (and thus their removal disconnects G with v as one component) or
(2) any colors whose removal disconnects $G - v$ also suffice to disconnect G.

If G has only three vertices, the claim is easily checked. Suppose then that $|V(G)| \geq 4$ and that v is connected to G by at least three colors.

Case 1: $G - v$ can be disconnected by the removal of a single color A.

Let $G_1, \ldots, G_k$ be the remaining components of $G - v$ upon removal of the color A. By assumption, there are vertices u and w in $V(G - v)$ and colors B and C, distinct from A, such that $B \in \overline{uv}$ and $C \in \overline{wv}$ (it could happen that $u = w$). To avoid a rainbow triangle, it must be the case that u and w fall in the same component of $G - v$, say G_1, and that v is connected to each of the remaining components by only the color A. Thus the removal of edges of color A disconnects G into components $G_1 + v, G_2, \ldots, G_k$.

Case 2: $G - v$ requires the removal of two colors, say A and B, to be disconnected.

Let $G_1, \ldots, G_k$ be the components of $G - v$ upon removal of colors A and B. Note that each pair of distinct components is connected by a single color A or B. If v is connected to one of these components, say G_1, by no color other than A or B, then the removal of colors A and B disconnects G with G_1 as one of the components. Otherwise, v must be connected to some vertex in G_i by a third color C_i distinct from A and B for each $1 \leq i \leq k$. Moreover, to avoid a rainbow triangle, it must be the case that $C_i = C_j$ for each $1 \leq i, j \leq k$. Let C be this common color.

To avoid rainbow triangles, every edge incident with v must be colored A, B, or C. Thus we may select $u \in V(G - v)$ such that $A \in \overline{vu}$. Without loss of generality, we may assume $u \in G_1$. Since $G - v$ cannot be disconnected by the removal of just the color A, there is another component, say G_2, such that G_1 and G_2 are connected by B. Fix $w \in G_2$, such that $\overline{vw} = C$. The vertices u, v, w now form a rainbow triangle.

It must then be the case that v and G_i are connected by just the colors A and B for some i and thus G can be disconnected by the removal of A and B.

$\square$

3. Decomposition of Gallai Multigraphs

We now develop an iterative decomposition of Gallai multigraphs into directed trees.

3.1. **Basic Notation.** We denote vertices using lowercase letters such as u, v, and w, sets of vertices using uppercase letters from the end of the alphabet such as U, V, and W, and colors using uppercase letters from the beginning of the alphabet such as A, B, and C. Given two sets of vertices, U and V, we write UV for the set of edges connecting vertices of U to vertices of V. This notation will also be used with singletons, u and v, to refer to the edges connecting u and v. To denote the set of colors present in a set of edges, say UV, we write $\overline{UV}$ when there is no risk of ambiguity. Otherwise we refer explicitly to the coloring at hand, i.e. $\rho[UV]$. If $\overline{UV} = \{A\}$, we will often shorten notation by writing $\overline{UV} = A$.

Many of our results will be stated in terms of mixed graphs. A *mixed graph* is a triple $M = (V, E, A)$ with vertices V, undirected edges E, and directed edges A. We say M is *complete* if every pair of distinct vertices is connected by exactly one directed or undirected edge. The *weak components* of a directed graph are the components of the graph that results from replacing each directed edge with an undirected edge. For our purposes, the *weak components* of a mixed graph $M = (V, E, A)$ will be the weak components of the directed graph (V, A). Note that this notion of component disregards undirected edges.

We use the term *rooted tree* to refer to a directed graph that is transitive and whose transitive reduction forms a tree in the usual sense. If (V, A) is a rooted tree, then its *root*, written 1_V, is the unique vertex having the property that there is a directed edge from 1_V to every other vertex in V.

3.2. **Basic Techniques: Maximality and Dominance.** Let $(G = (V, E), \rho)$ be a Gallai multigraph. We will in all cases assume that distinct edges connecting the same vertices are colored distinctly. We also think of $V \subseteq \mathbb{N}$ and thus having a natural ordering. We say that $(G = (V, E), \rho)$ is *uniformly colored* if $\rho(e_1) = \rho(e_2)$ for all $e_i \in E$.

We call *uv isolated* if for every $w \notin \{u, v\}$, $\overline{uw} = \overline{vw}$ and $|\overline{uw}| = 1$. Notice that if uv is isolated we can reduce the multigraph by collapsing the edge(s) uv. Likewise, given any multigraph, we can arbitrarily introduce new isolated edges without introducing rainbow triangles. We therefore call a multigraph *reduced* if it contains no isolated edges.

We call uv *maximal* if no new color can be added to $\overline{uv}$ without introducing a rainbow triangle. Here we allow the possibility that uv has "all possible colors" and thus is maximal. Likewise, $(G = (V, E), \rho)$ is *maximal* if uv is maximal for all $u, v \in V$.

Let $(G = (V, E), \rho)$ be a maximal Gallai multigraph. For $u, v \in V$, notice that $|\overline{uv}| \geq 3$ if and only if uv is isolated. Therefore, if G is reduced, $|\overline{uv}| = 1$ or 2 for all $u, v \in V$. Furthermore, if G is not reduced, we can reach a reduced Gallai multigraph by successively collapsing isolated edges of G.

Lemma 3.1. *Suppose $(G = (V, E), \rho)$ is a maximal Gallai multigraph. If $u, v \in V$ and $A \in \overline{uv}$, then for all $B \notin \overline{uv}$, there is $w \in V \setminus \{u, v\}$ and $C \notin \{A, B\}$ such that either $A \in \overline{uw}$ and $C \in \overline{wv}$ or $C \in \overline{uw}$ and $A \in \overline{wv}$.*

Proof. Note that if $|V| = 2$, then $\overline{uv}$ consists of "all possible colors" and we thus vacuously satisfy any claim about $B \notin \overline{uv}$. Assume then that $|V| \geq 3$.

Since G is maximal and $B \notin \overline{uv}$, we can find $w \neq u, v$ such that u, v, w would form a rainbow triangle if B were to be added to $\overline{uv}$. Thus we may find $X \in \overline{uw}$ and $Y \in \overline{vw}$ such that $X, Y,$ and B are distinct. However, since $A \in \overline{uv}$, $|\{X, Y, A\}| \leq 2$ and thus $A = X$ or $A = Y$. Let C be the other color. □

While Lemma 3.2 will follow from our general decomposition result, Theorem 3.3, we present it separately here because of its importance in understanding the most basic structure of a maximal reduced Gallai multigraph.

Lemma 3.2. *The vertices of a reduced maximal Gallai multigraph that are connected by two edges form uniformly colored cliques.*

Proof. Let $(G = (V, E), \rho)$ be a reduced maximal Gallai multigraph. Let $u, v, w \in V$. Suppose $\overline{uv} = \{A, B\}$ and $\overline{vw} = \{C, D\}$. If $\{A, B\} \neq \{C, D\}$, then we find a rainbow triangle no matter the colors of $\overline{uw}$. Suppose then that $\overline{uv} = \overline{vw} = \{A, B\}$. Certainly $\overline{uw} \subseteq \{A, B\}$. Suppose $\overline{uw} = A$. Since $B \notin \overline{uw}$, by Lemma 3.1, we may find $x \in V \setminus \{u, w\}$ such that, without loss of generality, $A \in \overline{ux}$ and $C \in \overline{wx}$. To avoid a rainbow triangle in v, x, w we must have $\overline{Vx} = C$. But this leads to a rainbow triangle in u, x, w. □

Our main result given in Theorem 3.3 is primarily an explanation of how each of these uniformly colored cliques are connected, and the following relation on sets of vertices plays a central role in this analysis. Let $(G = (\mathcal{V}, \mathcal{E}), \rho)$ be a Gallai multigraph. For $U, V \subseteq \mathcal{V}$ disjoint, we say that U *dominates* V and write $U \to V$ if and only if $|\overline{UV}| > 1$ and

(1) $U = \{u\}, V = \{v\}$, and $u < v$ or
(2) $|U| > 1$ or $|V| > 1$ and for every $u \in U$ and $v \in V$, $\overline{uv} = \overline{uV}$.

Given $U, V \subseteq \mathcal{V}$, we write $\Sigma(U, V)$ for the map from U to the powerset of $\overline{UV}$ defined by $u \mapsto \overline{uV}$. When $U \to V$, $\Sigma(U, V)$ completely describes the relationship between U and V and we call it the *signature of* $U \to V$.

Note that if $U \to V$ and $V \to U$, then every pair of vertices between U and V are connected by the same multiple colors. As we will see, our analysis will not encounter this situation because we will quickly deal only in cases when the vertices of U and V are connected by single edges. We also note the fact that if $U \to V$, then for every $v \in V$, we have $U \to v$.

Given a reduced maximal Gallai multigraph $(G = (\mathcal{V}, \mathcal{E}), \rho)$, we will describe its structure through a sequence of edge-colored mixed graphs $M_n(G) = (\mathcal{V}_n, \mathcal{E}_n, \mathcal{A}_n)$ defined as follows:

(1) $\mathcal{V}_0 := \mathcal{V}$,
(2) $\mathcal{A}_0 := \{(u, v) \in \mathcal{V}^2 : u \to v\}$, and
(3) $\mathcal{E}_0 := \{\{u, v\} \in [\mathcal{V}]^2 : |\rho[uv]| = 1\}$,

and for $n \geq 1$

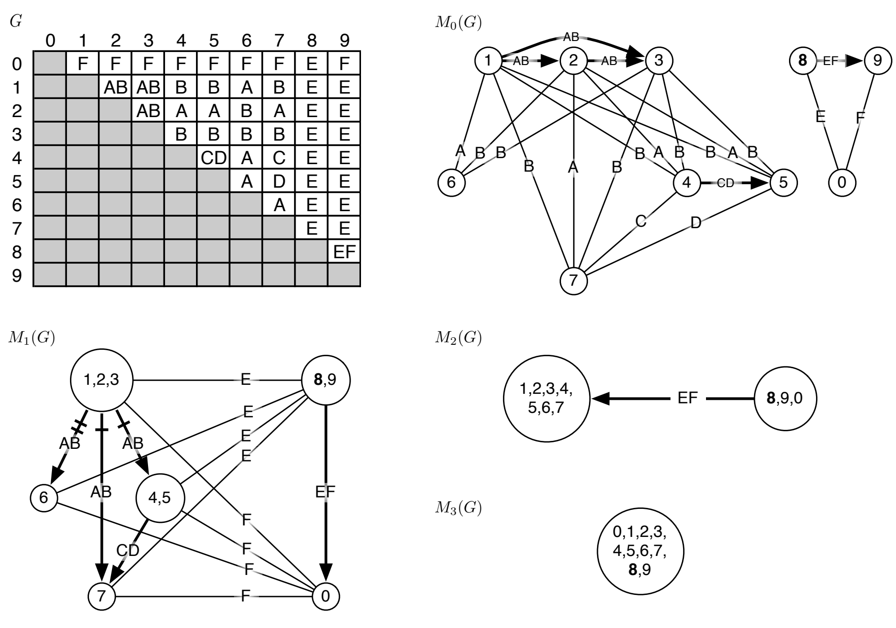
(1′) $\mathcal{V}_n$ is the partition of $\mathcal{V}$ induced by the weak components of $M_{n-1}(G)$,

(2′) $\mathcal{A}_n := \{(U,V) \in \mathcal{V}_n^2 : U \to V\}$, and

(3′) $\mathcal{E}_n := \{\{U,V\} \in [\mathcal{V}_n]^2 : |\rho[UV]| = 1\}$.

For each n, ρ induces a list-coloring, ρ', of $\mathcal{E}_n \cup \mathcal{A}_n$ by $\rho'(e) = \rho[UV]$ where $e = (U,V)$ or $e = \{U,V\}$. Likewise, Σ induces a partition of $\mathcal{A}_n$ by $(U_1,V_1) \sim_\Sigma (U_2,V_2)$ if and only if $\Sigma(U_1,V_1) = \Sigma(U_2,V_2)$. Note that $(U_1,V_1) \sim_\Sigma (U_2,V_2)$ if and only if $U_1 = U_2$ and $\rho[uV_1] = \rho[uV_2]$ for all $u \in U_1$.

Figure (1) shows an example of this sequence for a particular Gallai multigraph. For readability, we show only those edges in $M_n(G)$ that contribute to the formation of directed edges in $M_{n+1}(G)$. The hash marks on the directed edges in $M_1(G)$ indicate whether the signatures agree or disagree. Notice that vertex 8 is bolded. In the notation to be introduced immediately proceeding Lemma 3.6, this particular vertex will be identified as $\mathbf{1}_{V(G)}$ and has the property of being connected to the rest of G by two colors, E and F.



	0	1	2	3	4	5	6	7	8	9
0		F	F	F	F	F	F	F	E	F
1			AB	AB	B	B	A	B	E	E
2				AB	A	A	B	A	E	E
3					B	B	B	B	E	E
4						CD	A	C	E	E
5							A	D	E	E
6								A	E	E
7									E	E
8										EF
9										

FIGURE 1. Sequence of $M_n(G)$ for a Gallai multigraph.

3.3. **Decomposition of Maximal Gallai Multigraphs.** We may now state our main result.

Theorem 3.3. *Let G be a reduced maximal Gallai multigraph, H an induced subgraph of G, and $M_n(H) = (\mathcal{V}_n, \mathcal{E}_n, \mathcal{A}_n)$ the sequence described above. Then*

(1) *$M_n(H)$ is complete,*

(2) $|\rho'(e)| = \begin{cases} 1 & \text{if} \quad e \in \mathcal{E}_n \\ 2 & \text{if} \quad e \in \mathcal{A}_n \end{cases}$,

(3) *the weak components of $M_n(H)$ form rooted trees, and*

(4) *if $(U,V),(V,W) \in \mathcal{A}_n$, then $(U,V) \sim_\Sigma (U,W)$*

for all $n \geq 0$.

For convenience, if $M_k(H)$ has properties (1)-(4) for all $k \leq n$, we will say that H has the *tree property* for n.

3.4. **Proof of Theorem 3.3.** Throughout this section, we assume $(G = (\mathcal{V}, \mathcal{E}), \rho)$ is a reduced maximal Gallai multigraph and H an induced subgraph of G.

Lemma 3.4. *Suppose $U, V, W \subseteq \mathcal{V}$ disjoint, $U \to V$, and $\{A, B\} = \overline{UV}$.*

(1) *If $\overline{UW} = C \notin \{A, B\}$, then $\overline{VW} = C$.*

(2) *If $\overline{VW} = C \notin \{A, B\}$, then either $C \in \overline{UW}$ or $U \to W$ and $\Sigma(U,V) = \Sigma(U,W)$. If we also know that $U \to W, W \to U$, or $|\overline{UW}| = 1$ and we know that U always dominates with the same colors (i.e., whenever $U \to U'$, then $\overline{UU'} = \{A, B\}$), then either $\overline{UW} = C$ or $U \to W$ and $\Sigma(U,V) = \Sigma(U,W)$.*

(2′) *If W is a single vertex, we need only require $C \in \overline{VW}$ in (2).*

Proof. (1) Fix $v \in V$ and $w \in W$. Since $U \to V$, we may select $u_A, u_B \in U$ such that $A \in \overline{u_A v}$ and $B \in \overline{u_B v}$. Observe that the triangle w, u_A, v forces $\overline{vw} \subseteq \{A, C\}$ while w, u_B, v forces $\overline{vw} \subseteq \{B, C\}$. Thus $\overline{vw} = C$. Since v and w were arbitrary, $\overline{VW} = C$.

(2) Fix $u \in U, w \in W, v \in V$. Since $\overline{UV} = \{A, B\}$. We are in one of the following cases: $\overline{uv} = A$, $\overline{uv} = B$, or $\overline{uv} = \{A, B\}$. If $\overline{uv} = \{A, B\}$, then the fact that $\overline{vw} = C$ forces $\overline{uw} = C$ and thus $C \in \overline{UW}$. If $\overline{uv} = A$, then $\overline{uw} \subseteq \{A, C\}$ and thus if $C \notin \overline{UW}$, then $\overline{uw} = A$. Likewise, if $\overline{uv} = B$, then either $C \in \overline{uw}$ or $\overline{uw} = B$. We thus have either $C \in \overline{UW}$ or $U \to W$ and $\Sigma(U, V) = \Sigma(U, W)$.

Suppose we also know that we are in one of the following cases:

(i) $U \to W$ and $\overline{UW} = \{A, B\}$,

(ii) $W \to U$, or

(iii) $|\overline{UW}| = 1$.

Again, if $C \notin \overline{UW}$, then we must be in case (i). We would like to conclude that if $C \in \overline{UW}$, then we are in case (iii) and thus $\overline{UW} = C$. Suppose $W \to U$. To avoid a rainbow triangle, $\overline{UW} \subseteq \{A, B, C\}$. In particular, since $|\overline{UW}| \geq 2$, $A \in \overline{UW}$ or $B \in \overline{UW}$. Assume $A \in \overline{UW}$ and fix $w \in W$ such that $A \in \overline{wU}$. We may then choose $u \in U$ and $v \in V$ such that $B \in \overline{uv}$ but now u, v, w is a rainbow triangle. Thus $W \not\to U$ and $\overline{UW} = C$.

(2′) Let $W = \{w\}$ and $V = \{v\}$. Note that we are very nearly in the original setup of part (2). In particular, we still have $U \to V$, $\overline{UV} = \{A, B\}$, and each pair of vertices between V and W is connected by the color C. The only place where this relaxation might affect the proof in (2) is at the beginning where we consider $\overline{uv} = \{A, B\}$. We now know only that $C \in \overline{VW}$ but it follows immediately that $\overline{VW} = C$ and the proof follows as before.

$\square$

Lemma 3.5. *H has the tree property for* 0.

Proof. It is clear that $M_0(H)$ is complete. The rest of the claim is essentially a restatement of lemma 3.2. By the definition of dominance between single vertices, each complete, uniformly colored clique from lemma 3.2 becomes a linear ordered set of vertices and thus a rooted tree.

In this context, property (4) of Theorem 3.3 is simply the observation that these cliques are uniformly colored. □

Before proceeding, we introduce some convenient notation. Elements of $\mathcal{V}_n$ are by definition subsets of $V(H)$. We will however at times want to speak of their structure as rooted trees. For $U \in \mathcal{V}_n$, we write $\Upsilon(U)$ to refer to the set of elements of $\mathcal{V}_{n-1}$ contained in U and 1_U to refer to the root of $\Upsilon(U)$. Notice that $1_U \in \mathcal{V}_{n-1}$ has its own tree structure and thus we may refer to 1_{1_U}, $1_{1_{1_U}}$, etc. We may continue this recursion until we reach a single vertex. We write $\mathbf{1}_U$ to refer to this single vertex. Similarly, for $u \in V(H)$, we write $[u]_n$ to refer to the unique $U \in \mathcal{V}_n$ containing u. Lastly, we point out how this notation fits together. For $U \in \mathcal{V}_n$, $[\mathbf{1}_U]_n = U$, $[\mathbf{1}_U]_{n-1} = 1_U$, $[\mathbf{1}_U]_{n-2} = 1_{1_U}$, ..., and $[\mathbf{1}_U]_0 = \mathbf{1}_U$.

We also associate a set of colors with each member of $\mathcal{V}_n$ as follows. For $u \in \mathcal{V}_0$, $\widehat{u} := \cup_{u \to v} \overline{uv}$ and for $U \in \mathcal{V}_n$ with $n > 0$, $\widehat{U} := \widehat{1_U}$. Lemma 3.6 demonstrates the importance of this notation.

Lemma 3.6. *Suppose H has the tree property for n and $(U, V) \in \mathcal{A}_{n+1}$. Then U always dominates with the same two colors $\widehat{U}$, i.e.*

(1) $\overline{UV} = \widehat{U}$ *and*
(2) $|\widehat{U}| = 2$.

Proof. It is clear that $|\widehat{U}| = 2$ since $\widehat{U}$ is defined inductively and dominance between two vertices must be with exactly two colors. Likewise, (1) certainly holds for $U, V \in \mathcal{V}_0$.

Since H has the tree property for n and $U \to V$, $|\overline{U'V}| = 1$ for all $U' \in \Upsilon(U)$ and $|\overline{UV}| \geq 2$. Suppose we find $C \in \overline{UV} \setminus \widehat{U}$. Then fix $U_C \in \Upsilon(U)$ such that $\overline{U_C V} = C$. If $U_C = 1_U$, then for each $U' \in \Upsilon(U) \setminus \{1_U\}$, we may apply part (1) of Lemma 3.4 with $1_U \to U'$, $\overline{1_U U'} = \widehat{U}$, and $\overline{1_U V} = C \notin \widehat{U}$ to conclude that $\overline{U'V} = C$. We thus have $\overline{UV} = C$, a contradiction.

Suppose then that $1_U \neq U_C$ and thus $1_U \to U_C$. By induction, $\overline{1_U U_C} = \widehat{U}$. We may now apply part (2) of Lemma 3.4 with $1_U \to U_C$, $\overline{1_U U_C} = \widehat{U}$, and $\overline{U_C 1_V} = C \notin \widehat{U}$ to get that either $C \in \overline{1_U 1_V}$ or $1_U \to 1_V$. The former has already been ruled out while the latter contradicts the assumption that 1_U and 1_V were in different components of $\mathcal{V}_n$. □

The following lemma is useful because it allows us to locate a vertex in U that is connected to the rest of U by only the colors contained in $\widehat{U}$.

Lemma 3.7. *If H has the tree property for n and $U \in \mathcal{V}_{n+1}$, then $\overline{U\mathbf{1}_U} = \widehat{U}$.*

Proof. By Lemma 3.6, $|\widehat{U}| = 0$ or 2. Note that $|\widehat{U}| = 0$ if and only if $U = \{\mathbf{1}_U\}$. In this case, $\overline{U\mathbf{1}_U} = \widehat{U} = \emptyset$. We now proceed with the assumption that $|\widehat{U}| = 2$ and thus $|U| \geq 2$. For $n = 0$, U is a nontrivial uniformly colored clique, in which case $\widehat{U}$ is by definition $\overline{\mathbf{1}_U U}$.

For $n \geq 1$, by induction $\overline{1_U \mathbf{1}_{1_U}} = \widehat{1_U} = \widehat{U}$. But since $\mathbf{1}_{1_U} = \mathbf{1}_U$, we have $\overline{1_U \mathbf{1}_U} = \widehat{U}$ and thus $\widehat{U} \subseteq \overline{\mathbf{1}_U U}$. By Lemma 3.6, $\overline{1_U U'} = \widehat{U}$ for every $U' \in \Upsilon(U) \setminus \{1_U\}$. Finally, given that $\mathbf{1}_U \in 1_U$, we have $\overline{\mathbf{1}_U U'} \subseteq \overline{1_U U'} = \widehat{U}$. Thus $\overline{\mathbf{1}_U U} = \widehat{U}$. □

Lemmas 3.8 and 3.9 will be used in situations where a tree is connected to another tree or vertex by a color not present in the dominating colors of the first tree.

Lemma 3.8. *Suppose H has the tree property for n, $U, V \in \mathcal{V}_{n+1}$ distinct, and $V' \in \Upsilon(V)$ such that $C \in \overline{UV'} \setminus \widehat{U}$. Then $\overline{UV'} = C$.*

Proof. Suppose that $U' \in \Upsilon(U) \setminus \{1_U\}$ such that $\overline{U'V'} = C$. Applying part (2) of Lemma 3.4 with $1_U \to U'$ and $\overline{U'V'} = C \notin \overline{1_U U'}$, we get that $\overline{1_U V'} = C$ or $1_U \to V'$. Since 1_U and V' are in different components of $\mathcal{V}_{n+1}$, we must be in the case $\overline{1_U V'} = C$.

For each $U' \in \Upsilon(U) \setminus \{1_U\}$ we may apply part (1) of Lemma 3.4 with $1_U \to U'$ and $\overline{1_U V'} = C \notin \overline{1_U U'}$ to get that $\overline{U'V'} = C$ and thus $\overline{UV'} = C$. □

Lemma 3.9. *Suppose H has the tree property for n, $U \in \mathcal{V}_{n+1}$, and $v \in \mathcal{V}$ such that $\overline{\mathbf{1}_U v} = C \notin \widehat{U}$. Then $\overline{Uv} = C$.*

Proof. First observe that $v \notin U$ since by Lemma 3.7, $\overline{\mathbf{1}_U U} = \widehat{U}$. Next let k be maximal such that $\overline{[\mathbf{1}_U]_k v} = C$. If $k = n+1$, we are done. Suppose $k < n+1$ and select $u \in [\mathbf{1}_U]_{k+1} \setminus [\mathbf{1}_U]_k$. We may apply (1) of Lemma 3.4 with $[\mathbf{1}_U]_k \to u$ and $\overline{[\mathbf{1}_U]_k v} = C \notin \overline{[\mathbf{1}_U]_k u}$ to get that $\overline{uv} = C$. Thus $\overline{[\mathbf{1}_U]_{k+1} v} = C$, which violates the maximality of k. □

Note that in Lemma 3.9 we do not require that v be in $V(H)$ but rather in the larger set $\mathcal{V}$.

Lemma 3.10. *For $n \geq 0$, suppose H has the tree property for n and $U, V \in \mathcal{V}_{n+1}$ such that $\overline{UV} \subseteq \widehat{U} = \widehat{V} = \{A, B\}$ Then the following statements are equivalent:*

(1) *$U \to V$,*
(2) *there is $x \in \mathcal{V} \setminus (U \cup V)$ such that $U \to x$ and $\overline{Vx} = C \notin \{A, B\}$, and*
(3) *$V \not\to U$ and $\overline{UV} = \{A, B\}$.*

Furthermore, when the statements are true, $\Sigma(U, V) = \Sigma(U, x)$.

Proof. ($1 \Rightarrow 2$) We may assume $\overline{1_U 1_V} = A$. Since G is maximal and $B \notin \overline{\mathbf{1}_U \mathbf{1}_V}$, there is $x \in \mathcal{V}$ and $C \notin \{A, B\}$ such that $A \in \overline{\mathbf{1}_U x}$ and $C \in \overline{x \mathbf{1}_V}$ or $C \in \overline{\mathbf{1}_U x}$ and $A \in \overline{x \mathbf{1}_V}$. Notice that if $C \in \overline{\mathbf{1}_U x}$, since $C \notin \widehat{U}$, $|\mathbf{1}_U x| = 1$ and thus $C = \overline{\mathbf{1}_U x}$. Likewise, if $C \in \overline{\mathbf{1}_V x}$, then $C = \overline{\mathbf{1}_V x}$. Furthermore, in either case, since $C \notin \overline{\mathbf{1}_U U} = \overline{\mathbf{1}_V V}$, we must conclude that $x \in \mathcal{V} \setminus (U \cup V)$.

Suppose we are in the latter case, i.e. $C = \overline{\mathbf{1}_U x}$ and $A \in \overline{x \mathbf{1}_V}$. By Lemma 3.9, $\overline{Ux} = C$. We may then apply part (1) of Lemma 3.4 with $U \to V$ and $\overline{Ux} = C \notin \overline{UV}$ to get that $\overline{Vx} = C$, which contradicts our assumption that $A \in \overline{\mathbf{1}_V x}$.

We must then be in the former case, i.e. $A \in \overline{\mathbf{1}_U x}$ and $C = \overline{x \mathbf{1}_V}$ and, again by Lemma 3.9, $\overline{Vx} = C$. Note that if we can show that $C \notin \overline{Ux}$, we may then apply part (2) of Lemma

3.4 with $U \to V$ and $\overline{xV} = C \notin \overline{UV}$ to get that $U \to x$ and that $\Sigma(U,V) = \Sigma(U,x)$, which is exactly what we would like to prove.

To this end, suppose $C \in \overline{Ux}$ and let k be minimal such that $C \in \overline{[\mathbf{1}_U]_k x}$. If $k = 0$, we have that $C \in \overline{\mathbf{1}_U x}$ and it again follows that $\overline{Ux} = C$, which is a contradiction. Thus $k > 0$. Fix $u \in [\mathbf{1}_U]_k$ such that $C \in \overline{ux}$. Since k is minimal, $u \in [\mathbf{1}_U]_k \setminus [\mathbf{1}_U]_{k-1}$ and thus $[\mathbf{1}_U]_{k-1} \to u$. Recall that $\widehat{[\mathbf{1}_U]_i} = \widehat{U}$ for all $i \le n$ and thus $\overline{[\mathbf{1}_U]_{k-1} u} = \{A, B\}$.

We may now apply part (2′) of Lemma 3.4 with $[\mathbf{1}_U]_{k-1} \to u$ and $C \in \overline{ux}$ to get that either $C \in \overline{[\mathbf{1}_U]_{k-1} x}$ or $[\mathbf{1}_U]_{k-1} \to x$ and $\Sigma([\mathbf{1}_U]_{k-1}, x) = \Sigma([\mathbf{1}_U]_{k-1}, u)$. By the minimality of k, we must be in the latter case. Then we may apply part (2) of Lemma 3.4 with $[\mathbf{1}_U]_{k-1} \to x$ and $C = \overline{xV}$ to get that either $C \in \overline{[\mathbf{1}_U]_{k-1} V}$ or $[\mathbf{1}_U]_{k-1} \to V$. Both of theses cases contradict the assumption that $\overline{1_U V} = A$. Thus $C \notin \overline{Ux}$.

($2 \Rightarrow 3$) We may apply part (2) of Lemma 3.4 with $U \to x$ and $\overline{xV} = C \notin \overline{Ux}$ to get that either $C \in \overline{UV}$ or $U \to V$ and $\Sigma(U,x) = \Sigma(U,V)$. Since $C \notin \overline{UV} \subseteq \{A, B\}$, we are left in the case $U \to V$ and thus $\overline{UV} = \widehat{U} = \{A, B\}$ and $V \not\to U$.

We note here that we are using our earlier observation about dominance that except in trivial cases $U \to V$ implies $V \not\to U$.

($3 \Rightarrow 1$). Again we may assume $\overline{1_U 1_V} = A$. As argued in ($1 \Rightarrow 2$), we may find $x \in \mathcal{V} \setminus (U \cup V)$ such that either $A \in \overline{\mathbf{1}_U x}$ and $\overline{Vx} = C$ or $\overline{Ux} = C$ and $A \in \overline{\mathbf{1}_V x}$ for some $C \notin \{A, B\}$.

Suppose we are in the latter case. Since $V \not\to U$ and $\overline{UV} = \{A, B\}$, there must be $U_A, U_B \in \Upsilon(U)$ and $V' \in \Upsilon(V)$ such that $\overline{U_A V'} = A$ and $\overline{U_B V'} = B$. Then x, U_A, V' forces $\overline{xV'} \subseteq \{A, C\}$ while xU_BV' forces $\overline{xV'} \subseteq \{B, C\}$. Thus $\overline{xV'} = C$ and $C \in \overline{Vx}$.

We may then let k be minimal such that $C \in \overline{[\mathbf{1}_V]_k x}$. If $k = 0$, then $C \in \overline{\mathbf{1}_V x}$ and by Lemma 3.9 $\overline{Vx} = C$, which contradicts our assumption that $A \in \overline{\mathbf{1}_V x}$. Therefore $k > 0$. As before, we select $v \in [\mathbf{1}_V]_k \setminus [\mathbf{1}_V]_{k-1}$ such that $C \in \overline{vx}$. We now apply part (2′) of Lemma

3.4 with $[\mathbf{1}_V]_{k-1} \to v$ and $C \in \overline{xv}$ to get that either $C \in \overline{[\mathbf{1}_V]_{k-1}x}$ or $[\mathbf{1}_V]_{k-1} \to x$. By the minimality of k, we must be in the latter case and we may apply part (2) of Lemma 3.4 with $[\mathbf{1}_V]_{k-1} \to x$ and $C = \overline{1_U x}$ (recall our assumption that $\overline{Ux} = C$) to get that either $C \in \overline{1_U[\mathbf{1}_V]_{k-1}}$ or $[\mathbf{1}_V]_{k-1} \to 1_U$. Both of these cases contradict the assumption that $\overline{1_U 1_V} = A$.

We therefore may assume that $A \in \overline{\mathbf{1}_U x}$ and $\overline{Vx} = C$. It is either the case that $U \to V$ or $U \not\to V$. If we suppose that $U \not\to V$, then we are in the case just handled with the roles of U and V reversed. Since that assumption leads to a contradiction, we have that $U \to V$. $\square$

We have now developed sufficient technical tools to address the main points of Theorem 3.3.

Lemma 3.11. *If H has the tree property for n, then $M_{n+1}(H)$ is complete.*

Proof. Let $U, V \in \mathcal{V}_{n+1}$. If $|\overline{UV}| = 1$, then $\{U, V\} \in \mathcal{E}_{n+1}$. Suppose then that $|\overline{UV}| > 1$. Notice that if $\widehat{U} = \emptyset$, then U is a single vertex and thus $V \to U$ and $(V, U) \in \mathcal{A}_{n+1}$.

We may therefore assume $|\widehat{U}| = |\widehat{V}| = 2$ and consider the following cases:

Case 1: $\widehat{U} \neq \widehat{V}$ and $|\overline{UV}| > 2$. We may then select $C_U, C_V \in \overline{UV}$ distinct such that $C_U \notin \widehat{U}$ and $C_V \notin \widehat{V}$ and $U' \in \Upsilon(U), V' \in \Upsilon(V)$ such that $C_V \in \overline{U'V}$ and $C_U \in \overline{UV'}$. By Lemma 3.8, $\overline{U'V} = C_V$ and $\overline{UV'} = C_U$ and thus $C_V = \overline{U'V'} = C_U$, a contradiction.

Case 2: $\widehat{U} \neq \widehat{V}$ and $|\overline{UV}| = 2$. Without loss of generality, we may assume there is $C \in \overline{UV} \setminus \widehat{U}$. Let $\overline{UV} = \{C, D\}$. Select $V_C \in \Upsilon(V)$ such that $C \in \overline{UV_C}$. By Lemma 3.8, $\overline{UV_C} = C$. Now select $V_D \in \Upsilon(V)$ such that $D \in \overline{UV_D}$. Observe that if $C \in \overline{UV_D}$, then $D \notin \overline{UV_D} = C$. Thus $\overline{UV_D} = D$. Since $\overline{UV} = \{C, D\}$ and every element of $\Upsilon(V)$ is of the type V_C or V_D, we have accounted for every element of $\Upsilon(V)$ and thus $V \to U$, i.e. $(V, U) \in \mathcal{A}_{n+1}$.

Case 3: $\widehat{U} = \widehat{V} = \{A, B\}$ and $C \in \overline{UV} \setminus \{A, B\}$. Fix $U_C \in \Upsilon(U)$ such that $C \in \overline{U_C V}$. By Lemma 3.8, $\overline{U_C V} = C$ and thus for every $V' \in \Upsilon(V), C \in \overline{UV'}$. Applying Lemma 3.8 again, gives us that $\overline{UV'} = C$ and thus $\overline{UV} = C$. Thus $\{U, V\} \in \mathcal{E}_{n+1}$.

Case 4: $\overline{UV} = \widehat{U} = \widehat{V}$. If $V \not\to U$, apply $(3 \Rightarrow 1)$ from Lemma 3.10 to get that $U \to V$. $\square$

Lemma 3.12. *Suppose H has the tree property for n. The weak components of $M_{n+1}(H)$ are transitive, and if $(U, V), (V, W) \in \mathcal{A}_{n+1}$, then $(U, V) \sim_\Sigma (U, W)$.*

Proof. Let $U, V, W \in \mathcal{V}_{n+1}$ such that $U \to V$ and $V \to W$. By Lemma 3.6, $|\widehat{U}| = |\widehat{V}| = 2$. We consider two cases: $\widehat{U} \neq \widehat{V}$ and $\widehat{U} = \widehat{V}$.

Suppose $\widehat{U} \neq \widehat{V}$ and let $A \in \widehat{U} \setminus \widehat{V}$. Fix $U_A \in \Upsilon(U)$ such that $\overline{U_A V} = A$ and $V_1, V_2 \in \Upsilon(V)$ such that $\overline{V_1 W} \neq \overline{V_2 W}$. Fix $W' \in \Upsilon(W)$. We have that U_A, V_1, W' forces $\overline{U_A W'} \subseteq \{A, \overline{V_1 W'}\}$ while U_A, V_2, W' forces $\overline{U_A W'} \subseteq \{A, \overline{V_2 W'}\}$ and thus $\overline{U_A W'} = A$. Since W' was arbitrary, $\overline{U_A W} = A$. Note that since $|\overline{U_A W}| = 1$, we have ruled out the possibility that $W \to U$. By Lemma 3.11, we will be done if we can show that $|\overline{UW}| > 1$. Observe that we could also choose $U_B \in \Upsilon(U)$ such that $\overline{U_B V} = B \neq A$. If it happens that $B \notin \widehat{V}$, by the same reasoning as above $\overline{U_B W} = B$ so that $\{A, B\} \subseteq \overline{UW}$ and thus $U \to W$ and $\Sigma(U, V) = \Sigma(U, W)$.

Suppose then that $\widehat{U} = \{A, B\}$ and $\widehat{V} = \{B, C\}$. We can now find $U_A, U_B \in \Upsilon(U)$ such that $\overline{U_A W} = A$ and $\overline{U_B W} \subseteq \{B, C\}$. Therefore $|\overline{UW}| > 1$ and by Lemma 3.11 we have that $U \to W$. By Lemma 3.6, $\overline{UW} = \widehat{U} = \{A, B\}$ and thus $\overline{U_B W} = B$. Therefore, $\Sigma(U, V) = \Sigma(U, W)$.

We now consider the case $\widehat{U} = \widehat{V} = \{A, B\}$. Note that $\overline{UW} \subseteq \{A, B\}$ since we may otherwise easily form a rainbow triangle. We now have the setup for $(1 \Rightarrow 2)$ of Lemma 3.10 with $U \to V$ and have $x \in \mathcal{V} \setminus (U \cup V)$ such that $U \to x$, $\Sigma(U, V) = \Sigma(U, x)$, and $\overline{xV} = C \notin \{A, B\}$. Applying part (1) of Lemma 3.4 to $V \to W$ and $\overline{Vx} = C \notin \overline{VW}$, we have that $\overline{xW} = C$. Now apply part (2) of Lemma 3.4 with $U \to x$ and $\overline{xW} = C \notin \overline{Ux}$ to get that either $C \in \overline{UW}$ or $U \to W$ and $\Sigma(U,W) = \Sigma(U,x) = \Sigma(U,V)$. We have already ruled out the former while the latter is what we sought to prove. $\square$

Lemma 3.13. *Suppose H has the tree property for n. The weak components of $M_{n+1}(H)$ form rooted trees.*

Proof. After Lemma 3.12, we need only show that for $U_1, U_2, V \in \mathcal{V}_{n+1}$ distinct, if $U_1 \to V$ and $U_2 \to V$, then either $U_1 \to U_2$ or $U_2 \to U_1$. By Lemma 3.11, it suffices to show $|\overline{U_1U_2}| > 1$.

Suppose $|\overline{U_1U_2}| = 1$. Observe that if $|\overline{U_1V} \cup \overline{U_2V} \cup \overline{U_1U_2}| > 2$, then we must find a rainbow triangle in U_1, U_2, V. Thus we may assume $\overline{U_1U_2} = A \in \widehat{U_1} = \widehat{U_2} = \{A, B\}$. As in the proof of Lemma 3.10, since $\widehat{U_1} = \widehat{U_2}$, we may select $x \in \mathcal{V} \setminus (U_1 \cup U_2)$ such that $\overline{U_1x} = C \notin \{A, B\}$ and $A \in \overline{\mathbf{1}_{U_2}x}$ (or with the roles of U_1 and U_2 reversed). We may then apply part (1) of Lemma 3.4 with $U_1 \to V$ and $\overline{U_1x} = C \notin \overline{U_1V}$ to get that $\overline{Vx} = C$ and apply part (2) of Lemma 3.4 with $U_2 \to V$ and $\overline{Vx} = C \notin \overline{U_2V}$ to get that either $C \in \overline{U_2x}$ or $U_2 \to x$.

First we consider the case $C \in \overline{U_2x}$. As before, let k be minimal such that $C \in \overline{[\mathbf{1}_{U_2}]_k x}$. If $k = 0$, by Lemma 3.9, $\overline{U_2x} = C$, which contradicts our assumption that $A \in \overline{\mathbf{1}_{U_2}x}$. Thus $k > 0$ and we may select $u \in [\mathbf{1}_{U_2}]_k \setminus [\mathbf{1}_{U_2}]_{k-1}$ such that $C \in \overline{ux}$. We may apply part (2′) of Lemma 3.4 with $[\mathbf{1}_{U_2}]_{k-1} \to u$ and x to get that either $C \in \overline{[\mathbf{1}_{U_2}]_{k-1}x}$ or $[\mathbf{1}_{U_2}]_{k-1} \to x$. By the minimality of k, we must be in the latter case. Note that $B \in \overline{[\mathbf{1}_{U_2}]_{k-1}x} = \widehat{U_2} = \{A, B\}$. Thus by our assumptions that $\overline{U_1U_2} = A$ and $\overline{U_1x} = C$, we may locate a rainbow triangle in U_1, $[\mathbf{1}_{U_2}]_{k-1}$, x.

We turn now to the second case, $U_2 \to x$. We may apply part (2) of Lemma 3.4 with $U_2 \to x$ and $\overline{xU_1} = C \notin \overline{U_2x}$ to get that either $C \in \overline{U_1U_2}$ or $U_2 \to U_1$, which both contradict our assumption that $\overline{U_1U_2} = A$. Thus $U_1 \to U_2$ or $U_2 \to U_1$. $\square$

Taking Lemmas 3.6, 3.11, 3.12, and 3.13 together we have proved Theorem 3.3.

3.5. **Alternative proof of Lemma 2.3.** We now point out how it follows from Theorem 3.3 that every Gallai multigraph having more than two vertices can be disconnected by the removal two colors.

Proof. We first observe that it suffices to prove the claim for reduced Gallai multigraphs. Let G be Gallai multigraph with more than two vertices. If G is not reduced, then we may form a smaller Gallai multigraph G' by collapsing an isolated edge in G. If removing the colors A and B disconnects G', then it also disconnects G. We may continue collapsing isolated edges until we reach either a single vertex or a non-trivial reduced Gallai multigraph. In the former case, prior to collapsing the final edge, we had two vertices connected by at most two colors. Thus these two colors suffice to disconnect G.

We will thus be done if we can establish the claim for reduced Gallai multigraphs, and, in turn, it suffices to establish the claim for reduced maximal Gallai multigraphs. Suppose then that G is a reduced maximal Gallai multigraph. By Theorem 3.3 , the sequence $M_n(G)$ terminates at either a single vertex or a non-trivial Gallai graph. In the latter case, we are done by Lemma 2.1.

Suppose then that $M_n(G)$ terminates at a singe vertex. It then follows from Theorem 3.3 and Lemma 3.6 that $\mathbf{1}_{V(G)}$ is connected to G by two colors and thus the removal of these colors disconnects G. □

4. Related Open Problems

We have shown that every non-trivial Gallai multigraph can be disconnected by the removal of two colors and that such multigraphs decompose into a nested sequence of directed trees. Although Gallai multigraphs can be constructed from the simple iterative method presented in §2, we had hoped that by naïvely reversing the decomposition into directed trees we might construct precisely the maximal Gallai multigraphs. This approach can be shown to construct all Gallai multigraphs but we were unable to carry it out in such a way as to arrive at only the maximal ones. We thus leave open the following problem: is there a direct construction of all maximal Gallai multigraphs and, if so, can it be carried out in some canonical manner? An affirmative answer to the latter question may provide a powerful tool to answering certain extremal questions about Gallai multigraphs.

We came at the topic of Gallai multigraphs through our search for a construction of complete edge-colored graphs lacking rainbow 4-cycles. We remain interested in this problem and the following related questions. We call a finite, simple, complete, edge-colored graph 4-*Gallai* if it lacks rainbow 4-cycles.

- Is there a finite number of colors that always suffice to disconnect a 4-Gallai graph?
- If so, does it lead to an analogous construction for all 4-Gallai graphs?
- If not, is there some other simple construction?

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