

EMBEDDINGS OF FINITE DIMENSIONAL OPERATOR SPACES INTO THE SECOND DUAL

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ABSTRACT. We show that, if a finite dimensional operator space E is such that X contains E C -completely isomorphically whenever X^{**} contains E completely isometrically, then E is $2^{15}C^{11}$ -completely isomorphic to $\mathbf{R}_m \oplus \mathbf{C}_n$, for some $n, m \in \mathbb{N} \cup \{0\}$. The converse is also true: if X^{**} contains $\mathbf{R}_m \oplus \mathbf{C}_n$ λ -completely isomorphically, then X contains $\mathbf{R}_m \oplus \mathbf{C}_n$ $(2\lambda + \varepsilon)$ -completely isomorphically, for any $\varepsilon > 0$.

1. INTRODUCTION

Local reflexivity of Banach spaces was first discovered by J. Lindenstrauss and H. Rosenthal in [12]. Later, W. Johnson, H. Rosenthal, and M. Zippin [9] improved on this result, and obtained:

Theorem 1.1. *Suppose X is a Banach space, E and F are finite dimensional subspaces of X^{**} and X^* , respectively, and $\varepsilon > 0$. Then there exists an operator $u : E \rightarrow X$ s.t. $\|u\| < 1 + \varepsilon$, $u|_{E \cap X} = I_{E \cap X}$, and $\langle ue, f \rangle = \langle e, f \rangle$ for any $e \in E$ and $f \in F$.*

This immediately implies the result of [12]:

Corollary 1.2. *Suppose E and X are Banach spaces, E is a finite dimensional space, and E is contained in X^{**} C -isomorphically (that is, there exists $E' \hookrightarrow X^{**}$ such that $d(E, E') \leq C$). Then E is contained in X $(C + \varepsilon)$ -isomorphically, for any $\varepsilon > 0$.*

In the non-commutative case, the results quoted above do not hold in general. It is well known that an infinite dimensional operator space need not be locally reflexive. Moreover, for every $n > 2$ the space ℓ_1^n (equipped with the maximal operator space structure) is contained in $\mathbf{B} = \mathbf{K}^{**}$, while, by Theorem 21.5 of [16], $d_{cb}(\ell_1^n, E) \geq n/(2\sqrt{n-1})$ for any $E \hookrightarrow \mathbf{K}$ (here and below, \mathbf{B} and \mathbf{K} denote the spaces of bounded and compact operators on ℓ_2 , respectively).

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In this paper, we show that a “non-commutative” analogue of the Corollary 1.2 holds for a finite dimensional operator space E if and only if E is completely isomorphic to a direct sum of row and column spaces (Theorem 1.3). We say that X contains E C -completely isomorphically (C -ci) if there exists $F \hookrightarrow X$ s.t. $d_{cb}(E, F) \leq C$. X is said to contain E C_+ -ci if it contains E C_1 -ci for every $C_1 > C$. A finite dimensional operator space E is said to be C -bidually representable (C -BDR, for short), if an operator space X contains E C -completely isomorphically whenever X^{**} contains a completely isometric copy of E . E is said to be C_+ -BDR if it is C_1 -BDR for any $C_1 > C$.

Below, \mathbf{R}_n and \mathbf{C}_n stand for the spaces spanned by the first row and the first column of $n \times n$ matrices, respectively. \oplus means \oplus_∞ (the ℓ_∞ sum of spaces), unless specified otherwise.

Theorem 1.3. (1) If E is λ -completely isomorphic to $\mathbf{R}_m \oplus \mathbf{C}_n$ for some $m, n \in \mathbb{Z}_+$, then E is $2\lambda^2$ -BDR.

(2) If E is λ -BDR, then, for some $m, n \in \mathbb{Z}_+$, $d_{cb}(E, \mathbf{R}_m \oplus \mathbf{C}_n) \leq 2^{15}\lambda^{11}$.

Theorem 1.4. An operator space is 1-BDR if and only if it is 1-dimensional.

The rest of the paper is organized as follows: in Section 2, we gather some essential facts about non-dual local reflexivity, and prove item (1) of Theorem 1.3. The proof of Theorem 1.3(2) proceeds in two steps. First, we show that E embeds “nicely” into $\mathbf{R} \oplus \mathbf{C}$ (Section 3). In Section 4 we complete the proof of Theorem 1.3(2). Section 5 is devoted to proving Theorem 1.4. Finally, in Section 6, we consider our problem in the setting of C^* -algebras.

The notation used in this paper is, by and large, either standard, or explained above. The minimal (also called injective, or spatial) tensor product of operator spaces is denoted by \otimes . If $T : X \rightarrow Y$ is a finite rank operator, \tilde{T} stands for the corresponding element of $X^* \otimes Y$ (then $\|T\|_{cb} = \|\tilde{T}\|$). We often use \mathbf{M}_n , \mathbf{B} , \mathbf{K} , and \mathbf{K}_0 – the space of $n \times n$ matrices, the space of bounded operators on ℓ_2 , the space of compact operators on ℓ_2 , and the space of compact operators with matrices with finitely many non-zero entries, respectively.

In the proofs, we use the notion of exactness of an operator space, and the notion of a complete M -ideal in an operator space.

We say that an operator space Z is C -exact ($C > 0$) if, for any finite dimensional subspace $E \hookrightarrow Z$, and every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ and $F \hookrightarrow M_N$ with the property that $d_{cb}(E, F) < C + \varepsilon$. Z is said to be *exact* if it is C -exact for some C . The *exactness constant* of Z (denoted by $\text{ex}(Z)$) is the infimum of all the C 's with the above property. It is easy to see that the row space \mathbf{R} , the column space \mathbf{C} , and the space of compact operators \mathbf{K} are 1-exact. On the other hand, it is known that

\mathbf{B} is not exact. The reader is referred to [15], Chapter 17 of [16], or Chapter 14 of [6] for more information.

A subspace X of an operator space Y is called a *complete M -summand* if $Y = X \oplus_\infty Z$, for some $Z \hookrightarrow Y$. X is a *complete M -ideal* in Y if $X^{\perp\perp}$ is a complete M -summand in Y^{**} . We refer the reader to [5], or to Section 4.8 of [2], for information about complete M -ideals. For the theory of M -ideals in Banach spaces, see [8].

Finally, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ is the set of non-negative integers.

2. SOME REMARKS ON BIDUAL REPRESENTABILITY

To prove Theorem 1.3(1), begin with

Proposition 2.1. *Suppose E and F are finite dimensional operator spaces, X is an operator space, $u : E \rightarrow X^{**}$ and $v : X^{**} \rightarrow F$ are linear maps, and $\varepsilon > 0$. Then there exist linear maps $u_1 : E \rightarrow X$ and $v_1 : X \rightarrow F$ such that $\|u_1\|_{cb}\|v_1\|_{cb} < (1 + \varepsilon)\text{ex}(E^*)\text{ex}(F)\|u\|_{cb}\|v\|_{cb}$, and $v_1u_1 = vu$.*

Proof. Pick biorthogonal systems $(e_i, e_i^*)_{i=1}^n$ and $(f_j, f_j^*)_{j=1}^m$ in E and F , respectively. Write $\tilde{u} = \sum_i e_i^* \otimes x_i^{**}$ and $\tilde{v} = \sum_j f_j \otimes x_j^{***}$, with $x_i^{**} \in X^{**}$ and $x_j^{***} \in X^{***}$. By Proposition 3.2.1 of [6] $(\mathbf{M}_N(X))^{**} = \mathbf{M}_N(X^{**})$ isometrically, hence $\tilde{v} \in X^{***} \otimes F$ can be “approximated” by $\tilde{v}_1 = \sum_j f_j \otimes x_j^* \in F \otimes X^*$ in such a way that (1) $\|\tilde{v}_1\| < \sqrt{1 + \varepsilon}\text{ex}(F)\|\tilde{v}\|$, and (2) $\langle x_j^*, x_i^{**} \rangle = \langle x_j^{***}, x_i^{**} \rangle$ for any pair (i, j) . Similarly, there exists $\tilde{u}_1 \in E^* \otimes X$ s.t. $\|\tilde{u}_1\| \leq \sqrt{1 + \varepsilon}\text{ex}(E^*)\|\tilde{u}\|$, and $\langle x_j^*, x_i^{**} \rangle = \langle x_j^*, x_i \rangle$ for any pair (i, j) .

Now go back from tensor products to c.b. maps $u_1 : E \rightarrow X$ and $v_1 : X \rightarrow F$. By the above, $\|u_1\|_{cb} < \sqrt{1 + \varepsilon}\text{ex}(E^*)\|u\|_{cb}$, and $\|v_1\|_{cb} < \sqrt{1 + \varepsilon}\text{ex}(F)\|v\|_{cb}$. Moreover, for any i ,

$$v_1u_1e_i = \sum_j \langle x_j^*, x_i \rangle f_j = \sum_i \langle x_j^{***}, x_i^{**} \rangle f_j = v_1ue_j.$$

Therefore, $vu = v_1u_1$, and $\|u_1\|_{cb}\|v_1\|_{cb} < (1 + \varepsilon)\text{ex}(E^*)\text{ex}(F)\|u\|_{cb}\|v\|_{cb}$. ■

Proposition 2.1 implies:

Corollary 2.2. *Suppose X is an operator space, and $\delta > 0$. Then:*

- (1) *If X^{**} contains a λ -injective finite dimensional operator space E C -completely isomorphically, then X contains E $(\lambda C \text{ex}(E) \text{ex}(E^*) + \delta)$ -completely isomorphically.*
- (2) *If X^{**} contains $\mathbf{R}_m \oplus \mathbf{C}_n$ ($n, m \in \mathbb{Z}_+$) C -completely isomorphically, then X contains $\mathbf{R}_m \oplus \mathbf{C}_n$ $(2C + \delta)$ -completely isomorphically. Consequently, $\mathbf{R}_m \oplus \mathbf{C}_n$ is $2+$ -BDR.*
- (3) *Suppose $E = \mathbf{R}_m, \mathbf{C}_m, \ell_\infty^2, \mathbb{C} \oplus_\infty \mathbf{R}_m$, or $\mathbb{C} \oplus_\infty \mathbf{R}_m$ ($m \in \mathbb{N}$). If X^{**} contains E C -completely isomorphically, then X contains E $(C + \delta)$ -completely isomorphically. Consequently, E is $1+$ -BDR.*

Proof. (1) Consider a subspace $E' \hookrightarrow X^{**}$, for which there exists a completely contractive isomorphism $uE \rightarrow E'$ with $\|u^{-1}\|_{cb} \leq C$. As E is λ -injective, there exists $v : X^{**} \rightarrow E$, extending u^{-1} (that is, $v|_{E'} = u^{-1}$), of norm not exceeding λC . By Proposition 2.1, for every $\varepsilon > 0$ there exist $u_1 : E \rightarrow X$ and $v_1 : X \rightarrow E$, such that $\|u_1\|_{cb}\|v_1\|_{cb} \leq (1 + \varepsilon)\text{ex}(E)\text{ex}(E^*)\lambda C$, and $v_1u_1 = vu = I_E$. Therefore, $d_{cb}(E, u_1(E)) \leq (1 + \varepsilon)\text{ex}(E)\text{ex}(E^*)\lambda C$. Since ε can be arbitrarily small, we are done.

(2) The space $\mathbf{R}_m \oplus \mathbf{C}_n$ embeds into \mathbf{M}_{m+n} as 1-completely complemented subspace, hence $\text{ex}(\mathbf{R}_m \oplus \mathbf{C}_n) = 1$. Moreover, $(\mathbf{R}_m \oplus \mathbf{C}_n)^* = \mathbf{C}_n \oplus_1 \mathbf{R}_m$ is 2-completely isomorphic to $\mathbf{C}_n \oplus \mathbf{R}_m$, hence $\text{ex}((\mathbf{R}_m \oplus \mathbf{C}_n)^*) \leq 2$. An application of part (1) yields the desired result.

(3) Reason as in the proof of part (2), and recall that $\text{ex}(E) = \text{ex}(E^*) = 1$ for any E from the list (see Chapter 21 of [16]). \blacksquare

Remark 2.3. We return to the connections between injectivity, exactness, and BDR in Section 5. Meanwhile, note that by Theorem 1.4, \mathbb{C} is the only 1-BDR space. Hence, the condition of being 1-BDR is much stronger than being 1+-BDR.

The next result shows that bidual representability is an isomorphic property.

Proposition 2.4. *Suppose X is an operator space, E is a finite dimensional subspace of X^{**} , $d_{cb}(E, F) = C_1$, and $\text{ex}(F) < C_2$. Then there exists an operator space Y s.t. $d_{cb}(X, Y) < C_1C_2$, and Y^{**} contains a completely isometric copy of F .*

Proof. Suppose the map $u : E \rightarrow F$ is such that $\|u\|_{cb} = C_1$, and $\|u^{-1}\|_{cb} = 1$. By renorming X , we shall construct a space Y and a map $T : X \rightarrow Y$ s.t. $\|T\|_{cb} < C_1C_2$, and T^{-1} is completely contractive. To this end, find a subspace $F_1 \hookrightarrow \mathbf{M}_N$ and a map $v : F_1 \rightarrow F$ s.t. $\|v^{-1}\|_{cb} = 1$, and $\|v\|_{cb} < C_2$. Embedding F into \mathbf{B} completely isometrically, and applying Stinespring Extension Theorem, we obtain an operator $\tilde{v} : \mathbf{M}_N \rightarrow \mathbf{B}$ s.t. $\tilde{v}|_{F_1} = v$, and $\|\tilde{v}\|_{cb} = \|v\|_{cb}$. Let $\tilde{F} = \tilde{v}(\mathbf{M}_N)$.

Extend $w = v^{-1}u$ to $\tilde{w} : X^{**} \rightarrow \mathbf{M}_N$ s.t. $\|\tilde{w}\|_{cb} = \|w\|_{cb} \leq C_1$, and $\tilde{w}|_E = w$. Let $C'_1 = C_1C_2/\|v\|_{cb}$. The unit ball of $X^* \otimes \mathbf{M}_N$ is weak* dense in the unit ball of $X^{***} \otimes \mathbf{M}_N$, hence there exists an operator $w_1 : X \rightarrow \mathbf{M}_N$ s.t. $\|w_1\|_{cb} < C'_1$, and $w_1^{**}|_E = w$.

Define a new operator space structure Y on X by setting, for $x \in X \otimes \mathbf{K}_0$,

$$\|x\|_{Y \otimes \mathbf{K}} = \max\{\|x\|_{X \otimes \mathbf{K}}, \|(\tilde{v}w_1 \otimes I_{\mathbf{K}})x\|_{\tilde{F} \otimes \mathbf{K}}\}.$$

Denote by T the formal identity map from X to Y . Clearly, T^{-1} is a complete contraction, and $\|T\|_{cb} \leq \|\tilde{v}w_1\|_{cb} < C_1C_2$. Moreover, for $x^{**} \in X^{**} \otimes \mathbf{K}_0$,

$$\|x^{**}\|_{Y^{**} \otimes \mathbf{K}} = \max\{\|x^{**}\|_{X^{**} \otimes \mathbf{K}}, \|(\tilde{v}w_1^{**} \otimes I_{\mathbf{K}})x^{**}\|_{\tilde{F} \otimes \mathbf{K}}\},$$

which implies that, for $e \in E \otimes \mathbf{K}_0$, $\|e\|_{Y^{**} \otimes \mathbf{K}} = \|(u \otimes I_{\mathbf{K}})e\|_{F \otimes \mathbf{K}}$. \blacksquare

Proposition 2.5. *Suppose the space E is $C+$ -BDR. Then E is C -exact.*

Proof. Apply the definition of BDR to $X = \mathbf{K}$. ■

Corollary 2.6. *Suppose E and E' are operator spaces of the same (finite) dimension, E is C -BDR, $d_{cb}(E, E') = \lambda$, and $\text{ex}(E) < \mu$. Then an operator space X contains E $C\lambda\mu$ -completely isomorphically whenever X^{**} contains E' completely isometrically. Consequently, E' is $C\lambda^2\mu$ -BDR.*

Proof. Suppose E' is contained in X^{**} . Consider a map $u : E' \rightarrow E$, s.t. $\|u\|_{cb} = \lambda$, and $\|u^{-1}\|_{cb} = 1$. By the proof of Proposition 2.4, there exists an operator space Y and a map $T : X \rightarrow Y$ s.t. $\|T\|_{cb} < \lambda\mu$, $\|T^{-1}\|_{cb} \leq 1$, and $T^{**}(E') = E$. There exists a subspace $F \hookrightarrow Y$, with $d_{cb}(E, F) \leq C$. Let $F' = T^{-1}(F) \hookrightarrow X$. Then

$$d_{cb}(E, F') \leq d_{cb}(E, F)d_{cb}(F, F') \leq d_{cb}(E, F)\|T\|_{cb}\|T^{-1}\|_{cb} < C\lambda\mu$$

Therefore, $d_{cb}(E', F') \leq d_{cb}(E', E)d_{cb}(E, F') < C\lambda^2\mu$. ■

3. PROOF OF THEOREM 1.3(2): E IS A SUBSPACE OF $\mathbf{R} \oplus \mathbf{C}$

In this section, we make the first step toward proving Theorem 1.3(2) by showing that every C -BDR space embeds “neatly” into $\mathbf{R} \oplus \mathbf{C}$. More precisely, we prove:

Theorem 3.1. *For every $N \in \mathbb{N}$, there exists a separable operator space X such that:*

- (1) X^{**} contains \mathbf{B} as a complete M -summand.
- (2) Suppose E is a finite dimensional operator space, such that $d_{cb}(E, E') < C$ for some $E' \hookrightarrow \ell_\infty(\mathbf{M}_N)$, and X contains E c -completely isomorphically for some $c < C$. Then E is $4\sqrt{2}C^3$ -completely isomorphic to a subspace of $\mathbf{R} \oplus \mathbf{C}$.

Consequently, any C -BDR space is $4\sqrt{2}C^3$ -completely isomorphic to a subspace of $\mathbf{R} \oplus \mathbf{C}$.

We start the proof by constructing the space X . At the Banach space level, let $X = (\oplus_{n>N} \mathbf{M}_n)_c$, be the space of all sequences whose elements are $n \times n$ matrices, and which have a limit (in \mathbf{K}). Denote by P_n (once again, $n > N$) the canonical “truncation” from X to $(\sum_{k=N+1}^n \mathbf{M}_k)_\infty$. Let $X_n = \text{MAX}_n((\sum_{k=N+1}^n \mathbf{M}_k)_\infty)$ (see [13] or [11] for the definition and properties of the functor MAX_n), and set, for $x \in X \otimes \mathbf{K}_0$,

$$(3.1) \quad \|x\| = \sup_n \|(P_n \otimes I_{\mathbf{K}})x\|_{X_n \otimes \mathbf{K}}.$$

It is easy to notice that the Banach space structure of X is as described above. Denote by Y the space $(\oplus_{n>N} \mathbf{M}_n)_{c_0}$, with the operator space structure inherited from X .

For further use, we state the following easy consequence of (3.1).

Lemma 3.2. *Suppose x is an element of $X \otimes \mathbf{M}_m$ ($m \in \mathbb{N}$). Write $x = (x_i)_{i>N}$, with $x_i \in \mathbf{M}_i \otimes \mathbf{M}_m$. Then*

$$\|x\|_{X \otimes \mathbf{M}_m} = \max\{\|x\|_{(\sum_{n>N} \mathbf{M}_n)_c \otimes \mathbf{M}_m}, \max_{N < n \leq m} \|(P_n \otimes I_{\mathbf{M}_m})x\|_{X_n \otimes \mathbf{M}_m}\}.$$

Lemma 3.3. *Y is a complete M -ideal in X .*

Proof. For $n > N$, define the map $T_n : X \rightarrow Y$ by setting

$$T_n((x_i)_{i>N}) = (x_{N+1}, \dots, x_n, 0, 0, \dots).$$

We need to show that the sequence (T_n) is an M -complete approximate identity (see Definition 1.1 of [1]). Clearly, $T_n y \rightarrow y$ for any $y \in Y$. Moreover, suppose $x = (x_i)$ and $z = (z_i)$ are elements of $X \otimes \mathbf{M}_m$. By Lemma 3.2, for $n \geq m$,

$$\begin{aligned} & \|(T_n \otimes I_{\mathbf{M}_m})(x) + ((I - T_n) \otimes I_{\mathbf{M}_m})(z)\| = \|(x_{N+1}, \dots, x_n, z_{n+1}, z_{n+2}, \dots)\| \\ & = \max\{\|(x_{N+1}, \dots, x_n, z_{n+1}, z_{n+2}, \dots)\|_{(\sum_{n>N} \mathbf{M}_n)_c \otimes \mathbf{M}_m}, \\ & \quad \max_{N < k \leq m} \|(x_{N+1}, \dots, x_k)\|_{X_k \otimes \mathbf{M}_m}\} \\ & = \max\{\max_{N < k \leq n} \|x_k\|, \sup_{k > n} \|z_k\|, \max_{N < k \leq m} \|(x_{N+1}, \dots, x_k)\|_{X_k \otimes \mathbf{M}_m}\} \\ & \leq \max\{\|x\|, \|z\|\}. \end{aligned}$$

Thus, (T_n) is indeed an M -completely approximate identity, and therefore, by Theorem 1.1 of [1], Y is a complete M -ideal in X . \blacksquare

Lemma 3.4. *The quotient X/Y is completely isometric to \mathbf{K} .*

Proof. Define the map $U : X/Y \rightarrow \mathbf{K}$ by setting $U([(x_i)_{i>N}]) = \lim_i x_i$. To show that it is a complete isometry, fix $m \in \mathbb{N}$, and consider $x = (x_i)_{i>N} \in X \otimes \mathbf{M}_m$. By Lemma 3.2, $\|[x]\|_{X/Y \otimes \mathbf{M}_m} = \lim_i \|x_i\| = \|U[x]\|_{\mathbf{K} \otimes \mathbf{M}_m}$. \blacksquare

Conclusion of the proof of Theorem 3.1. Part (1) of the Theorem follows from Lemma 3.4. Suppose E is a finite dimensional operator space as in part (2), $u : E \rightarrow X$ is a complete contraction, $F = u(E)$, and $\|u^{-1}\|_{cb} < C$. Let $F_n = P_n(F)$ be a subspace of X_n (here, P_n and X_n are as in (3.1)), and let $u_n = P_n u$. By the definition of MAX_n ,

$$\|(x_{N+1}, \dots, x_n)\|_{X_n \otimes \mathbf{M}_N} = \max_{N < k \leq n} \|x_k\|,$$

hence there exists $n > N$ s.t. $\|(u_n \otimes I_{\mathbf{M}_N})e\| > C^{-1}\|e\|$ for any $e \in E \otimes \mathbf{M}_N$. By Smith's lemma, $\|u_n^{-1}\|_{cb} < C\|u_n^{-1} \otimes I_{\mathbf{M}_N}\| < C^2$.

X_n^* is 1-exact. Hence, by [17], there exist operators $v : E \rightarrow \mathbf{R} \oplus \mathbf{C}$ and $w : \mathbf{R} \oplus \mathbf{C} \rightarrow X_n$, so that $\|v\|_{cb}\|w\|_{cb} \leq 4\sqrt{2}C$, and $u_n = wv$. Let $G = v(E)$. Then $d_{cb}(E, G) \leq \|v\|_{cb}\|u_n^{-1}w\|_{cb} < 4\sqrt{2}C^3$.

To prove the last assertion, note that, by the reasoning above and Proposition 2.5, any N -dimensional $C+$ -BDR space E embeds into $\mathbf{R}_N \oplus \mathbf{C}_N$ $4\sqrt{2}(C + \varepsilon)^3$ -completely

isomorphically, for any $\varepsilon > 0$. Letting ε approach 0, and applying a classical compactness argument, we complete the proof. \blacksquare

4. PROOF OF THEOREM 1.3(2): E IS $\mathbf{R}_m \oplus \mathbf{C}_n$

In this section, we complete the proof of Theorem 1.3(2). For the convenience of working with Hilbert spaces as much as possible, we use the sum \oplus_2 : if X and Y are operator space, and x and y are elements of $X \otimes \mathbf{K}_0$ and $Y \otimes \mathbf{K}_0$, respectively, define

$$(4.1) \quad \|x \oplus y\|_{(X \oplus_2 Y) \otimes \mathbf{K}} = \max\{\|x\|, \|y\|, \|x \oplus y\|_{\text{MIN}(X \oplus_2 Y) \otimes \mathbf{K}}\}.$$

Clearly, Ruan's axioms are satisfied, so $X \oplus_2 Y$ is indeed an operator space.

Note that any N -dimensional subspace E of $\mathbf{R} \oplus_2 \mathbf{C}$ is contained in $\mathbf{R}_N \oplus_2 \mathbf{C}_N$. In view of Theorem 3.1, the proof of Theorem 1.3(2) follows from:

Theorem 4.1. *Suppose E is an N -dimensional subspace of $\mathbf{R}_N \oplus_2 \mathbf{C}_N$, which is λ -BDR. Then $d_{cb}(E, \mathbf{R}_m \oplus_2 \mathbf{C}_n) \leq 32\sqrt{2}\lambda$ for suitable $m, n \in \mathbb{Z}_+$.*

Denote by $A_{\mathbf{R}}$ and $A_{\mathbf{C}}$ the orthogonal projections from E on \mathbf{R}_N and \mathbf{C}_N , respectively. By polar decomposition, there exists an orthonormal basis $(e_i)_{i=1}^N$ in E s.t. $A_{\mathbf{R}}e_i = a_i^{(\mathbf{R})}e_i^{(\mathbf{R})}$, with $0 \leq a_N^{(\mathbf{R})} \leq \dots \leq a_1^{(\mathbf{R})} \leq 1$, and $e_i^{(\mathbf{R})}$ being an orthonormal basis for \mathbf{R}_N . Then, for $1 \leq i \leq N$, $A_{\mathbf{C}}e_i = a_i^{(\mathbf{C})}e_i^{(\mathbf{C})}$, where $e_i^{(\mathbf{C})}$ is a unit vector, and $a_i^{(\mathbf{C})} = \sqrt{1 - (a_i^{(\mathbf{R})})^2}$. Note that, for $i \neq j$,

$$\langle e_i, e_j^{(\mathbf{R})} \rangle = \langle a_i^{(\mathbf{R})}e_i^{(\mathbf{R})} + a_i^{(\mathbf{C})}e_i^{(\mathbf{C})}, e_j^{(\mathbf{R})} \rangle = 0.$$

Moreover,

$$0 = \langle e_i, e_j \rangle = \langle a_i^{(\mathbf{R})}e_i^{(\mathbf{R})} + a_i^{(\mathbf{C})}e_i^{(\mathbf{C})}, a_j^{(\mathbf{R})}e_j^{(\mathbf{R})} + a_j^{(\mathbf{C})}e_j^{(\mathbf{C})} \rangle = a_i^{(\mathbf{C})}a_j^{(\mathbf{C})}\langle e_i^{(\mathbf{C})}, e_j^{(\mathbf{C})} \rangle,$$

and therefore, the vectors $(e_i^{(\mathbf{C})})_{i=1}^N$ form an orthonormal basis in \mathbf{C}_N (certain minor changes to this construction need to be made if $a_i^{(\mathbf{C})} = 0$ for some i 's). One can also show that $\langle e_i, e_j^{(\mathbf{C})} \rangle = 0$ for $i \neq j$.

Lemma 4.2. *$(e_i)_{i=1}^N$ is a 1-completely unconditional basis in E .*

Proof. Suppose $\lambda_1, \dots, \lambda_N$ are complex numbers, of absolute value not exceeding 1. We have to show that the operator $\Lambda = \text{diag}((\lambda_i)_{i=1}^N)$ is completely contractive. To this end, consider an operator $\tilde{\Lambda}$ on $\mathbf{R}_N \oplus_2 \mathbf{C}_N$, mapping $e_i^{(\mathbf{R})}$ (or $e_i^{(\mathbf{C})}$) into $\lambda_i e_i^{(\mathbf{R})}$ (resp. $\lambda_i e_i^{(\mathbf{C})}$), for $1 \leq i \leq N$. By the discussion preceding the statement of this lemma, the restrictions of $\tilde{\Lambda}$ to \mathbf{R}_N and \mathbf{C}_N are contractive, hence completely contractive (row and column spaces are 1-homogeneous). Thus, by the homogeneity of minimal spaces, and by (4.1), $\tilde{\Lambda}$ is completely contractive. To complete the proof, observe that the restriction of $\tilde{\Lambda}$ to E coincides with Λ . \blacksquare

This lemma, together with (4.1), yields:

Corollary 4.3. *Suppose \mathcal{I} is a subset of $\{1, \dots, N\}$. Let $E_{\mathcal{I}} = \text{span}[e_i \mid i \in \mathcal{I}]$, and $E_{\mathcal{I}}^{\perp} = \text{span}[e_i \mid i \notin \mathcal{I}]$. Then the formal identity map $\text{id} : E \rightarrow E_{\mathcal{I}} \oplus_2 E_{\mathcal{I}}^{\perp}$ is completely contractive, and $\|\text{id}^{-1}\|_{cb} \leq \sqrt{2}$.*

Proof. For simplicity, denote the space $E_{\mathcal{I}} \oplus_2 E_{\mathcal{I}}^{\perp}$ by F . Consider $x = \sum_i e_i \otimes x_i \in E \otimes \mathbf{K}$. Then

$$\|x\|_{E \otimes \mathbf{K}} = \max \left\{ \left\| \sum_{i=1}^N (a_i^{(\mathbf{R})})^2 x_i^* x_i \right\|^{1/2}, \left\| \sum_{i=1}^N (a_i^{(\mathbf{C})})^2 x_i x_i^* \right\|^{1/2}, \left\| \sum_{i=1}^N e_i \otimes x_i \right\|_{\text{MIN}(\ell_2^N) \otimes \mathbf{K}} \right\},$$

while

$$\begin{aligned} \|x\|_{F \otimes \mathbf{K}} = \max \left\{ \left\| \sum_{i \in \mathcal{I}} (a_i^{(\mathbf{R})})^2 x_i^* x_i \right\|^{1/2}, \left\| \sum_{i \in \mathcal{I}^c} (a_i^{(\mathbf{R})})^2 x_i^* x_i \right\|^{1/2}, \right. \\ \left. \left\| \sum_{i \in \mathcal{I}} (a_i^{(\mathbf{C})})^2 x_i x_i^* \right\|^{1/2}, \left\| \sum_{i \in \mathcal{I}^c} (a_i^{(\mathbf{C})})^2 x_i x_i^* \right\|^{1/2}, \left\| \sum_{i=1}^N e_i \otimes x_i \right\|_{\text{MIN}(\ell_2^N) \otimes \mathbf{K}} \right\}, \end{aligned}$$

Comparing the two centered expressions yields the result. \blacksquare

Turning back to the proof of Theorem 4.1, denote by m the largest number i for which $a_i^{(\mathbf{R})} \geq 1/\sqrt{2}$ (if $a_1^{(\mathbf{R})} < 1/\sqrt{2}$, set $m = 0$), and let $n = N - m$. Let $E_{\mathbf{R}} = \text{span}[e_i \mid 1 \leq i \leq m]$, $E_{\mathbf{C}} = \text{span}[e_i \mid m < i \leq N]$. For a compact operator $T \in B(H, K)$ (H and K are Hilbert spaces), we denote by $\|T\|_2$ its Hilbert-Schmidt norm. That is, $\|T\|_2 = (\sum_n t_n^2)^{1/2}$, where $t_1 \geq t_2 \geq \dots \geq 0$ are the singular numbers of T . Equivalently, $\|T\|_2^2 = \sum_{i,j} |\langle T e_i, f_j \rangle|^2$, where (e_i) and (f_j) are orthonormal bases in H and K , respectively.

To complete the proof, it suffices to show that

$$\max\{\|A_{\mathbf{C}}|_{E_{\mathbf{R}}}\|_2, \|A_{\mathbf{R}}|_{E_{\mathbf{C}}}\|_2\} \leq 16\sqrt{2}\lambda.$$

Indeed, this would imply that $E_{\mathbf{R}}$ and $E_{\mathbf{C}}$ are 32λ -completely isomorphic to \mathbf{R}_m and \mathbf{C}_n , respectively. An application of Corollary 4.3 would then yield the result. Thus, it remains to prove:

Proposition 4.4. *In the above notation, $\|A_{\mathbf{C}}|_{E_{\mathbf{R}}}\|_2 \leq 16\sqrt{2}\lambda$.*

Proof. If $m = 0$, there is nothing to prove. If $m \geq 1$, denote $A_{\mathbf{C}}|_{E_{\mathbf{R}}}$ by A for simplicity of notation. Let

$$X_1 = \left(\sum_{n>N} \text{MAX}_n(E_{\mathbf{R}}) \right)_c, \quad Y = \left(\sum_{n>N} \text{MAX}_n(E_{\mathbf{R}}) \right)_{c_0}, \quad X = X_1 \oplus_2 E_{\mathbf{C}}.$$

By Proposition 3.2 of [1], Y is a complete M -ideal in X_1 . Imitating the proof of Lemma 3.4, one can show that $X_1/Y = E_{\mathbf{R}}$ completely isometrically. Therefore, X_1^{**} contains $E_{\mathbf{R}}$ as a complete M -summand. Finally, $\mathbf{M}_s(X^{**}) = (\mathbf{M}_s(X))^{**}$ for any $s \in \mathbb{N}$, hence $X^{**} = X_1^{**} \oplus_2 E_{\mathbf{C}}$, and therefore, X^{**} contains E $\sqrt{2}$ -completely isomorphically. By Corollary 2.6, X contains E $\sqrt{2}\lambda$ -ci.

Pick $C > \lambda$, and consider a complete contraction $u : E \rightarrow X$, satisfying $\|u^{-1}\|_{cb} \leq \sqrt{2}C$. Denote the ‘‘natural truncation’’ of u to $E_{\mathbf{C}}$ (or the n -th summand of X_1 , $n > N$) by u_0 (respectively, u_n). More precisely, we view u_0 (resp. u_n) as a map from E to $E_{\mathbf{C}}$ (resp. $\text{MAX}_n(E_{\mathbf{R}})$). In this notation, $F = \ker u_0$ is an M -dimensional subspace of E ($M \geq m$). Let v be an isometry from \mathbf{R}_M onto F . To complete the proof, it suffices to show that

$$(4.2) \quad \begin{aligned} (1) \quad & \|v\|_{cb} \geq \max\{1, \|A\|_2/2\} \\ (2) \quad & \|u_n v\|_{cb} \leq 8 \text{ for any } n > N \end{aligned}$$

Indeed, then $\|uv\|_{cb} = \sup_n \|u_n v\|_{cb} \leq 8$. On the other hand, $v = u^{-1} \circ (uv)$, hence the above inequalities imply

$$\frac{\|A\|_2}{2} \leq \|v\|_{cb} < \sqrt{2}C \|uv\|_{cb} \leq 8\sqrt{2}C.$$

Since $C > \lambda$ is arbitrary, we conclude that $\|A_2\|_2 \leq 16\sqrt{2}\lambda$.

We start by proving (4.2(1)). Denote by Q and Q^\perp the orthogonal projections from F onto $E_{\mathbf{R}}$ and $E_{\mathbf{C}}$, respectively. Reasoning as in the proof of Lemma 4.2 (see also the discussion preceding it), we can find an orthonormal basis $(f_i)_{i=1}^M$ in F , s.t. $\langle Qf_i, Qf_j \rangle = \langle Q^\perp f_i, Q^\perp f_j \rangle = 0$ if $i \neq j$. By changing the enumeration if necessary, assume that $\|Q^\perp f_i\| \geq 1/\sqrt{2}$ for $1 \leq i \leq \ell$, and $\|Qf_i\| > 1/\sqrt{2}$ for $\ell < i \leq M$. Let $F_1 = \text{span}[f_i \mid 1 \leq i \leq \ell]$, $F_2 = \text{span}[f_i \mid \ell < i \leq M]$, and $G_s = v^{-1}(F_s)$ for $s = 1, 2$. We can identify G_1 and G_2 with \mathbf{R}_ℓ and $\mathbf{R}_{M-\ell}$, respectively. Note that

$$\|v\|_{cb} \geq \max\{\|v|_{G_1}\|_{cb}, \|v|_{G_2}\|_{cb}\} \geq (\|v|_{G_1}\|_{cb}^2 + \|v|_{G_2}\|_{cb}^2)^{1/2}/\sqrt{2}.$$

However,

$$\|v|_{G_1}\|_{cb} \geq \|Q^\perp v|_{G_1}\|_{CB(\mathbf{R}_\ell, \mathbf{C}_M)} = \|Q^\perp v|_{G_1}\|_2 \geq \sqrt{\ell/2}.$$

Similarly,

$$\|v|_{G_2}\|_{cb} \geq \|AQv|_{G_2}\|_{CB(\mathbf{R}_{M-\ell}, \mathbf{C}_M)} = \|AQv|_{G_2}\|_2 = \left(\sum_{i=\ell+1}^M \|AQf_i\|^2 \right)^{1/2}.$$

To evaluate $\|A\|_2$, introduce the vectors $f'_i \in E_{\mathbf{R}}$ ($1 \leq i \leq M$) in such a way that $E_{\mathbf{R}} = \text{span}[f'_i \mid 1 \leq i \leq M]$, and, for each i , $\|f'_i\|$ equals 0 or 1, and $f'_i = Qf_i/\|Qf_i\|$ provided $Qf_i \neq 0$ (this is possible, since $M \geq m = \dim E_{\mathbf{R}}$). A is a contraction, hence

$$\|A\|_2^2 = \sum_{i=1}^M \|Af'_i\|^2 \leq \ell + 2 \sum_{i=\ell+1}^M \|AQf_i\|^2 \leq 2(\|v|_{G_1}\|_{cb}^2 + \|v|_{G_2}\|_{cb}^2) \leq 4\|v\|_{cb}^2.$$

Moreover, v is an isometry, thus (4.2(1)).

Next we tackle (4.2(2)). Fix n . By [17], there exist operators $T_{\mathbf{R}} : F \rightarrow \mathbf{R}_M$, $T_{\mathbf{C}} : F \rightarrow \mathbf{C}_M$, $S_{\mathbf{R}} : \mathbf{R}_M \rightarrow \text{MAX}_n(E_{\mathbf{R}})$, and $S_{\mathbf{C}} : \mathbf{C}_M \rightarrow \text{MAX}_n(E_{\mathbf{R}})$, so that

$u_n|_F = S_{\mathbf{R}}T_{\mathbf{R}} + S_{\mathbf{C}}T_{\mathbf{C}}$, and $\max\{\|T_{\mathbf{R}}\|_{cb}\|S_{\mathbf{R}}\|_{cb}, \|T_{\mathbf{C}}\|_{cb}\|S_{\mathbf{C}}\|_{cb}\} \leq 2\sqrt{2}$. Then

$$\|S_{\mathbf{R}}T_{\mathbf{R}}v\|_{cb} \leq \|S_{\mathbf{R}}\|_{cb}\|T_{\mathbf{R}}v\|_{cb} = \|S_{\mathbf{R}}\|_{cb}\|T_{\mathbf{R}}v\| \leq 2\sqrt{2}.$$

Moreover,

$$\|S_{\mathbf{C}}\|_{cb} \geq \|A_{\mathbf{R}}S_{\mathbf{C}}\|_{CB(\mathbf{C}_M, \mathbf{R}_N)} = \|A_{\mathbf{R}}S_{\mathbf{C}}\|_2 \geq \|S_{\mathbf{C}}\|_2/\sqrt{2},$$

hence

$$\|S_{\mathbf{C}}T_{\mathbf{C}}v\|_{cb} \leq \|S_{\mathbf{C}}T_{\mathbf{C}}v\|_2 \leq \|S_{\mathbf{C}}\|_2\|T_{\mathbf{C}}v\| \leq 4.$$

Thus,

$$\|u_nv\|_{cb} \leq \|S_{\mathbf{R}}T_{\mathbf{R}}v\|_{cb} + \|S_{\mathbf{C}}T_{\mathbf{C}}v\|_{cb} \leq 8.$$

This establishes (4.2(2)). \blacksquare

Proof of Theorem 1.3(2). Suppose an N -dimensional space E is λ -BDR. Pick $C > \lambda$. By Theorem 3.1, there exists a subspace F of $\mathbf{C}_N \oplus_2 \mathbf{R}_N$, s.t. $d_{cb}(E, F) < 8C^3$. By Corollary 2.6 and Proposition 2.5, F is 2^6C^8 -BDR. By Theorem 4.1, F is $2^{23/2}C^8$ -ci to $G_C = \mathbf{R}_{m(C)} \oplus_2 \mathbf{C}_{n(C)}$. Thus, $d_{cb}(E, G_C) \leq 2^{29/2}C^{11}$. Find a sequence (C_j) , decreasing to λ , s.t. $m = m(C_j)$ for any j (then $n = N - m(C_j) = n(C_j)$). Clearly, $d_{cb}(E, \mathbf{R}_m \oplus_{\infty} \mathbf{C}_n) \leq 2^{15}\lambda^{10}$. \blacksquare

5. PROOF OF THEOREM 1.4, AND SIMILAR LOWER ESTIMATES

This section is devoted to the proof of Theorem 1.4. Clearly, \mathbb{C} is 1-BDR. A series of lemmas helps us rule out other spaces. The first lemma seems to be a partly folklore.

Lemma 5.1. (1) *Suppose $(E_k)_{k=1}^{\infty}$ is a sequence of Banach spaces, and E is a finite dimensional subspace of $(\sum_k E_k)_{c_0}$. Then there exists $N \in \mathbb{N}$ such that E is isometric to a finite dimensional subspace of $(\sum_{k=1}^N E_k)_{\infty}$.*

(2) *Suppose $(E_k)_{k=1}^{\infty}$ is a sequence of operator spaces, and E is a finite dimensional subspace of $(\sum_k E_k)_{c_0}$. Then there exists $N \in \mathbb{N}$ such that E is completely isometric to a finite dimensional subspace of $(\sum_{k=1}^N E_k)_{\infty}$.*

Proof. We prove part (2), since part (1) can be tackled in a similar manner. Let $n = \dim E$. Suppose $(e_i)_{i=1}^n$ is an Auerbach basis in E – that is, $\max_i |\alpha_i| \leq \|\sum_i \alpha_i e_i\| \leq \sum_i |\alpha_i|$ for each sequence of scalars (α_i) . In particular, the projection $R_j : E \rightarrow E : \sum_i \alpha_i e_i \mapsto \alpha_j e_j$ is contractive for every j . Since it is a rank one projection, it must also be completely contractive. Thus, for any $(a_i)_{i=1}^n \subset \mathbf{K}$, $\|\sum_i a_i \otimes e_i\| \geq \max_i \|a_i\|$.

Now write $e_i = (e_{ik})_{k=1}^{\infty}$, with $e_{ik} \in E_k$. There exists $N \in \mathbb{N}$ s.t. $\|e_{ik}\| < 1/n$ for any $1 \leq i \leq n$, and any $k > N$. For such k , and for any $(a_i)_{i=1}^n \subset \mathbf{K}$, $\|\sum_i a_i \otimes e_{ik}\| < \max_i \|a_i\|$. Therefore,

$$\left\| \sum_i a_i \otimes e_i \right\| = \sup_k \left\| \sum_i a_i \otimes e_{ik} \right\| = \max_{1 \leq k \leq N} \left\| \sum_i a_i \otimes e_{ik} \right\|,$$

which implies that E is completely isometric to a subspace of $(\sum_{k=1}^N E_k)^N$, spanned by the vectors $\tilde{e}_i = (e_{ik})_{k=1}^N$ ($1 \leq i \leq n$). ■

We next apply this lemma to our situation.

Lemma 5.2. *Suppose a finite dimensional operator space E is 1-BDR. Then, for some $N \in \mathbb{N}$, E embeds into ℓ_∞^N isometrically, and into \mathbf{M}_N completely isometrically.*

Proof. Suppose E is a finite dimensional 1-BDR space. For $n \in \mathbb{N}$, find $e_{1n}^*, \dots, e_{M_n n}^* \in E^*$ of norm 1 so that, for any $e \in E$,

$$\max_{1 \leq k \leq M_n} |\langle e_{kn}^*, e \rangle| \geq (1 - 2^{-n}) \|e\|.$$

Define the operator space E_n by setting, for $e \in E \otimes \mathbf{K}_0$,

$$\|e\|_{E_n \otimes \mathbf{K}} = \max \left\{ (1 - 2^{-n}) \|e\|_{E \otimes \mathbf{K}}, \max_{1 \leq k \leq M_n} \|(e_{kn}^* \otimes I_{\mathbf{K}})e\|_{\mathbf{K}} \right\}.$$

Note that, for $e \in E$,

$$(5.1) \quad \|e\|_{E_n} = \max_{1 \leq k \leq M_n} |\langle e_{kn}^*, e \rangle|.$$

Let $X = (\sum_{n=1}^\infty E_n)_{c_0}$. Then the map

$$u : E \rightarrow X^{**} = \left(\sum_{n=1}^\infty E_n \right)_{\ell_\infty} : e \mapsto (e, e, \dots)$$

is a complete isometry. On the other hand, if E embeds into X isometrically, then, by Lemma 5.1, it must embed into $(\sum_{n=1}^K E_n)_\infty$, for some $K \in \mathbb{N}$. By (5.1), E_n embeds isometrically into $\ell_\infty^{M_n}$. Therefore, E embeds isometrically into ℓ_∞^N ($N = \sum_{n=1}^K M_n$).

Next test 1-bidual representability with $X = (\sum_{k=1}^\infty \mathbf{M}_k)_{c_0}$. The space $X^{**} = (\sum_{k=1}^\infty \mathbf{M}_k)_\infty$ contains \mathbf{B} , hence it also contains E . Therefore, X contains E . By Lemma 5.1, E embeds into $(\sum_{k=1}^M \mathbf{M}_k)_\infty$, for some M . *A fortiori*, E embeds completely isometrically into \mathbf{M}_N , with $N = M(M+1)/2$. ■

Corollary 5.3. *If a 1-BDR space E has dimension at least 2, then E is not strictly convex.*

Recall that a Banach space E is called *strictly convex* if, for any $e_1, e_2 \in E$, the equality $\|e_1 + e_2\|^2 = 2(\|e_1\|^2 + \|e_2\|^2)$, implies $e_1 = e_2$.

Proof. By Lemma 5.2, there exist linear functionals $f_1, \dots, f_N \in E^*$ s.t. $\|e\| = \max_{1 \leq i \leq N} |\langle f_i, e \rangle|$ for any $e \in E$. Therefore, there exist i and two distinct elements e_1, e_2 of the unit ball of E , satisfying

$$\|e_1\| = \|e_2\| = |\langle f_i, e_1 \rangle| = |\langle f_i, e_2 \rangle| = 1.$$

Then $4 = \|e_1 + e_2\|^2 = 2(\|e_1\|^2 + \|e_2\|^2)$. ■

Proof of Theorem 1.4. Suppose, for the sake of contradiction, that $\dim E > 1$, and E is 1-BDR. By Lemma 5.2, E embeds into \mathbf{M}_N completely isometrically. We then construct an operator space X , isomorphic to $c_0(\mathbf{M}_N)$, which is strictly convex (hence, by Corollary 5.3, X cannot contain E isometrically), but such that X^{**} contains \mathbf{M}_N completely isometrically.

To this end, find a dense sequence $(f_j)_{j=1}^\infty$ in the unit ball of \mathbf{M}_N^* . For $x = (x_1, x_2, \dots) \in c_0(\mathbf{M}_N)$, set

$$(5.2) \quad \|x\|_X = \sup_n \|x\|_n, \quad \text{where } \|x\|_n = \left(\|x_n\|^2 + \sum_{j,k=1}^\infty 10^{-(n+j+k)} |\langle f_j, x_k \rangle|^2 \right)^{1/2}.$$

Observe that

$$\begin{aligned} \sum_{j,k=1}^\infty 10^{-(n+j+k)} |\langle f_j, x_k \rangle|^2 &\leq 10^{-(n+1)} \sum_{j,k=1}^\infty 10^{-(j+k-1)} = 10^{-(n+1)} \sum_{\ell=0}^\infty \ell 10^{-\ell} \\ &= 10^{-(n+1)} \left(\frac{1}{1-1/10} \right)^2 < 1.25 \cdot 10^{-(n+1)}. \end{aligned}$$

$$(5.3) \quad \|x\|_n < \|x_n\| + 10^{-n} \|x\|_{c_0(\mathbf{M}_N)},$$

hence $\|\cdot\|_X$ is well defined. To show that $\|\cdot\|_n$ is a norm, note that $\|x\|_n = \|J_n x\|_{\mathbf{M}_N \oplus_2 \ell_2(\mathbb{N} \times \mathbb{N})}$, where

$$J_n : c_0(\mathbf{M}_N) \rightarrow \mathbf{M}_N \oplus_2 \ell_2(\mathbb{N} \times \mathbb{N}) : x \mapsto (x_n, (10^{-(n+j+k)} \langle f_j, x_k \rangle)_{j,k}).$$

By (5.3), $\lim_n \|J_n x\| = 0$ for any x , hence (5.2) describes X as a subspace of $(\sum(\mathbf{M}_N \oplus_2 \ell_2))_{c_0}$, and $\|\cdot\|_X$ is a norm, too. Thus, X^{**} is isomorphic to $\ell_\infty(\mathbf{M}_N)$, and the norm is also given by (5.2).

Finally, note that the supremum in (5.2) is attained. Indeed, if $\|x\|_{c_0(\mathbf{M}_N)} = 1$, find $K \in \mathbb{N}$ s.t. $\|x_n\|^2 + 10^{-n} < 1/4$ for $n \geq K$. For such n , $\|x\|_n < \frac{1}{2}$, while $\|x\|_X > 1$. Therefore, $\|x\|_X = \max_{n < K} \|x\|_n$.

To show that X is strictly convex, suppose $\|x^{(1)} + x^{(2)}\|^2 = 2(\|x^{(1)}\|^2 + \|x^{(2)}\|^2)$ for some $x^{(1)}, x^{(2)} \in X$. By the observation above, there exists $n \in \mathbb{N}$ s.t. $\|x^{(1)} + x^{(2)}\| = \|x^{(1)} + x^{(2)}\|_n$. Therefore (writing $x^{(s)} = (x_n^{(s)})_{n=1}^\infty$),

$$\begin{aligned} 0 &\geq 2(\|x^{(1)}\|_n^2 + \|x^{(2)}\|_n^2) - \|x^{(1)} + x^{(2)}\|_n^2 = 2(\|x_n^{(1)}\|^2 + \|x_n^{(2)}\|^2) - \|x_n^{(1)} + x_n^{(2)}\|^2 \\ &+ \sum_{j,k=1}^\infty 10^{-(n+j+k)} \left(2(|\langle f_j, x_k^{(1)} \rangle|^2 + |\langle f_j, x_k^{(2)} \rangle|^2) - |\langle f_j, x_k^{(1)} + x_k^{(2)} \rangle|^2 \right). \end{aligned}$$

But, by the triangle inequality,

$$2(\|x_n^{(1)}\|^2 + \|x_n^{(2)}\|^2) \geq \|x_n^{(1)} + x_n^{(2)}\|^2,$$

and

$$2(|\langle f_j, x_k^{(1)} \rangle|^2 + |\langle f_j, x_k^{(2)} \rangle|^2) \geq |\langle f_j, x_k^{(1)} + x_k^{(2)} \rangle|^2.$$

In the last centered line, the equality holds if and only if $x_k^{(1)} = x_k^{(2)}$ for every k . Thus, $x^{(1)} = x^{(2)}$.

Define the operator space structure on X by setting, for $x \in X \otimes \mathbf{K}_0$,

$$\|x\| = \max\{\|x\|_{\text{MIN}(X) \otimes \mathbf{K}_0}, \|x\|_{c_0(\mathbf{M}_N) \otimes \mathbf{K}_0}\}.$$

In other words, the operator space structure on X is generated by its ‘‘natural’’ embedding into $\text{MIN}(X) \oplus c_0(\mathbf{M}_N)$. Then X^{**} embeds into $\text{MIN}(X^{**}) \oplus \ell_\infty(\mathbf{M}_N)$, and, for $x \in X^{**} \otimes \mathbf{K}_0$,

$$(5.4) \quad \|x\| = \max\{\|x\|_{\text{MIN}(X^{**}) \otimes \mathbf{K}_0}, \|x\|_{\ell_\infty(\mathbf{M}_N) \otimes \mathbf{K}_0}\}.$$

Define $U : \mathbf{M}_N \rightarrow X^{**} : a \mapsto ((1 - 10^{-n})a)_{n=1}^\infty$. By (5.2) and the discussion following it, and by (5.4), U is a complete contraction. Furthermore, for any $a \in \mathbf{M}_N \otimes \mathbf{K}_0$,

$$\|(U \otimes I_{\mathbf{K}})a\|_{X^{**} \otimes \mathbf{K}} \geq \sup_n (1 - 10^{-n}) \|a\|_{\mathbf{M}_N \otimes \mathbf{K}} = \|a\|,$$

hence U is a complete isometry. ■

Remark 5.4. In a similar way one can prove the commutative version of Theorem 1.4: if a finite dimensional Banach space E is such that a Banach space X contains E isometrically whenever X^{**} contains E isometrically, then E is 1-dimensional. Indeed, one can imitate the proof of Lemma 5.2 to show that such an E embeds into ℓ_∞^N , for some N . Then, as above, one constructs a strictly convex Banach space Z , whose dual contains ℓ_∞^N .

Next we investigate the smallest C for which the given space E is C -BDR.

Theorem 5.5. *Suppose E is a finite dimensional C -injective operator space, which is λ -BDR. Then $\lambda \geq \text{ex}(E^*) / (C \text{ex}(E))$.*

Proof. Suppose, for the sake of contradiction, that $\lambda < \text{ex}(E^*) / (C \text{ex}(E))$. Pick $C_1 > \text{ex}(E)$, $C_2 < \text{ex}(E^*)$, and $C_3 > \lambda$, s.t. $C_2 / (C_1 C_3) > C$. Following the proof of Theorem 3.1, find $E_1 \hookrightarrow \mathbf{M}_N$, s.t. $d_{cb}(E, E_1) < C_1$. Also, define the spaces X_n ($n > N$) and X as in the proof of that theorem. We know that X^{**} contains \mathbf{B} as a complete M -summand. Suppose X contains E C_3 -ci. As in the proof of Theorem 3.1, we conclude that, for some $n > N$, X_n contains E $C_1 C_3$ -ci. That is, there exists a subspace $F \hookrightarrow X_n$ and a complete contraction $u : F \rightarrow E$, with $\|u^{-1}\|_{cb} \leq C_1 C_3$. E is C -injective, thus there exists a map $\tilde{u} : X_n \rightarrow E$ s.t. $\tilde{u}|_F = u$, and $\|\tilde{u}\|_{cb} \leq C$.

As $\text{ex}(E^*) > C_2$, there exists an operator $v : E \rightarrow \mathbf{B}$ s.t. $\|v\|_{cb} > C_2$, and $\|v \otimes I_{\mathbf{M}_n}\| < 1$ (to see this, apply Theorem 18 of [15] to E^* , and dualize). Note that

$$\|v\tilde{u}\|_{cb} \geq \|vu\|_{cb} \geq \frac{\|v\|_{cb}}{\|u^{-1}\|_{cb}} \geq \frac{C_2}{C_1 C_3}.$$

On the other hand, by definition of X_n ,

$$\|v\tilde{u}\|_{cb} = \|v\tilde{u} \otimes I_{\mathbf{M}_n}\| \leq \|v \otimes I_{\mathbf{M}_n}\| \|\tilde{u}\|_{cb} < C.$$

This contradicts our assumption that $C_2/(C_1C_3) > C$. \blacksquare

Corollary 5.6. *If ℓ_∞^n is C -BDR, then $C \geq n/(2\sqrt{n-1})$.*

Proof. We know that $E = \ell_\infty^n$ is 1-injective, and 1-exact. Moreover (see Theorem 21.5 of [16]), $\text{ex}(E^*) \geq n/(2\sqrt{n-1})$. We complete the proof by applying Theorem 5.5. \blacksquare

6. C^* -ALGEBRAS AND BIDUAL REPRESENTABILITY

Passing from operator spaces to C^* -algebras, we obtain:

Proposition 6.1. *Suppose X is a C^* -algebra, and $n \in \mathbb{N}$. Then the following are equivalent:*

- (1) X^{**} contains \mathbf{M}_n completely isometrically.
- (2) X^{**} contains \mathbf{M}_n as a sub- C^* -algebra.
- (3) X contains \mathbf{M}_n completely isometrically.
- (4) X^{**} contains \mathbf{M}_n c -completely isomorphically, for some $c < n/(n-1)$.
- (5) There exists an irreducible representation of X into $B(H)$, with $\dim H \geq n$.

Remark 6.2. J. Roydor [18] proved a related result: if a C^* -algebra X embeds into $C(\Omega, \mathbf{M}_n)$ (for some Ω) completely isometrically, then X is a C^* -subalgebra of $C(\Omega', \mathbf{M}_n)$, for some set Ω' .

Below, we denote by $(E_{ij})_{i,j=1}^n$ the canonical matrix units in \mathbf{M}_n .

Lemma 6.3. *Suppose $n > m$, and $T : \mathbf{M}_n \rightarrow \mathbf{M}_m$ is a complete contraction. Then*

$$\|T \otimes I_{\mathbf{M}_n} \left(\sum_{i=1}^n E_{ij} \otimes E_{ij} \right)\|_{\mathbf{M}_m \otimes \mathbf{M}_n} \leq m.$$

Corollary 6.4. *If \mathcal{I} is a set, $m < n$, and E is an n^2 -dimensional subspace of $\ell_\infty(\mathcal{I}, \mathbf{M}_m)$, then $d_{cb}(\mathbf{M}_n, E) \geq n/m$.*

Proof. Consider a complete contraction $T : \mathbf{M}_n \rightarrow E$. By Lemma 6.3, $\|T \otimes I_{\mathbf{M}_n}(\sum_{i=1}^n E_{ij} \otimes E_{ij})\| \leq m$. However, $\|\sum_{i,j=1}^n E_{ij} \otimes E_{ij}\|_{\mathbf{M}_n \otimes \mathbf{M}_n} = n$. \blacksquare

Proof of Lemma 6.3. Consider a complete contraction $T : \mathbf{M}_n \rightarrow \mathbf{M}_m$. By Stinespring Representation Theorem, there exists a Hilbert space \tilde{H} , a unital representation $\pi : \mathbf{M}_n \rightarrow B(\tilde{H})$, and contractions $U \in B(\tilde{H}, \ell_2^m)$, $V \in B(\ell_2^m, \tilde{H})$, such that $Ta = U\pi(a)V$ for any $a \in \mathbf{M}_n$. Note that, for each i , $p_i = \pi(E_{ii})$ is a projection. Denote the range of this projection by H_i . Then $\pi(E_{ij})$ is a partial isometry, with initial and terminal projections H_i and H_j , respectively. Let $H = H_1$, and

$u_i = \pi(E_{i1}) : H \rightarrow H_i$ (once again, $1 \leq i \leq n$). Then $\ell_2^n(H)$ can be identified with \tilde{H} , via $(\xi_i)_{i=1}^n \mapsto \sum_{i=1}^n u_i \xi_i$.

For $1 \leq i \leq n$, let $U_i = U u_i \in B(H, \ell_2^m)$, and $V_i = u_i p_i V \in B(\ell_2^m, H)$. Then we can identify U with $\sum_{i=1}^n E_{1i} \otimes U_i$ (viewed as an operator from $\ell_2^n(H)$ to ℓ_2^m). Similarly, we identify V with $\sum_{i=1}^n E_{i1} \otimes V_i \in B(\ell_2^m, \ell_2^n(H))$. Then

$$\begin{aligned} \|V\| &= \left\| \sum_{i=1}^n E_{i1} \otimes V_i \right\| = \left\| \sum_{i=1}^n V_i^* V_i \right\|^{1/2} \leq 1, \text{ and} \\ \|U\| &= \left\| \sum_{i=1}^n E_{1i} \otimes U_i \right\| = \left\| \sum_{i=1}^n U_i U_i^* \right\|^{1/2} \leq 1. \end{aligned}$$

Moreover, $T E_{ij} = U_i V_j$ for any i and j . Therefore,

$$\begin{aligned} \|T \otimes I_{\mathbf{M}_n} \left(\sum_{i,j=1}^n E_{ij} \otimes E_{ij} \right)\|_{\mathbf{M}_m \otimes \mathbf{M}_n} &= \left\| \sum_{i,j=1}^n U_i V_j \otimes E_{ij} \right\| \\ &= \left\| \left(\sum_i U_i \otimes E_{i1} \right) \left(\sum_j V_j \otimes E_{1j} \right) \right\|. \end{aligned}$$

However,

$$\begin{aligned} \left\| \sum_i U_i \otimes E_{i1} \right\| &= \left\| \sum_{i=1}^n U_i^* U_i \right\|^{1/2} \leq \left(\text{Tr} \left(\sum_{i=1}^n U_i^* U_i \right) \right)^{1/2} \\ &= \left(\text{Tr} \left(\sum_{i=1}^n U_i U_i^* \right) \right)^{1/2} \leq \sqrt{m} \left\| \sum_{i=1}^n U_i U_i^* \right\| \leq \sqrt{m}, \end{aligned}$$

and similarly, $\left\| \sum_j V_j \otimes E_{1j} \right\| \leq \sqrt{m}$. Thus,

$$\|T \otimes I_{\mathbf{M}_n} \left(\sum_{i,j=1}^n E_{ij} \otimes E_{ij} \right)\| \leq \left\| \sum_i U_i \otimes E_{i1} \right\| \cdot \left\| \sum_j V_j \otimes E_{1j} \right\| \leq m.$$

■

Remark 6.5. In a similar fashion, one can show that $\left\| \sum_{i=1}^n E_{i1} \otimes E_{i1} \right\|_{\mathbf{C}_n \otimes \mathbf{C}_n} = \sqrt{n}$, and

$$\|T \otimes I_{\mathbf{C}_n} \left(\sum_{i=1}^n E_{i1} \otimes E_{i1} \right)\|_{\mathbf{M}_m \otimes \mathbf{C}_n} \leq \sqrt{m},$$

whenever $T : \mathbf{C}_n \rightarrow \mathbf{M}_m$ is a complete contraction. From this, one concludes that $d_{cb}(\mathbf{C}_n, E) \geq \sqrt{n/m}$ for every n -dimensional subspace E of $\ell_\infty(\mathcal{I}, \mathbf{M}_m)$ (provided $n > m$).

Lemma 6.6. *Suppose $\pi : Y \rightarrow B(H)$ is an irreducible representation of a C^* -algebra Y on a Hilbert space H of dimension at least n . Then Y contains \mathbf{M}_n completely isometrically as an operator system.*

Proof. If p is a projection of rank n on H , then, by the transitivity of irreducible representations (Theorem II.4.18 of [20]), $\pi(Y)p = B(H)p$. Denote by A the set of all $y \in Y$, satisfying $\pi(y)p = p\pi(y)$ ($= p\pi(y)p$). Clearly, A is a C^* -subalgebra of Y , and $\pi(A)$ can be identified with $B(p(H)) \sim \mathbf{M}_n$. Moreover, A has a separable subalgebra (call it A_1), s.t. $\pi(A_1)$ can again be identified with \mathbf{M}_n . In other words, $\mathbf{M}_n = A_1/J$, where $J = \ker \pi \cap A_1$ is a closed two-sided ideal. By Theorem 3.10 of [3], \mathbf{M}_n lifts to A_1 completely positively. Thus, A_1 contains \mathbf{M}_n completely isometrically. ■

Remark 6.7. A similar lifting technique was used in [7]. Earlier, lifting methods were used in [19, 21] to prove that a C^* -algebra which is not $(n-1)$ -subhomogeneous contains a completely positive copy of \mathbf{M}_n .

Proof of Proposition 6.1. The implications (2) \Rightarrow (1) \Rightarrow (4) and (3) \Rightarrow (1) are clear.

(1) \Rightarrow (2): By Section 6.4 of [10], we can decompose the von Neumann algebra X^{**} into a direct sum of components of type I_k ($k \in \mathbb{N} \cup \{\infty\}$), II , and III . By Lemma 6.4, at least one of the summands of type I_k ($k \geq n$), II , and III is non-trivial. All such summands contain \mathbf{M}_n as a subalgebra (see e.g. Comparison Theorem for projections in a von Neumann algebras, Theorem 6.2.7 of [10]).

(1) \Rightarrow (5): Suppose, for the sake of contradiction, that (5) fails. Viewing X^{**} as the enveloping algebra of X , we embed it into $\ell_\infty(\mathcal{I}, \mathbf{M}_{n-1})$ for some set \mathcal{I} . Therefore, by Lemma 6.4, X^{**} cannot contain \mathbf{M}_n c -completely isomorphically with $c < n/(n-1)$.

(4) \Rightarrow (1) and (5) \Rightarrow (3): If there are no irreducible representation of X^{**} (or X) on Hilbert spaces with dimension $\geq n$, then X^{**} (respectively, X) embeds into $\ell_\infty(\mathcal{I}, \mathbf{M}_{n-1})$, hence, by Lemma 6.4, X^{**} cannot contain \mathbf{M}_n c -completely isomorphically when $c < n/(n-1)$. Otherwise, X^{**} (or X) contains \mathbf{M}_n completely isometrically, by Lemma 6.6. ■

Remark 6.8. By Lemma 6.6, items (1) and (4) of Proposition 6.1 guarantee that X (or X^{**}) contains \mathbf{M}_n as an operator system. However, X need not contain \mathbf{M}_n as a subalgebra. Indeed, let X be the left regular algebra of a free group on two generators. By [4], X has no non-trivial projections, hence it cannot contain \mathbf{M}_n as a subalgebra.

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