IRREDUCIBLE RESIDUATED SEMILATTICES AND FINITELY BASED VARIETIES

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Abstract. This paper deals with axiomatization problems for varieties of residuated meet semilattice-ordered monoids (RSs). An internal characterization of the finitely subdirectly irreducible RSs is proved, and it is used to investigate the varieties of RSs within which the finitely based subvarieties are closed under finite joins. It is shown that a variety has this closure property if its finitely subdirectly irreducible members form an elementary class. A syntactic characterization of this hypothesis is proved, and examples are discussed.

1. Introduction

Residuated lattices (RLs) algebraize the associative full Lambek calculus $\mathbf{FL}^+$, while residuated meet semilattices (RSs) algebraize the disjunction-free fragment of this system. In both cases there is a lattice anti-isomorphism between the axiomatic extensions of the logic and the *subvarieties* of the algebraic class, which preserves and reflects finite axiomatizability (cf. [9]). Every RS can be embedded into an RL, but it is not known which equations in the language of RSs persist in suitable RL–extensions. Thus, in general, results about *varieties* of RLs do not transfer effortlessly to RSs. Some reasons for studying RSs in their own right can be found in [17].

In [6], Galatos showed how to transform the equational bases for two varieties of RLs into an equational basis for their varietal join. In [16], Olson showed how to axiomatize the varieties generated by certain universal positive classes of commutative RSs. The arguments in these two papers are rather different. Indeed, in [16], our capacity to get by with join-free axioms appears to depend on an analysis of the subdirectly irreducible algebras that breaks down in the noncommutative case, and that has no explicit analogue in [6].

In the present paper we prove an internal characterization of the finitely subdirectly irreducible RSs (Theorem 5) that allows us to unify and extend the approaches of [6] and [16]. It turns out that in any variety V of RSs, if the finitely subdirectly irreducible algebras are closed under ultraproducts then they form an elementary class, in which case the finitely based subvarieties of V are closed under finite joins (Theorems 29, 31). The hypothesis in this assertion will be characterized syntactically. The conclusion exceeds what could be predicted from a general finite basis theorem of Jónsson (Theorem 6). The result applies

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to all RSs satisfying a weak form of commutativity, as well as to lattice-ordered groups and to subdirect products of residuated chains.

2. Residuated Semilattices

Definition 1. A *residuated semilattice* (briefly, an *RS*) is an algebra $\boldsymbol{A} = \langle A; \cdot, \backslash, /, \wedge, e \rangle$ such that $\langle A; \wedge \rangle$ is a semilattice, $\langle A; \cdot, e \rangle$ is a monoid and $\backslash, /$ are binary residual operations, i.e., for all $a, b, c \in A$,

$$(1) \qquad a \cdot c \leq b \quad \text{iff} \quad c \leq a \backslash b \quad \text{iff} \quad a \leq b/c. \qquad (\textit{residuation})$$

Here, and whenever a semilattice operation denoted by $\wedge$ is under discussion, $x \leq y$ means $x \wedge y = x$ (and $x \geq y$ is the inverse relation). It follows from (1) that the order of an RS is *compatible* with the monoid operation, i.e.,

$$\text{if } a \leq b \text{ and } c \leq d \text{ then } a \cdot c \leq b \cdot d.$$

To verify that a semilattice-ordered monoid $\langle A; \cdot, \wedge, e \rangle$ is residuated (i.e., that it admits an RS–structure), we need only check that $\leq$ is compatible with $\cdot$ and that $\max \{z : a \cdot z \leq b\}$ and $\max \{z : z \cdot a \leq b\}$ both exist for all $a, b \in A$. These maxima become $a \backslash b$ and b/a, respectively, so the RS–expansion is unique.

Another consequence of (1) is that in every RS, we have

$$a \leq b \text{ iff } e \leq a \backslash b \text{ iff } e \leq b/a.$$

The monoid identity e need not be the greatest element of the semilattice order, but we always have $e \backslash a = a = a/e$.

The class of all residuated semilattices is a finitely based variety, which we denote by RS. This variety is congruence distributive [10].

A *residuated lattice* (*RL*) is a lattice-ordered RS whose binary join operation $\vee$ is appended to the type as a new basic operation. For background on residuated lattices, see [8, 11].

Given a partially ordered set $\langle P; \leq \rangle$, with $a \in P$ and $X \subseteq P$, we use the abbreviations $[\, a \,) = \{b \in P : b \geq a\}$ and $[\, X \,) = \bigcup_{x \in X} [\, x \,)$.

Suppose $\boldsymbol{A}$ is an RS (or an RL). The congruence lattice of $\boldsymbol{A}$ is isomorphic to the lattice of convex normal subalgebras of $\boldsymbol{A}$, under the map $\theta \mapsto e/\theta$ (cf. [3]). It is also isomorphic, under the map $\theta \mapsto [\, e/\theta \,)$, to the lattice of *deductive filters* of $\boldsymbol{A}$ (cf. [9]); these subsets may be defined as the semilattice filters F of $\langle A; \wedge \rangle$ that are also submonoids of $\langle A; \cdot, e \rangle$ with the following closure property:

$$\text{whenever } a \in F \text{ then } c \backslash (a \cdot c) \in F \text{ and } (c \cdot a)/c \in F \text{ for every } c \in A.$$

The least deductive filter of $\boldsymbol{A}$ is always $[\, e \,)$.

When $\boldsymbol{A}$ is understood and $X \subseteq A$, we use $Fg(X)$ to denote the deductive filter of $\boldsymbol{A}$ *generated* by X, i.e., the intersection of all deductive filters containing X. We write $Fg(a)$ for $Fg(\{a\})$.

An element a of an RS $\boldsymbol{A}$ is said to be *negative* if $a \leq e$. Let $a, b \in A$, with a negative. We say that b is a *conjugate* of a if

$$b = [\, c \backslash (a \cdot c) \,] \wedge e \quad \text{or} \quad b = [\, (c \cdot a)/c \,] \wedge e$$

for some $c \in A$. In particular, a is a conjugate of itself (set $c = e$). Let W be the smallest subset of A such that $a \in W$ and all conjugates of elements of W belong to W. The elements of W are called the *iterated conjugates* of a.

Note that iterated conjugates are always negative, and that e is the only iterated conjugate of itself, because $e \leq c\backslash c$ and $e \leq c/c$ for all $c \in A$.

Theorem 2. (cf. [9, Thm. 4.8(3)]) *Let $\boldsymbol{A}$ be a residuated semilattice and let $a, b \in A$. Then $b \in Fg(a)$ iff $b \geq \gamma_1 \cdot \ldots \cdot \gamma_m$ for some $m \in \omega$ and some iterated conjugates $\gamma_1, \ldots, \gamma_m$ of $a \wedge e$.*

(We allow $m = 0$, interpreting the empty product as e.)

3. Finitely Subdirectly Irreducible Algebras

Although an RS $\boldsymbol{A}$ need not be lattice-ordered, we shall need to consider subsets of A that do have least upper bounds (lubs). Generalizing [6, Lem. 3.2], we have:

Lemma 3. *Let $\boldsymbol{A}$ be a residuated semilattice and let $A_1, \ldots, A_m$ be sets of negative elements of A.*

If e is the lub of $a_1, \ldots, a_m$ whenever each a_i belongs to the corresponding A_i, then e is also the lub of $p_1, \ldots, p_m$ whenever each p_i is a finite product of elements of A_i. (The factors of p_i are not assumed distinct.)

Proof. Suppose e is the lub of $X \cup \{b\}$, and also of $X \cup \{c\}$, where $a, b, c \leq e$ for all $a \in X$. We show that e is the lub of $X \cup \{b \cdot c\}$. Then the general result will follow inductively. Certainly, $b \cdot c \leq e \cdot e = e$, so e is an upper bound of $X \cup \{b \cdot c\}$. Suppose d is an upper bound of $X \cup \{b \cdot c\}$. We must show that $e \leq d$. From $b \cdot c \leq d$ we get $c \leq b\backslash d$. Also, for all $a \in X$, we have $a \leq b\backslash d$, because $b \cdot a \leq e \cdot a = a \leq d$. Then because e is the lub of $X \cup \{c\}$, it follows that $e \leq b\backslash d$, i.e., $b \leq d$. Now $a, b \leq d$ for all $a \in X$, so $e \leq d$, because e is the lub of $X \cup \{b\}$. □

Recall that in any partially ordered set, an element c is said to be *join-irreducible* if, whenever c is the lub of elements a, b, then $a = c$ or $b = c$.

Definition 4. Let $\boldsymbol{A}$ be a residuated semilattice. We shall say that e is *weakly join-irreducible* provided that the following is true for all negative elements $a, b \in A$: If e is the lub of γ, γ' for every iterated conjugate γ of a and every iterated conjugate γ' of b, then a or b is e.

Recall that an algebra $\boldsymbol{A}$ is said to be *finitely subdirectly irreducible* (briefly, *FSI*) if the identity relation is meet-irreducible in the congruence lattice of $\boldsymbol{A}$. Consequently, an RS $\boldsymbol{A}$ is FSI iff $[\,e\,)$ is meet-irreducible in the lattice of deductive filters of $\boldsymbol{A}$.

Theorem 5. *A residuated semilattice $\boldsymbol{A}$ is FSI iff its identity element is weakly join-irreducible.*

In this case, for any positive integer k and negative elements $a_1, \ldots, a_k \in A$, if e is the lub of $\gamma_1, \ldots, \gamma_k$ whenever each γ_i is an iterated conjugate of a_i, then $e = a_i$ for some i.

Proof. (⇐) Assume that e is weakly join-irreducible and let F, G be deductive filters of $\boldsymbol{A}$ with $F \cap G = [\,e\,)$. We must show that $F = [\,e\,)$ or $G = [\,e\,)$. Suppose, on the contrary, that $a \in F$ and $b \in G$, where $e \not\leq a$ and $e \not\leq b$, i.e., $a \wedge e < e$ and $b \wedge e < e$. Of course, $a \wedge e \in F$ and $b \wedge e \in G$. Let γ and γ' be iterated conjugates of $a \wedge e$ and $b \wedge e$, respectively, so $\gamma, \gamma' \leq e$. If $\gamma, \gamma' \leq u \in A$, then $u \in F \cap G = [\,e\,)$, so $e \leq u$. Thus, e is the lub of γ, γ' for all such iterated conjugates. Since e is weakly join-irreducible, $a \wedge e = e$ or $b \wedge e = e$, a contradiction.

(⇒) Let $\boldsymbol{A}$ be FSI. It suffices to prove the second claim in the theorem's statement. So let $a_1, \ldots, a_k \leq e$, and assume that e is the lub of $\gamma_1, \ldots, \gamma_k$ whenever each γ_i is an iterated conjugate of a_i. If $f \in Fg(a_1) \cap \ldots \cap Fg(a_k)$ then, by Theorem 2, there exist $p_1, \ldots, p_k \leq f$ such that each p_i is a product of iterated conjugates of a_i. Using the assumption and Lemma 3, we deduce that e is the lub of $p_1, \ldots, p_k$, so $f \in [\,e\,)$. Consequently, $Fg(a_1) \cap \ldots \cap Fg(a_k) = [\,e\,)$. Since $\boldsymbol{A}$ is FSI, $Fg(a_i) = [\,e\,)$ for some $i \in \{1, \ldots, k\}$, that is, $a_i = e$. □

Recall that a class of similar structures is said to be *elementary* if it can be axiomatized by a set of first order sentences. It is called *strictly elementary* if it can be axiomatized by a *finite* set of first order sentences (or equivalently, by one such sentence).

Problem 1. *Do the FSI residuated semilattices form an elementary class?*

Theorem 5 suggests that a negative answer is more likely, but we have not proved this. The corresponding problem for RLs is also open. The following finite basis theorem is due to Jónsson [13] (cf. [4, Thm. V.4.17]). For any class of algebras K, we use K_{FSI} to denote the class of all FSI members of K.

Theorem 6. (Jónsson) *If* V *is a congruence distributive variety of finite type and* V_{FSI} *is a strictly elementary class then* V *has a finite equational basis.*

Jónsson's more famous 'lemma' has the consequence that in the varietal join $\mathsf{V}_1 + \mathsf{V}_2$ of two subvarieties of a congruence distributive variety V, every FSI algebra belongs to one of the two subvarieties [12, Lem. 4.1]. Thus, if both subvarieties are finitely based and V_{FSI} is strictly elementary then $(\mathsf{V}_1 + \mathsf{V}_2)_{FSI} = (\mathsf{V}_1 \cup \mathsf{V}_2)_{FSI}$ is strictly elementary. In this case $\mathsf{V}_1 + \mathsf{V}_2$ is finitely based, by Theorem 6, provided the type is finite. Since RS is congruence distributive, we obtain

Theorem 7. *For any variety* V *of residuated semilattices, if* V_{FSI} *is strictly elementary, then the finitely based subvarieties of* V *are closed under finite joins.*

We shall see later that the adverb 'strictly' can be dropped from this statement (Corollary 32). Consequently, the open problem below would be solved affirmatively if Problem 1 has an affirmative solution (and similarly for RLs).

Problem 2. *Are the finitely based varieties of residuated semilattices closed under finite joins?*

In the next section we shall identify a large class of residuated semilattices within which the FSI algebras form a strictly elementary class.

4. Stability and Subcommutativity

An RS is said to be *commutative* if its monoid operation $\cdot$ is commutative. In this case, $a\backslash b = b/a$ for all elements a, b, and it is customary to omit / from the type, writing $a\backslash b$ as $a \to b$. We shall see presently that for any variety V of commutative RSs, the class V_{FSI} is strictly elementary. But here the demand of commutativity is unnecessarily strong. In this section we consider some weak variants of commutativity.

In an RS $\boldsymbol{A}$, we define $a^0 := e$ and $a^{m+1} := a^m \cdot a$ for all $a \in A$ and $m \in \omega$.

Definition 8. A negative element a of an RS $\boldsymbol{A}$ will be called *stable* if the following is true: for each $c \in A$ there exist $m, n \in \omega$ such that

$$c \cdot a^m \leq a \cdot c \quad \text{and} \quad a^n \cdot c \leq c \cdot a.$$

Theorem 9. *In a residuated semilattice $\boldsymbol{A}$, a negative element a is stable iff for each iterated conjugate γ of a, there exists $m \in \omega$ such that $\gamma \geq a^m$.*

Proof. ($\Rightarrow$) Assuming that a is stable, we can prove the following claim by induction on k.

For each $c \in A$ and each $k \in \omega$, there exist $r(c,k), l(c,k) \in \omega$ such that $c \cdot a^{l(c,k)} \leq a^k \cdot c$ and $a^{r(c,k)} \cdot c \leq c \cdot a^k$.

This is clear for $k \leq 1$. And if the claim holds for some $k \geq 1$ then, defining $l(c, k+1) = l(c,k) + l(c,1)$ and $r(c,k+1) = r(c,k) + r(c,1)$, we get

$$c \cdot a^{l(c,k+1)} = c \cdot a^{l(c,k)} \cdot a^{l(c,1)} \leq a^k \cdot c \cdot a^{l(c,1)} \leq a^k \cdot a \cdot c = a^{k+1} \cdot c,$$

and similarly, $a^{r(c,k+1)} \cdot c \leq c \cdot a^{k+1}$. Note that every power of a is negative, since a is negative. Thus, the conclusion of the claim can be restated as

$$a^{l(c,k)} \leq (c\backslash(a^k \cdot c)) \wedge e \quad \text{and} \quad a^{r(c,k)} \leq ((c \cdot a^k)/c) \wedge e. \tag{2}$$

Setting $k = 1$ in (2), we see that every 'depth 1' conjugate of a dominates a power of a. Assume now that for some iterated conjugate γ of a, there exists $m \in \omega$ such that $a^m \leq \gamma$. For all c, the function $x \mapsto (c\backslash(x \cdot c)) \wedge e$ is clearly order preserving, so from (2) we get $a^{l(c,m)} \leq (c\backslash(a^m \cdot c)) \wedge e \leq (c\backslash(\gamma \cdot c)) \wedge e$. Likewise, $a^{r(c,m)} \leq ((c \cdot \gamma)/c) \wedge e$, and the result follows by induction.

($\Leftarrow$) Conversely, given $c \in A$, the condition on iterated conjugates implies that $(c\backslash(a \cdot c)) \wedge e$ dominates a^m for some $m \in \omega$. So $a^m \leq c\backslash(a \cdot c)$, i.e., $c \cdot a^m \leq a \cdot c$. Similarly, $a^n \cdot c \leq c \cdot a$ for some $n \in \omega$. □

The next result generalizes observations in [5] and [9].

Corollary 10. *For any residuated semilattice $\boldsymbol{A}$, the following are equivalent.*

(i) *Every negative element of $\boldsymbol{A}$ is stable.*
(ii) *For any $a, b \in A$, we have $b \in Fg(a)$ iff $b \geq (a \wedge e)^m$ for some $m \in \omega$.*
(iii) *The deductive filters of $\boldsymbol{A}$ are just the submonoids of $\langle A; \cdot, e\rangle$ that are filters of the semilattice $\langle A; \wedge\rangle$.*

Proof. (i) $\Rightarrow$ (ii) follows from Theorems 2 and 9. Also, (ii) $\Rightarrow$ (i) follows from Theorem 9, because iterated conjugates of a negative element a belong to $Fg(a)$.

(ii) $\Rightarrow$ (iii): Let H be a submonoid of $\langle A; \cdot, e\rangle$ that is a filter of $\langle A; \wedge\rangle$. For all $a, b \in A$, we have $Fg(a) \cup Fg(b) \subseteq Fg(a \wedge b)$, and when $a, b \in H$ then $a \wedge b \in H$. It follows that $\bigcup_{a \in H} Fg(a)$ is a deductive filter of $\boldsymbol{A}$, so $Fg(H) = \bigcup_{a \in H} Fg(a)$. But $\bigcup_{a \in H} Fg(a) \subseteq H$, by (ii). Thus H is a deductive filter.

(iii) $\Rightarrow$ (ii) is straightforward and just like the commutative case. □

Lemma 11. *Let $\boldsymbol{A}$ be a residuated semilattice in which every negative element is stable. Then e is weakly join-irreducible iff it is join-irreducible. Consequently, $\boldsymbol{A}$ is FSI iff e is join-irreducible.*

Proof. Suppose e is the lub of $a, b \in A$. By Lemma 3, e is also the lub of a^m, b^n for all $m, n \in \omega$. So, since negative elements are stable, Theorem 9 shows that e is the lub of γ, γ' whenever γ, γ' are iterated conjugates of a, b, respectively. Thus, e will be join-irreducible if it is weakly join-irreducible. The second claim follows from the first, by Theorem 5. □

The stability of negative elements does not seem to be a first order property, but it holds in many simply defined varieties of RSs. In particular,

Definition 12. For any positive integer n, an RS will be called *n–subcommutative* if it satisfies $x \leq e \implies x^n \cdot y \approx y \cdot x^n$, or equivalently,

$$(x \wedge e)^n \cdot y \approx y \cdot (x \wedge e)^n.$$

A class of RSs is said to be *n–subcommutative* if its members are. It is said to be *subcommutative* if it is n–subcommutative for some fixed finite n.

Obviously, n–subcommutative implies kn–subcommutative for every positive integer k, so the union of two subcommutative classes of RSs is subcommutative. In particular, 1–subcommutative is equivalent to 'n–subcommutative for all finite n'. The next result shows that there is an abundant supply of subcommutative RSs, other than commutative ones.

Proposition 13. *Let $\boldsymbol{A}$ be any residuated semilattice. Then the e–free reduct $\langle A; \cdot, \backslash, /, \wedge\rangle$ of $\boldsymbol{A}$ can be embedded into a 1–subcommutative residuated semilattice.*

Proof. We add to A a new element e', extending the definitions of $\cdot$ and $\wedge$ to $A \cup \{e'\}$ as follows: $e' \cdot x = x = x \cdot e'$ for all $x \in A \cup \{e'\}$ and $e' < a$ for all $a \in A$ (really $e' \wedge x = e' = x \wedge e'$ for all $x \in A \cup \{e'\}$). Then we add two further elements $\bot, \top$, defining that $\bot < x < \top$ for all $x \in A \cup \{e'\}$, and $\bot \cdot y = \bot = y \cdot \bot$ for all $y \in A \cup \{e', \bot, \top\}$ and $\top \cdot z = \top = z \cdot \top$ for all $z \in A \cup \{e', \top\}$. Now $\langle A \cup \{e', \bot, \top\}; \cdot, \wedge\rangle$ is a meet semilattice-ordered semigroup with identity e', extending the structure of $\langle A; \cdot, \wedge\rangle$. It is residuated, and residuals within $\boldsymbol{A}$ continue to function as such in the superstructure. The superstructure is 1–subcommutative, as the value of $x \wedge e'$ is always $\bot$ or e'. □

(This proof also establishes the corresponding proposition for RLs.)

For any negative element a of an RS $\boldsymbol{A}$, if $0 < n \in \omega$ then $a \geq a^n$, and so $a \cdot c \geq a^n \cdot c$. If, in addition, $\boldsymbol{A}$ is n–subcommutative, then $a^n \cdot c = c \cdot a^n$, whence $a \cdot c \geq c \cdot a^n$, and similarly $c \cdot a \geq a^n \cdot c$. Thus,

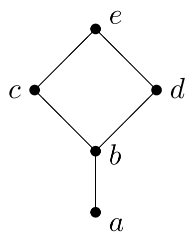
Lemma 14. *In a subcommutative residuated semilattice, every negative element is stable.*

From Lemmas 11 and 14, we infer

Corollary 15. *A subcommutative residuated semilattice is FSI iff its identity element is join-irreducible.*

For commutative RSs, the implication from left to right was proved in [16], and it has antecedents in the theory of BCK-algebras: see [18]. (BCK–algebras are the pure $\{\rightarrow, e\}$–subreducts of *integral* commutative RLs, where *integral* means that e is the greatest element.) An example of a commutative RS that is FSI but not subdirectly irreducible is the reduct of the Heyting chain with order type $\omega + 1$, where $\cdot$ and $\wedge$ are both interpreted as minimum, $\rightarrow$ is relative pseudo-complementation, and e is the top element. The next example shows that in Corollary 15, we cannot drop the hypothesis of subcommutativity. The algebra in this example is taken from [19, p. 436].

Example 16. Consider the integral ordered monoid $\langle A; \cdot, \wedge, e\rangle$ whose Hasse diagram and binary operation $\cdot$ are indicated below. Since $\leq$ is compatible with $\cdot$, and since $\max\{z : x \cdot z \leq y\}$ and $\max\{z : z \cdot x \leq y\}$ both exist for all $x, y \in A$, this structure is the reduct of a unique RS $\boldsymbol{A}$.



$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	b	a	b
c	a	a	c	a	c
d	a	b	b	d	d
e	a	b	c	d	e

Now c and d are negative non-commuting idempotents, so $\boldsymbol{A}$ is not subcommutative. Note that $d\backslash(c \cdot d) = d\backslash a = a$ and $(c \cdot d)/c = a/c = a$. Since deductive filters are upward closed and closed under conjugation, this shows that $Fg(c) = Fg(d) = A$, whence $\boldsymbol{A}$ is simple (and therefore FSI), but e is not join-irreducible. Notice that c and d are not stable, since the conjugate a does not dominate any of their powers.

Notation. Let n–RS denote the class of all n–subcommutative RSs.

Since n–RS is a finitely based variety and the join-irreducibility of e can be expressed as a first order sentence about e and meets, Corollary 15 implies that the FSI algebras in n–RS form a strictly elementary class, for each $n > 0$. These are not universal classes, since join-irreducibility of e may be lost in subalgebras (unlike the case of subcommutative RLs). Still, Theorem 7 yields

Theorem 17. *The varietal join of any two finitely based subcommutative varieties of residuated semilattices is finitely based.*

An RS is said to be *idempotent* if $a^2 = a$ for all elements a. It is well known that every idempotent *integral* RS is commutative; these are the *Brouwerian semilattices* of [14]. The next result partially generalizes this fact to the non-integral case. Here an RS is called *e–comparable* if, for all elements a, we have $e \leq a$ or $a \leq e$.

Theorem 18. *For any idempotent e–comparable residuated semilattice* $\boldsymbol{A}$, *the following conditions are equivalent.*

(i) *Every negative element of* $\boldsymbol{A}$ *is stable.*
(ii) $\boldsymbol{A}$ *satisfies* $x\backslash e \approx e/x$.
(iii) $\boldsymbol{A}$ *satisfies* $x \cdot y \leq e \iff y \cdot x \leq e$.
(iv) $\boldsymbol{A}$ *is commutative.*

Proof. (i) $\Rightarrow$ (ii): Let $b = a\backslash e$, where $a \in A$. Then $a \cdot b \leq e$, so we cannot have $a, b > e$, as that would imply $a \cdot b \geq a, b$. So $a \leq e$ or $b \leq e$, by e–comparability. If $a \leq e$ then, by (i), there exists $m \in \omega$ such that $a \cdot b \geq b \cdot a^m \geq b \cdot a$, where the last inequality follows from idempotence (or from the negativity of a when $m = 0$). Similarly, if $b \leq e$ then $a \cdot b \geq b^n \cdot a \geq b \cdot a$ for some $n \in \omega$. In both cases, $b \cdot a \leq e$, i.e., $b \leq e/a$. We have shown that $\boldsymbol{A}$ satisfies $x\backslash e \leq e/x$, so by symmetry, it satisfies $x\backslash e \approx e/x$.

(ii) $\Rightarrow$ (iii) follows from the definition of residuation.

(iii) $\Rightarrow$ (iv): Idempotence alone ensures that when $a, b \geq e$ then $a \cdot b$ is the lub of a, b. For in this case $a \cdot b \geq a, b$, and if $u \geq a, b$ then $u = u^2 \geq a \cdot b$. Thus, elements above e commute. A dual argument shows that elements below e commute, the product of a and b being $a \wedge b$.

By e–comparability and symmetry, it remains only to consider the case $a < e < b$. In this case, $a \leq a \cdot b \leq b$ and $a \leq b \cdot a \leq b$. Suppose first that $a \cdot b > e$. Then $b \cdot a > e$, by (iii) and e–comparability. By idempotence, $a \cdot b = a \cdot b^2 = (a \cdot b) \cdot b \geq b$, hence $a \cdot b = b$. Similarly, $b \cdot a = b$, so a and b commute. By e–comparability, we may now assume that $a \cdot b \leq e$. Then a dual argument gives $a \cdot b = a = b \cdot a$, completing the proof of commutativity.

(iv) $\Rightarrow$ (i) is obvious. $\square$

In the light of the above proof, it is easy to see that the e–comparable idempotent RLs are just the RLs satisfying $\forall x \, \forall y \; (x \cdot y \approx x \wedge y$ or $x \cdot y \approx x \vee y)$. Some noncommutative totally ordered idempotent RLs are exhibited in [7].

5. Constructive Axiomatization

Theorems 7 and 17 do not give us a practical method of axiomatizing the join of two subcommutative varieties of RSs for which finite equational bases are known, because Jónsson's finite basis theorem has a non-constructive proof that invokes the Compactness Theorem of first order logic. For varieties with equationally definable principal congruences, a constructive proof can be given: see [1]. But n–RS lacks even first order-definable principal congruences, since an ultraproduct of simple commutative (integral) RSs need not be simple. Indeed, the additive monoid of non-positive integers with the conventional total order is a simple commutative RS with no simple non-principal ultrapower: see [2].

Galatos [6] proved constructively that the varietal join of any two recursively axiomatized varieties of RLs is recursively axiomatized, and that the joins of certain *finitely* based varieties of RLs are finitely based—including the case of subcommutative varieties. The arguments made use of both lattice operations. Using the theory developed above, we shall obtain the corresponding results for RSs constructively.

Lemma 19. *Let* $\boldsymbol{A}$ *be a residuated semilattice and* $a_1, \dots, a_k \in A$*. Then* e *is the lub of* $a_1, \dots, a_k$ *iff* $(a_1 \backslash b) \wedge \cdots \wedge (a_k \backslash b) = b$ *for all* $b \in A$*.*

Proof. Suppose e is the lub of $a_1, \dots, a_k$, and let $b \in A$. Then for each i, we have $a_i \leq e$, hence $a_i \cdot b \leq b$, i.e., $b \leq a_i \backslash b$. Thus, $b \leq (a_1 \backslash b) \wedge \cdots \wedge (a_k \backslash b)$. On the other hand, since $a_i \cdot ((a_1 \backslash b) \wedge \cdots \wedge (a_k \backslash b)) \leq a_i \cdot (a_i \backslash b) \leq b$, we have $a_i \leq b/((a_1 \backslash b) \wedge \cdots \wedge (a_k \backslash b))$. Since i was arbitrary and e is the lub of $a_1, \dots, a_k$, it follows that $e \leq b/((a_1 \backslash b) \wedge \cdots \wedge (a_k \backslash b))$, i.e., $(a_1 \backslash b) \wedge \cdots \wedge (a_k \backslash b) \leq b$. Thus, $(a_1 \backslash b) \wedge \cdots \wedge (a_k \backslash b) = b$.

Conversely, suppose $(a_1 \backslash b) \wedge \cdots \wedge (a_k \backslash b) = b$ for all $b \in A$. In particular, $(a_1 \backslash e) \wedge \cdots \wedge (a_k \backslash e) = e$, so for each i, we have $e \leq a_i \backslash e$, i.e., $a_i \leq e$. Thus, e is an upper bound of $a_1, \dots, a_k$. Suppose b is another upper bound. For each i, we infer from $a_i \leq b$ that $e \leq a_i \backslash b$, hence $e \leq (a_1 \backslash b) \wedge \cdots \wedge (a_k \backslash b)$. The right hand side of this inequality is b, by assumption, so $e \leq b$. This shows that e is the lub of $a_1, \dots, a_k$. □

We need to recall the following lemma from [6].

Lemma 20. *Let* α *be any universal positive sentence in the first order language with equality determined by* $\cdot, \backslash, /, \wedge, e$*. Then* α *can be transformed systematically into the universal closure* α' *of a disjunction of atomic formulas* $e \leq r$*,* r *an* RS*–term, where* α *and* α' *are logically equivalent over* RS *and all variables occurring in the terms* r *already occur in* α*.*

Proof. Since $\mathsf{RS} \models x \leq y \iff e \leq x \backslash y$, each equation $p \approx q$ is logically equivalent over RS to $e \leq p \backslash q \;\&\; e \leq q \backslash p$. Also, a conjunction $e \leq r_1 \;\&\; \cdots \;\&\; e \leq r_k$ is clearly equivalent to $e \leq r_1 \wedge \cdots \wedge r_k$. Now the result follows because any universal positive sentence can be transformed systematically into the logically equivalent universal closure of a disjunction of conjunctions of equations, without introducing new variables. □

Notation. For each RS–term r, for any set of variables Var, and for each $m \in \omega$, we define the following sets of terms.

$$\begin{aligned}
\Gamma^0_{\mathit{Var}}(r) &= \{r \wedge e\}\,; \\
\Gamma^{m+1}_{\mathit{Var}}(r) &= \{\, [\, v \backslash (s \cdot v)\,] \wedge e \,:\, v \in \mathit{Var} \text{ and } s \in \Gamma^m_{\mathit{Var}}(r)\,\} \\
&\quad \cup \{\, [\, (v \cdot s)/v \,] \wedge e \,:\, v \in \mathit{Var} \text{ and } s \in \Gamma^m_{\mathit{Var}}(r)\,\}\,; \\
\Gamma_{\mathit{Var}}(r) &= \textstyle\bigcup_{n \in \omega} \Gamma^n_{\mathit{Var}}(r)\,.
\end{aligned}$$

Definition 21. Let Ψ be a set of universal positive sentences in the language of RS. Expanding the set of variables if necessary, we choose a denumerable set of variables Y and a variable $z \notin Y$ such that no variable in $Y \cup \{z\}$ occurs in any sentence from Ψ. Suppose that $\alpha \in \Psi$ and that the transformation of α, according to Lemma 20, is

$$\forall\, \overline{x}\;\; (\, e \leq r_1 \;\;\mathsf{or}\;\; \cdots \;\;\mathsf{or}\;\; e \leq r_k \,), \tag{3}$$

so no variable in $Y \cup \{z\}$ occurs in any of the terms r_i. Then $\widetilde{\alpha}_0$ shall denote the singleton consisting of the equation

$$[\,(r_1 \wedge e)\backslash z\,] \;\wedge\; \cdots \;\wedge\; [\,(r_k \wedge e)\backslash z\,] \;\approx\; z.$$

For each integer $m > 0$, let $\widetilde{\alpha}_m$ be the set of all equations of the form

$$(\gamma_1\backslash z) \;\wedge\; \cdots \;\wedge\; (\gamma_k\backslash z) \;\approx\; z$$

such that $\gamma_i \in \Gamma_Y^m(r_i)$ for each $i \in \{1, \ldots, k\}$. Let $\widetilde{\alpha} = \bigcup_{n\in\omega} \widetilde{\alpha}_n$. Finally, for each $m \in \omega$, we define

$$\widetilde{\Psi}_m \;=\; \bigcup\nolimits_{\alpha\in\Psi} \widetilde{\alpha}_m \qquad \text{and} \qquad \widetilde{\Psi} \;=\; \bigcup\nolimits_{\alpha\in\Psi} \widetilde{\alpha} \quad (= \bigcup\nolimits_{n\in\omega} \widetilde{\Psi}_n).$$

Theorem 22. *Let α be a universal positive sentence in the language of* RS*, and let $\boldsymbol{A}$ be a residuated semilattice that is FSI. Then*

(i) $\boldsymbol{A} \models \alpha$ *iff* $\boldsymbol{A} \models \widetilde{\alpha}$.

If every negative element of $\boldsymbol{A}$ is stable, then

(ii) $\boldsymbol{A} \models \alpha$ *iff* $\boldsymbol{A} \models \widetilde{\alpha}_0$.

Proof. (i) The implication from left to right does not depend on finite subdirect irreducibility. Suppose that $\boldsymbol{A} \models \alpha$. By Lemma 20, we may assume that α has the form displayed in (3). Consider an interpretation in $\boldsymbol{A}$ of the variables $\overline{x}$, and for each term t, let t^* denote the induced interpretation of t. As $\boldsymbol{A} \models \alpha$, we can choose an $i \in \{1, \ldots, k\}$ such that $e \leq r_i^*$, i.e., $r_i^* \wedge e = e$. Since e is the only iterated conjugate of itself, we have $\gamma_i^* = e$ for every $\gamma_i \in \Gamma_Y(r_i)$. On the other hand, if $\gamma_j \in \Gamma_Y(r_j)$ for each $j \in \{1, \ldots, k\}$, then $\gamma_j^* \leq e$ for all j, so e is the lub of $\gamma_1^*, \ldots, \gamma_k^*$. Then $\boldsymbol{A} \models \widetilde{\alpha}$, by Lemma 19.

For the converse, note first that e is weakly join-irreducible, by Theorem 5, because $\boldsymbol{A}$ is FSI. Suppose $\boldsymbol{A} \not\models \alpha$. Then in $\boldsymbol{A}$, there is an interpretation of the variables of $r_1, \ldots, r_k$ which simultaneously falsifies all of the disjuncts $e \leq r_i$. Fixing one such interpretation, we adopt the t^* notation as before. For each i, we have $r_i^* \wedge e < e$. So, because e is weakly join-irreducible, Theorem 5 shows that under a suitable extension to Y of our interpretation $v \mapsto v^*$, e fails to be the least upper bound of some $\gamma_1^*, \ldots, \gamma_k^* \in A$, where $\gamma_i \in \Gamma_Y(r_i)$ for each i. (We use here the fact that no variable in Y occurs in any of the r_i.) Then by Lemma 19, there exists $b \in A$ such that

$$(\gamma_1^*\backslash b) \;\wedge\; \cdots \;\wedge\; (\gamma_k^*\backslash b) \;\neq\; b. \tag{4}$$

Since $z \notin Y$ and z does not occur in any of the terms r_i, we may again extend our interpretation so that b interprets z. Thus, (4) witnesses that $\boldsymbol{A} \not\models \widetilde{\alpha}$.

(ii) The implication from left to right follows from (i). For the converse, note that e is join-irreducible, by Lemma 11, and we can simply replace each γ_i^* by $r_i^* \wedge e$ in the argument. □

Theorem 23. *Let* K *be the class of all RSs that satisfy a given set* Ψ *of universal positive sentences. Then* $\mathrm{HSP}(\mathsf{K})$ *is axiomatized, relative to* RS, *by* $\widetilde{\Psi}$.

If negative elements are stable in all members of $\mathrm{HSP}(\mathsf{K})$, *e.g., if* K *is subcommutative, then* $\mathrm{HSP}(\mathsf{K})$ *is axiomatized, relative to* RS, *by* $\widetilde{\Psi}_0$.

Proof. Let $\boldsymbol{A} \in \mathrm{HSP}(\mathsf{K})$ be subdirectly irreducible. Since RS is congruence distributive, Jónsson's Lemma [12, Cor. 3.2] implies that $\boldsymbol{A} \in \mathrm{HSP_U}(\mathsf{K})$.[1] As universal positive sentences persist under the class operators H, S, and $\mathrm{P_U}$, it follows that $\boldsymbol{A} \models \Psi$. By Theorem 22, $\boldsymbol{A} \models \widetilde{\Psi}$. But Theorem 22 also shows that every subdirectly irreducible RS which satisfies $\widetilde{\Psi}$ satisfies Ψ, and so is in K already, hence in $\mathrm{HSP}(\mathsf{K})$. The second statement follows similarly. □

Recall that $\widetilde{\Psi}_0$ and Ψ have the same cardinality. In particular, if K is subcommutative and axiomatized by a given *finite* set of universal positive sentences, we may construct a finite equational basis for $\mathrm{HSP}(\mathsf{K})$. For instance,

Example 24. Let K be the class of all RSs $\boldsymbol{A}$ with a least element $\bot$ such that $A = \{\bot\} \cup [\,e\,)$. Then K is 1–subcommutative, so we may apply the second statement of Theorem 23. Now K is axiomatized relative to RS by $\forall x\,\forall y\ (e \leq x \ \textsf{or}\ x \leq y)$. The second disjunct becomes $e \leq x\backslash y$. Then the algorithm presented above produces the identity

$$[\,(x \wedge e)\backslash z\,] \ \wedge\ [\,((x\backslash y) \wedge e)\backslash z\,] \ \approx\ z.$$

So this identity axiomatizes the variety generated by K, relative to RS.

Remark 25. For residuated *lattices*, variants of Definition 21 and the last two theorems appear in [6], where equations of the form $\gamma_1 \vee \cdots \vee \gamma_k \approx e$ were used instead of $(\gamma_1\backslash z) \wedge \cdots \wedge (\gamma_k\backslash z) \approx z$. Despite Lemma 19, we could not have presented Theorems 22 and 23 as corollaries of the corresponding results for RLs, as there is no evidence that an arbitrary FSI RS can be embedded into an RL satisfying all the same join-free identities. Our proof of Theorem 22 made use of Theorem 5, which has no analogue in [6]. We could instead have adapted the proof in [6]. But the present approach reveals what that proof has in common with the treatment of commutative RSs in [16], where the commutative case of Corollary 15 was used to get restricted versions of Theorems 22 and 23.

Remark 26. Suppose V_1 and V_2 are varieties of RSs, where V_1 is axiomatized by equations δ_i and V_2 by equations ε_j, and no variable occurs both in some δ_i and in some ε_j. Then the universal positive class $\mathsf{V}_1 \cup \mathsf{V}_2$ is axiomatized by the universal closure of $((\,\&_i\ \delta_i\,)\ \textsf{or}\ (\,\&_j\ \varepsilon_j\,))$. This is not generally a first order sentence, but it is equivalent in infinitary logic to the set Ψ of all universally

[1] The proof of Jónsson's Lemma shows that only finite subdirect irreducibility is needed, hence we could replace 'subdirectly irreducible' by 'FSI' throughout the present proof.

quantified first order formulas of the form $(\,\delta_i \text{ or } \varepsilon_j\,)$. So Theorem 23 shows that the varietal join $\mathsf{V}_1 + \mathsf{V}_2$ is axiomatized by $\widetilde{\Psi}$. It also shows that if both varieties were subcommutative then $\mathsf{V}_1 + \mathsf{V}_2$ is axiomatized by $\widetilde{\Psi}_0$. If we start with only finitely many equations δ_i, ε_j then $\widetilde{\Psi}_0$ is finite, so this constructively proves Theorem 17.

In any variety, the set of finitely based subvarieties is obviously closed under finite intersections. So the finitely based subvarieties of n–RS form a sublattice of the lattice of all varieties of RSs. The *commutative* RSs algebraize the disjunction-free fragment of the system $\mathbf{FL}_{\mathbf{e}}^{+}$, discussed for instance in [9], which is itself a fragment of linear logic. Thus we infer

Corollary 27. *Over the disjunction-free fragment of* $\mathbf{FL}_{\mathbf{e}}^{+}$, *the finitely based axiomatic extensions form a sublattice of the lattice of all axiomatic extensions.*

6. Elementarity of V_{FSI}

In this section we shall characterize the demand 'V_{FSI} is an elementary class', where V is any variety of RSs. We also show that when V satisfies this condition, then its finitely based subvarieties are closed under varietal joins. This result covers many cases in which V is not subcommutative (see Section 7). We shall need the following abbreviations.

Notation. $\lambda_a(b) := [\,a\backslash(b \cdot a)\,] \wedge e$, and $\rho_a(b) := [\,(a \cdot b)/a\,] \wedge e$.

We make the convention that λ_a and ρ_a bind more strongly than the basic operations, e.g., $\lambda_a(b \wedge c)\backslash d$ abbreviates $(\lambda_a(b \wedge c))\backslash d$.

In Definition 21, the sets $\widetilde{\Psi}_m$ are infinite for all $m \geq 1$, even when Ψ is a finite set of sentences. Nevertheless, adapting [6], we may replace each $\widetilde{\Psi}_m$ by a *finite* set of equations that serves the same purpose. Indeed, let α be a universal positive sentence in the form

$$\forall\,\overline{x}\ \ (\,e \leq r_1 \ \text{ or } \ \cdots \ \text{ or } \ e \leq r_k\,),$$

and choose a denumerable set of variables $Y = \{y_1, y_2, \dots\}$ and a variable $z \notin Y$, where no variable in $Y \cup \{z\}$ occurs in any of the terms r_i. The set $\widetilde{\alpha}_m$ consists of equations $(\gamma_1\backslash z) \wedge \cdots \wedge (\gamma_k\backslash z) \approx z$ where, for instance, γ_1 is an expression of the form $\mu_1\mu_2 \,\dots\, \mu_m(r_1 \wedge e)$ in which each μ_j is either λ_y or ρ_y for some conjugating variable $y \in Y$.

From now on, let us insist that the indices of the conjugating variables in $\mu_1, \dots, \mu_m$ are $y_1, \dots, y_m$, respectively, and similarly that the conjugating variables in γ_2 are $y_{m+1}, \dots, y_{2m}$, etc., so that the conjugating variables in $\gamma_1, \dots, \gamma_k$ (in that order) are $y_1, \dots, y_{km}$. This re-definition of $\widetilde{\alpha}_m$ makes $\widetilde{\alpha}_m$ a finite set with 2^{km} elements. For instance, when $k = 1$ then $\widetilde{\alpha}_2$ consists of

$$\lambda_{y_1}\lambda_{y_2}(r_1 \wedge e)\backslash z \;\approx\; z\,, \qquad \lambda_{y_1}\rho_{y_2}(r_1 \wedge e)\backslash z \;\approx\; z\,,$$
$$\rho_{y_1}\lambda_{y_2}(r_1 \wedge e)\backslash z \;\approx\; z\,, \qquad \rho_{y_1}\rho_{y_2}(r_1 \wedge e)\backslash z \;\approx\; z\,,$$

and when $k = 2$ then $\widetilde{\alpha}_1$ consists of

$$\begin{aligned}
[\lambda_{y_1}(r_1\wedge e)\backslash z] \ &\wedge\ [\lambda_{y_2}(r_2\wedge e)\backslash z] \ \approx\ z\,,\\
[\lambda_{y_1}(r_1\wedge e)\backslash z] \ &\wedge\ [\rho_{y_2}(r_2\wedge e)\backslash z] \ \approx\ z\,,\\
[\rho_{y_1}(r_1\wedge e)\backslash z] \ &\wedge\ [\lambda_{y_2}(r_2\wedge e)\backslash z] \ \approx\ z\,,\\
[\rho_{y_1}(r_1\wedge e)\backslash z] \ &\wedge\ [\rho_{y_2}(r_2\wedge e)\backslash z] \ \approx\ z\,.
\end{aligned}$$

We define $\widetilde{\alpha}$, $\widetilde{\Psi}_m$ and $\widetilde{\Psi}$ in terms of the sets $\widetilde{\alpha}_m$, as in Definition 21. Thus, if Ψ is a finite set of sentences then $\widetilde{\Psi}_m$ is finite for each $m\in\omega$. It is clear that a residuated semilattice satisfies $\widetilde{\Psi}_m$ in its new sense iff it satisfies $\widetilde{\Psi}_m$ in the original sense, and similarly for $\widetilde{\Psi}$.

Notation. Let β denote the first order sentence $\forall x_1\,\forall x_2\ (e\leq x_1 \text{ or } e\leq x_2)$.

In view of Lemma 19, the demand that e is the lub of γ_1,γ_2 for all iterated conjugates γ_1 of $x_1\wedge e$ and γ_2 of $x_2\wedge e$ is captured by the infinitary formula

$$\forall\,\overline{y}\ \forall z\ \ \&\ \textstyle\bigcup_{n\in\omega}\widetilde{\beta}_n,$$

where $\overline{y}$ abbreviates $y_1,y_2,\ldots$. The free variables of this formula are just x_1,x_2. Since $x_i\wedge e\approx e$ may be rewritten as $e\leq x_i$, Theorem 5 may be paraphrased as

Proposition 28. *A residuated semilattice is FSI iff it satisfies the infinitary sentence*

$$\forall x_1\,\forall x_2\ [\,(\,\forall\,\overline{y}\ \forall z\ \ \&\ \textstyle\bigcup_{n\in\omega}\widetilde{\beta}_n\,)\implies(\,e\leq x_1 \text{ or } e\leq x_2\,)\,].$$

The converse of the implication in Proposition 28 is always true, as was essentially shown in the proof of Theorem 22(i). Note that every RS satisfies

$$(5)\qquad \forall x_1\,\forall x_2\ [\,(\,\forall\,\overline{y}\ \forall z\ \ \&\ \widetilde{\beta}_{m+1}\,)\implies(\,\forall\,\overline{y}\ \forall z\ \ \&\ \widetilde{\beta}_m\,)\,]$$

for all $m\in\omega$, because it satisfies $x\leq e\implies\lambda_e(x)\approx x\approx\rho_e(x)$.

Theorem 29. *For any variety* V *of residuated semilattices, the following conditions are equivalent.*

(i) V_{FSI} *is an elementary class.*
(ii) V_{FSI} *is closed under ultraproducts.*
(iii) *There exists* $m\in\omega$ *such that* V_{FSI} *satisfies*

$$\forall x_1\,\forall x_2\ [\,(\,\forall\,\overline{y}\ \forall z\ \ \&\ \widetilde{\beta}_m\,)\implies(\,e\leq x_1 \text{ or } e\leq x_2\,)\,].$$

(iv) *There exists* $m\in\omega$ *such that* V *satisfies*

$$\forall x_1\,\forall x_2\ [\,(\,\forall\,\overline{y}\ \forall z\ \ \&\ \widetilde{\beta}_m\,)\implies(\,\forall\,\overline{y}\ \forall z\ \ \&\ \widetilde{\beta}_{m+1}\,)\,].$$

In this case, V_{FSI} *is strictly elementary iff* V *is finitely axiomatized.*

Proof. (i) $\Rightarrow$ (ii) is an instance of Łos' Theorem.

(ii) $\Rightarrow$ (iii) is proved by contradiction, using a standard argument. Suppose that for each $m\in\omega$, we can find elements x_1^m and x_2^m which witness failure of the sentence in (iii) in some FSI algebra $\boldsymbol{A}_m\in\mathsf{V}$. Then for any non-principal ultrafilter $\mathcal{U}$ over ω, the elements $(x_1^0,x_1^1,x_1^2,\ldots)/\mathcal{U}$ and $(x_2^0,x_2^1,x_2^2,\ldots)/\mathcal{U}$ witness failure of the sentence in Proposition 28 in the ultraproduct $\prod_{m\in\omega}\boldsymbol{A}_m/\mathcal{U}$ (in view of (5)), whence this ultraproduct is not FSI. This contradicts (ii).

(iii) $\Rightarrow$ (iv): It follows from (iii) and the converse of the bracketed implication in Proposition 28 that for some $m \in \omega$,

$$\mathsf{V}_{FSI} \models \forall x_1 \forall x_2 \, [\, (\forall \overline{y} \, \forall z \, \, \& \, \widetilde{\beta}_m \,) \implies (\forall \overline{y} \, \forall z \, \, \& \, \widetilde{\beta}_{m+1} \,) \,]. \tag{6}$$

Note that $\forall \overline{y} \, \forall z \, \& \, \widetilde{\beta}_m$ is a positive formula and $\widetilde{\beta}_{m+1}$ consists of atomic formulas, viz. equations. So (6) is logically equivalent to a special Horn sentence in the sense of Lyndon [15], whence it persists in subdirect products. Then (iv) follows because every algebra in V is a subdirect product of ones in V_{FSI}.

(iv) $\Rightarrow$ (i): Let $m \in \omega$ and assume that

$$\forall x_1 \forall x_2 \, [\, (\forall \overline{y} \, \forall z \, \, \& \, \widetilde{\beta}_m \,) \implies (\forall \overline{y} \, \forall z \, \, \& \, \widetilde{\beta}_{m+1} \,) \,] \tag{7}$$

is true in V. Here $\overline{y}$ may be taken to abbreviate $y_1, \ldots, y_{2m+2}$, of which only $y_1, \ldots, y_{2m}$ occur in the premise of the implication. Consider the sentence

$$\forall x_1 \forall x_2 \, [\, (\forall \overline{y} \, \forall z \, \, \& \, \widetilde{\beta}_{m+1} \,) \implies (\forall \overline{y} \, \forall z \, \, \& \, \widetilde{\beta}_{m+2} \,) \,], \tag{8}$$

in which $\overline{y}$ now abbreviates $y_1, \ldots, y_{2m+4}$. Observe that (8) is equivalent to a finite conjunction of instances of (7), each of which is got by first re-labeling $y_{m+1}, \ldots, y_{2m+2}$ as $y_{m+2}, \ldots, y_{2m+3}$ (respectively), and then replacing x_1 by $\lambda_{y_{m+1}}(x_1)$ or by $\rho_{y_{m+1}}(x_1)$, and x_2 by $\lambda_{y_{2m+4}}(x_2)$ or by $\rho_{y_{2m+4}}(x_2)$. So (8) is also true in V. By induction, and in view of (5), V satisfies

$$\forall x_1 \forall x_2 \, [\, (\forall \overline{y} \, \forall z \, \, \& \, \widetilde{\beta}_m \,) \implies (\forall \overline{y} \, \forall z \, \, \& \textstyle\bigcup_{n \in \omega} \widetilde{\beta}_n \,) \,],$$

where $\overline{y}$ is again the full enumeration of Y. Then by Proposition 28, V_{FSI} is axiomatized, relative to V, by the first order sentence

$$\forall x_1 \forall x_2 \, [\, (\forall \overline{y} \, \forall z \, \, \& \, \widetilde{\beta}_m \,) \implies (\, e \leq x_1 \text{ or } e \leq x_2 \,) \,],$$

where $\overline{y}$ is $y_1, \ldots, y_{2m}$. Consequently, V_{FSI} is an elementary class, and it is strictly elementary if V is finitely axiomatized. The converse of this last assertion follows from Jónsson's finite basis theorem (Theorem 6), because RS is congruence distributive. ☐

An analysis of the above proof shows that the smallest m witnessing condition (iii) is also the smallest m witnessing (iv).

Notation. From now on, we use the expression $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{m}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\boldsymbol{m+1}}$ to abbreviate

$$\forall x_1 \forall x_2 \, [\, (\forall \overline{y} \, \forall z \, \, \& \, \widetilde{\beta}_m \,) \implies (\forall \overline{y} \, \forall z \, \, \& \, \widetilde{\beta}_{m+1} \,) \,].$$

In particular, $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{0}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\boldsymbol{1}}$ amounts to the demand that when e is the lub of x_1, x_2, then e is also the lub of γ_1, γ_2, provided that each γ_i is an iterated conjugate of x_i (of arbitrary depth). Using Theorem 5 and the fact that e is the only conjugate of itself, we infer

Proposition 30. *For any variety V of residuated semilattices, the following conditions are equivalent.*

(i) V *satisfies* $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{0}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\boldsymbol{1}}$.
(ii) V_{FSI} *is the class of all members of* V *in which* e *is join-irreducible.*
(iii) e *is join-irreducible in every member of* V_{FSI}.

In Remark 26, the sets $\widetilde{\Psi}_m$ involved in the axiomatization of the varietal join $\mathsf{V}_1 + \mathsf{V}_2$ are finite, provided the equational bases for V_1 and V_2 were finite. We can now give a sufficient condition for the varietal join to be axiomatized by $\widetilde{\Psi}_m$, for a given m, together with a finite set of equations.

Theorem 31. *Let* V_1 *and* V_2 *be two varieties of residuated semilattices that satisfy* $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{m}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\boldsymbol{m+1}}$. *Then*:

(i) *The varietal join* $\mathsf{V}_1 + \mathsf{V}_2$ *is axiomatized by a finite set of equations together with* $\widetilde{\Psi}_m$, *where* Ψ *is the set defined in Remark* 26.

(ii) *If* V_1 *and* V_2 *are finitely axiomatized then so is* $\mathsf{V}_1 + \mathsf{V}_2$.

Proof. (i) By the congruence distributivity of RS and Jónsson's Lemma, the FSI members of $\mathsf{V}_1 + \mathsf{V}_2$ belong to $\mathsf{V}_1 \cup \mathsf{V}_2$, hence they satisfy $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{m}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\boldsymbol{m+1}}$. It follows, as in the proof of Theorem 29 [(iii) $\Rightarrow$ (iv)], that $\mathsf{V}_1 + \mathsf{V}_2$ satisfies $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{m}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\boldsymbol{m+1}}$. Then by the Compactness Theorem of first order logic, there is a finite set B of equations, valid in $\mathsf{V}_1 + \mathsf{V}_2$, from which the sentence $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{m}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\boldsymbol{m+1}}$ already follows. By Remark 26, $\mathsf{V}_1 + \mathsf{V}_2$ is axiomatized by $\widetilde{\Psi}$ and, with the help of $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{m}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\boldsymbol{m+1}}$, $\widetilde{\Psi}_m$ implies $\widetilde{\Psi}_n$ for all finite $n > m$, just as in the proof of Theorem 29 [(iv) $\Rightarrow$ (i)]. Consequently, $\mathsf{V}_1 + \mathsf{V}_2$ is axiomatized by $\widetilde{\Psi}_m \cup B$. This proves (i), and (ii) follows because $\widetilde{\Psi}_m$ can be made finite when V_1 and V_2 are finitely based. □

Theorems 29 and 31 combine to give the following stronger version of Theorem 7.

Corollary 32. *For any variety* V *of residuated semilattices, if* V_{FSI} *is an elementary class, then the finitely based subvarieties of* V *are closed under finite joins.*

7. Examples

We have seen that the varieties characterized in Proposition 30 include all those in which negative elements are stable (e.g., all subcommutative varieties). An independent instance of Proposition 30 is:

Example 33. An RS is said to be *representable* (or *semilinear*) if it is a subdirect product of totally ordered RSs. In this case, it is lattice-ordered, since joins can be defined by

$$x \vee y \;=\; [\,x/((x\backslash x) \wedge (y\backslash x))\,] \;\wedge\; [\,y/((x\backslash y) \wedge (y\backslash y))\,].$$

(A proof of this claim in the commutative case can be found in [17]; the noncommutative case is similar.) Because joins are definable, we can infer from [3] that the representable RSs form a finitely based variety V, and then from [6] that they satisfy $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{0}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\boldsymbol{1}}$. Alternatively, since V is a congruence distributive variety generated by algebras with an equationally definable total order, its FSI members are totally ordered, by Jónsson's Lemma. Thus, their identity elements are join-irreducible and Proposition 30 applies. The negative elements of a representable RS need not be stable, in view of Theorem 18 and noncommutative examples in [7].

Example 34. Lattice-ordered groups form a finitely based variety of RLs in which $x\backslash y = x^{-1} \cdot y$ and $y/x = y \cdot x^{-1}$. In these algebras, joins are eliminable from the signature, because $x \mapsto x^{-1}$ is an involution. Conjugates of negative elements are just conjugates in the group-theoretic sense, because $y^{-1} \cdot x \cdot y \leq e$ whenever $x \leq e$. Lattice-ordered groups are not generally subcommutative or representable, but they satisfy $\widetilde{\boldsymbol{\beta}}_{\mathbf{1}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\mathbf{2}}$, because any iterated conjugate of x is already a conjugate of x.

Example 35. Let $\boldsymbol{A}$ be the algebra in Example 16. The variety $\mathsf{V} = \mathrm{HSP}(\boldsymbol{A})$ satisfies $\widetilde{\boldsymbol{\beta}}_{\mathbf{1}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\mathbf{2}}$ but not $\widetilde{\boldsymbol{\beta}}_{\mathbf{0}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\mathbf{1}}$ (and the cardinality of $\boldsymbol{A}$ is minimal in this respect). Failure of $\widetilde{\boldsymbol{\beta}}_{\mathbf{0}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\mathbf{1}}$ follows from Proposition 30, as $\boldsymbol{A}$ is FSI but e is not join-irreducible. Recall that a is a common (depth 1) conjugate of the only pair of incomparable elements whose lub is e. Since every element is a conjugate of itself, it follows that for each $x_1, x_2 \in A \setminus \{e\}$, there are respective conjugates γ_1, γ_2 of x_1, x_2 such that e is not the lub of γ_1, γ_2. So $\boldsymbol{A}$ satisfies $\widetilde{\boldsymbol{\beta}}_{\mathbf{1}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\mathbf{2}}$ because the premise holds only when x_1 or x_2 is e (and because e is the only iterated conjugate of itself). Then V_{FSI} satisfies $\widetilde{\boldsymbol{\beta}}_{\mathbf{1}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\mathbf{2}}$, by an easy application of Jónsson's Lemma. It follows, as in the proof of Theorem 29 [(iii) $\Rightarrow$ (iv)], that V satisfies $\widetilde{\boldsymbol{\beta}}_{\mathbf{1}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\mathbf{2}}$.

We exhibit a variety satisfying $\widetilde{\boldsymbol{\beta}}_{\mathbf{0}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\mathbf{1}}$, which is not encompassed by the above examples.

Example 36. Given $n, k \in \omega$ and variables x, y, let $p_1, \ldots, p_{2^n}$ be all 2^n possible products of the form $s_1 \cdot \ldots \cdot s_n$, where each s_i is x or y. We interpret $p_1, \ldots, p_{2^n}$ as e when $n = 0$. We denote by $t_n(x, y, z)$ the term

$$(p_1\backslash z) \wedge \ldots \wedge (p_{2^n}\backslash z)$$

and by $\varphi_{n,k}$ the first order formula

$$\forall x \, \forall y \; (e \leq t_n(x, y, x) \text{ or } e \leq t_k(x, y, y)),$$

where $k \in \omega$. For RLs, $\varphi_{n,k}$ is equivalent to the formula

$$\forall x \, \forall y \; ((x \vee y)^n \leq x \text{ or } (x \vee y)^k \leq y).$$

We denote by $\mathsf{V}_{n,k}$ the variety generated by the RSs that satisfy $\varphi_{n,k}$.

Note that $\mathsf{V}_{0,1}$ is the variety of RSs in Example 24 and $\mathsf{V}_{1,1}$ is the variety of representable RSs.

Theorem 37. *For each $n, k \in \omega$, $\mathsf{V}_{n,k}$ is axiomatized, relative to* RS, *by*

$$[\,\lambda_v(t_n(x, y, x))\backslash z\,] \wedge [\,\rho_w(t_k(x, y, y))\backslash z\,] \approx z. \tag{9}$$

Proof. By Theorem 23, $\mathsf{V}_{n,k}$ is axiomatized by the set of all equations

$$(\gamma_1\backslash z) \wedge (\gamma_2\backslash z) \approx z, \tag{10}$$

where γ_1 and γ_2 range over all iterated conjugates of $t_n(x,y,x) \wedge e$ and of $t_k(x,y,y) \wedge e$, respectively. In particular, $\mathsf{V}_{n,k}$ satisfies the equation

$$[\,\lambda_v(t_n(x,y,x) \wedge e)\backslash z\,] \wedge [\,\rho_w(t_k(x,y,y) \wedge e)\backslash z\,] \approx z. \tag{11}$$

So, because RSs satisfy $\lambda_v(u \wedge e) \leq \lambda_v(u)$ and $\rho_w(u \wedge e) \leq \rho_w(u)$, it follows that $\mathsf{V}_{n,k}$ satisfies $[\,\lambda_v(t_n(x,y,x))\backslash z\,] \wedge [\,\rho_w(t_k(x,y,y))\backslash z\,] \leq z$. The reverse inequality is also true, because λ_v and ρ_w are negative-valued: in particular, RSs satisfy $\lambda_v(u) \cdot z \leq e \cdot z \approx z$, i.e., $z \leq \lambda_v(u)\backslash z$, and similarly, $z \leq \rho_w(u)\backslash z$. So $\mathsf{V}_{n,k}$ satisfies (9).

Conversely, replacing v and w by e in (9), we get

$$[\,(t_n(x,y,x) \wedge e)\backslash z\,] \wedge [\,(t_k(x,y,y)) \wedge e)\backslash z\,] \approx z. \tag{12}$$

Also, the variety axiomatized by (9) satisfies the implication

$$[\forall z\ ((x\backslash z) \wedge (y\backslash z) \approx z)\,] \implies \forall z\ ((\lambda_v(x)\backslash z) \wedge (\rho_w(y)\backslash z) \approx z). \tag{13}$$

Indeed, for any x, y, if $(x\backslash z) \wedge (y\backslash z) = z$ holds for all z, then e is the lub of x, y, by Lemma 19. In this case, e is also the lub of $p_1, \ldots, p_{2^n}$, by Lemma 3, so $t_n(x,y,z) = z$ for all z, by Lemma 19 again. Thus, $\forall z\,((x\backslash z) \wedge (y\backslash z) \approx z)$ entails $t_n(x,y,x) \approx x$ and $t_k(x,y,y) \approx y$, whence by (9), it entails the right hand side of (13), as claimed.

Now all of the equations schematized in (10) can be derived from (12) by repeated judicious application of (13). (We replace v or w by e in (13) whenever we want conjugation to have no effect.) This completes the proof. □

The implication (13) and Theorem 31(ii) give the following result, which generalizes Example 33.

Corollary 38. *For all $n, k \in \omega$, the variety $\mathsf{V}_{n,k}$ satisfies $\widetilde{\boldsymbol{\beta}}_{\mathbf{0}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\mathbf{1}}$. Consequently, the varietal join of any two finitely based subvarieties of $\mathsf{V}_{n,k}$ is also finitely based.*

In view of Theorem 31, the second claim in Corollary 38 remains true if we replace $\mathsf{V}_{n,k}$ by its join with any variety of RSs that is known to satisfy $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{m}} \Rightarrow \widetilde{\boldsymbol{\beta}}_{\boldsymbol{m+1}}$ for some finite m.

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