# SEQUENTIAL PRODUCTS OF QUANTUM SUBTESTS

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**Abstract**

The sequential product $\mathcal{A} \circ \mathcal{B}$ of two subtests $\mathcal{A}$ and $\mathcal{B}$ on a sequential $\sigma$-effect algebra $E$ with an order determining set of $\sigma$-states is defined. The concepts of sharp, compatible and coexistent subtests are defined. It is shown that a subtest is repeatable if and only if it is sharp. It is also shown that if $\mathcal{A}$ is sharp and $\mathcal{A}$, $\mathcal{B}$ coexist, then $\mathcal{A}$ and $\mathcal{B}$ are compatible. A partial order on subtests is defined using the notion of refinement. It is shown that if $\mathcal{A}$ is sharp, then the infimum $\mathcal{A} \wedge \mathcal{B}$ exists if and only if $\mathcal{A}$ and $\mathcal{B}$ are compatible and in this case $\mathcal{A} \wedge \mathcal{B} = \mathcal{A} \circ \mathcal{B}$. A generalization of a stochastic matrix called a transition effect matrix (TEM) on $E$ is defined. A TEM is then used to describe a quantum Markov chain. These chains possess two types of natural dynamics called state and matrix dynamics. It is demonstrated that these two types of dynamics are statistically equivalent.

## 1 Introduction

A (discrete) quantum measurement is an experimental procedure $\mathcal{A}$ that results in a discrete set of possible outcomes. The fuzzy quantum event that is observed when $\mathcal{A}$ results in a specific outcome is an operator called an effect on a complex Hilbert space $H$ [1, 2, 3, 14, 15, 16]. In a recent article the author introduced the concept of a sequential product $\mathcal{A} \circ \mathcal{B}$ of two measurements $\mathcal{A}$ and $\mathcal{B}$ on $H$ [9]. In the present paper this concept is generalized to a sequential product of subtests on a sequential $\sigma$-effect algebra $E$ with an order determining set of $\sigma$-states. It is shown that some of the results in [9] extend to this more general framework.

We first define two subtests $\mathcal{A}$ and $\mathcal{B}$ to be equivalent and write $\mathcal{A} \approx \mathcal{B}$ if they have the same outcome effects. The concepts of sharpness, compatibility and coexistence for effects are generalized in a natural way to subtests on $E$. It is shown that $\mathcal{A}$ is repeatable (that is, $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$) if and only if $\mathcal{A}$ is sharp. It is also shown that if $\mathcal{A}$ is sharp and $\mathcal{A}$, $\mathcal{B}$ coexist, then $\mathcal{A}$ and $\mathcal{B}$ are compatible. A partial order is defined on (equivalence classes) of subtests in terms of the notion of refinement. It is demonstrated that if $\mathcal{A}$ is sharp, then the infimum $\mathcal{A} \wedge \mathcal{B}$ exists if and only if $\mathcal{A}$ and $\mathcal{B}$ are compatible and in this case $\mathcal{A} \wedge \mathcal{B} = \mathcal{A} \circ \mathcal{B}$. Various other results are proved. For example, if $\mathcal{A}$ is sharp and $\mathcal{B} \leq \mathcal{A}$, then $\mathcal{B} \approx \mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$.

Generalizations of the author's work in [10] are discussed next. An extension of the concept of a stochastic matrix called a transition effect matrix (TEM) is introduced. The product of two TEMs and the product of a TEM with a vector density are defined. It is shown that the product of two TEMs is again a TEM and the product of a TEM with a vector density is again a vector density. It is noted that the product of two TEMs generalizes the sequential product of subtests. An alternative characterization of TEMs in terms of discrete effect kernels is provided.

A TEM is used to describe a quantum Markov chain. These chains possess two types of natural dynamics called state and matrix dynamics. For a state dynamics, the vector density evolves in time and the TEM is considered fixed while for a matrix dynamics, the TEM evolves in time and the vector density is considered fixed. Although these two types of dynamics are not identical, it is demonstrated that they are statistically equivalent. That is, they provide the same probability distributions. Various examples and unsolved problems are given.

The reason that the author considers this research important is that it frees the investigator from the trappings of Hilbert space. The central role of the operator product in Hilbert space is replaced by the sequential product of effects and measurements. Unlike the operator product, the sequential product has physical significance. In this way the basic axioms of measurement theory come to the forefront and their roles in the theory become more transparent [1, 2, 14, 16].

## 2 Background and Notations

An effect is thought of as a fuzzy quantum event or a two-valued quantum measurement. There is an orthogonality relation $\perp$ on the set of effects for a quantum system and if $a \perp b$ then an orthosum $a \oplus b$ is defined. We assume that the set of effects for a quantum system can be organized into a structure called an effect algebra [4, 5, 7]. An **effect algebra** is an algebraic system $(E, 0, 1, \oplus)$ where $0, 1$ are distinct elements of $E$ and $\oplus$ is a partial binary operation on $E$ that satisfies the following conditions:

(E1) If $a \perp b$, then $b \perp a$ and $b \oplus a = a \oplus b$.

(E2) If $a \perp b$ and $c \perp (a \oplus b)$, then $b \perp c$, $a \perp (b \oplus c)$ and
$$a \oplus (b \oplus c) = (a \oplus b) \oplus c$$

(E3) For every $a \in E$ there exists a unique $a' \in E$ such that $a \perp a$ and $a \oplus a' = 1$.

(E4) If $a \oplus 1$, then $a = 0$.

In the sequel, whenever we write $a \oplus b$ we are implicitly assuming that $a \perp b$. We define $a \leq b$ if there exists a $c \in E$ such that $a \oplus c = b$. In this case $c$ is unique and we write $c = b \ominus a$. It can be shown that $(E, \leq, ')$ is a partially ordered set with $0 \leq a \leq 1$ for all $a \in E$, $a'' = a$ and $a \leq b$ implies $b' \leq a'$. Moreover, $a \perp b$ if and only if $a \leq b'$. If the $n$-fold orthosum $a \oplus a \oplus \cdots \oplus a$ is defined in $E$ we denote this element by $na$. We say that $E$ is **isotropically finite** if for every $a \neq 0$ in $E$ there is a largest $n \in \mathbb{N}$ such that $na$ exists. An effect $a$ is **sharp** if $a \wedge a' = 0$. Sharp effects correspond to precise two-valued measurements.

If $E$ and $F$ are effect algebras, we say that $\phi\colon E \to F$ is **additive** if $a \perp b$ implies $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. If $\phi\colon E \to F$ is additive and $\phi(1) = 1$, then $\phi$ is a **morphism**. It is clear that the unit interval $[0, 1] \subseteq \mathbb{R}$ is an effect algebra where $a \perp b$ if $a + b \leq 1$ and in this case $a \oplus b = a + b$. A **state** on $E$ is a morphism $s\colon E \to [0, 1]$. We interpret $s(a)$ as the probability that the effect $a$ is observed (has the value yes) when the system is in the state $s$. A set of states $S$ is **order determining** if $s(a) \leq s(b)$ for all $s \in S$ implies that $a \leq b$.

An effect algebra is mathematically and physically a weak structure. Roughly speaking, the orthosum $a \oplus b$ only describes parallel combinations of effects. It is also desirable to describe series combinations or sequential products of effects. For this reason we introduce the concept of a sequential effect algebra [11, 12, 13].

For a binary operation $\circ$, if $a \circ b = b \circ a$ we write $a \mid b$. A **sequential effect algebra** (SEA) is a system $(E, 0, 1, \oplus, \circ)$ where $(E, 0, 1, \oplus)$ is an effect algebra and $\circ\colon E \times E \to E$ is a binary operation that satisfies the following conditions:

(S1) $b \mapsto a \circ b$ is additive for all $a \in E$.

(S2) $1 \circ a = a$ for all $a \in E$.

(S3) If $a \circ b = 0$, then $a \mid b$.

(S4) If $a \mid b$, then $a \mid b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$ for all $c \in E$.

(S5) If $c \mid a$ and $c \mid b$, then $c \mid a \circ b$ and $c \mid (a \oplus b)$.

We call an operation that satisfies (S1)–(S5) a **sequential product** on $E$. If $a \mid b$ for all $a, b \in E$ we call $E$ a **commutative** SEA. We now discuss some of the common examples of SEAs. The effect algebra $[0, 1] \subseteq \mathbb{R}$ is a SEA with sequential product $a \circ b = ab$. For a Boolean algebra $\mathcal{B}$, define $a \perp b$ if $a \wedge b = 0$ and in this case $a \oplus b = a \vee b$. Under the operation $a \circ b = a \wedge b$, $(\mathcal{B}, 0, 1, \oplus, \circ)$ becomes a SEA. In particular, if $X$ is a nonempty set, then $(2^X, \emptyset, X, \oplus, \circ)$ is a SEA. For the function space $[0, 1]^X$ on the unit interval $[0, 1] \subseteq \mathbb{R}$ define the functions $f_0, f_1$ by $f_0(x) = 0$, $f_1(x) = 1$ for all $x \in X$. We call a subset $\mathcal{F} \subseteq [0, 1]^X$ a **fuzzy set system** on $X$ if $f_0, f_1 \in \mathcal{F}$, if $f \in \mathcal{F}$ then $f_1 - f \in \mathcal{F}$, if $f, g \in \mathcal{F}$ with $f + g \leq 1$ then $f + g \in \mathcal{F}$ and if $f, g \in \mathcal{F}$ then $fg \in \mathcal{F}$. Then $(\mathcal{F}, f_0, f_1, \oplus, \circ)$ becomes a SEA when $f \oplus g = f + g$ for $f + g \leq 1$ and $f \circ g = fg$. If $\mathcal{F} = [0, 1]^X$ we call $\mathcal{F}$ a **full fuzzy set system**. All of these examples are commutative SEAs.

The most important example for our purposes is the noncommutative SEA $\mathcal{E}(H)$. If $H$ is a complex Hilbert space, $\mathcal{E}(H)$ is the set of operators on $H$ that satisfy $0 \leq \langle Ax, x \rangle \leq \langle x, x \rangle$ for all $x \in H$. For $A, B \in \mathcal{E}(H)$ we define $A \perp B$ if $A + B \in \mathcal{E}(H)$ and in this case $A \oplus B = A + B$. Moreover, for all $A, B \in \mathcal{E}(H)$ define the operation $A \circ B = A^{1/2} B A^{1/2}$ where $A^{1/2}$ is the unique positive square root of $A$. Then $(\mathcal{E}(H), 0, I, \oplus, \circ)$ is a SEA which is called a **Hilbert space** SEA [6, 13]. The sharp elements of $\mathcal{E}(H)$ are the projection operators on $H$. It can be shown that $A \circ B = B \circ A$ if and only if $AB = BA$ [12]. More generally, if $\mathcal{N}$ is a von Neumann algebra then

$\mathcal{E}(\mathcal{N}) = \{A \in \mathcal{N} \colon 0 \le A \le I\}$ is a SEA under the above defined $\oplus$ and $\circ$. We denote the set of sharp elements of $E$ by $E_S$. It is clear that $0, 1 \in E_S$ and $a' \in E_S$ whenever $a \in E_S$. The following lemma summarizes some of the properties of a SEA that we shall find useful [11].

**Lemma 2.1.** (i) *$a \circ 0 = 0 \circ a = 0$ and $a \circ 1 = 1 \circ a = a$ for all $a \in E$.* (ii) *$a \circ b \le a$ for all $a, b \in E$.* (iii) *If $a \le b$, then $c \circ a \le c \circ b$ for all $c \in E$.* (iv) *$a \in E_S$ if and only if $a \circ a = a$.* (v) *For $a \in E$ and $b \in E_S$, $a \le b$ if and only if $a \circ b = b \circ a = a$ and $b \le a$ if and only if $a \circ b = b \circ a = b$.* (vi) *If $a, b \in E_S$ and $a \perp b$, the $a \oplus b \in E_S$.*

We shall also need the next result which was not proved previously.

**Lemma 2.2.** *If $a \circ b = a$, then $a \mid b$ and $a \le b$.*

*Proof.* If $a \circ b = a$, then by Condition (S1)

$$a = a \circ b \oplus a \circ b' = a \oplus a \circ b'$$

Hence, $a \circ b' = 0$. By Conditions (S3) and (S4) we conclude that $a \mid b'$ so that $a \mid b$. Applying Lemma 2.1(ii) we have $a = b \circ a \le b$. $\square$

It is frequently useful to consider SEAs that are closed under countable operations. A $\sigma$-SEA is a SEA $E$ that satisfies the following conditions whenever $a_1 \le a_2 \le \cdots$ in $E$.

($\sigma$1) $\vee a_i$ exists in $E$.

($\sigma$2) $b \circ (\vee a_i) = \vee (b \circ a_i)$ for every $b \in E$.

($\sigma$3) If $b \mid a_i$, $i = 1, 2, \ldots$, then $b \mid \vee a_i$.

It is well known that $\mathcal{E}(\mathcal{N})$ and in particular $\mathcal{E}(H)$ are $\sigma$-SEAs. Moreover, $[0,1]$, $2^X$, $[0,1]^X$ are $\sigma$-SEAs and the other examples considered previously can be easily modified to become $\sigma$-SEAs.

Let $a_i$, $i = 1, 2, \ldots$, be a sequence in the $\sigma$-SEA $E$ such that $b_n = a_1 \oplus a_2 \oplus \cdots \oplus a_n$ is defined for all $n \in \mathbb{N}$. Then $b_1 \le b_2 \le \cdots$, so by Condition ($\sigma$1) $\vee b_i$ exists in $E$. We then use the notation $\vee b_i = \oplus a_i$ and whenever we write $\oplus a_i$ we are implicitly assuming that this expression exists. It follows from Conditions ($\sigma$2) and ($\sigma$3) that $b \circ (\oplus a_i) = \oplus (b \circ a_i)$ and $b \mid \oplus a_i$ whenever $b \mid a_i$, $i = 1, 2, \ldots$. A state $s$ on a $\sigma$-SEA is a **$\sigma$-state** if

$a_1 \le a_2 \le \cdots$ implies that $\vee s(a_i) = s(\vee a_i)$. We conclude that if $s$ is a $\sigma$-state, then $s(\oplus a_i) = \sum s(a_i)$; that is, $s$ is countably additive. It follows from Gleason's theorem [4, 8] that $\sigma$-states on $\mathcal{E}(H)$ have the form $s(A) = \mathrm{tr}(\rho A)$ for a unique density operator $\rho$. All the examples of $\sigma$-SEAs that we have considered possess an order determining set of $\sigma$-states.

**Lemma 2.3.** *Let $E$ be a SEA that possesses an order determining set of states.* (i) *$E$ is isotropically finite.* (ii) *For $a \in E$, if $a \circ a = 0$, then $a = 0$.*

*Proof.* (i) Suppose $a \in E$ with $a \ne 0$ and $na$ exists for all $n \in \mathbb{N}$. Let $s$ be a state such that $s(a) \ne 0$. Then

$$ns(a) = s(na) \le 1$$

for all $n \in \mathbb{N}$. Hence, $s(a) \le 1/n$ for all $n \in \mathbb{N}$ so $s(a) = 0$ which is a contradiction. Thus, there is a largest $n$ such that $na$ exists so $E$ is isotropically finite. (ii) If $a \circ a = 0$ then by Lemma 2.1(ii) we have

$$a = a \circ a \oplus a \circ a' = a \circ a' = a' \circ a \le a'$$

Hence, $2a$ exists. Since $(2a) \circ (2a) = 0$, we conclude that $4a$ exists. Continuing in this way, it follows that $na$ exists for all $n \in \mathbb{N}$. Applying Part (i), we conclude that $a = 0$. □

# 3 Tests and Subtests

In the sequel $E$ will always be a $\sigma$-SEA that possesses an order determining set of $\sigma$-states. A **subtest** is a finite or infinite sequence $\mathcal{A} = \{a_i\}$ in $E$ such that $\oplus a_i$ exists. We call $a_1, a_2, \ldots$ **elements** of $\mathcal{A}$. We can also view $\mathcal{A}$ as a multiset in which elements can be repeated. If $a_i \in E_S$ for all $i$, then $\mathcal{A}$ is a **sharp** subtest. Notice that a sharp element $a \ne 0$ of a subtest cannot be repeated. Indeed, if $a$ is repeated then $a \oplus a$ exists so that $a \le a'$. Hence, $a \wedge a' = a \ne 0$ which is a contradiction. In particular, the nonzero elements of a sharp subtest must be distinct. If $\mathcal{A} = \{a_i\}$ and $\oplus a_i = 1$, then $\mathcal{A}$ is a **test**. We denote the set of subtests, sharp subtests, tests and sharp tests on $E$ by $\mathcal{T}(E)$, $\mathcal{T}_S(E)$, $\overline{\mathcal{T}}(E)$, $\overline{\mathcal{T}}_S(E)$, respectively.

We view a test $\mathcal{A} = \{a_i\}$ as a measurement with outcomes $\omega_1, \omega_2, \ldots$ and $a_i$ is observed when $\mathcal{A}$ is preformed and the result is outcome $\omega_i$. If the system is in the $\sigma$-state $s$ and $\mathcal{A}$ is performed, then the probability that the result is $\omega_i$ is given by $s(a_i)$. Since $\sum s(a_i) = 1$, one of the outcomes must occur. We may think of a subtest $\mathcal{A} = \{a_i\}$ as a incomplete measurement in general because $\sum s(a_i) \le 1$. Note that any subtest can be extended to a test in various ways. For example, if $\mathcal{A} = \{a_i\} \in \mathcal{T}(E)$, then we can adjoin the element $(\oplus a_i)'$ to $\mathcal{A}$ to obtain a test.

If $\mathcal{A}, \mathcal{B} \in \mathcal{T}(E)$ with $\mathcal{A} = \{a_i\}$ and $\mathcal{B} = \{b_j\}$, we define the **sequential product** of $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A} \circ \mathcal{B} = \{a_i \circ b_j\}$. Then $\mathcal{A} \circ \mathcal{B} \in \mathcal{T}(E)$ because in the infinite case (the finite case is trivial) for every $N \in \mathbb{N}$ we have

$$\bigoplus_{i,j=1}^{N}(a_i \circ b_j) = \bigoplus_{i=1}^{N} a_i \circ \left(\bigoplus_{j=1}^{N} b_j\right) \le \bigoplus_{i=1}^{N} a_i \le \bigoplus a_i$$

so that $\oplus(a_i \circ b_j)$ exists. In particular, if $\mathcal{A}, \mathcal{B} \in \overline{\mathcal{T}}(E)$ then

$$\oplus(a_i \circ b_j) = \oplus a_i \circ (\oplus b_j) = \oplus a_i = 1$$

so that $\mathcal{A} \circ \mathcal{B} \in \overline{\mathcal{T}}(E)$. This is the natural definition for the sequential product of subtests. For example, if $\mathcal{A} = \{a, a'\}$ and $\mathcal{B} = \{b, b'\}$ are two-outcome tests, then $\mathcal{A} \circ \mathcal{B}$ is the four-outcome test

$$\mathcal{A} \circ \mathcal{B} = \{a \circ b, a \circ b', a' \circ b, a' \circ b'\}$$

Defining the **identity test** $\mathcal{I} = \{1\}$ we have that $\mathcal{I} \circ \mathcal{A} = \mathcal{A} \circ \mathcal{I} = \mathcal{A}$ for all $\mathcal{A} \in \mathcal{T}(E)$.

For $\mathcal{A}, \mathcal{B} \in \mathcal{T}(E)$ we write $\mathcal{A} \approx \mathcal{B}$ if the nonzero elements of $\mathcal{A}$ are a permutation of the nonzero elements of $\mathcal{B}$. Viewed as multisets, this would say that the multisets $\mathcal{A}$ and $\mathcal{B}$ are identical except possibly for their zero elements. Of course, $\approx$ is an equivalence relation and when $\mathcal{A} \approx \mathcal{B}$ we say that $\mathcal{A}$ and $\mathcal{B}$ are **equivalent**. It is natural to identify equivalent subtests because they coincide except for an ordering of their outcomes.

We say that subtests $\mathcal{A} = \{a_i\}$ and $\mathcal{B} = \{b_j\}$ are **compatible** if $a_i \mid b_j$ for all $i, j$. It is clear that if $\mathcal{A}$ and $\mathcal{B}$ are compatible, then $\mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$. The converse does not hold. Indeed, $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A} \circ \mathcal{A}$ and yet $\mathcal{A}$ need not be compatible with itself. We conjecture that if $\mathcal{A} \in \overline{\mathcal{T}}(E)$, $\mathcal{B} \in \overline{\mathcal{T}}_S(E)$ and $\mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$, then $\mathcal{A}$ and $\mathcal{B}$ are compatible. This conjecture holds in $\mathcal{E}(H)$ [9] but is in an open problem in general.

**Theorem 3.1.** *For $\mathcal{A}, \mathcal{B} \in \overline{\mathcal{T}}(E)$ we have $\mathcal{A} \circ \mathcal{B} \in \overline{\mathcal{T}}_S(E)$ if and only if $\mathcal{A}, \mathcal{B} \in \overline{\mathcal{T}}_S(E)$ and $\mathcal{A}$, $\mathcal{B}$ are compatible.*

*Proof.* If $\mathcal{A}, \mathcal{B} \in \overline{\mathcal{T}}_S(E)$ and $\mathcal{A}$, $\mathcal{B}$ are compatible, then by Lemma 2.1(iv) we have that $\mathcal{A} \circ \mathcal{B} \in \overline{\mathcal{T}}_S(E)$. Conversely, suppose $\mathcal{A} = \{a_i\}$, $\mathcal{B} = \{b_j\}$ and $\mathcal{A} \circ \mathcal{B} \in \overline{\mathcal{T}}_S(E)$. Then for all $i, j$ we have $a_i \circ b_j = c_{ij} \in E_S$. It follows from Lemma 2.1(vi) that

$$a_i = a_i \circ \left( \bigoplus_j b_j \right) = \bigoplus_j (a_i \circ b_j) = \bigoplus_j c_{ij} \in E_S$$

Hence, $\mathcal{A} \in \overline{\mathcal{T}}_S(E)$. Now $c_{ij} = a_i \circ b_j \leq a_i$ so by Lemma 2.1(v) we have

$$a_i \circ c_{ij} = c_{ij} \circ a_i = c_{ij}$$

Thus, by Condition (S4) we have

$$c_{ij} \circ b_j = c_{ij} \circ (a_i \circ b_j) = c_{ij}$$

It follows from Lemma 2.2 that $b_j \mid c_{ij}$. Applying Lemma 2.1(v) we conclude that

$$(a_i \ominus c_{ij}) \circ b_j = (a_i \ominus c_{ij}) \circ (a_i \circ b_j) = (a_i \ominus c_{ij}) \circ c_{ij} = 0$$

By Conditions (S3) and (S5) we have that $b_j \mid a_i \ominus c_{ij}$ so that $b_j \mid a_i$. Hence, $\mathcal{A}$ and $\mathcal{B}$ are compatible. We then have

$$b_j = b_j \circ \left( \bigoplus_i a_i \right) = \bigoplus_i (b_j \circ a_i) = \bigoplus_i (a_i \circ b_j) \in E_S$$

Therefore, $\mathcal{B} \in \overline{\mathcal{T}}_S(E)$. □

**Lemma 3.2.** (i) *If $\{a_i\} \in \mathcal{T}(E)$ and $a_j \in E_S$ for some $j$, then $a_j \circ a_k = 0$ for all $k \neq j$.* (ii) *If $\mathcal{A} \in \mathcal{T}_S(E)$, then $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$ and $\mathcal{A}$ is compatible with itself.* (iii) *Any $\mathcal{A} \in \mathcal{T}(E)$ has a maximal element.*

*Proof.* (i) Since

$$\begin{aligned} a_j &= a_j \circ 1 \geq a_j \circ \left( \bigoplus_k a_k \right) \\ &= (a_j \circ a_j) \oplus \left( \bigoplus_{k \neq j} a_j \circ a_k \right) = a_j \oplus \left( \bigoplus_{k \neq j} a_j \circ a_k \right) \end{aligned}$$

we have that $\bigoplus_{k\neq j}(a_j \circ a_k) = 0$. Hence, $a_j \circ a_k = 0$ for all $k \neq j$.
(ii) Applying (i), it is clear that $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$. Since $a_j \circ a_k = 0$, $j \neq k$, implies that $a_j \mid a_k$, $j \neq k$, we have that $\mathcal{A}$ is compatible with itself.
(iii) We first show that there are no infinite increasing chains in $\mathcal{A}$. Suppose that $b_1 \leq b_2 \leq \cdots$ are nonzero elements of $\mathcal{A}$ and let $s$ be a $\sigma$-state satisfying $s(b_1) \neq 0$. Then $s(b_1) \leq s(b_2) \leq \cdots$ and since

$$\sum s(b_i) = s(\oplus b_i) \leq 1$$

we have a contradiction. Let $\mathcal{A} = \{a_i\}$. If $a_1$ is not maximal, there exists an element $b_1$ of $\mathcal{A}$ with $a_1 < b_1$. If $b_1$ is not maximal, there exists an element $b_2$ of $\mathcal{A}$ with $a_1 < b_1 < b_2$. If this process does not stop, we obtain an infinite chain $a_1 < b_1 < b_2 < \cdots$. But this contradicts our previous result. Hence, we have an element $b_n$ of $\mathcal{A}$ such that

$$a_1 < b_1 < b_2 < \cdots < b_n$$

and there is no element $a$ of $\mathcal{A}$ satisfying $b_n < a$. Therefore, $b_n$ is a maximal element of $\mathcal{A}$. $\square$

The next theorem shows that sharp subtests are precisely the idempotents of $\mathcal{T}(E)$. In physical terms, this says that a subtest is repeatable if and only if it is sharp.

**Theorem 3.3.** *For $\mathcal{A} \in \mathcal{T}(E)$ we have that $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$ if and only if $\mathcal{A} \in \mathcal{T}_S(E)$.*

*Proof.* It follows from Lemma 3.2(ii) that if $\mathcal{A} \in \mathcal{T}_S(E)$, then $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$. Conversely, suppose that $\mathcal{A} = \{a_i\}$ and $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$. Then $\{a_i \circ a_j\} \approx \{a_i\}$ and we may assume that $a_i \neq 0$ for all $i$. Notice that if $a_i, a_j \notin E_S$, then $a_k = a_i \circ a_j \notin E_S$ because if $a_k \in E_S$, then $a_k = a_k \circ a_k$ would appear as an element of $\{a_i \circ a_j\}$ twice and this would contradict Lemma 3.2(i). Let $b_1, b_2, \ldots$ be the sharp elements of $\mathcal{A}$. By Lemma 3.2(i), $b_i \circ a_j = 0$ whenever $a_j \neq b_i$. Let $\mathcal{A}_1$ be the subtest $\mathcal{A} \smallsetminus \{b_i \colon i = 1, 2, \ldots\}$ and assume that $\mathcal{A}_1 \neq \emptyset$. Writing $\mathcal{A}_1 = \{c_i\}$ we have that $\mathcal{A}_1 \circ \mathcal{A}_1 \approx \mathcal{A}_1$ so that $\{c_i \circ c_j\} \approx \{c_i\}$. By Lemma 3.2(iii), $\{c_i \circ c_j\}$ has a maximal element say $c_i \circ c_j$. Since $c_i \circ c_j \leq c_i$ and $c_i$ is an element of $\{c_i \circ c_j\}$ we have that $c_i \circ c_j = c_i$. It follows from Lemma 2.2 that

$$c_i \circ c_j = c_j \circ c_i = c_i$$

But now $c_j \circ c_i$ is a maximal element and $c_j \circ c_i \leq c_j$. Hence,

$$c_j = c_j \circ c_i = c_i$$

We conclude that $c_i \circ c_i = c_i$ so by Lemma 2.1(iv), $c_i \in E_S$. Since this is a contradiction, $\mathcal{A}_1 = \emptyset$. Hence, $a_i \in E_S$ for all $i$ so that $\mathcal{A} \in \mathcal{T}_S(E)$. $\square$

A subtest is **finite** if it has a finite number of elements. We can apply Lemma 2.3(ii) to give a simple proof of Theorem 3.3 for finite subtests. Indeed, let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a finite subtest where $a_i \neq 0$, $i = 1, \ldots, n$ and suppose that $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$. Then $\{a_i \circ a_j\} \approx \{a_i\}$ and by Lemma 2.3(ii), $a_i \circ a_i \neq 0$, $i = 1, \ldots, n$. Since $\mathcal{A} \circ \mathcal{A}$ and $\mathcal{A}$ have the same number of nonzero elements, we conclude that $a_i \circ a_j = 0$, $i \neq j$. It follows that

$$a_i = a_i \circ 1 = a_i \circ (\oplus a_j) = \bigoplus_j (a_i \circ a_j) = a_i \circ a_i$$

So by Lemma 2.1(iv), $a_i \in E_S$, $i = 1, \ldots, n$. Hence, $\mathcal{A} \in \mathcal{T}_S(E)$.

# 4 Refinements and Coexistence

For $\mathcal{A}, \mathcal{B} \in \mathcal{T}(E)$ with $\mathcal{A} = \{a_i\}$, $\mathcal{B} = \{b_j\}$ we call $\mathcal{A}$ a **refinement** of $\mathcal{B}$ and write $\mathcal{A} \leq \mathcal{B}$ if we can adjoin 0s to $\mathcal{A}$ if necessary and organize the elements of $\mathcal{A}$ so that $\mathcal{A} \approx \{a_{ij}\}$ and $b_i = \oplus_j a_{ij}$ for every $i$. For example, $\mathcal{A} \circ \mathcal{B} \leq \mathcal{A}$. Indeed, $\mathcal{A} \circ \mathcal{B} = \{a_i \circ b_j\}$ and $a_i = \oplus_j a_i \circ b_j$ for every $i$. The converse does not hold. That is, $\mathcal{B} \leq \mathcal{A}$ does not imply that $\mathcal{B} = \mathcal{A} \circ \mathcal{C}$ for some $\mathcal{C} \in \mathcal{T}(E)$ [9]. We now show that $\leq$ gives a partial order on $\mathcal{T}(E)$. Strictly speaking, we are considering equivalence classes because we use $\approx$ instead of equality.

**Theorem 4.1.** *$(\mathcal{T}(E), \leq)$ is a poset in which $\mathcal{A} \leq \mathcal{B}$ implies $\mathcal{C} \circ \mathcal{A} \leq \mathcal{C} \circ \mathcal{B}$ for all $\mathcal{C} \in \mathcal{T}(E)$.*

*Proof.* It is clear that $\leq$ is reflexive and it is straightforward to show that $\leq$ is transitive. To prove anti-symmetry, let $\mathcal{A} = \{a_i\}$, $\mathcal{B} = \{b_j\}$ and suppose that $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{A}$. Eliminate from $\mathcal{A}$ and $\mathcal{B}$ pairwise equal elements $a_i = b_j$ until the remaining subtests $\mathcal{A}_1$ and $\mathcal{B}_1$ have no elements in common. Letting $\mathcal{A}_1 = \{c_i\}$, $\mathcal{B}_1 = \{d_j\}$, since $\oplus a_i = \oplus b_j$ it follows that $\oplus c_i = \oplus d_j$. We then conclude that $\mathcal{A}_1 \leq \mathcal{B}_1$ and $\mathcal{B}_1 \leq \mathcal{A}_1$. Assuming that $\mathcal{A}_1 \neq \emptyset$, by Lemma 3.2(iii), $\mathcal{A}_1$ has a maximal element $c_i$. Then some $d_j \in \mathcal{B}_1$ satisfies

$$d_j = c_i \oplus c_{j_1} \oplus c_{j_2} \oplus \cdots$$

Hence, $d_j \geq c_i$. Now there exists a $c_k \in \mathcal{A}_1$ such that

$$c_k = d_j \oplus d_{k_1} \oplus d_{k_2} \oplus \cdots$$

Thus, $c_k \geq d_j \geq c_i$. Since $c_i$ is maximal in $\mathcal{A}_1$, we have that $c_k = d_j = c_i$. Hence, $\mathcal{A}_1$ and $\mathcal{B}_1$ have the common element $c_i = d_j$ which is a contradiction. We conclude that $\mathcal{A}_1 = \mathcal{B}_1 = \emptyset$ and $\mathcal{A} \approx \mathcal{B}$. Finally, suppose that $\mathcal{A} \leq \mathcal{B}$. By definition, we have that $b_i = \oplus_j a_{ij}$ for every $i$. Letting $\mathcal{C} = \{c_i\} \in \mathcal{T}(E)$ we conclude that

$$c_k \circ b_i = c_k \circ \left( \bigoplus_j a_{ij} \right) = \bigoplus_j (c_k \circ a_{ij})$$

It follows that $\mathcal{C} \circ \mathcal{A} \leq \mathcal{C} \circ \mathcal{B}$. □

**Theorem 4.2.** (i) *If $\mathcal{A} \in \mathcal{T}_S(E)$, $\mathcal{B} \in \mathcal{T}(E)$ and $\mathcal{B} \leq \mathcal{A}$, then $\mathcal{B} \approx \mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$.* (ii) *If $\mathcal{A} \in \mathcal{T}_S(E)$, $\mathcal{B} \in \mathcal{T}(E)$ and $\mathcal{A} \leq \mathcal{B}$, then $\mathcal{B} \in \mathcal{T}_S(E)$ and $\mathcal{A} \approx \mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$.* (iii) *If $\mathcal{A}, \mathcal{B} \in \overline{\mathcal{T}}(E)$ and $\mathcal{A} \circ \mathcal{B} \approx \mathcal{A}$, then $\mathcal{A}$, $\mathcal{B}$ are compatible, $\mathcal{B} \in \overline{\mathcal{T}}_S(E)$ and $\mathcal{A} \leq \mathcal{B}$.*

*Proof.* (i) Since $\mathcal{B} \leq \mathcal{A}$, we can write $\mathcal{B} = \{b_{ij}\}$, $\mathcal{A} = \{a_i\}$ where $a_i = \oplus_j b_{ij}$ for every $i$. Since $b_{ij} \leq a_i$, by Lemma 2.1(v) we have

$$b_{ij} \circ a_i = a_i \circ b_{ij} = b_{ij}$$

for every $i$ and $b_{ij} \circ a_k = 0$ for every $j$ and $k \neq i$. It follows that $\mathcal{B} \approx \mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$. (ii) It is clear that $\mathcal{B} \in \mathcal{T}_S(E)$ and the result follows from Part (i). (iii) Letting $\mathcal{A} = \{a_i\}$, $\mathcal{B} = \{b_j\}$, by Lemma 3.2(iii), $\mathcal{A}$ has a maximal element $a_k$. Now $a_k = a_i \circ b_j$ for some $i$ and $j$. Since $a_k = a_i \circ b_j \leq a_i$ we have that $a_k = a_i$. Hence, $a_k = a_k \circ b_j$. Applying Lemma 2.2 we have that $a_k \mid b_j$. Since

$$a_k \circ b_j = a_k = \bigoplus_r (a_k \circ b_r)$$

we have that $a_k \circ b_r = 0$ for $r \neq j$. Call $a_k$ type 1 if $a_k \circ b_j = a_k$ for some $j$. Let $\mathcal{A}_1 = \{a_{i_1}, a_{i_2}, \ldots\}$ be the non-type 1 elements of $\mathcal{A}$. If $\mathcal{A}_1 \neq \emptyset$, then by Lemma 3.2(iii), $\mathcal{A}_1$ has a maximal element $a_{i_k}$. As before, there is a $b_s$ such that $a_{i_k} \circ b_s = a_{i_k}$ which is a contradiction. Hence, all elements of $\mathcal{A}$

are type 1. Now for every $b_j$ we have $a_{j_1} \circ b_j = a_{j_1}$, $a_{j_2} \circ b_j = a_{j_2}, \ldots$ and $a_i \circ b_j = 0$, $i \neq j_1, j_2, \ldots$. Hence,

$$b_j = \bigoplus_i (b_j \circ a_{j_i}) = \bigoplus_i (a_{j_i} \circ b_j) = \bigoplus_i a_{j_i}$$

Therefore, $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{A}$, $\mathcal{B}$ are compatible. Moreover,

$$b_j \circ b_j = \bigoplus_i b_j \circ a_{j_i} = b_j$$

By Lemma 2.1(iv), $b_j \in E_S$ so that $\mathcal{B} \in \overline{\mathcal{T}}_s(E)$. $\square$

Notice that Theorem 4.2(iii) implies that Theorem 3.3 holds for the particular case $\mathcal{A} \in \overline{\mathcal{T}}(E)$. Two effects $a, b \in E$ **coexist** if there exist effects $c, d, e \in E$ such that $c \oplus d \oplus e$ exists and $a = c \oplus d$, $b = d \oplus e$ [15, 16]. We say that $a \in E$ is **associated** with $\mathcal{A} \in \overline{\mathcal{T}}(E)$ if $\mathcal{A} \leq \{a, a'\}$.

**Lemma 4.3.** *Two effects $a, b \in E$ coexist if and only if $a$ and $b$ are associated with a common test $\mathcal{A} \in \overline{\mathcal{T}}(E)$.*

*Proof.* If $a$ and $b$ coexist, let $c, d, e$ be as in the definition. Letting $f = (c \oplus d \oplus e)'$ we obtain the test $\mathcal{A} = \{c, d, e, f\}$. Since $a = c \oplus d$, $b = d \oplus e$, $a$ and $b$ are both associated with $\mathcal{A}$. Conversely, suppose that $a$ and $b$ are both associated with $\mathcal{A} = \{a_i\} \in \overline{\mathcal{T}}(E)$. We can then write

$$a = \left(\bigoplus_r a_{i_r}\right) \oplus \left(\bigoplus_s a_{j_s}\right)$$
$$b = \left(\bigoplus_s a_{j_s}\right) \oplus \left(\bigoplus_t a_{k_t}\right)$$

where $i_r \neq k_t$ for every $r$, $t$. Letting $c = \oplus_r a_{i_r}$, $d = \oplus_s a_{j_s}$, $e = \oplus_t a_{k_t}$ we have that $a = c \oplus d$, $b = d \oplus e$ and $c \oplus d \oplus e$ exists. Hence, $a$, $b$ coexist. $\square$

Lemma 4.3 shows that $a$, $b$ coexist if and only if $\{a, a'\}$ and $\{b, b'\}$ can be measured together using a common refinement. We say that $\mathcal{A}, \mathcal{B} \in \mathcal{T}(E)$ **coexist** if they have a common refinement $\mathcal{C} \leq \mathcal{A}, \mathcal{B}$. Notice that if $\mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$, then $\mathcal{A} \circ \mathcal{B} \leq \mathcal{A}, \mathcal{B}$ so $\mathcal{A}$ and $\mathcal{B}$ coexist. In particular, compatible subtests coexist. Simple examples show that the converse does not hold.

Moreover, any subtest coexists with itself. It follows from Lemma 4.3 that this definition generalizes the definition of coexistence of effects.

Notice that if $\{a_i\}$, $\{b_j\}$ coexist, then for any fixed $i$ or $j$, the two-valued tests $\{a_i, a_i'\}$, $\left\{b_j, b_j'\right\}$ coexist so $a_i$, $b_j$ coexist for all $i$, $j$.There are examples which show that the converse does not hold [9]. Since $\mathcal{A} \leq \mathcal{B}$ implies that $\mathcal{C} \circ \mathcal{A} \leq \mathcal{C} \circ \mathcal{B}$, it follows that if $\mathcal{A}$, $\mathcal{B}$ coexist, then $\mathcal{C} \circ \mathcal{A}$, $\mathcal{C} \circ \mathcal{B}$ coexist. In contrast, if $\mathcal{A}$, $\mathcal{B}$ are compatible, then $\mathcal{C} \circ \mathcal{A}$, $\mathcal{C} \circ \mathcal{B}$ need not be compatible.

**Lemma 4.4.** *If $\mathcal{A} \in \mathcal{T}(E)$, $\mathcal{B} \in \mathcal{T}_S(E)$ coexist, then $\mathcal{A}$, $\mathcal{B}$ are compatible.*

*Proof.* Letting $\mathcal{A} = \{a_i\}$, $\mathcal{B} = \{b_j\}$, it follows that $a_i$, $b_j$ coexist for all $i$, $j$. We conclude that $a_i$, $b_j$ are compatible for all $i$, $j$ [11]. Hence, $\mathcal{A}$, $\mathcal{B}$ are compatible. $\square$

The next result shows that the poset $(\mathcal{T}(E), \leq)$ is not a lattice.

**Theorem 4.5.** *For $\mathcal{A} \in \mathcal{T}(E)$ and $\mathcal{B} \in \mathcal{T}_S(E)$, $\mathcal{A} \wedge \mathcal{B}$ exists if and only if $\mathcal{A}$ and $\mathcal{B}$ are compatible. In this case $\mathcal{A} \wedge \mathcal{B} = \mathcal{A} \circ \mathcal{B}$.*

*Proof.* If $\mathcal{A} \wedge \mathcal{B}$ exists, then $\mathcal{A} \wedge \mathcal{B} \leq \mathcal{A}, \mathcal{B}$ so $\mathcal{A}$, $\mathcal{B}$ coexist. By Lemma 4.4, $\mathcal{A}$ and $\mathcal{B}$ are compatible. Conversely, suppose that $\mathcal{A}$ and $\mathcal{B}$ are compatible. Then $\mathcal{B} \circ \mathcal{A} \leq \mathcal{A}, \mathcal{B}$. If $\mathcal{C} \leq \mathcal{A}, \mathcal{B}$, then by Theorem 4.2(i) $\mathcal{C} \approx \mathcal{C} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{C}$. Since $\mathcal{C} \leq \mathcal{A}$ we have

$$\mathcal{C} \approx \mathcal{B} \circ \mathcal{C} \leq \mathcal{B} \circ \mathcal{A} \approx \mathcal{A} \circ \mathcal{B}$$

Hence, $\mathcal{A} \circ \mathcal{B} = \mathcal{A} \wedge \mathcal{B}$. $\square$

The characterization of pairs $\mathcal{A}, \mathcal{B} \in \mathcal{T}(E)$ such that $\mathcal{A} \wedge \mathcal{B}$ (or $\mathcal{A} \vee \mathcal{B}$) exist is an open problem.

# 5 Transition Effect Matrices

For a $\sigma$-SEA $E$, a map $\tau \colon E \to [0, \infty]$ is $\sigma$-**additive** if $\tau\left(\oplus a_i\right) = \sum \tau(a_i)$. A $\sigma$-additive map $\tau$ on $E$ is called a **trace** if

(T1) $\tau\left[(a \circ b) \circ c\right] = \tau\left[b \circ (a \circ c)\right]$ for all $a, b, c \in E$;

(T2) $\tau(a \circ c) = \tau(b \circ c) < \infty$ for all $c \in E$ implies that $a = b$.

In the sequel we shall assume that $\tau$ is a fixed trace on $E$. Notice that letting $c = 1$ in Condition (T1) gives that $\tau(a \circ b) = \tau(b \circ a)$ for all $a, b \in E$. An effect $a \in E$ is **trace class** if $\tau(a) < \infty$. We denote the set of trace class effects by $\mathrm{Tr}(E)$.

We now show that the usual trace $\mathrm{tr}(A)$ on $\mathcal{E}(H)$ satisfies the above conditions. It is well known that tr is countably additive. For $x \in H$ with $\|x\| = 1$, denote the one-dimensional projection onto the span of $x$ by $P_x$. If $A, B \in \mathcal{E}(H)$ satisfy

$$\mathrm{tr}(A \circ C) = \mathrm{tr}(B \circ C) < \infty$$

for every $C \in \mathcal{E}(H)$ then letting $C = P_x$ we obtain $\langle Ax, x\rangle = \langle Bx, x\rangle$ for every $x \in H$ with $\|x\| = 1$. It follows that $A = B$. Thus, Condition (T2) holds. To verify Condition (T1), we have for every $A, B, C \in \mathcal{E}(H)$

$$\begin{aligned}\mathrm{tr}\left[(A \circ B) \circ C\right] &= \mathrm{tr}\left(A^{1/2}BA^{1/2}C\right) = \mathrm{tr}\left(BA^{1/2}CA^{1/2}\right)\\ &= \mathrm{tr}\left[B \circ (A \circ C)\right]\end{aligned}$$

**Lemma 5.1.** (i) *If* $\mathrm{Tr}(E) \neq \emptyset$*, then* $\tau(0) = 0$. (ii) *If* $a, b \in \mathrm{Tr}(E)$ *and* $a \perp b$*, then* $a \oplus b \in \mathrm{Tr}(E)$. (iii) *If* $b \in \mathrm{Tr}(E)$ *and* $a \leq b$*, then* $a \in \mathrm{Tr}(E)$. (iv) *If* $a \in \mathrm{Tr}(E)$ *and* $b \in E$*, then* $a \circ b, b \circ a \in \mathrm{Tr}(E)$. (v) *If* $\rho \in \mathrm{Tr}(E)$ *with* $\rho \neq 0$*, then* $s(a) = \tau(\rho \circ a)/\tau(\rho)$ *is a* $\sigma$*-state.*

*Proof.* (i) If $a \in \mathrm{Tr}(E)$, then

$$\tau(a) = \tau(0 \oplus a) = \tau(0) + \tau(a)$$

Since $\tau(a) < \infty$ we have that $\tau(0) = 0$.
(ii) $\tau(a \oplus b) = \tau(a) + \tau(b) < \infty$.
(iii) If $a \leq b$, then there exists a $c \in E$ such that $a \oplus c = b$. Hence,

$$\tau(a) + \tau(c) = \tau(a \oplus c) = \tau(b) < \infty$$

Thus, $\tau(a) \leq \tau(b) < \infty$ so that $c \in \mathrm{Tr}(E)$.
(iv) Since $a \circ b \leq a$, applying (iii) gives $a \circ b \in \mathrm{Tr}(E)$. Since $\tau(b \circ a) = \tau(a \circ b) < \infty$, $b \circ a \in \mathrm{Tr}(E)$.
(v) If $\tau(\rho) = 0$, then $\tau(\rho \circ c) = \tau(0 \circ c) = 0$ for all $c \in E$. Hence, $\rho = 0$ which is a contradiction. We conclude that $\tau(\rho) \neq 0$. Now $s(1) = 1$ and we have

$$\begin{aligned}s(\oplus a_i) &= \frac{\tau\left[\rho \circ (\oplus a_i)\right]}{\tau(\rho)} = \frac{1}{\tau(\rho)}\tau\left[\oplus(\rho \circ a_i)\right]\\ &= \frac{1}{\tau(\rho)}\sum \tau(\rho \circ a_i) = \sum s(a_i)\end{aligned}$$

It follows that $s$ is a $\sigma$-state. □

If $\rho \in \mathrm{Tr}(E)$ with $\tau(\rho) = 1$, we call $\rho$ a **density**. It follows from Lemma 5.1(v) that if $\rho$ is a density, then $a \mapsto \tau(\rho \circ a)$ is a $\sigma$-state. A $\sigma$-state $s$ is **tracial** if there exists a $\rho \in \mathrm{Tr}(E)$ such that $s(a) = \tau(\rho \circ a)$ for all $a \in E$. Then $\rho$ is a density, by Condition (T2) $\rho$ is unique and is called the density for $s$. By Gleason's theorem every $\sigma$-state on $\mathcal{E}(H)$ is tracial relative to the usual trace.

If $s$ is tracial with density $\rho$ and $s(a) \neq 0$, then the $\sigma$-state $s \mid a$ given by

$$(s \mid a)(b) = \frac{\tau\left[(a \circ \rho) \circ b\right]}{s(a)} = \frac{\tau\left[(a \circ \rho) \circ b\right]}{\tau(a \circ \rho)}$$

is the $\sigma$-state $s$ **conditioned by the effect** $a$. If $\mathcal{A} = \{a_i\} \in \overline{\mathcal{T}}(E)$ and $\rho$ is a density, we define $\mathcal{A}(\rho) = \oplus(a_i \circ \rho)$. Notice that

$$\begin{aligned}\tau\left[\mathcal{A}(\rho)\right] &= \tau\left[\oplus(a_i \circ \rho)\right] = \sum \tau(\rho \circ a_i) = \tau\left[\rho \circ (\oplus a_i)\right]\\ &= \tau(\rho) = 1\end{aligned}$$

so $\mathcal{A}(\rho)$ is again a density. If $s$ has density $\rho$, then $s$ **conditioned by** $\mathcal{A}$ is the tracial state with density $\mathcal{A}(\rho)$ and is denoted by $s \mid \mathcal{A}$. An important axiom of quantum measurement theory says that if a quantum system is initially in the tracial state $s$, a test $\mathcal{A} = \{a_i\}$ is performed and the result is not observed (no result is registered) then the post-test state is $s \mid \mathcal{A}$. Moreover, if the $i$th outcome is observed then the post-test state is $s \mid a_i$ [17].

A **transition effect matrix** (TEM) is a finite or infinite square matrix $[a_{ij}]$ such that $a_{ij} \in E$ and $\oplus_j a_{ij} = 1$ for every $i$. Thus, a TEM is a matrix of effects whose rows are tests. A TEM generalizes a stochastic matrix which is a matrix whose rows are probability distributions. Hence, a TEM on the $\sigma$-SEA $[0,1] \subseteq \mathbb{R}$ is a stochastic matrix. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are TEMs of the same size on $E$, we define their **product** $A \circ B = [c_{ij}]$ by $c_{ij} = \oplus_k (a_{ik} \circ b_{kj})$. Notice that this is a generalization of the usual matrix multiplication.

**Lemma 5.2.** *If $A$ and $B$ are TEMs of the same size on $E$ then $A \circ B$ is a TEM on $E$.*

*Proof.* Let $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ where $C = A \circ B$. Since $N \mapsto \oplus_{k=1}^{N}(a_{ik} \circ b_{kj})$ is an increasing sequence, $c_{ij} = \oplus_{k=1}^{\infty}(a_{ik} \circ b_{kj})$ is defined in the infinite case. We have for all $i$ that

$$\bigoplus_j c_{ij} = \bigoplus_j \bigoplus_k (a_{ik} \circ b_{kj}) = \bigoplus_k a_{ik} \circ \left(\bigoplus_j b_{kj}\right) = \bigoplus_k a_{ik} = 1$$

which shows that $C$ is a TEM. $\square$

Notice that the product of TEMs gives a generalization of the sequential product of tests. For example, let $\mathcal{A} = \{a, a'\}$, $\mathcal{B} = \{b, b'\}$ be two-valued tests. Form the TEMs

$$A = \begin{bmatrix} a & a' & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} b & b' & 0 & 0 \\ 0 & 0 & b & b' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Their product becomes

$$A \circ B = \begin{bmatrix} a \circ b & a \circ b' & a' \circ b & a' \circ b' \\ 0 & 0 & b & b' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This last matrix has $\mathcal{A} \circ \mathcal{B}$ in the first row and other tests in the remaining rows. Of course, this can be generalized to tests with more values.

We now give an alternative formulation for a TEM. Let $I$ be a finite or countable index set and let $\mathcal{P}(I) = 2^I$ be the power set on $I$. A **discrete effect kernel** is a map $K\colon I \times \mathcal{P}(I) \to E$ such that $K(i, \cup\Delta_j) = \oplus_j K(i, \Delta_j)$ whenever $\Delta_j \cap \Delta_k = \emptyset$, $j \neq k$, and $K(i, I) = 1$ for all $i \in I$. Thus, $K(i, \bullet)$ is an effect valued measure on $\mathcal{P}(I)$ for all $i \in I$. There is a natural bijection between TEMs and discrete effect kernels. Indeed, if $A = [a_{ij}]$ is a TEM then $K(i, \Delta) = \oplus_{j \in \Delta} a_{ij}$ is a discrete effect kernel. Conversely, if $K$ is a discrete effect kernel and we define $a_{ij} = K(i, \{j\})$, then $A = [a_{ij}]$ is a TEM. If $J$ and $K$ as discrete effect kernels, we define their product $L = J \circ K$ by

$$L(i, \Delta) = \bigoplus_k J(i, \{k\}) \circ J(k, \Delta)$$

This last definition can be written in the integral form

$$L(i, \Delta) = \int J(i, d\lambda) \circ K(\lambda, \Delta)$$

which is similar to the usual equation for classical Markov kernels. Moreover, by Lemma 5.2, $L$ is again a discrete effect kernel.

A **vector density** is a vector $\rho = (\rho_1, \rho_2, \dots)$ where $\rho_i \in \mathrm{Tr}(E)$ and $\sum \tau(\rho_i) = 1$. If $A = [a_{ij}]$ is a TEM and $\rho$ is a vector density of the same size, we define $A * \rho = A^T \circ \rho$ to be the vector given by

$$(A * \rho)_i = \bigoplus_j (a_{ji} \circ \rho_j)$$

Thinking of $\rho$ as a column vector, this is again a generalization of matrix multiplication where $A^T$ is the transpose of $A$.

**Lemma 5.3.** *If $A$ is a TEM and $\rho$ is a vector density of the same size, then $A * \rho$ is a vector density.*

*Proof.* Letting $A = [a_{ij}]$ and $\rho = (\rho_i)$ we have

$$\begin{aligned}
\sum_i \tau\left[(A * \rho)_i\right] &= \sum_i \tau\left[\bigoplus_j (a_{ji} \circ \rho_j)\right] = \sum_{i,j} \tau(a_{ji} \circ \rho_j) \\
&= \sum_{i,j} \tau(\rho_j \circ a_{ji}) = \sum_j \tau\left[\bigoplus_i (\rho_j \circ a_{ji})\right] \\
&= \sum_j \tau\left[\rho_j \circ \left(\bigoplus_i a_{ji}\right)\right] = \sum_j \tau(\rho_j) = 1
\end{aligned}$$

Hence, $A * \rho$ is a vector density. □

# 6 Quantum Markov Chains

For a vector density $\rho = (\rho_1, \rho_2, \dots)$, we define $\tau(\rho) = (\tau(\rho_1), \tau(\rho_2), \dots)$. Then $\tau(\rho)$ becomes a probability distribution. We now show that this formalism can be used to describe a quantum generalization of a Markov chain. A **quantum Markov chain** is a finite or countable directed graph $G$ in which the edge from vertex $i$ to vertex $j$ is labeled $a_{ij}$ (if there is no edge from vertex $i$ to vertex $j$, $a_{ij} = 0$) and $A = [a_{ij}]$ forms a TEM. This definition in the particular case of $\mathcal{E}(H)$ was given in [10] and the following formalism generalizes the work there. We may think of the vertices of $G$ as sites that a quantum system can occupy and $a_{ij}$ is the effect that is observed when there is a transition from site $i$ to site $j$ in one time step. Alternatively, the kernel $K(i, \Delta)$ is the effect that is observed when there is a transition from site $i$ to some site in $\Delta$ in one time step. A quantum Markov chain in the particular case of the $\sigma$-SEA $[0,1]\subseteq\mathbb{R}$ is a classical Markov chain.

The quantum system is initially described by a vector density $\rho = (\rho_1,\rho_2,\dots)$. Then $\tau(\rho_i)$ is the probability that the system is initially at site $i$ and $\tau(\rho)$ is the initial probability distribution of the system. If the system is described by the quantum Markov chain $(G,A)$, then by Lemma 5.3, $A*\rho$ is a vector density that we interpret as the vector density at one time step. The probability distribution at one time step is $\tau(A*\rho)$. In general, the vector density at $n$ time steps is

$$A_{(n)}(\rho) = A*(\cdots A*(A*\rho))$$

and the probability distribution at $n$ time steps is $\tau\left[A_{(n)}(\rho)\right]$.

For a TEM $A$, we denote the set of vector densities of the proper size for matrix multiplication by $A$ as $\mathcal{D}_A(E)$. For simplicity we write $\mathcal{D}(E)=\mathcal{D}_A(E)$ and no confusion should result. The maps $A_{(n)}\colon\mathcal{D}(E)\to\mathcal{D}(E)$, $n=1,2,\dots$, are called the **state dynamics**. We now introduce another type of dynamics called the matrix dynamics. For the state dynamics, the vector density $\rho$ evolves and the TEM $A$ is considered fixed. For the matrix dynamics, the TEM evolves and the vector density i considered fixed. Roughly speaking this is analogous to the Schrödinger and Heisenberg pictures for quantum dynamics. In the present framework the two types of dynamics are not identical. However, as in the usual quantum dynamics they produce the same probability distributions; that is, they are statistically equivalent.

If $A_1,\dots,A_n$ are TEMs of the same size on $E$, by Lemma 5.2 their product is again a TEM on $E$. However, the product $\circ$ is nonassociative in general and when we write $A_n\circ\cdots\circ A_2\circ A_1$ we mean

$$A_n\circ\cdots\circ\{A_4\circ[A_3\circ(A_2\circ A_1)]\}$$

That is,

$$A_n\circ\cdots\circ A_2\circ A_1 = A_n\circ(A_{n-1}\circ\cdots\circ A_1)$$

If $A$ is a TEM and $n\in\mathbb{N}$, we define the $n$-**step** TEM $A^{(n)}=A\circ\cdots\circ A$ ($n$ factors). Of course, $A^{(n)}$ is indeed a TEM and we interpret the $ij$ entry $A_{ij}^{(n)}$ as the effect that the system evolves from site $i$ to site $j$ in $n$ time steps. The maps $A^{(n)}\colon\mathcal{D}(E)\to\mathcal{D}(E)$, $n=1,2,\dots$, given by $\rho\mapsto A^{(n)}*\rho$ are called the **matrix dynamics**. As we already mentioned, $A^{(n)}\neq A_{(n)}$ in general [10]. One reason for introducing the matrix dynamics is because the state dynamics $A_{(n)}(\rho)$ depends on $\rho \in \mathcal{D}(E)$ while $A^{(n)}$ is independent of the vector density. Thus, if a general form for $A^{(n)}$ can be derived, it can be applied to any $\rho \in \mathcal{D}(E)$.

**Lemma 6.1.** *If $A$ and $B$ are TEMs of the same size on $E$, then*

$$\tau\left[(A \circ B) * \rho\right] = \tau\left[B * (A * \rho)\right]$$

*for every $\rho \in \mathcal{D}(E)$.*

*Proof.* Letting $A = [a_{ij}]$ and $B = [b_{ij}]$ we have

$$\left[(A \circ B) * \rho\right]_i = \bigoplus_j (A \circ B)_{ji} \circ \rho_j = \bigoplus_j \left(\bigoplus_k a_{jk} \circ b_{ki}\right) \circ \rho_j$$

We then obtain

$$\begin{aligned}
\tau\left\{\left[(A \circ B) * \rho\right]_i\right\} &= \sum_j \tau\left[\left(\bigoplus_k a_{ik} \circ b_{ki}\right) \circ \rho_j\right] \\
&= \sum_j \tau\left[\rho_j \circ \left(\bigoplus_k a_{jk} \circ b_{ki}\right)\right] \\
&= \sum_{j,k} \tau\left[\rho_j \circ (a_{jk} \circ b_{ki})\right] \\
&= \sum_{j,k} \tau\left[b_{ki} \circ (a_{jk} \circ \rho_j)\right]
\end{aligned}$$

Since

$$\begin{aligned}
\left[B * (A * \rho)\right]_i &= \bigoplus_k b_{ki} \circ (A * \rho)_k = \bigoplus_k b_{ki} \circ \left(\bigoplus_j a_{jk} \circ \rho_j\right) \\
&= \bigoplus_{j,k} \left[b_{ki} \circ (a_{jk} \circ \rho_j)\right]
\end{aligned}$$

we have that

$$\tau\left\{\left[B * (A * \rho)\right]_i\right\} = \sum_{j,k} \tau\left[b_{ki} \circ (a_{jk} \circ \rho_j)\right]$$

and the result follows. □

The next result shows that the two types of dynamics are statistically equivalent.

**Theorem 6.2.** *If $A$ is a TEM, then $\tau\left[A^{(n)} * \rho\right] = \tau\left[A_{(n)}(\rho)\right]$ for every $\rho \in \mathcal{D}(E)$.*

*Proof.* We prove the result by induction on $n$. The result clearly holds for $n = 1$. Suppose the result holds for $n \in \mathbb{N}$. Applying Lemma 6.1 and the induction hypothesis gives

$$\begin{aligned}\tau\left[A^{(n+1)} * \rho\right] &= \tau\left\{\left[A \circ A^{(n)}\right] * \rho\right\} = \tau\left[A^{(n)} * (A * \rho)\right] \\ &= \tau\left[A_{(n)}(A * \rho)\right] = \tau\left[A_{(n+1)}(\rho)\right]\end{aligned}$$

The result follows by induction. □

We close this section with two simple examples taken from [10]. These examples illustrate why quantum Markov chains are much more general than classical Markov chains. The $\sigma$-SEA employed in these examples is $\mathcal{E}(H)$.

**Example 1**. Let $P, Q$ be projection operators on $H$ and form the TEM

$$A = \begin{bmatrix} P & P' \\ Q & Q' \end{bmatrix}$$

We then have

$$A^{(2)} = A \circ A = \begin{bmatrix} P & P' \\ Q & Q' \end{bmatrix} \circ \begin{bmatrix} P & P' \\ Q & Q' \end{bmatrix} = \begin{bmatrix} P + P' \circ Q & P' \circ Q' \\ Q \circ P & Q \circ P' + Q' \end{bmatrix}$$

If $S = (S_1, S_2)$ is the initial vector density, then in the matrix dynamics, the vector density at two time steps is

$$A^{(2)} * S = \begin{bmatrix} P + P' \circ Q & Q \circ P \\ P' \circ Q' & Q \circ P' + Q' \end{bmatrix} \circ \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

$$= \begin{bmatrix} (P + P' \circ Q) \circ S_1 + (Q \circ P) \circ S_2 \\ (P' \circ Q') \circ S_1 + (Q \circ P' + Q') \circ S_2 \end{bmatrix}$$

To find the vector density at two time steps in the state dynamics, we first compute the vector density at one time step

$$A * S = \begin{bmatrix} P & Q \\ P' & Q' \end{bmatrix} \circ \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} P \circ S_1 + Q \circ S_2 \\ P' \circ S_1 + Q' \circ S_2 \end{bmatrix}$$

At two time steps, $A_{(2)}(S)$ becomes

$$\begin{aligned} A * (A * S) &= \begin{bmatrix} P & Q \\ P' & Q' \end{bmatrix} \circ \begin{bmatrix} P \circ S_1 + Q \circ S_2 \\ P' \circ S_1 + Q' \circ S_2 \end{bmatrix} \\ &= \begin{bmatrix} P \circ S_1 + P \circ (Q \circ S_2) + Q \circ (P' \circ S_1) \\ P' \circ (Q \circ S_2) + Q' \circ (P' \circ S_1) + Q' \circ S_2 \end{bmatrix} \end{aligned}$$

These two expressions, $A^{(2)} * S$ and $A_{(2)}(S)$ are quite different. For example, if $\dim H = 2$ and $S = (I/2, 0)$ then

$$A^{(2)} * S = \frac{1}{2} \begin{bmatrix} P + P' \circ Q \\ P' \circ Q' \end{bmatrix}, \qquad A_{(2)}(S) = \frac{1}{2} \begin{bmatrix} P + Q \circ P' \\ Q' \circ P' \end{bmatrix}$$

These agree if and only if $P' \circ Q = Q \circ P'$ which is equivalent to $PQ = QP$ [13]. Of course, this latter condition does not hold in general.

**Example 2**. We now consider a quantum random walk with absorbing barriers. In this example, $P$ is a projection operator on $H$ and the sites are labeled by $-2, -1, 0, 1, 2$. Suppose the quantum random walk is governed by the TEM

$$A = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ P' & 0 & P & 0 & 0 \\ 0 & P' & 0 & P & 0 \\ 0 & 0 & P' & 0 & P \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

We then have

$$A^{(2)} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ P' & 0 & 0 & P & 0 \\ P' & 0' & 0 & 0 & P \\ 0 & P' & 0 & 0 & P \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \qquad A^{(3)} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ P' & 0 & 0 & 0 & P \\ P' & 0 & 0 & 0 & P \\ P' & 0 & 0 & 0 & P \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

and $A^{(3)} = A^{(4)} = A^{(5)} = \cdots$. Suppose the initial vector density is $S = (0, 0, P_x, 0, 0)$. Then

$$\begin{aligned}\operatorname{tr}(A * S) &= (0, \langle P'x, x\rangle, 0, \langle Px, x\rangle, 0)\\ \operatorname{tr}(A^{(2)} * S) &= (\langle P'x, x\rangle, 0, 0, 0\langle Px, x\rangle)\end{aligned}$$

and $\operatorname{tr}(A^{(2)} * S) = \operatorname{tr}(A^{(3)} * S) = \cdots$. The dynamics shows that if the system is initially at the site 0, it moves directly to the right or to the left and is absorbed at the boundary sites $\pm 2$ in two time steps. There is no classical random walk that would produce this type of dynamics.

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