CANCELLATION IN SKEW LATTICES

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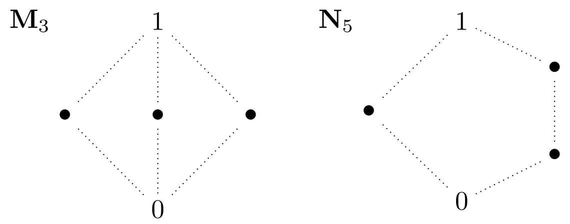
Abstract. Distributive lattices are well known to be precisely those lattices that possess cancellation: $x \vee y = x \vee z$ and $x \wedge y = x \wedge z$ imply $y = z$. Cancellation, in turn, occurs whenever a lattice has neither of the 5-element lattices $\mathbf{M}_3$ or $\mathbf{N}_5$ as sublattices. In this paper we examine cancellation in skew lattices, where the involved objects are in many ways lattice-like, but the operations $\wedge$ and $\vee$ no longer need be commutative. In particular, we find necessary and sufficient conditions involving the *nonoccurrence* of potential sub-objects similar to $\mathbf{M}_3$ or $\mathbf{N}_5$ that insure that a skew lattice is *left cancellative* (satisfying the above implication) *right cancellative* ($x\vee z = y\vee z$ and $x\wedge z = y\wedge z$ imply $x = y$) or just cancellative (satisfying both implications). We also present systems of identities showing that left [right or fully] cancellative skew lattices form varieties. Finally, we give some positive characterizations of cancellation.

1. Introduction

Recall that a lattice $\mathbf{L} := \langle L; \wedge, \vee\rangle$ is ***distributive*** if the identity $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ holds on $\mathbf{L}$. One of the first results in lattice theory is the equivalence of this identity to its dual, $x\vee(y\wedge z) = (x\vee y)\wedge(x\vee z)$. Distributive lattices are also characterized as being ***cancellative***: $x \vee y = x \vee z$ and $x \wedge y = x \wedge z$ jointly imply $y = z$. Another characterization is that neither of the following 5-element lattices can be embedded in the given lattice. (The relevant discussion is given in any introduction to lattice theory. See, *e.g.*, [5].)



Distribution is also basic in the study of ***skew lattices***, that is, algebras $\langle S; \vee, \wedge\rangle$, where $\vee$ and $\wedge$ are associative, idempotent binary operations satisfying the absorption identities:

$$x \wedge (x \vee y) = x = (y \vee x) \wedge x\,, \tag{1.1}$$

$$x \vee (x \wedge y) = x = (y \wedge x) \vee x\,. \tag{1.2}$$

Give that both operations are associative and idempotent, these identities imply (and indeed are equivalent to) the following pair of dualities:

$$u \wedge v = u \quad \text{iff} \quad u \vee v = v\,, \tag{1.3}$$

$$u \wedge v = v \quad \text{iff} \quad u \vee v = u\,. \tag{1.4}$$

For instance, given $u \wedge v = u$, then by (1.2), $u \vee v = (u \wedge v) \vee v = v$.

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For skew lattices the distributive identities of primary interest have proven to be the identities

$$x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)\,, \tag{1.5}$$

$$x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)\,. \tag{1.6}$$

Unlike the case for lattices, these identities need not be equivalent; however Spinks in 1998 obtained a computer-generated proof of their equivalence for symmetric skew lattices (defined below). A precursor was the 1992 result of Leech that the strengthened first identity $x \wedge (y \vee z) \wedge w = (x \wedge y \wedge w) \vee (x \wedge z \wedge w)$ implied the second identity. (See M. Spinks [18, 19], K. Cvetko-Vah [7] and J. Leech [13].)

What about cancellation and the role of $\mathbf{M}_3$ and $\mathbf{N}_5$ in all this? It is remarkable how disentangled cancellation can be from distribution in the noncommutative context. In this paper we study ***left cancellative*** skew lattices that satisfy the implication

$$x \vee y = x \vee z \quad \text{and} \quad x \wedge y = x \wedge z \quad \text{imply} \quad y = z\,, \tag{1.7}$$

their ***right cancellative*** duals which satisfy the implication

$$x \vee z = y \vee z \quad \text{and} \quad x \wedge z = y \wedge z \quad \text{imply} \quad x = y \tag{1.8}$$

and ***cancellative*** skew lattices which by definition are both left and right cancellative. Cancellative skew lattices are of special interest since the two classes of skew lattices studied the most over the last twenty years, skew lattices in rings and skew Boolean algebras, are both cancellative. (See [1]–[4], [6], [9], [11], [12], [16] and [20].) Our main goals are (1) to show that all three classes of cancellative skew lattices are varieties and (2) to characterize each class by a short list of forbidden subalgebras, much as excluding $\mathbf{M}_3$ and $\mathbf{N}_5$ characterizes distributive lattices.

This study is carried out in the five following sections, the next of which provides some background on skew lattices. In Section 3 we study preconditions of full and partial cancellation. In particular, we introduce a primitive form of cancellation called ***simple cancellation*** that includes the three cases already given and is characterized by the implication

$$x \wedge z \wedge x = y \wedge z \wedge y \quad \text{and} \quad x \vee z \vee x = y \vee z \vee y \quad \text{imply} \quad x = y\,. \tag{1.9}$$

$\mathbf{M}_3$ and $\mathbf{N}_5$ play a role in characterizing simple cancellation as do two other skew lattices of small order. In Section 4, we consider symmetry, or bi-conditional commutativity ($x \vee y = y \vee x$ iff $x \wedge y = y \wedge x$). Symmetry along with forms of partial symmetry bridges the gap between the simple case and higher forms of cancellation. Again, we find skew lattices of small order that allow us to characterize symmetry and its partial forms in terms of forbidden subalgebras. Left, right and full cancellation are analyzed and characterized in Section 5. Identities that characterize the corresponding varieties are given, along with lists of forbidden subalgebras for each case. We conclude the section by examining some seemingly alternative forms of cancellation. In the final section we consider positive characterizations of cancellation that, in contrast to lists of forbidden algebras, describe how certain subalgebras must behave.

2. Background

The introductory remarks and first sections of [11] and [15] are good background resources for both the motivation of and basic results about skew lattices. We recall those results of greatest relevance to our concerns in this paper.

Given a skew lattice $(S; \vee, \wedge)$, both algebraic reducts $(S; \vee)$ and $(S; \wedge)$ are bands, that is, semigroups whose elements are idempotent ($x^2 = x$). Bands form a variety of algebras, with an important subvariety being ***regular*** bands satisfying the "middle inclusion-exclusion identity", $xyxzx = xyzx$. Thanks to [11], Theorem 1.15, we have:

Lemma 2.1. *Both band reducts $(S; \vee)$ and $(S; \wedge)$ of a skew lattice $(S; \vee, \wedge)$ are regular. Thus skew lattices satisfy the identities:*

$$x \vee y \vee x \vee z \vee x = x \vee y \vee z \vee x \quad \textit{and} \quad x \wedge y \wedge x \wedge z \wedge x = x \wedge y \wedge z \wedge x\,. \tag{2.1}$$

Lemma 2.2. *Given a skew lattice $(S; \vee, \wedge)$, for all $x, y \in S$, one has:*

$$x \vee y = x = y \vee x \quad \textit{iff} \quad x \wedge y = y = y \wedge x\,. \tag{2.2}$$

$$x \vee y \vee x = x \quad \textit{iff} \quad y \wedge x \wedge y = y\,. \tag{2.3}$$

Proof. Here (2.2) follows easily from (1.3) and (1.4). For (2.3), given $x \vee y \vee x = x$, both $(x \vee y) \wedge x = x \vee y$ and $x \wedge (y \vee x) = y \vee x$ follow by (1.3) and (1.4). One thus has $(x \vee y) \wedge x \wedge y = (x \vee y) \wedge y = y$ and likewise $y \wedge x \wedge (y \vee x) = y$. Thus (2.1) combined with (1.1) gives

$$y = y \wedge x \wedge (y \vee x) \wedge (x \vee y) \wedge x \wedge y \overset{(2.1)}{=} y \wedge x \wedge (y \vee x) \wedge x \wedge (x \vee y) \wedge x \wedge y \overset{(1.1)}{=} y \wedge x \wedge y\,.$$

The dual implication, that $y \wedge x \wedge y = y$ implies $x \vee y \vee x = x$, is similarly verified. □

Given a skew lattice $\mathbf{S}$, the ***natural partial order*** on S is defined by

(2.4) $$x \geq y \quad \text{if} \quad x \wedge y = y = y \wedge x\,, \quad \text{or equivalently,} \quad x \vee y = x = y \vee x\,.$$

The relation $\geq$ is easily seen to be reflexive, antisymmetric and transtive. The ***natural quasiorder*** is defined on S by

(2.5) $$x \succeq y \quad \text{if} \quad y \wedge x \wedge y = y\,, \quad \text{or equivalently,} \quad x \vee y \vee x = x\,.$$

The relation $\succeq$ is clearly reflexive. Given $x \succeq y \succeq z$, we get $x \succeq z$ as follows:

$$z \wedge x \wedge z = z \wedge y \wedge z \wedge x \wedge z \wedge y \wedge z \overset{(2.1)}{=} z \wedge y \wedge x \wedge y \wedge z = z \wedge y \wedge z = z\,.$$

Thus $\succeq$ is a quasi-order. Its induced equivalence relation, denoted by $\mathcal{D}$, is given by: $x\,\mathcal{D}\,y$ if and only if $x \succeq y \succeq x$, so that $x \wedge y \wedge x = x$, $y \wedge x \wedge y = y$ and their duals hold. Equivalence classes are called ***$\mathcal{D}$-classes***.

Lemma 2.3. *For any elements x, y of a skew lattice $\mathbf{S}$, $x\,\mathcal{D}\,y$ if and only if $x \vee y = y \wedge x$.*

Proof. If $x\,\mathcal{D}\,y$, then

(2.6) $$x \vee y \overset{(1.1)}{=} (y \vee x \vee y) \wedge (x \vee y) = y \wedge (x \vee y)\,.$$

Thus $y \wedge x \overset{(1.1)}{=} y \wedge x \wedge (x \vee y) \overset{(2.6)}{=} y \wedge x \wedge [y \wedge (x \vee y)] = y \wedge (x \vee y) \overset{(2.6)}{=} x \vee y$, where we have used $x\,\mathcal{D}\,y$ in the third equality. Conversely, if $x \vee y = y \wedge x$, then $x \wedge y \wedge x = x \wedge (x \vee y) \overset{(1.1)}{=} x$ and $x \vee y \vee x = (y \wedge x) \vee x \overset{(1.2)}{=} x$, and thus $x\,\mathcal{D}\,y$. □

Lemma 2.4. *In every skew lattice, the identities*

(2.7) $$(x \wedge y) \vee (y \wedge x) = y \wedge x \wedge y \qquad \textit{and} \qquad (x \vee y) \wedge (y \vee x) = y \vee x \vee y$$

hold for all x, y.

Proof. Since $x \wedge y \mathcal{D} y \wedge x$, Lemma 2.3 gives $(x \wedge y) \vee (y \wedge x) = (y \wedge x) \wedge (x \wedge y) = y \wedge x \wedge y$ and $(x \vee y) \wedge (y \vee x) = (y \vee x) \vee (x \vee y) = y \vee x \vee y$. □

Lemma 2.5. *$\mathcal{D}$ is a congruence and $\mathbf{S}/\mathcal{D}$ is a lattice. Given any congruence C on $\mathbf{S}$ such that $\mathbf{S}/C$ is a lattice, $\mathcal{D} \subseteq C$. Thus $\mathbf{S}/\mathcal{D}$ is the maximal lattice image of S.*

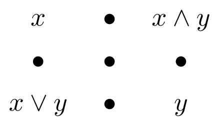
Proof. Given $x\mathcal{D}y$ and $u \in S$, $u \vee x \vee u \vee y \vee u \vee x \overset{(2.1)}{=} u \vee x \vee y \vee u \vee x \overset{(2.1)}{=} u \vee x \vee y \vee x \vee u \vee x = u \vee x \vee u \vee x = u \vee x$, and likewise, $u \vee y \vee u \vee x \vee u \vee y = u \vee y$, so that $u \vee x\,\mathcal{D}\,u \vee y$. Similarly, $u \wedge x\,\mathcal{D}\,u \wedge y$, $x \vee u\,\mathcal{D}\,y \vee u$ and $x \wedge u\,\mathcal{D}\,y \wedge u$ so that $\mathcal{D}$ is indeed a congruence. Since both $x \vee y\,\mathcal{D}\,y \vee x$ and $x \wedge y\,\mathcal{D}\,y \wedge x$ for all $x, y \in S$, $\mathbf{S}/\mathcal{D}$ is commutative. Suppose now that C is a congruence such that $\mathbf{S}/C$ is a lattice, and suppose that $x\,\mathcal{D}\,y$. Then $x \vee y \;\; C \;\; y \vee x$ implies $(x \vee y \vee x) \;\; C \;\; y \vee x \vee x$, which simplifies to $x \;\; C \;\; y \vee x$. Again, $x \vee y \;\; C \;\; y \vee x$ implies $y \vee x \vee y \;\; C \;\; y \vee y \vee x$, which simplifies to $y \;\; C \;\; y \vee x$. Thus $x \;\; C \;\; y$, and so $\mathcal{D} \subseteq C$. □

A skew lattice $\mathbf{S}$ for which $\mathcal{D}$ is the universal relation is said to be ***rectangular***. By Lemma 2.3, this is equivalent to the identity

(2.8) $$x \vee y = y \wedge x$$

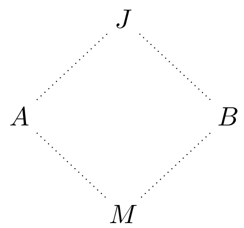
holding for all $x, y \in S$. If $\mathbf{T}$ is a subalgebra of a skew lattice $\mathbf{S}$, then $\mathbf{T}$'s own relation $\mathcal{D}$ is just the restriction of the relation $\mathcal{D}$ in $\mathbf{S}$ to $T \times T$. Thus if $\mathbf{T}$ itself is a rectangular subalgebra of $\mathbf{S}$, it follows that T is entirely contained in a $\mathcal{D}$-class of $\mathbf{S}$. Summarizing, *each $\mathcal{D}$-class of a skew lattice $\mathbf{S}$ is a maximal rectangular subalgebra of $\mathbf{S}$.*

A rectangular skew lattice can be envisioned as a (possibly infinite) rectangular array of points. Given points x and y, point $x \wedge y$ is the unique point in the row of x and the column of y, while the point $x \vee y$ is the unique point in the column of x and the row of y. Every rectangular skew lattice is isomorphic to such an array with just such operations.



Thus a skew lattice $\mathbf{S}$ is roughly the result of taking a lattice $\mathbf{L}$ and hanging a rectangular skew lattice $\mathbf{M}_x$ at each of its points x. This is the *Clifford-McLean Theorem*, originally proved for bands, but also holding for skew lattices: *every skew lattice is a lattice* ($\mathbf{S}/\mathcal{D}$) *rectangular skew lattices* (its $\mathcal{D}$-class subalgebras).

When restricted to a common $\mathcal{D}$-class, the operations $\vee$ and $\wedge$ in a skew lattice are quite trivial (thanks to the rectangular description above). Applying them to elements in distinct $\mathcal{D}$-classes is another matter. For some insight, consider a ***skew diamond*** consisting of a pair of incomparable $\mathcal{D}$-classes, A and B, and their join class J and their meet class M in the lattice of $\mathcal{D}$-classes $\mathbf{S}/\mathcal{D}$.



Lemma 2.6. *Given the above skew diamond with $a \in A$, $b \in B$, $j \in J$ and $m \in M$:*

i) $m \leq$ *both* a, b *iff* $a \wedge b = m = b \wedge a$.

ii) $j \geq$ *both* a, b *iff* $a \vee b = j = b \vee a$.

iii)

$$J = \{a \vee b \mid a \in A, b \in B, \text{ and } a \vee b = b \vee a\},$$
$$M = \{a \wedge b \mid a \in A, b \in B, \text{ and } a \wedge b = b \wedge a\}.$$

Proof. Since $a \wedge b \in M$ by Lemma 2.5,

$$a \wedge b = (a \wedge b) \wedge m \wedge (a \wedge b) = a \wedge (b \wedge m \wedge a) \wedge b = a \wedge m \wedge b = m$$

if $m \leq$ both a and b. The converse is trivial. Thus (i) holds and the proof of (ii) is similar. (iii) asserts that all elements of J and M are obtained this way. Indeed, given, say, $m \in M$ and arbitrary $u \in A$, $v \in B$, upon setting $a = m \vee u \vee m \in A$ and $b = m \vee v \vee m \in B$, we get $m \leq a, b$. □

A pair of natural congruences, $\mathcal{L}$ and $\mathcal{R}$, refine $\mathcal{D}$. We say that x is ***$\mathcal{L}$-related*** to y (denoted $x \,\mathcal{L}\, y$) if $x \wedge y = x$ and $y \wedge x = y$, or dually by (1.3) and (1.4), $x \vee y = y$ and $y \vee x = x$. Likewise, x and y are ***$\mathcal{R}$-related*** ($x \,\mathcal{R}\, y$) if $x \wedge y = y$ and $y \wedge x = x$, or dually, $x \vee y = x$ and $y \vee x = y$.

A skew lattice is ***left-handed*** if $\mathcal{D} = \mathcal{L}$ so that $x \wedge y = x = y \vee x$ on each rectangular subalgebra. Left-handed skew lattices are characterized by various equivalent identities:

(2.9) $$x \wedge y \wedge x = x \wedge y \quad \text{or} \quad x \wedge (y \vee x) = x \quad \text{or} \quad x \vee y \vee x = y \vee x \quad \text{or} \quad (x \wedge y) \vee x = x\,.$$

For instance, if $x \wedge y \wedge x = x \wedge y$ holds identically, then $x \wedge (y \vee x) = x \wedge (y \vee x) \wedge x \overset{(1.1)}{=} x \wedge x = x$. If $x \wedge (y \vee x) = x$ holds, then $y \vee x \overset{(1.2)}{=} (x \wedge (y \vee x)) \vee y \vee x = x \vee y \vee x$. Similar arguments show that the third identity implies the fourth and that the fourth implies the first.

Dually a skew lattice is ***right-handed*** if $\mathcal{D} = \mathcal{R}$ so that $x \wedge y = y = y \vee x$ on each rectangular subalgebra. Right-handed skew lattices are characterized by the following equivalent identities:

(2.10) $$x \wedge y \wedge x = y \wedge x \quad \text{or} \quad (x \vee y) \wedge x = x \quad \text{or} \quad x \vee y \vee x = x \vee y \quad \text{or} \quad x \vee (y \wedge x) = x\,.$$

In the rectangular array described above, the rows are $\mathcal{R}$-classes of mutually $\mathcal{R}$-related elements while the columns are $\mathcal{L}$-classes. The quotient algebra $\mathbf{S}/\mathcal{R}$ is the maximal left-handed image of $\mathbf{S}$, while $\mathbf{S}/\mathcal{L}$ is the maximal right-handed image of $\mathbf{S}$. In terms of rectangular arrays, $\mathbf{S}/\mathcal{R}$ is a lattice of column subalgebras satisfying $x \wedge y = x = y \vee x$, while $\mathbf{S}/\mathcal{L}$ is a lattice of row subalgebras satisfying $x \wedge y = y = y \vee x$. Since $\mathcal{L} \cap \mathcal{R}$ is the identity relation, $\mathbf{S}$ is isomorphic to a subalgebra of the direct product $\mathbf{S}/\mathcal{R} \times \mathbf{S}/\mathcal{L}$.

A subalgebra $\mathbf{T}$ is a skew lattice $\mathbf{S}$ is a ***sublattice*** of $\mathbf{S}$ if it is commutative and thus forms a lattice. Any sublattice $\mathbf{T}$ intersects each $\mathcal{D}$-class of $\mathbf{S}$ in *at most* one point. If $\mathbf{T}$ meets each $\mathcal{D}$-class of $\mathbf{S}$ in *exactly* one point, then $\mathbf{T}$ is called a ***lattice section*** of $\mathbf{S}$. As such it is a maximal sublattice that is also an internal copy inside $\mathbf{S}$ of the maximal lattice image $\mathbf{S}/\mathcal{D}$. Every skew lattice $\mathbf{S}$ that is a finite chain of $\mathcal{D}$-classes $A_1 > A_2 > \cdots > A_k$ has a lattice section. (First pick arbitrary elements $a_i \in A_i$, set $a'_1 = a_1$ and then $a'_{i+1} = a'_i \wedge a_{i+1} \wedge a'_i \in A_{i+1}$ for $i = 1, \ldots, k$. The chain of elements $a'_1 > a'_2 > \cdots a'_k$ is a lattice section of $\mathbf{S}$.

Proposition 2.7. (Cvetko-Vah [8]) *Let $\mathbf{T}$ be a lattice section of a skew lattice $\mathbf{S}$ intersecting each $\mathcal{D}$-class A of S at some (necessarily unique) point t_A. Then the union $\mathbf{T}[\mathcal{L}] = \bigcup_A \mathcal{L}(t_A)$ of the $\mathcal{L}$-classes of the t_A is a maximal left-handed subalgebra of $\mathbf{S}$ that is isomorphic to $\mathbf{S}/\mathcal{R}$ upon restricting to $\mathbf{T}[\mathcal{L}]$ the natural map from $\mathbf{S}$ to $\mathbf{S}/\mathcal{R}$. Dually, $\mathbf{T}[\mathcal{R}] = \bigcup_A \mathcal{R}(t_A)$ is a maximal right-handed subalgebra of $\mathbf{S}$ that is isomorphic to $\mathbf{S}/\mathcal{L}$.*

Lemma 2.8. *A skew diamond $\mathbf{S}$ has a lattice section and thus also has internal copies of both $\mathbf{S}/\mathcal{R}$ and $\mathbf{S}/\mathcal{L}$.*

Proof. Given $S = A \cup B \cup J \cup M$ as before with $a \in A$ and $b \in B$, set $j = a \vee b \vee a \in J$, $m = a \wedge b \wedge a \in M$ and $b' = m \vee (j \wedge b \wedge j) \vee m \in B$. Since $m \leq a, b' \leq j$, it follows that $\{m, a, b', j\}$ is indeed a lattice section of $\mathbf{S}$ by Lemma 2.6. □

Since any skew lattice $\mathbf{S}$ is embedded in the product $\mathbf{S}/\mathcal{R} \times \mathbf{S}/\mathcal{L}$, joint properties of $\mathbf{S}/\mathcal{R}$ and $\mathbf{S}/\mathcal{L}$ are *often* passed on to $\mathbf{S}$, and conversely. In particular, $\mathbf{S}/\mathcal{R}$ and $\mathbf{S}/\mathcal{L}$ both belong to a variety if and only if $\mathbf{S}$ does. We also have:

Proposition 2.9. *If $\mathcal{V}$ is a variety of skew lattices, then so are the following:*

- *the class $\mathcal{V}(\mathrm{mod}\mathcal{R})$ of skew lattices $\mathbf{S}$ for which $\mathbf{S}/\mathcal{R} \in \mathcal{V}$,*
- *the class $\mathcal{V}(\mathrm{mod}\mathcal{L})$ of skew lattices $\mathbf{S}$ for which $\mathbf{S}/\mathcal{L} \in \mathcal{V}$, and*
- *the class $\mathcal{V}(\mathrm{mod}\mathcal{D})$ of skew lattices $\mathbf{S}$ for which $\mathbf{S}/\mathcal{D}$ lies in $\mathcal{V}$.*

Proof. If the identities $u_i(x_j) = v_i(x_j)$, $i \in I$ characterize $\mathcal{V}$, then $\mathcal{V}(\mathrm{mod}\mathcal{R})$ is characterized by the relations $u_i(x_j) \mathrel{\mathcal{R}} v_i(x_j)$, $i \in I$ each of which decomposes into $u_i(x_j) \wedge v_i(x_j) = v_i(x_j)$ and $v_i(x_j) \wedge u_i(x_j) = u_i(x_j)$. Similar sets of identities characterize $\mathcal{V}(\mathrm{mod}\mathcal{L})$ and $\mathcal{V}(\mathrm{mod}\mathcal{D})$. □

We conclude the section with the following useful extension of Lemma 2.1:

Lemma 2.10. *In any skew lattice $\mathbf{S}$,*

$$a, b \succeq c \quad \textit{implies} \quad a \vee c \vee b = a \vee b\,, \tag{2.11}$$

$$a, b \preceq c \quad \textit{implies} \quad a \wedge c \wedge b = a \wedge b\,. \tag{2.12}$$

Proof. $a \vee c \vee b \overset{(2.5)}{=} (a \vee c \vee a) \vee c \vee (b \vee c \vee b) \overset{(2.1)}{=} (a \vee c \vee a) \vee (b \vee c \vee b) \overset{(2.5)}{=} a \vee b$. This establishes (2.11) and the proof of (2.12) is similar. □

3. Simple cancellation

As indicated in the introduction, we begin our study of cancellation with a form that includes left and right (and hence, full) cancellation as special cases. One obvious, if inelegant way of doing this is to pack everything into the antecedent of the implication like so:

$$\left\{ \begin{array}{ll} x \vee z = y \vee z, & z \vee x = z \vee y, \\ x \wedge z = y \wedge z, & z \wedge x = z \wedge y \end{array} \right\} \quad \Rightarrow \quad x = y\,. \tag{3.1}$$

This turns out to be equivalent to the more satisfactory form given in the following definition.

A skew lattice $\mathbf{S}$ is ***simply cancellative*** if for all $x, y, z \in S$,

$$x \vee z \vee x = y \vee z \vee y \quad \text{and} \quad x \wedge z \wedge x = y \wedge z \wedge y \quad \Rightarrow \quad x = y\,. \tag{SC}$$

To see the equivalence of (SC) with (3.1), note that from $x \vee z = y \vee z$ and $z \vee x = z \vee y$, $x \vee z \vee x = y \vee z \vee y$ must follow. Conversely, from $x \vee z \vee x = y \vee z \vee y$ we get $x \vee z = x \vee z \vee x \vee z = y \vee z \vee y \vee z = y \vee z$, and similarly, $z \vee x = z \vee y$. Similar remarks hold for the equalities involving $\wedge$. From the equivalence, we have the following observation.

Lemma 3.1. *Every left [right, fully] cancellative skew lattice is simply cancellative.*

We next examine necessary conditions for simple cancellation.

A skew lattice $\mathbf{S}$ is ***quasidistributive*** if its lattice image $\mathbf{S}/\mathcal{D}$ is distributive. By Proposition 2.9, *quasidistributive skew lattices form a variety of skew lattices.* Indeed, since $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$ holds for all lattices, these skew lattices are characterized by $x \wedge (y \vee z) \preceq (x \wedge y) \vee (x \wedge z)$, which is expressed by the identity

$$[x \wedge (y \vee z)] \wedge [(x \wedge y) \vee (x \wedge z)] \wedge [x \wedge (y \vee z)] = x \wedge (y \vee z)\,. \tag{3.2}$$

Distributivity and cancellativity are equivalent in the lattice $\mathbf{S}/\mathcal{D}$, and are characterized by the nonoccurrence of $\mathbf{M}_3$ and $\mathbf{N}_5$ as subalgebras. We can lift these characterizations to $\mathbf{S}$ itself.

Theorem 3.2. *For a skew lattice* $\mathbf{S}$*, the following are equivalent.*

i) $\mathbf{S}$ *is quasidistributive.*
ii) $\mathbf{S}$ *satisfies the implication*

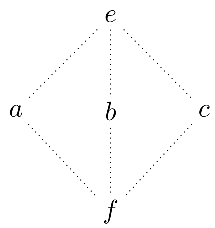
$$x \vee z \vee x = y \vee z \vee y \quad \textit{and} \quad x \wedge z \wedge x = y \wedge z \wedge y \quad \Rightarrow \quad x\,\mathcal{D}\,y\,. \tag{QD}$$

iii) *Neither* $\mathbf{M}_3$ *nor* $\mathbf{N}_5$ *occur as subalgebras of* $\mathbf{S}$.

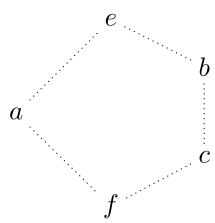
Proof. (i)⇒(ii): Suppose $\mathbf{S}$ is quasidistributive and suppose $x, y, z \in S$ satisfy $x \vee z \vee x = y \vee z \vee y$ and $x \wedge z \wedge x = y \wedge z \wedge y$. Then in $\mathbf{S}/\mathcal{D}$, $x \vee z\,\mathcal{D}\,x \vee y$ and $x \wedge z\,\mathcal{D}\,x \wedge y$. Since $\mathbf{S}/\mathcal{D}$ is cancellative, $y\,\mathcal{D}\,z$. Thus (QD) holds.

(ii)⇒(iii): Neither $\mathbf{M}_3$ nor $\mathbf{N}_5$ satisfy (QD).

(iii)⇒(i) If $\mathbf{S}/\mathcal{D}$ is not distributive, then it contains a copy of $\mathbf{M}_3$ or $\mathbf{N}_5$. We can lift this copy to a sublattice copy in $\mathbf{S}$. To see this, suppose first that a copy of $\mathbf{M}_3$ lies in $\mathbf{S}/\mathcal{D}$. Let



be inverse images of this copy in $\mathbf{S}$, where the dotted lines indicate the quasiordering between the elements. We convert this diagram to a copy of $\mathbf{M}_3$ in $\mathbf{S}$ as follows. First replace a, b, c and f by $e \wedge a \wedge e$, $e \wedge b \wedge e$, $e \wedge c \wedge e$ and $e \wedge f \wedge e$ respectively. Renaming elements, we have $e > a, b, c, f$. Next replace a, b, c, by $f \vee a \vee f$, $f \vee b \vee f$, $f \vee c \vee f$. Renaming again we get $e > a, b, c > f$ with a, b, c being $\mathcal{D}$-incomparable in $\mathbf{S}$. (Note that in changing and then renaming we do not change the relevant $\mathcal{D}$-classes!) Since by Lemma 2.6, *e.g.*, $a \vee b = e$ and $a \wedge b = f$, $\{e, a, b, c, f\}$ is the desired copy of $\mathbf{M}_3$ in $\mathbf{S}$. Next suppose that a copy of $\mathbf{N}_5$ lies in $\mathbf{S}/\mathcal{D}$ and let

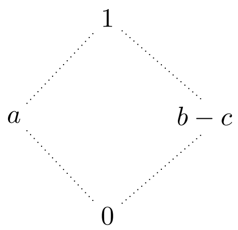


be inverse images of this copy in $\mathbf{S}$. We similarly convert this to a copy of $\mathbf{N}_5$ in $\mathbf{S}$ by again replacing a, b, c and f by $e \wedge a \wedge e$, $e \wedge b \wedge e$, $e \wedge c \wedge e$ and $e \wedge f \wedge e$ and renaming, then replacing a, b, c, by $a \vee f \vee a$, $b \vee f \vee b$, $c \vee f \vee c$ and renaming, thus making $e > a, b, c > f$ with a being $\mathcal{D}$-incomparable with b and c, and finally replacing c by $b \wedge c \wedge b$ and again renaming to obtain $b > c$ and thus a copy of $\mathbf{N}_5$ in $\mathbf{S}$. □

Corollary 3.3. *Every simply (and hence, left, right or fully) cancellative skew lattice is quasidistributive.*

Proof. The implication (QD) is trivially implied by (SC). □

As (QD) suggests, quasi-distributivity is not sufficient for simple cancellativity. Minimal examples of quasidistributive skew lattices which are not simply cancellative are given by a (horizontally) dual pair of skew diamonds with the Hasse diagram below. We denote the right-handed case by $\mathbf{NC}_5^{\mathcal{R}}$ and its left-handed dual by $\mathbf{NC}_5^{\mathcal{L}}$. The Cayley tables for the operations of $\mathbf{NC}_5^{\mathcal{R}}$ are also given. Transposing them gives the tables for $\mathbf{NC}_5^{\mathcal{L}}$.



$\vee$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	1	1	1
b	b	1	b	b	1
c	c	1	c	c	1
1	1	1	1	1	1

$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	c	b
c	0	0	b	c	c
1	0	a	b	c	1

We say that a skew lattice $\mathbf{S}$ is ***weakly simply cancellative*** if every skew diamond within $\mathbf{S}$ is simply cancellative.

Lemma 3.4. *Given a skew diamond $\mathbf{T}$ with incomparable $\mathcal{D}$-classes A and B, join class J and meet class M, the following are equivalent:*

i) *$\mathbf{T}$ is simply cancellative.*
ii) *Given $e \in J$, $f \in M$ with $e > f$, unique $a \in A$ and $b \in B$ exist such that both $e > a > f$ and $e > b > f$.*
iii) *Given $e \in J$, $f \in M$ with $e > f$, unique $a \in A$ and $b \in B$ exist such that $e = a \vee b = b \vee a$ and $f = a \wedge b = b \wedge a$.*
iv) *$\mathbf{T}$ contains no copy of $\mathbf{NC}_5^{\mathcal{L}}$ or $\mathbf{NC}_5^{\mathcal{R}}$.*

Proof. (ii)⇔(iii): This follows from Lemma 2.6.

(ii), (iii)⇒(iv): Neither $\mathbf{NC}_5^{\mathcal{L}}$ nor $\mathbf{NC}_5^{\mathcal{R}}$ satisfies the conditions of (ii) and (iii).

(iv)⇒(ii): Let $e \in J$ and $f \in M$ be such that $e > f$. Then given any $c \in A$ and $d \in B$, setting $a = f \vee (e \wedge c \wedge e) \vee f$ and $b = f \vee (e \wedge d \wedge e) \vee f$ gives a pair a, b satisfying the inequalities of (ii). To show uniqueness, suppose $e > a, a' > f$ where $a \,\mathcal{D}\, a'$. Then $a \,\mathcal{R}\, a \wedge a' \,\mathcal{L}\, a'$. Thus if $a \neq a'$ then either $a \neq a \wedge a'$ or $a \wedge a' \neq a'$. Moreover, $e > a \wedge a' > f$ so that either $\{a, a \wedge a', b, e, f\}$ or $\{a', a \wedge a', b, e, f\}$ is a copy of $\mathbf{NC}_5^{\mathcal{L}}$ or $\mathbf{NC}_5^{\mathcal{R}}$ in $\mathbf{T}$, contrary to (iv). Thus a is unique and similarly so is b. Thus (ii) holds.

(i)⇒(ii): Let $e \in J$ and $f \in M$ be as in (ii) and (iii). As before, $a \in A$ and $b \in B$ exist such that $e > a, b > f$. If $a' \in A$ is such that $e > a' > f$ also, then $a \vee b \vee a = e = a' \vee b \vee a'$ and $a \wedge b \wedge a = f = a' \wedge b \wedge a$. Simple cancellation yields $a = a'$. Similarly b is unique and (ii) follows.

(ii),(iii),(iv)⇒(i): Suppose $a, a', b \in \mathbf{T}$ satisfy $a \vee b \vee a = a' \vee b \vee a'$, $a \wedge b \wedge a = a' \wedge b \wedge a'$ and $a \,\mathcal{D}\, a'$. Again, if a and b are comparable under $\succeq$, then $a = a'$, as seen above. So we may assume they are incomparable and hence, say $a \in A$ and $b \in B$. Setting $e = a \vee b \vee a = a' \vee b \vee a'$ in J and $f = a \wedge b \wedge a = a' \wedge b \wedge a'$ in M, we have $e > a, a' > f$. By (ii), $a = a'$ and (i) follows. □

Theorem 3.5. *For a skew lattice $\mathbf{S}$, the following are equivalent.*

i) *$\mathbf{S}$ is weakly simply cancellative.*
ii) *$\mathbf{S}$ satisfies the implication*

$$\text{(WSC)} \qquad x \vee z \vee x = y \vee z \vee y\,, \quad x \wedge z \wedge x = y \wedge z \wedge y \quad \text{and} \quad x \,\mathcal{D}\, y \quad \Rightarrow \quad x = y\,.$$

iii) *Neither $\mathbf{NC}_5^{\mathcal{R}}$ nor $\mathbf{NC}_5^{\mathcal{L}}$ occur as subalgebras of $\mathbf{S}$.*

Proof. (i)⇒(ii): Suppose a, b, c satisfy $b \vee a \vee b = c \vee a \vee c$, $b \wedge a \wedge b = c \wedge a \wedge c$ and $b \,\mathcal{D}\, c$. By Lemma 2.10, if b and c were comparable to a under $\succeq$, then $b = c$. Otherwise, $A = \{a\}$, $B = \{b, c\}$, $M = \{b \wedge a \wedge b\}$ and $J = \{b \vee a \vee b\}$ forms a skew diamond.

(ii)⇒(iii): Neither $\mathbf{NC}_5^{\mathcal{R}}$ nor $\mathbf{NC}_5^{\mathcal{L}}$ satisfy (WSC).

(iii)⇔(i): This follows from Lemma 3.4. □

Theorem 3.6. *The class of weakly simply cancellative skew lattices forms a variety of skew lattices with an equational base given by the following identity where where $e = x \vee (y \wedge z) \vee x$ and $f = x \wedge (y \wedge z) \wedge x$:*

$$f \vee (e \wedge y \wedge z \wedge e) \vee f = f \vee (e \wedge z \wedge y \wedge e) \vee f\,. \tag{3.3}$$

Proof. Let u and v denote the left and right sides of (3.3). Since $y \wedge z \,\mathcal{D}\, z \wedge y$ and $\mathcal{D}$ is a congruence (Lemma 2.5), we have $u \,\mathcal{D}\, v$. Also, one has $e \geq x \geq f$ and $e \geq u, v \geq f$.

If x is $\succeq$-comparable with $y \wedge z$ and $z \wedge y$, then $u = v$ follows. Indeed if $x \preceq y \wedge z, z \wedge y$, then $e \,\mathcal{D}\, y \wedge z$ and $f = x$ so that $u = e = v$. If $y \wedge z, z \wedge y \preceq x$, then $f \,\mathcal{D}\, y \wedge z$ and $e = x$ so that $u = f = v$. Thus (3.3) really pertains to the incomparable case.

Given $\mathbf{NC}_5^{\mathcal{L}}$ or $\mathbf{NC}_5^{\mathcal{R}}$, assigning a to x, b to y and c to z leads to $\{u, v\} = \{b, c\}$, so that neither algebra satisfies (3.3). Thus all skew lattices satisfying (3.3) contain copies of neither $\mathbf{NC}_5^{\mathcal{L}}$ nor $\mathbf{NC}_5^{\mathcal{R}}$ and so are weakly simply cancellative by Theorem 3.5.

Conversely, a weakly simply cancellative skew lattice satisfies (3.3) in the case where the choice of x is $\succeq$-incomparable with the outcomes for $y \wedge z$ and $z \wedge y$. For Lemma 2.6 implies that e is the commuting join of x with both u and v while f is the commuting meet of x with both u and v and thus $u = v$ by (WSC) (Theorem 3.5(ii)). □

Theorem 3.7. *For a skew lattice* $\mathbf{S}$*, the following are equivalent.*

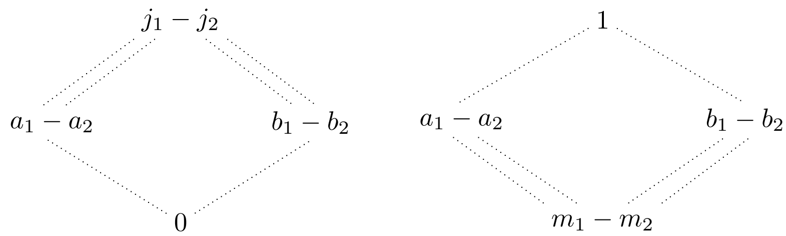
i) $\mathbf{S}$ *is simply cancellative.*
ii) $\mathbf{S}$ *is quasidistributive and weakly simply cancellative.*
iii) *None of* $\mathbf{M}_3$, $\mathbf{N}_5$, $\mathbf{NC}_5^{\mathcal{R}}$ *nor* $\mathbf{NC}_5^{\mathcal{L}}$ *occur as subalgebras of* $\mathbf{S}$.

Further, the class of all simply cancellative skew lattices forms a variety.

Proof. Evidently, (SC) is equivalent to the conjunction of (QD) and (WSC), and then Theorems 3.2 and 3.5 apply to give (i)⇔(ii)⇔(iii). For the rest, we have already noted that quasidistributive skew lattices form a variety, and then Theorem 3.6 applies. □

4. Symmetry

Having characterized simple cancellation, we now wish to fill the gap between that notion and full cancellation. To this end, consider the following pair of Hasse diagrams corresponding to two right-handed skew diamonds and their left-handed duals. (The dotted lines denote the natural partial order $\geq$ and the horizontal lines '–' denote the relation $\mathcal{D}$.)



The induced right-handed algebras are denoted respectively by $\mathbf{NS}_7^{\mathcal{R},0}$ and $\mathbf{NS}_7^{\mathcal{R},1}$, and their left-handed duals by $\mathbf{NS}_7^{\mathcal{L},0}$ and $\mathbf{NS}_7^{\mathcal{L},1}$. Cayley tables for $\mathbf{NS}_7^{\mathcal{R},0}$ and $\mathbf{NS}_7^{\mathcal{R},1}$ are as follows, with tables for $\mathbf{NS}_7^{\mathcal{L},0}$ and $\mathbf{NS}_7^{\mathcal{L},1}$ obtained by transposing these:

$\mathbf{NS}_7^{\mathcal{R},0}$:

$\vee$	0	a_n	b_n	j_n
0	0	a_n	b_n	j_n
a_m	a_m	a_m	j_m	j_m
b_m	b_m	j_m	b_m	j_m
j_m	j_m	j_m	j_m	j_m

$\wedge$	0	a_n	b_n	j_n
0	0	0	0	0
a_m	0	a_n	0	a_n
b_m	0	0	b_n	b_n
j_m	0	a_n	b_n	j_n

where $m, n = 1, 2$ and $r, s = 1, 2$.

Note that $\mathbf{NS}_7^{\mathcal{R},0}$ and $\mathbf{NS}_7^{\mathcal{L},0}$ are ***horizontal duals*** (new $x \vee y$ = old $y \vee x$, new $x \wedge y$ = old $y \wedge x$), and so are $\mathbf{NS}_7^{\mathcal{R},1}$ and $\mathbf{NS}_7^{\mathcal{L},1}$. Also, $\mathbf{NS}_7^{\mathcal{R},0}$ and $\mathbf{NS}_7^{\mathcal{L},1}$ are ***vertical duals*** (new $x \vee y$ = old $x \wedge y$, new $x \wedge y =$

$\mathbf{NS}_7^{\mathcal{R},1}$:

$\vee$	1	a_s	b_s	m_s
1	1	1	1	1
a_r	1	a_r	1	a_r
b_r	1	1	b_r	b_r
m_r	1	a_r	b_r	m_r

$\wedge$	1	a_s	b_s	m_s
1	1	a_s	b_s	m_s
a_r	a_r	a_s	m_s	m_s
b_r	b_r	m_s	b_s	m_s
m_r	m_r	m_s	m_s	m_s

old $x \vee y$ with 0 and 1 switched), and so are $\mathbf{NS}_7^{\mathcal{R},1}$ and $\mathbf{NS}_7^{\mathcal{L},0}$. Thus any observation we make about one of these algebras implies a variant observation about the other three. We will use this repeatedly in the discussion that follows.

Lemma 4.1. *Each of* $\mathbf{NS}_7^{\mathcal{R},0}$*,* $\mathbf{NS}_7^{\mathcal{L},0}$*,* $\mathbf{NS}_7^{\mathcal{R},1}$ *and* $\mathbf{NS}_7^{\mathcal{L},1}$ *is simply cancellative. Further,*

i) $\mathbf{NS}_7^{\mathcal{R},0}$ *and* $\mathbf{NS}_7^{\mathcal{L},1}$ *are right cancellative, but not left cancellative.*

ii) $\mathbf{NS}_7^{\mathcal{L},0}$ *and* $\mathbf{NS}_7^{\mathcal{R},1}$ *are left cancellative, but not right cancellative.*

Proof. The first assertion follows from Theorem 3.7 because none of the four algebras contains a copy of either $\mathbf{NC}_5^{\mathcal{L}}$ or $\mathbf{NC}_5^{\mathcal{R}}$. That $\mathbf{NS}_7^{\mathcal{R},0}$ is right cancellative follows from directly checking its Cayley tables, while that it is not left cancellative is because, for instance, $a_1 \vee b_1 = j_1 = a_1 \vee b_2$ and $a_1 \wedge b_1 = 0 = a_1 \wedge b_2$ but $b_1 \neq b_2$. As noted above, the remaining cases follow by various dualities. □

The preceding discussion leads us to recall a concept that is central in the study of skew lattices. A skew lattice $\mathbf{S}$ is ***symmetric*** if for all $x, y \in S$, $x \vee y = y \vee x$ if and only if $x \wedge y = y \wedge x$. It is ***upper symmetric*** if $x \wedge y = y \wedge x$ implies $x \vee y = y \vee x$. Dually, a skew lattice is ***lower symmetric*** if $x \vee y = y \vee x$ implies $x \wedge y = y \wedge x$.

Combined results of Leech ([11], 2.3.) and Spinks [18] give:

Theorem 4.2. *Let* $\mathbf{S}$ *be a skew lattice.*

i) $\mathbf{S}$ *is upper symmetric if and only if it satisfies:*

(US) $$x \vee y \vee (x \wedge y) = (y \wedge x) \vee y \vee x\,.$$

ii) $\mathbf{S}$ *is lower symmetric if and only if it satisfies:*

(LS) $$x \wedge y \wedge (x \vee y) = (y \vee x) \wedge y \wedge x\,.$$

Thus the class of [*upper, lower*] *symmetric skew lattices forms a variety.*

Proof. We will prove (i); the proof of (ii) follows similarly by vertical duality. It is easy to see that if $\mathbf{S}$ satisfies (US), then it is upper symmetric. Indeed if $x \wedge y = y \wedge x$, then $x \vee y \overset{(1.2)}{=} x \vee y \vee (y \wedge x) = x \vee y \vee (x \wedge y) \overset{(\mathrm{US})}{=} (y \wedge x) \vee y \vee x = (x \wedge y) \vee y \vee x \overset{(1.2)}{=} y \vee x$.

Now given $x, y \in \mathbf{S}$, set $a = (x \wedge y \wedge x) \vee y \vee (x \wedge y \wedge x)$. We claim that x and a have commuting meet $x \wedge y \wedge x$. Indeed, if $x \,\mathcal{D}\, y$ then $a = x$, in which case our claim is trivial; otherwise, we have both x and $a \geq x \wedge y \wedge x$, and so $x \wedge a = a \wedge x = x \wedge y \wedge x$ by Lemma 2.6(i).

Next note that $a \overset{(2.7)}{=} (x \wedge y \wedge x) \vee y \vee (y \wedge x) \vee (x \wedge y) \overset{(1.2)}{=} (x \wedge y \wedge x) \vee y \vee (x \wedge y)$, and similarly, $a = (y \wedge x) \vee y \vee (x \wedge y \wedge x)$.

Now suppose $\mathbf{S}$ is upper symmetric. It follows that $x \vee a = a \vee x$. Using the two different expressions for a in the preceding paragraph, this is $x \vee [(x \wedge y \wedge x) \vee y \vee (x \wedge y)] = [(y \wedge x) \vee y \vee (x \wedge y \wedge x)] \vee x$. Dropping the brackets and applying (1.2) on both sides, we have $x \vee y \vee (x \wedge y) = (y \wedge x) \vee y \vee x$, that is, (US) holds. □

The above four 7-element algebras are the skew lattices of smallest order that are not fully symmetric, although $\mathbf{NS}_7^{\mathcal{R},0}$ and $\mathbf{NS}_7^{\mathcal{L},0}$ are lower symmetric and $\mathbf{NS}_7^{\mathcal{R},1}$ and $\mathbf{NS}_7^{\mathcal{L},1}$ are upper symmetric. This is due to the following result.

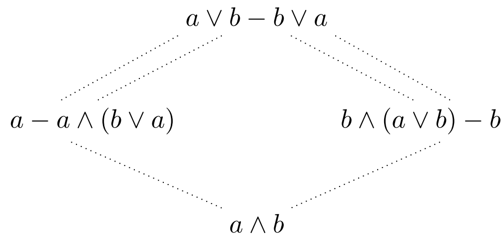
Theorem 4.3. *Let* $\mathbf{S}$ *be a right-*[*left-*]*handed skew lattice on two generators* a *and* b*. If* $a \wedge b = b \wedge a$ *but* $a \vee b \neq b \vee a$*, then* $\mathbf{S}$ *is isomorphic to* $\mathbf{NS}_7^{\mathcal{R},0}$ [*to* $\mathbf{NS}_7^{\mathcal{L},0}$]*. Dually, if* $a \vee b = b \vee a$ *but* $a \wedge b \neq b \wedge a$*, then* $\mathbf{S}$ *is isomorphic to* $\mathbf{NS}_7^{\mathcal{R},1}$ [*to* $\mathbf{NS}_7^{\mathcal{L},1}$].

Proof. Thanks to duality, we need only prove the right-handed case of the first assertion, doing so in the following steps:

1) **S** *is a skew diamond whose incomparable $\mathcal{D}$-classes separately contain a and b.* In particular, a and b are $\succeq$-incomparable. For suppose otherwise, say $a \succeq b$. Then $a \wedge b = b \wedge a = b \wedge a \wedge b \overset{(2.9)}{=} b$ so that $a \vee b = b \vee a = b$ by (2.2), contradicting our assumption.

2) *For all $x \in S$, $a \wedge b \leq x$.* Since $a \wedge b = b \wedge a$, we have $a \wedge b \leq a, b$. In general, $u \leq v, w$ implies $u \leq$ both $v \wedge w$ and $v \vee w$. (Indeed, (2.4) implies $u \wedge v \wedge w = (u \wedge v) \wedge w = u \wedge w \wedge u$ and similarly, $v \wedge w \wedge u = u$. Likewise, $u \vee v \vee w = (u \vee v) \vee w = v \vee w$ and similarly, $v \vee w \vee u = v \vee w$.) *Henceforth we denote $a \wedge b$ by* 0. We have shown that $0 \leq x$ for all $x \in \mathbf{S}$.

We obtain at least the following elements in the four $\mathcal{D}$-classes of **S**.



3) *All seven elements above are distinct, lie in the displayed $\mathcal{D}$-classes and have precisely the indicated partial order relationships.* Indeed a, b, 0, $a \vee b$ and $b \vee a$ are distinct by our assumptions on a and b. Since the images of $a \vee b$ and $b \vee a$, of a and $a \wedge (b \vee a)$ and of b and $b \wedge (a \vee b)$ agree in the lattice $\mathbf{S}/\mathcal{D}$, each of these three pairs are $\mathcal{D}$-related in **S**. By right-handedness, $a \vee b \overset{(2.10)}{=} a \vee b \vee a \geq a$ and $b \vee a \geq (b \vee a) \wedge a \wedge (b \vee a) \overset{(2.10)}{=} a \wedge (b \vee a)$. Likewise, $b \vee a \geq b$ and $a \vee b \geq b \wedge (a \vee b)$. Suppose next that $a = a \wedge (b \vee a)$. Then both $a, b \leq b \vee a$ so that a and b would have a commuting join by Lemma 2.6, thus forcing $a \vee b = b \vee a$. Thus $a \neq a \wedge (b \vee a)$ and likewise $b \neq b \wedge (a \vee b)$. Finally, there is no more to the partial order $\geq$ between the displayed elements in the top $\mathcal{D}$-class and any of the four displayed elements in the middle. Since $a \vee b \neq b \vee a$, $a \vee b \geq b$ and $b \vee a \geq a$ cannot occur. Suppose that $a \vee b \geq a \wedge (b \vee a)$. Then by definition of $\geq$ and (2.7), $a \wedge (b \vee a) = a \wedge (b \vee a) \wedge (a \vee b) = a \wedge (a \vee b \vee a) = a$. Thus $a \vee b \geq a \wedge (b \vee a)$ cannot happen and neither can $b \vee a \geq b \wedge (a \vee b)$.

4) *These seven elements are closed under* $\wedge$. Closure clearly occurs when one of the elements is 0 or both elements lie in a common ($\mathcal{R} =$) $\mathcal{D}$-class where $x \wedge y = y$ holds. When they lie in distinct $\mathcal{D}$-classes in the middle, $x \wedge y = 0$ by Lemma 2.6. Suppose one element is in $\{a \vee b, b \vee a\}$ and the other in, say, $\{a, a \wedge (b \vee a)\}$. From $\geq$, we have

$$(a \vee b) \wedge a = a = a \wedge (a \vee b) \quad \text{and} \quad (b \vee a) \wedge [a \wedge (b \vee a)] = a \wedge (b \vee a) = [a \wedge (b \vee a)] \wedge (b \vee a)\,.$$

Elements a and $b \vee a$ meet to either $a \wedge (b \vee a)$ or to $(b \vee a) \wedge a \overset{(1.1)}{=} a$. Next since $a \vee b \geq a$,

$$(a \vee b) \wedge [a \wedge (b \vee a)] = [(a \vee b) \wedge a] \wedge (b \vee a) = a \wedge (b \vee a)\,,$$

and

$$[a \wedge (b \vee a)] \wedge (a \vee b) = a \wedge [(b \vee a) \wedge (a \vee b)] = a \wedge (a \vee b) \overset{(1.1)}{=} a$$

because $x \wedge y = y$ holds on $\{a \vee b, b \vee a\}$. Likewise closure under $\wedge$ occurs between $\{a \vee b, b \vee a\}$ and $\{b, b \wedge (a \vee b)\}$.

5) *They are also closed under* $\vee$. Again this occurs when one of the elements is 0 or both elements lie in a common ($\mathcal{R} =$) $\mathcal{D}$-class where $x \vee y = x$ holds. Consider the case of comparable, nontrivial $\mathcal{D}$-classes with one element in $\{a \vee b, b \vee a\}$ and the other in, say, $\{a, a \wedge (b \vee a)\}$. Again due to $\geq$,

$$(a \vee b) \vee a = a \vee b = a \vee (a \vee b) \qquad \text{and} \qquad (b \vee a) \vee [a \wedge (b \vee a)] = b \vee a = [a \wedge (b \vee a)] \vee (b \vee a)\,.$$

Likewise a and $b \vee a$ join to either $a \vee b \vee a = a \vee b$ by (2.10) or to $b \vee a \vee a = b \vee a$. Next since $a \vee b \geq a$,

$$(a \vee b) \vee [a \wedge (b \vee a)] \overset{(2.10)}{=} (a \vee b \vee a) \vee [a \wedge (b \vee a)] \overset{(1.2)}{=} a \vee b \vee a \overset{(2.10)}{=} a \vee b\,.$$

Also

$$(4.1) \qquad [a \wedge (b \vee a)] \vee (a \vee b) = ([a \wedge (b \vee a)] \vee a) \vee b = [a \wedge (b \vee a)] \vee b = b \vee a$$

since $a \wedge (b \vee a) \,\mathcal{R}\, a$ and since $b \vee a \geq b,\ a \wedge (b \vee a)$ so that Lemma 2.6 applies. Thus closure under $\vee$ occurs on $\{a \vee b, b \vee a, a, a \wedge (b \vee a)\}$ and similarly on $\{a \vee b, b \vee a, b, b \wedge (a \vee b)\}$.

Finally consider joins from the middle classes, $\{a, a \wedge (b \vee a)\}$ and $\{b, b \wedge (a \vee b)\}$. Many outcomes are trivial. We directly get $a \vee b$ and $b \vee a$ from a and b. The partial order also gives $a \vee b$ from a and $b \wedge (a \vee b)$, and gives $b \vee a$ from $a \wedge (b \vee a)$ and b. Finally we join $b \wedge (a \vee b)$ and $a \wedge (b \vee a)$. Since $b \wedge (a \vee b) \vee (a \wedge (b \vee a))$ lies in the ($\mathcal{R} =$) $\mathcal{D}$-class of $b \vee a$, we have

$$[b \wedge (a \vee b)] \vee [a \wedge (b \vee a)] = [b \wedge (a \vee b)] \vee [a \wedge (b \vee a)] \vee (b \vee a) \overset{(1.2)}{=} [b \wedge (a \vee b)] \vee (b \vee a)$$

which reduces to $a \vee b$ by the a-b dual of (4.1) above. Similarly, $[a \wedge (b \vee a)] \vee [b \wedge (a \vee b)]$ reduces to $b \vee a$.
6) Thus given $a \wedge b = b \wedge a$ but $a \vee b \neq b \vee a$ in *any* right-handed skew lattice, the elements a and b generate a 7-element subalgebra $\mathbf{T}_{\{a,b\}}$ whose elements have *exactly* the description in terms of a and b given in part (3) above, and whose binary outcomes are *exactly* as described in parts (4) and (5) above. *Hence all such (sub)algebras are isomorphic.* In particular they are isomorphic to $\mathbf{NS}_7^{\mathcal{R},0}$. An isomorphism of $\mathbf{NS}_7^{\mathcal{R},0}$ with $\mathbf{T}_{\{a,b\}}$ is given by:

$$j_1 \to a \vee b \qquad j_2 \to b \vee a$$
$$a_1 \to a \qquad a_2 \to a \wedge (b \vee a) \qquad b_1 \to b \wedge (a \vee b) \qquad b_2 \to b$$
$$0 \to a \wedge b$$

(In both algebras $x \wedge y$ is the unique element m in the meet $\mathcal{D}$-class such that $y \geq m$, while $x \vee y = y$ if $x = 0$ or else it is the unique element u in the join $\mathcal{D}$-class such that $u \geq x$.) □

Theorem 4.4. *Let* $\mathbf{S}$ *be a skew lattice.*

i) $\mathbf{S}$ *is upper symmetric if and only if neither* $\mathbf{NS}_7^{\mathcal{R},0}$ *nor* $\mathbf{NS}_7^{\mathcal{L},0}$ *occurs as a subalgebra.*
ii) $\mathbf{S}$ *is lower symmetric if and only if neither* $\mathbf{NS}_7^{\mathcal{R},1}$ *nor* $\mathbf{NS}_7^{\mathcal{L},1}$ *occurs as a subalgebra.*
iii) $\mathbf{S}$ *is symmetric if and only if none of these four skew lattices occur as subalgebras.*

Proof. Since neither $\mathbf{NS}_7^{\mathcal{R},0}$ nor $\mathbf{NS}_7^{\mathcal{L},0}$ is upper symmetric, any skew lattice containing a copy of one of them cannot be upper symmetric. Conversely, if $\mathbf{S}$ is not upper symmetric, elements $a, b \in \mathbf{S}$ exist such that $a \wedge b = b \wedge a$ but $a \vee b \neq b \vee a$. Let $\mathbf{T}$ be the subalgebra of $\mathbf{S}$ generated by a and b. $\mathbf{T}$ is also a subalgebra of the product $\mathbf{T}/\mathcal{L} \times \mathbf{T}/\mathcal{R}$ with both factor algebras generated also from pairs of generators, with each pair being the images of a and b. Moreover, $x \wedge y = y \wedge x$ but $x \vee y \neq y \vee x$ must be passed on to at least one of these pairs of generators, else it would not be the case for a and b. By Theorem 4.3, either $\mathbf{T}/\mathcal{L}$ or $\mathbf{T}/\mathcal{R}$ is a copy of $\mathbf{NS}_7^{\mathcal{R},0}$ or $\mathbf{NS}_7^{\mathcal{L},0}$ respectively. By Lemma 2.8, $\mathbf{T}$ itself and hence also $\mathbf{S}$ must have a copy of one of these algebras. Thus (i) follows by double contraposition. Part (ii) follows by duality and (iii) follows from (i) and (ii). □

Corollary 4.5. *Cancellative skew lattices are symmetric.*

Proof. Such a skew lattice cannot contain a copy of $\mathbf{NS}_7^{\mathcal{L},0}$, $\mathbf{NS}_7^{\mathcal{L},1}$, $\mathbf{NS}_7^{\mathcal{R},0}$ or $\mathbf{NS}_7^{\mathcal{R},1}$. □

Upper and lower symmetry can be bisected in turn in left-right fashion to give a four-fold partition of symmetry that will prove useful in the next section.

Theorem 4.6. *Given a skew lattice* $\mathbf{S}$*:*

i) $\mathbf{S}/\mathcal{L}$ *being upper symmetric is equivalent to either of the following:*
  a) $\mathbf{S}$ *contain no copy of* $\mathbf{NS}_7^{\mathcal{R},0}$.
  b) $\mathbf{S}$ *satisfies* $x \vee y \vee x = (y \wedge x) \vee y \vee x$.
ii) $\mathbf{S}/\mathcal{R}$ *being upper symmetric is equivalent to either of the following:*
  a) $\mathbf{S}$ *contain no copy of* $\mathbf{NS}_7^{\mathcal{L},0}$.
  b) $\mathbf{S}$ *satisfies* $x \vee y \vee x = x \vee y \vee (x \wedge y)$.
iii) $\mathbf{S}/\mathcal{R}$ *being lower symmetric is equivalent to either of the following:*
  a) $\mathbf{S}$ *contain no copy of* $\mathbf{NS}_7^{\mathcal{L},1}$.
  b) $\mathbf{S}$ *satisfies* $x \wedge y \wedge x = (y \vee x) \wedge y \wedge x$.

iv) $\mathbf{S}/\mathcal{L}$ *being lower symmetric is equivalent to either of the following:*
 a) $\mathbf{S}$ *contain no copy of* $\mathbf{NS}_7^{\mathcal{R},1}$.
 b) $\mathbf{S}$ *satisfies* $x \wedge y \wedge x = x \wedge y \wedge (x \vee y)$.

The above four conditions hence determine four varieties of skew lattices.

Proof. Being right-handed, $\mathbf{S}/\mathcal{L}$ is upper symmetric if and only if $\mathbf{S}/\mathcal{L}$ itself contains no copy of $\mathbf{NS}_7^{\mathcal{R},0}$, or put otherwise, no skew diamond in $\mathbf{S}/\mathcal{L}$ contains a copy of $\mathbf{NS}_7^{\mathcal{R},0}$. Thus the equivalence of (i)(a) with the upper symmetry of $\mathbf{S}/\mathcal{L}$ follows from Lemma 2.8. As for (i)(b), by (US), $\mathbf{S}/\mathcal{L}$ is upper symmetric if and only if $(y \wedge x) \vee y \vee x \,\mathcal{L}\, x \vee y \vee (x \wedge y)$ holds. Using the definition of $\mathcal{L}$, the latter can be expressed by

$$[x \vee y \vee (x \wedge y)] \vee [(y \wedge x) \vee y \vee x] = (y \wedge x) \vee y \vee x \tag{4.2}$$

and

$$[(y \wedge x) \vee y \vee x] \vee [x \vee y \vee (x \wedge y)] = x \vee y \vee (x \wedge y)\,. \tag{4.3}$$

Since both $x \vee y$ and $y \vee x \succeq x \wedge y$ and $y \wedge x$, (4.2) reduces to (i)(b) by Lemma 2.10. Upon replacing $(y \wedge x) \vee y \vee x$ by $x \vee y \vee x$ in (4.3), we obtain the redundant triviality $x \vee y \vee x \vee x \vee y \vee (x \wedge y) = x \vee y \vee (x \wedge y)$ which we drop. The other three cases are similar. The final assertion is now clear. □

Corollary 4.7. *A skew lattice* $\mathbf{S}$ *is upper (lower, fully) symmetric if and only if* $\mathbf{S}/\mathcal{L}$ *and* $\mathbf{S}/\mathcal{R}$ *are thus.*

Proof. This follows from Theorems 4.2 and 4.6. □

5. Characterizing cancellative skew lattices

We begin by stating the main result of this paper, which in spirit extends Theorem 3.6.

Theorem 5.1. (*The Main Theorem*)

1) *The following are equivalent for a skew lattice* $\mathbf{S}$.
 i) $\mathbf{S}$ *is left cancellative.*
 ii) *None of* $\mathbf{M}_3$, $\mathbf{N}_5$, $\mathbf{NC}_5^{\mathcal{L}}$, $\mathbf{NC}_5^{\mathcal{R}}$, $\mathbf{NS}_7^{\mathcal{R},0}$ *nor* $\mathbf{NS}_7^{\mathcal{L},1}$ *occur as subalgebras of* $\mathbf{S}$.
 iii) $\mathbf{S}$ *is simply cancellative,* $\mathbf{S}/\mathcal{L}$ *is upper symmetric, and* $\mathbf{S}/\mathcal{R}$ *is lower symmetric.*
2) *The following are equivalent for a skew lattice* $\mathbf{S}$.
 i) $\mathbf{S}$ *is right cancellative.*
 ii) *None of* $\mathbf{M}_3$, $\mathbf{N}_5$, $\mathbf{NC}_5^{\mathcal{L}}$, $\mathbf{NC}_5^{\mathcal{R}}$, $\mathbf{NS}_7^{\mathcal{R},1}$ *nor* $\mathbf{NS}_7^{\mathcal{L},0}$ *occur as subalgebras of* $\mathbf{S}$.
 iii) $\mathbf{S}$ *is simply cancellative,* $\mathbf{S}/\mathcal{R}$ *is upper symmetric, and* $\mathbf{S}/\mathcal{L}$ *is lower symmetric.*
3) *The following are equivalent for a skew lattice* $\mathbf{S}$.
 i) $\mathbf{S}$ *is cancellative.*
 ii) *None of the above* 5*- or* 7*-element algebras occur as subalgebras of* $\mathbf{S}$.
 iii) S *is simply cancellative and symmetric.*
4) *Left [right, fully] cancellative skew lattices form a variety.*

Proof. We begin with (1). If $\mathbf{S}$ is left cancellative, then $\mathbf{S}$ cannot contain copies of any of the algebras listed in (ii) since none of them are left cancellative. Thus (i) implies (ii). The equivalence of (ii) and (iii) follows from Theorems 3.6 and 4.6. Now assume $\mathbf{S}$ satisfies (iii), and suppose $a, b, c \in S$ satisfy $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$. Since $\mathbf{S}/\mathcal{L}$ is upper symmetric, Theorem 4.6 gives

$$b \vee a \vee b = (a \wedge b) \vee a \vee b = (a \wedge c) \vee a \vee c = c \vee a \vee c\,.$$

Since $\mathbf{S}/\mathcal{R}$ is lower symmetric, Theorem 4.6 gives

$$b \wedge a \wedge b = (a \vee b) \wedge a \wedge b = (a \vee c) \wedge a \wedge c = c \wedge a \wedge c\,.$$

By simple cancellativity, $b = c$. Therefore $\mathbf{S}$ is left cancellative, that is, (i) holds. The proof of (2) is similar. Indeed, (2) follows from (1) by horizontal duality (new $x \vee y =$ old $y \vee x$; new $x \wedge y =$ old $y \wedge x$). As for (3), the equivalence of (i) with (ii) follows from the corresponding equivalences of (1) and (2). The equivalence of (ii) with (iii) follows from the corresponding equivalences of (1) and (2) and Corollary 4.7. Finally, (4) now follows from parts (iii) and (1), (2) and (3) plus Theorems 3.7, 4.2 and 4.6. □

This theorem has an immediate consequence. Recall that a skew lattice $\mathbf{S}$ is ***totally quasi-ordered*** if for all $x, y \in \mathbf{S}$, either $a \succeq b$ or $b \succeq a$.

Corollary 5.2. *Totally quasi-ordered skew lattices are cancellative.*

Equational bases. To reprise what has been done, equational bases for some of the varieties of cancellative skew lattice encountered in this paper are as follows:
Simply cancellative skew lattices:

$$[x \wedge (y \vee z)] \wedge [(x \wedge y) \vee (x \wedge z)] \wedge [x \wedge (y \vee z)] = x \wedge (y \vee z) \tag{5.1}$$

$$f \vee (e \wedge y \wedge z \wedge e) \vee f = f \vee (e \wedge z \wedge y \wedge e) \vee f \tag{5.2}$$

where $e = x \vee (y \wedge z) \vee x$ and $f = x \wedge (y \wedge z) \wedge x$.
Left [right] cancellative skew lattices: (5.1) and (5.2) plus respectively from Theorem 4.6,

(LC) $\quad x \vee y \vee x = (y \wedge x) \vee y \vee x \quad$ and dually $\quad x \wedge y \wedge x = (y \vee x) \wedge y \wedge x\,,$

(RC) $\quad x \wedge y \wedge x = x \wedge y \wedge (x \vee y) \quad$ and dually $\quad x \vee y \vee x = x \vee y \vee (x \wedge y)\,.$

Cancellative skew lattices: all of the above, with possibly (LC) and (RC) replaced by the identities for symmetry, (US) and (LS), given in Theorem 4.2.

Given non-commutativity, other forms of cancellation can be formulated. Consider, for instance, the following variant forms of cancellation.

(5.3) $\quad x \wedge y = x \wedge z \quad$ and $\quad y \vee x = z \vee x \quad$ imply $\quad y = z$

(5.4) $\quad x \wedge z = y \wedge z \quad$ and $\quad z \vee x = z \vee y \quad$ imply $\quad x = y$

(5.5) $\quad x \wedge y = z \wedge x \quad$ and $\quad y \vee x = x \vee z \quad$ imply $\quad y = z$

(5.6) $\quad x \wedge y = z \wedge x \quad$ and $\quad x \vee y = z \vee x \quad$ imply $\quad y = z\,.$

It turns out that none of these forms of cancellation introduce anything substantially new; they do, however, provide succinct criteria for cases of cancellation already encountered.

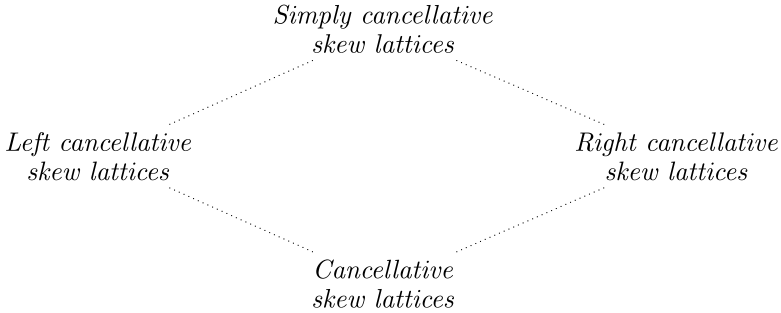
Theorem 5.3. *Given a skew lattice* $\mathbf{S}$*, the following hold:*
i) $\mathbf{S}$ *is right-handed and simply cancellative if and only if it satisfies* (5.3).
ii) $\mathbf{S}$ *is left-handed and simply cancellative if and only if it satisfies* (5.4).
iii) $\mathbf{S}$ *is a distributive lattice if and only if it satisfies* (5.5).
iv) $\mathbf{S}$ *is cancellative if and only if it satisfies* (5.6).

Proof. To begin, if $\mathbf{S}$ satisfies any of the four implications introduced above, it is at least simply cancellative since all four special 5-element (sub)algebras are ruled out. When restricted to $\mathcal{D}$-classes, (5.3) forces them to be right-handed and thus $\mathbf{S}$ to be the same. Conversely, if $\mathbf{S}$ is both right-handed and simply cancellative, then the implication $y \vee x \vee y = z \vee x \vee z$ and $y \wedge x \wedge y = z \wedge x \wedge z$ imply $y = z$ reduces to: $x \vee y = x \vee z$ and $y \wedge x = z \wedge x$ imply $y = z$. This establishes (i), and (ii) is similar. For (5.5), observe that the two equalities of the condition are equivalent on any $\mathcal{D}$-class. On a $\mathcal{D}$-class, $x \wedge y = z \wedge x$ holds precisely when $y \,\mathcal{L}\, x$ and $z \,\mathcal{R}\, x$, which would force $y = z$, making that class trivial. Thus $\mathbf{S}$ is a cancellative, and hence distributive, lattice. The converse is clear and so (iii) holds. Finally, (5.6) clearly rules out $\mathbf{M}_3$, $\mathbf{N}_5$, $\mathbf{NC}_5^{\mathcal{L}}$ and $\mathbf{NC}_5^{\mathcal{R}}$. In $\mathbf{NS}_7^{\mathcal{R},1}$ one has $a_1 \wedge b_1 = m_1 = b_2 \wedge a_1$ and $a_1 \vee b_1 = 1 = b_2 \vee a_2$, but $b_1 \neq b_2$. Thus $\mathbf{NS}_7^{\mathcal{R},1}$ is also ruled out. Similarly, so are $\mathbf{NS}_7^{\mathcal{R},0}$, $\mathbf{NS}_7^{\mathcal{L},1}$ and $\mathbf{NS}_7^{\mathcal{L},0}$. Hence any skew lattice satisfying (5.6) is fully cancellative. Conversely, given full cancellation, $\mathbf{S}$ is symmetric. Let $a \wedge b = c \wedge a$ and $a \vee b = c \vee a$ in $\mathbf{S}$ with both sides respectively equaling $a \wedge b \wedge a$ (and $a \wedge c \wedge a$) and $a \vee b \vee a$ (and $a \vee c \vee a$). We get

$$a \vee b \vee a = a \vee c \vee a = (c \wedge a) \vee c \vee a = (a \wedge b) \vee a \vee b = b \vee a \vee b\,,$$

where we have used Theorem 4.6(i(b)) in the second and fourth equalities. Thus $a \vee b \vee a = b \vee a \vee b$, from which $a \vee b = b \vee a$ follows. Similarly $a \wedge b = b \wedge a$ and we get both $b \wedge a = c \wedge a$ and $b \vee a = c \vee a$. Since $\mathbf{S}$ is cancellative, $b = c$ and (5.6) is demonstrated. □

Consider the following lattice of varieties:

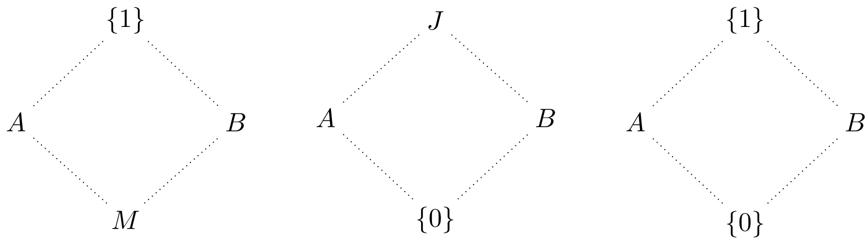


While the bottom variety is the intersection of the middle varieties, we do not know if the top variety is generated from the middle varieties and thus is their join in the lattice of all skew lattice varieties. The bottom variety is, of course, the intersection of each variety with the variety of symmetric skew lattices. Using the model builder Mace4 [17], we found four distinct minimal cases of order 12 exist that are simply cancellative, but are neither left nor right cancellative. These turn out to be the fibered product $\mathbf{NS}_7^{\mathcal{L},0} \times_{\mathbf{2}\times\mathbf{2}} \mathbf{NS}_7^{\mathcal{L},1}$ of $\mathbf{NS}_7^{\mathcal{L},0}$ and $\mathbf{NS}_7^{\mathcal{L},1}$ over their maximal lattice image $\mathbf{2} \times \mathbf{2}$; the splice of $\mathbf{NS}_7^{\mathcal{L},0}$ with $\mathbf{NS}_7^{\mathcal{L},1}$ obtained by identifying the join class of $\mathbf{NS}_7^{\mathcal{L},0}$ with the meet class of $\mathbf{NS}_7^{\mathcal{L},1}$; the two right-handed duals.

6. Positive characterizations

Much of our discussion has occurred in the context of skew diamonds (often viewed as potential subalgebras of skew lattices) since a *quasidistributive skew lattice is (simply, left-, right- or fully) cancellative precisely when all of its skew diamonds are such.* But the emphasis has thus far been on what cannot occur in a skew diamond if it is to have a type of cancellation. But what, positively, *must* occur?

A skew diamond $\mathbf{S}$ with $\mathcal{D}$-classes $J > A, B, > M$ is ***pointed above*** if $J = \{j\}$. It is ***pointed below*** if $M = \{m\}$. $\mathbf{S}$ is ***doubly pointed*** if J and M are singletons. The unique element in the involved class(es) is often denoted by 1 or 0, respectively.



Clearly $\mathbf{NS}_7^{\mathcal{L},0}$ and $\mathbf{NS}_7^{\mathcal{R},0}$ are pointed below, $\mathbf{NS}_7^{\mathcal{L},1}$ and $\mathbf{NS}_7^{\mathcal{R},1}$ are pointed above and $\mathbf{NC}_5^{\mathcal{L}}$ and $\mathbf{NC}_5^{\mathcal{R}}$ are doubly pointed. Lemma 3.4 gives us

Theorem 6.1. *A quasidistributive skew lattice is simply cancellative if and only if all of its doubly pointed skew diamond subalgebras are sublattices.*

Now we consider full cancellation. Recall that a skew lattice $\mathbf{S}$ is ***primitive*** if it consists of just two distinct $\mathcal{D}$-classes $C > D$. Given primitive skew lattices $\mathbf{S}_1$ and $\mathbf{S}_2$, the product $\mathbf{S}_1 \times \mathbf{S}_2$ is a skew diamond that must be cancellative since both factors are cancellative by Corollary 5.2. While this is an easy way to obtain cancellative skew diamonds, not all cancellative skew diamonds factor this way. We do have, however:

Theorem 6.2. *A quasidistributive skew lattice is cancellative if and only if all of its pointed skew diamond subalgebras factor as products of primitive skew lattices.*

Proof. Given a quasidistributive skew lattice S, the condition on its pointed subalgebras excludes copies of $\mathbf{NC}_5^{\mathcal{L}}$, $\mathbf{NC}_5^{\mathcal{R}}$, $\mathbf{NS}_7^{\mathcal{L},0}$, $\mathbf{NS}_7^{\mathcal{R},0}$, $\mathbf{NS}_7^{\mathcal{L},1}$ and $\mathbf{NS}_7^{\mathcal{R},1}$ from being subalgebras since in each of these six cases the order of either the join or meet classes is inconsistent with factorization. Thus $\mathbf{S}$ is cancellative by Theorem 5.1. Conversely, given a cancellative skew lattice, we show that its pointed subalgebras factor as stated. Our task quickly reduces to the case of a cancellative pointed skew diamond. So let **S** consisting of $\mathcal{D}$-classes $J > A, B > M$ be cancellative and pointed (say) below, with $M = \{0\}$. A pair of primitive subalgebras are given by $A^0 = A \cup \{0\}$ and $B^0 = B \cup \{0\}$. We define $\sigma : A^0 \times B^0 \to S$ by

$$\sigma[(x, y)] = x \vee y \qquad \text{for} \quad x \in A^0 \quad \text{and} \quad y \in B^0 .$$

σ is surjective by Lemma 2.6(iii). It is clearly bijective from $\{0\} \times \{0\}$ to $\{0\}$, from $A \times \{0\}$ to A and from $\{0\} \times B$ to B. Thus we need only show it is bijective from $A \times B$ to J. Since **S** is symmetric, $a \vee b = b \vee a$ for every $(a, b) \in A \times B$, since $a \wedge b = 0 = b \wedge a$. Hence bijectivity at the top must follow from Lemma 3.4(iii).

Finally for all $x_1, x_2 \in A^0$ and $y_1, y_2 \in B^0$,

$$\begin{aligned}\sigma[(x_1, y_1) \vee (x_2, y_2)] &= \sigma[(x_1 \vee x_2, y_1 \vee y_2)] = x_1 \vee x_2 \vee y_1 \vee y_2 \\ &= x_1 \vee y_1 \vee x_2 \vee y_2 \quad = \sigma[(x_1, y_1)] \vee \sigma[(x_2, y_2)] ,\end{aligned}$$

since elements from A^0 commute with elements from B^0. Expanding $\sigma[(x_1, y_1) \wedge (x_2, y_2)]$ and $\sigma[(x_1, y_1)] \wedge \sigma[(x_2, y_2)]$, we get respectively:

$$(x_1 \wedge x_2) \vee (y_1 \wedge y_2) \qquad \text{and} \qquad (x_1 \vee y_1) \wedge (x_2 \vee y_2) .$$

Case 1) At least one of the x_i and one of the y_j is 0. Here both expressions reduce to 0.
Case 2) Just one of the x_i or y_j is 0, say one of the $y_j = 0$. We get $x_1 \wedge x_2$ on the left and $(x_1 \vee y_1) \wedge x_2$ or $x_1 \wedge (x_2 \vee y_2)$ on the right. But

$$(x_1 \vee y_1) \wedge x_2 = (x_1 \vee y_1) \wedge x_2 \wedge (x_1 \vee y_1)] \wedge x_2 = x_1 \wedge x_2 .$$

Indeed both x_1, $[(x_1 \vee y_1) \wedge x_2 \wedge (x_1 \vee y_1)] < x_1 \vee y_1$. Since both also lie in A, they are equal by Lemma 3.4(ii). Likewise $x_1 \wedge (x_2 \vee y_2) = x_1 \wedge x_2$. The case where some $x_i = 0$ and the outcome is $y_1 \wedge y_2$ is similar.
Case 3) None of the x_i or y_j is 0. Since $u \vee v = v \wedge u$ holds in every $\mathcal{D}$-class, we have:

$$(x_1 \wedge x_2) \vee (y_1 \wedge y_2) = x_2 \vee x_1 \vee y_2 \vee y_1 \quad \text{and} \quad (x_1 \vee y_1) \wedge (x_2 \vee y_2) = x_2 \vee y_2 \vee x_1 \vee y_1 .$$

By $A - B$ commutation in the middle, $x_2 \vee x_1 \vee y_2 \vee y_1 = x_2 \vee y_2 \vee x_1 \vee y_1$. □

In general, *every skew diamond S is a union of maximal pointed subalgebras in two ways*: $S = \bigcup_{m \in M}(m \vee S \vee m) = \bigcup_{j \in J}(j \wedge S \wedge j)$ where

$$m \vee S \vee m = \{m \vee x \vee m | x \in S\} = \{x \in S | x \geq m\} \qquad \text{and} \tag{6.1}$$

$$j \wedge S \wedge j = \{j \wedge x \wedge j | x \in S\} = \{x \in S | j \geq x\} \tag{6.2}$$

are pointed respectively below and above. One can show that *if **S** is cancellative, then all the $m \vee S \vee m$ are mutually isomorphic and all the $j \wedge S \wedge j$ are mutually isomorphic.* Thus in this case, instances of each $m \vee S \vee m$ or $j \wedge S \wedge j$ serve as a parameter for the entire skew diamond.

We next consider a related criterion for skew diamonds to be cancellative. To begin, given comparable $\mathcal{D}$-classes $C > D$ in any skew lattice, with $c \in C$ and $d \in D$:

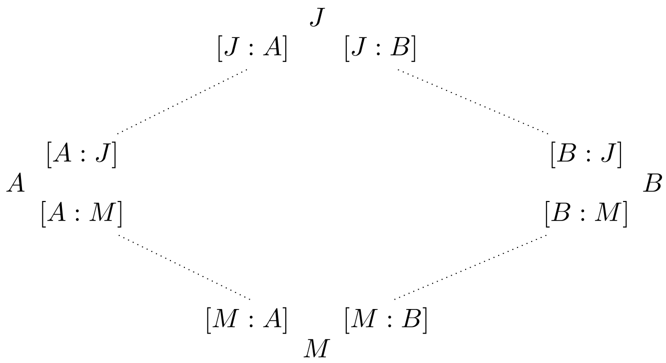
$$d \vee C \vee d = \{d \vee c \vee d \mid c \in C\} = \{c \in C \mid c \geq d\}$$

is the ***image*** under $\geq$ of d in C, and

$$c \wedge D \wedge c = \{c \wedge d \wedge c \mid d \in D\} = \{d \in D \mid c \geq d\}$$

is the ***image*** under $\geq$ of c in D. One easily shows that $|d \vee C \vee d| = |d' \vee C \vee d'|$ for all $d, d' \in D$ and likewise $|c \wedge D \wedge c| = |c' \wedge D \wedge c'|$ for all $c, c' \in C$. For instance, defining $f : d \vee C \vee d \to d' \vee C \vee d'$ and $g : d' \vee C \vee d' \to d \vee C \vee d$ by $f(x) = d' \vee x \vee d'$ and $g(y) = d \vee y \vee d$ gives an inverse pair of functions. This is due to regularity (2.1) plus the fact that $x \vee y \vee x = x$ in every $\mathcal{D}$-class. The ***index*** $[C : D]$ of D in C is the common size $|d \vee C \vee d|$ for all d; dually, the index $[D : C]$ of C in D is the common size $|c \wedge D \wedge c|$ for all c. No relationship need exist between $[C : D]$ and $[D : C]$. In particular, they need not be equal. But together they measure the behavior of $\geq$ on the primitive subalgebra $C \cup D$.

Our interest here is with opposite sides of a skew diamond, *i.e.*, $A > M$ and $J > B$ or $B > M$ and $J > A$ The following "Parallelogram Laws" of Cvetko-Vah [6] state that opposite sides of a cancellative skew lattice always share the same indices.



Theorem 6.3. *Let* $\mathbf{S}$ *be a cancellative skew diamond* $\{J > A, B > M\}$. *Then:*

i) $[A : M] = [J : B]$ *and* $[M : A] = [B : J]$.

ii) $[B : M] = [J : A]$ *and* $[M : B] = [A : J]$.

In detail, given $j \in J$, $a \in A$, $b \in B$ *and* $m \in M$ *such that* $j > a, b > m$, *isomorphisms*

$$\alpha : \{x \in A \mid x > m\} \cong \{y \in J \mid y > b\} \text{ and } \beta : \{u \in B \mid u < j\} \cong \{v \in M \mid v < a\}$$

are defined by $\alpha(x) = x \vee b \vee x$ *and* $\beta(u) = u \wedge a \wedge u$. *Isomorphisms between other pairs of index sets are defined similarly. Conversely, if* $\mathbf{S}$ *is a skew diamond for which all such maps are isomorphisms, then* $\mathbf{S}$ *is cancellative.*

Proof. We show α to be an isomorphism. The case for β is dual. The immediate context of α is the pointed subalgebra $S' = m \vee S \vee m$ with class structure $J' > A', B' > M'$ where $J' = m \vee J \vee m, \ldots,$ $M' = m \vee M \vee m = \{m\}$. Thus, at the outset we may assume that $\mathbf{S}$ is cancellative, pointed and of the form $A^0 \times B^0$ of the previous proof. Thus upon resetting m as $(0, 0)$, $\{x \in A | x > m\}$ as $A \times \{0\}$, b as $(0, b)$ and $\{y \in J | y > b\}$ as $A \times \{b\}$, α is just the evident homomorphism $(a, 0) \to (a, b)$ from $A \times \{0\}$ to $A \times \{b\}$. The situation for β is similar. Conversely, if all such maps are bijections in a skew diamond $\mathbf{S}$, then $\mathbf{NC}_5^{\mathcal{R}}$, $\mathbf{NC}_5^{\mathcal{L}}$ and the four 7-element algebras cannot arise in $\mathbf{S}$, making it cancellative. □

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