MOUFANG LOOPS WITH COMMUTING INNER MAPPINGS

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Abstract. We investigate the relation between the structure of a Moufang loop and its inner mapping group. Moufang loops of odd order with commuting inner mappings have nilpotency class at most two. 6-divisible Moufang loops with commuting inner mappings have nilpotency class at most two. There is a Moufang loop of order 2^{14} with commuting inner mappings and of nilpotency class three.

1. Introduction

Problems in loop theory are often attacked by tools of group theory, and it is therefore a question of considerable interest in nonassociative algebra to understand the relationship between the nilpotency class of a loop Q and the nilpotency class of its inner mapping group $\operatorname{Inn} Q$.

When Q is a group, the situation is transparent thanks to the textbook isomorphism $Q/Z(Q) \cong \operatorname{Inn} Q$. Hence for a nontrivial group Q we have

$$c\ell(Q) = c\ell(\operatorname{Inn} Q) + 1. \tag{1.1}$$

There are other varieties of loops, not necessarily associative, for which the equality (1.1) is satisfied, for instance in the case of commutative Moufang loops [2, Theorem VIII.11.5].

Does (1.1) hold for all loops? No, (1.1) fails badly already for loops of nilpotency class 3, as there is a loop Q of order 18 with $c\ell(Q) = 3$ for which $\operatorname{Inn} Q$ is not nilpotent. Such a loop was for the first time constructed by Vesanen in 1995, and it can be found in [9].

But there are some good news. Bruck observed in [1] that $\operatorname{Inn} Q$ is abelian if $c\ell(Q) = 2$. Niemenmaa and Kepka showed in [13] that a finite loop Q with $\operatorname{Inn} Q$ abelian is nilpotent, without giving a bound on $c\ell(Q)$. More generally, a recent result of Niemenmaa [14] (based on [11]) states that if Q is a finite loop with $\operatorname{Inn} Q$ nilpotent then Q is nilpotent.

The converse of Bruck's observation does not hold. Using the technique of connected transversals, Csörgő [3] constructed a loop of order 128 and nilpotency class 3 with abelian inner mapping group. We will therefore call loops Q with abelian $\operatorname{Inn} Q$ satisfying $c\ell(Q) > 2$ *loops of Csörgő type*.

How prevalent are loops of Csörgő type in loop theory? As we have already mentioned, there are no loops of Csörgő type in the variety of commutative Moufang loops. By [4], there are no left conjugacy closed loops of Csörgő type. On the other

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hand, numerous loops of Csörgő type have been constructed in [6] by a combinatorial approach based on symmetric trilinear forms. There is a Buchsteiner loop of Csörgő type of order 128 [5].

But until now, no loop of Csörgő type has been found in a mainstream variety of loops. In this paper we construct a Moufang loop of Csörgő type of order 2^{14} and nilpotency class three. On the other hand, we show that there is no uniquely 6-divisible Moufang loop of Csörgő type and no odd order Moufang loop of Csörgő type.

For a list of open problems concerning loops of Csörgő type, see [6].

Our method is mostly loop theoretical, similar to that of Bruck [2]—we apply heavy commutator and associator calculus. It is known that the category of Moufang loops is equivalent to the category of groups with triality, however, this functorial equivalence is not fully understood, even when restricted to solvable or nilpotent loops. A translation of the results obtained here into the language of groups with triality could elucidate the connection between (nilpotent) Moufang loops and (nilpotent) groups with triality.

2. Prerequisites

2.1. **Notation.** For an element x of a groupoid Q, denote by $L_x : Q \to Q$, $yL_x = xy$ the *left translation by x in Q*, and by $R_x : Q \to Q$, $yR_x = yx$ the *right translation by x in Q*.

Quasigroup is a groupoid in which all translations are bijections. For x, y in a quasigroup Q, we denote by $x\backslash y$ the unique solution $a \in Q$ to the equation $xa = y$, and by y/x the unique solution $b \in Q$ to the equation $bx = y$.

Loop is a quasigroup Q with *neutral element* $1 \in Q$, that is, $x1 = x = 1x$ holds for every $x \in Q$. From now on, Q always denotes a loop.

The *inner mapping group* $\mathrm{Inn}\, Q$ of Q is the permutation group generated by all *middle inner mappings* (*conjugations*) $T(x) = R_x L_x^{-1}$, *left inner mappings* $L(x, y) = L_x L_y L_{yx}^{-1}$, and *right inner mappings* $R(x, y) = R_x R_y R_{xy}^{-1}$, where x, $y \in Q$.

A subloop H of Q is *normal*, $H \trianglelefteq Q$, if $H\varphi = H$ for every $\varphi \in \mathrm{Inn}\, Q$. If H is normal in Q, Q/H is the factor loop defined in the usual way.

The *center* $Z(Q) = \{x \in Q;\ x\varphi = x$ for every $\varphi \in \mathrm{Inn}\, Q\}$ consists of all those elements of Q that commute and associate with all other elements of Q.

The *iterated centra* of Q are defined by $Z_0(Q) = 1$, $Z_{i+1}(Q)/Z_i(Q) = Z(Q/Z_i(Q))$. A loop Q is *nilpotent of class m*, $c\ell(Q) = m$, if m is the least integer satisfying $Z_m(Q) = Q$.

For x, $y \in Q$, let $[x, y] \in Q$ be the *commutator* of x and y, defined by $xy = (yx)[x, y]$. For x, y, $z \in Q$, let $[x, y, z] \in Q$ be the *associator* of x, y and z, defined by $(xy)z = (x(yz))[x, y, z]$. Note that the commutator and associator are well-defined modulo $Z(Q)$, that is, $[xc_1, yc_2] = [x, y]$, $[xc_1, yc_2, zc_3] = [x, y, z]$ for every x, y, $z \in Q$ and c_1, c_2, $c_3 \in Z(Q)$.

The *associator subloop* $A(Q)$ of Q is the least normal subloop of Q such that $Q/A(Q)$ is a group. Hence $A(Q)$ is the least normal subloop of Q containing all associators $[x, y, z]$. The *commutator-associator subloop* Q', also called the *derived subloop*, is the least normal subloop of Q such that Q/Q' is an abelian group. Hence Q' is the least normal subloop of Q containing all commutators $[x, y]$ and all associators $[x, y, z]$.

Note that we have $c\ell(Q) \le 2$ if and only if $Q/Z(Q)$ is an abelian group, i.e., $Q' \le Z(Q)$, i.e., $[x, y] \in Z(Q)$ and $[x, y, z] \in Z(Q)$ for every x, y, $z \in Q$.

The *nucleus* of a loop Q is the subloop $N(Q) = \{x \in Q;\ [x,y,z] = [y,x,z] = [y,z,x] = 1 \text{ for every } y,\ z \in Q\}$.

We denote by $\langle S\rangle$ the subloop of Q generated by $S \subseteq Q$, and write $\langle x\rangle$ instead of $\langle\{x\}\rangle$, etc. For a loop Q, let $rank(Q)$ be the least cardinal κ such that Q is generated by a subset of size κ.

Let $m > 1$ be an integer. A loop Q is *m-divisible* if for every $x \in Q$ there is $y \in Q$ such that $y^m = x$. A loop Q is *uniquely m-divisible* if for every $x \in Q$ there is a unique $y \in Q$ such that $y^m = x$, that is, if the power map $\varphi : Q \mapsto Q$, $z \mapsto z^m$ is a bijection of Q. In such a case we denote the m-th root $x\varphi^{-1}$ of x by $x^{1/m}$. When Q is finite, it is uniquely m-divisible if and only if it is m-divisible.

A loop is said to be *Moufang* if it satisfies the identity $x(y(xz)) = ((xy)x)z$. It is well-known that Moufang loops are *diassociative*, that is, any two elements generate an associative subloop. In particular, Moufang loops are *power-associative*, that is, every element generates an associative subloop. This also means that for every element x of a Moufang loop Q there is $x^{-1} \in Q$ such that $xx^{-1} = 1 = x^{-1}x$, and the inverses satisfy $(xy)^{-1} = y^{-1}x^{-1}$, $x^{-1}(xy) = y$, $(xy)y^{-1} = x$, $[x,y] = x^{-1}y^{-1}xy$, $[x,y]^{-1} = [y,x]$, and so on.

The famous Moufang theorem [12] states that three elements x, y, z of a Moufang loop generate an associative subloop if and only if $[x,y,z] = 1$, in which case $[y,x,z] = [x,z,y] = 1$, too. Thus, a Moufang loop Q satisfies $c\ell(Q) \le 2$ if and only if $[[x,y],z] = [[x,y,z],u] = [[x,y],z,u] = [[x,y,z],u,v] = 1$ for every x, y, z, u, $v \in Q$.

From now on we will employ the *dot-convention*, which uses $\cdot$ to indicate priority of multiplication. For instance, the product $xy \cdot z$ is to be read as $(xy)z$.

2.2. **Non-generators.** We say that $x \in Q$ is a *non-generator* of Q if $\langle S\rangle = Q$ whenever $\langle S \cup \{x\}\rangle = Q$.

It follows from [2, Theorems 2.1 and 2.2] that in a finite nilpotent loop Q the derived subloop Q' consists of non-generators. This allows us to ignore commutators and associators in any generating subset S of a finite nilpotent loop Q, as long as $\langle S\rangle = Q$.

To illustrate the technique, if Q is a finite nilpotent loop with $rank(Q) = 3$ and x, y, z, u, $v \in Q$ then $\langle x, y, [x,z], [y,u,v]\rangle$ is a proper subloop of Q. Of course, we are generally not allowed to remove commutators and associators from sets S that generate a proper subloop of Q. For instance, we cannot conclude that $\langle x, [x,y]\rangle$ has rank 1 in the above example.

2.3. **Inner mappings and pseudo-automorphisms of Moufang loops.** A permutation φ of a loop Q is a *pseudo-automorphism* if there exists an element $c \in Q$ such that $(x\varphi)(y\varphi \cdot c) = (xy)\varphi \cdot c$ for every x, $y \in Q$. The element c is then referred to as a *companion* of φ.

Note that if φ is a pseudo-automorphism with companion $c \in N(Q)$ then φ is an automorphism.

By [2, Lemma VII.2.2], the left inner mapping $L(x,y)$ has companion $[y,x]$, the conjugation $T(x)$ has companion x^3, and $R(x,y) = L(x^{-1}, y^{-1})$ for all elements x, y of a Moufang loop Q. Moreover, by [2, Lemma VII.5.4], we have

$$xL(z,y) = x[x,y,z]^{-1}$$

in a Moufang loop.

2.4. **Center automorphisms.** An automorphism φ of a loop Q is called a *center automorphism* if for every $x \in Q$ we have $xZ(Q) = (x\varphi)Z(Q)$.

Lemma 2.1. *Let φ be a center automorphism of a loop Q. Define $\psi : Q \to Q$ by $x\psi = x\backslash(x\varphi)$. Then ψ is a homomorphism from Q to $Z(Q)$.*

Proof. Since $xZ(Q) = (x\varphi)Z(Q)$, we have $x\backslash(x\varphi) \in Z(Q)$, and ψ is a mapping from Q to $Z(Q)$. Now, $xy \cdot (xy)\psi = (xy)\varphi = x\varphi \cdot y\varphi = x(x\psi) \cdot y(y\psi) = xy \cdot (x\psi \cdot y\psi)$, where we have used $x\psi$, $y\psi \in Z(Q)$ in the last equality. Thus $(xy)\psi = x\psi \cdot y\psi$. □

2.5. **The associated Bruck loop.** Let Q be a uniquely 2-divisible Moufang loop. Define $Q(1/2) = (Q, *)$ by

$$x * y = x^{1/2}yx^{1/2}.$$

Then, by [2, Theorem VII.5.2], $Q(1/2)$ is a power-associative loop with the same identity and powers as Q, and $x * (y * (x * z)) = (x * (y * x)) * z$ holds in $Q(1/2)$. We call $Q(1/2)$ the *Bruck loop associated with* Q.

Lemma 2.2. *Let Q be a uniquely 2-divisible Moufang loop. Then $Q(1/2)$ is commutative (hence a commutative Moufang loop) if and only if every subloop H of Q with $rank(H) \le 2$ satisfies $c\ell(H) \le 2$.*

Proof. Note that $rank(H) \le 2$ implies that H is a group. The following four identities are equivalent: $x * y = x^{1/2}yx^{1/2} = y^{1/2}xy^{1/2} = y * x$, $xy^2x = yx^2y$ (by unique 2-divisibility), $[x,y]yx = yx[x,y]$, and $[x,y]x = x[x,y]$. The first identity says that $Q(1/2)$ is commutative, while the last identity says that every subloop H with $rank(H) \le 2$ satisfies $c\ell(H) \le 2$. □

The following result is [2, Lemma VII.5.6]:

Lemma 2.3. *Let Q be a uniquely 2-divisible Moufang loop. Then $Q(1/2)$ is an abelian group if and only if $[x,y] \in N(Q)$, $[x,y,z] \in Z(Q)$, and $[[x,y],z] = [x,y,z]^2$ for every x, y, $z \in Q$.*

Corollary 2.4. *Let Q be a uniquely 2-divisible group with $c\ell(Q) \le 2$. Then $Q(1/2)$ is an abelian group.*

In a uniquely 2-divisible Moufang loop we have $(x^{1/2})^{-1} = (x^{-1})^{1/2}$, and we denote this element by $x^{-1/2}$.

Recall that in a group of nilpotency class 2 we have $[xy,z] = [x,z][y,z]$.

Lemma 2.5. *Let Q be a uniquely 2-divisible Moufang loop with $c\ell(Q) \le 2$. Let $Q(1/2) = (Q, *)$ be the associated Bruck loop. Assume that $c\ell(Q) \le 2$. Then:*

(i) *$y^{-1}xy = x * [x,y]$ for every x, $y \in Q$,*

(ii) *$[x^{1/2}, y] = [x,y]^{1/2}$ for every x, $y \in Q$,*

where the commutators are calculated in Q.

Proof. Part (i) is equivalent to

$$y^{-1}xy = x^{1/2}x^{-1}y^{-1}xyx^{1/2} = x^{-1/2}y^{-1}xyx^{1/2},$$

or $[y^{-1}xy, x^{1/2}] = 1$. But $[y^{-1}xy, x^{1/2}] = [y^{-1}, x^{1/2}][x, x^{1/2}][y, x^{1/2}] = [y^{-1}y, x^{1/2}] = 1$.

Part (ii) is equivalent to $[x,y] = [x^2,y]^{1/2}$, or $[x,y]^2 = [x^2,y]$, which certainly holds. □

3. Uniquely 2-divisible Moufang loops

Let Q be a loop. We say that a mapping $f : Q^n \to Q$ is *(multi)linear* if the identities

$$\begin{aligned}(x_1x_1', x_2, \dots, x_n)f &= (x_1, x_2, \dots, x_n)f \cdot (x_1', x_2, \dots, x_n)f,\\ (x_1, x_2x_2', \dots, x_n)f &= (x_1, x_2, \dots, x_n)f \cdot (x_1, x_2', \dots, x_n)f,\\ &\ \,\vdots\\ (x_1, x_2, \dots, x_nx_n')f &= (x_1, x_2, \dots, x_n)f \cdot (x_1, x_2, \dots, x_n')f\end{aligned}$$

are satisfied for every x_1, x_1', $\dots$, x_n, $x_n' \in Q$.

For a loop Q with two-sided inverses, a mapping $f : Q^n \to Q$ is *alternating* if for every x_1, $\dots$, $x_n \in Q$ and every permutation π of $\{1, \dots, n\}$ we have

$$(x_{1\pi}, \dots, x_{n\pi})f = ((x_1, \dots, x_n)f)^{\pi\,\mathrm{sgn}},$$

where $\pi\,\mathrm{sgn} = 1$ if π is an even permutation, and $\pi\,\mathrm{sgn} = -1$ if π is an odd permutation.

For a uniquely 2-divisible loop Q, it is easy to see that a linear mapping $f : Q^n \to Q$ is alternating if and only if $(x_1, \dots, x_n)f$ vanishes whenever $x_i = x_j$ for some $i \neq j$.

Proposition 3.1. *Let Q be a uniquely 2-divisible Moufang loop with $c\ell(Q) \le 2$. Then:*

(i) $x^3 \in N(Q)$ *for every* $x \in Q$,
(ii) $[x, y, z]^3 = 1$ *for every* x, y, $z \in Q$,
(iii) *the commutator mapping* $Q^2 \to Q$, $(x, y) \mapsto [x, y]$ *is alternating and linear,*
(iv) *the associator mapping* $Q^3 \to Q$, $(x, y, z) \mapsto [x, y, z]$ *is alternating and linear.*

Proof. By [8, Theorem 3.3], in every Moufang loop H with $c\ell(H) \le 2$ the associator mapping is alternating and linear, and the following identity holds:

$$[xy, z] = [x, z][y, z] \cdot [x, y, z]^3 = [x, z][y, z] \cdot [x^3, y, z].$$

Furthermore, by [8, Theorem A], we have $x^6 \in N(H)$. Since Q is uniquely 2-divisible, so is $N(Q)$. As $x^3 \cdot x^3 = x^6 \in N(Q)$, it follows that $x^3 \in N(Q)$ and thus $[x^3, y, z] = 1$. Hence the commutator mapping in Q is linear, and it is obviously alternating. □

Lemma 3.2. *Let Q be a uniquely 2-divisible Moufang loop with $c\ell(Q) \le 2$. Then the associators in Q and $Q(1/2)$ agree.*

Proof. We need to show that $(x(yz))^{-1} \cdot (xy)z = (x * (y * z))^{-1} * ((x * y) * z)$. Since $Q(1/2)$ is commutative by Lemma 2.2, we can rewrite the right hand side as

$$(x^{-1} * (y^{-1} * z^{-1})) * (z * (y * x)).$$

Our task is therefore to show that

$$(x(yz))^{-1} \cdot (xy)z = u^{1/2}vu^{1/2}, \tag{3.1}$$

where

$$u = x^{-1/2}(y^{-1/2}z^{-1}y^{-1/2})x^{-1/2}, \quad v = z^{1/2}(y^{1/2}xy^{1/2})z^{1/2}.$$

The commutator mapping is linear by Proposition 3.1, and all commutators and associators are central by $c\ell(Q) \le 2$. We can therefore rewrite the commutator $[v, u]$ as

$$[z(yx), x^{-1}(y^{-1}z^{-1})] = [z(yx), (x^{-1}y^{-1})z^{-1}] = [z(yx), (z(yx))^{-1}] = 1.$$

Then $[v, u^{1/2}] = [v, u]^{1/2} = 1$ by Lemma 2.5(ii) and unique 2-divisibility, which means that we can interchange the factors $u^{1/2}$, v in the right hand side of (3.1). It remains to check that

$$(x(yz))^{-1} \cdot (xy)z = uv = x^{-1/2}(y^{-1/2}z^{-1}y^{-1/2})x^{-1/2} \cdot z^{1/2}(y^{1/2}xy^{1/2})z^{1/2}.$$

Now,

$$\begin{aligned} x^{-1/2}(y^{-1/2}z^{-1}y^{-1/2})x^{-1/2} &= x^{-1/2}(z^{-1}y^{-1})x^{-1/2} \cdot [y^{-1/2}, z^{-1}] \\ &= (z^{-1}y^{-1})x^{-1} \cdot [y^{-1/2}, z^{-1}][x^{-1/2}, z^{-1}y^{-1}] \\ &= (x(yz))^{-1} \cdot [y^{-1/2}, z^{-1}][x^{-1/2}, z^{-1}y^{-1}] \end{aligned}$$

and, similarly,

$$z^{1/2}(y^{1/2}xy^{1/2})z^{1/2} = (xy)z \cdot [y^{1/2}, x][z^{1/2}, xy].$$

By Lemma 2.5(ii) and the linear and alternating properties of the commutator,

$$\begin{aligned} &[y^{-1/2}, z^{-1}][x^{-1/2}, z^{-1}y^{-1}][y^{1/2}, x][z^{1/2}, xy] \\ &\qquad = [y, z]^{1/2}[x, z]^{1/2}[x, y]^{1/2}[y, x]^{1/2}[z, x]^{1/2}[z, y]^{1/2} = 1. \end{aligned}$$

□

Proposition 3.3. *Let Q be a uniquely 2-divisible Moufang loop with $c\ell(Q) \le 3$. Assume further that every proper subloop H of Q has $c\ell(H) \le 2$. Then the mappings*

$$\begin{aligned} Q^3 &\to Q, \quad (x, y, z) \mapsto [[x, y], z], \\ Q^4 &\to Q, \quad (x, y, z, u) \mapsto [[x, y, z], u] \\ Q^4 &\to Q, \quad (x, y, z, u) \mapsto [x, y, [z, u]], \\ Q^5 &\to Q, \quad (x, y, z, u, v) \mapsto [x, y, [z, u, v]] \end{aligned}$$

are linear. Moreover, if $rank(Q) \ge 3$ then $(x, y, z) \mapsto [[x, y], z]$ is alternating.

Proof. If $rank(Q) \ge 3$ then $[[x, y], y] = 1$, so $(x, y, z) \mapsto [[x, y], z]$ will be alternating the moment it is linear.

When Q is a group, the mapping $(x, y, z) \mapsto [[x, y], z]$ is linear, and there is nothing else to prove. Thanks to diassociativity, we can assume that $rank(Q) \ge 3$.

Since $c\ell(Q/Z(Q)) \le 2$, Proposition 3.1 implies $[xx', y] = [x, y][x', y]c_1$, $[x, yy'] = [x, y][x, y']c_2$, $[xx', y, z] = [x, y, z][x', y, z]c_3$, and so on, where c_1, c_2, $c_3 \in Z(Q)$. Recall that the commutator and the associator are well-defined modulo $Z(Q)$.

Thus $[[xx', y], z] = [[x, y][x', y]c_1, z] = [[x, y][x', y], z]$. Now, $H = \langle [x, y], [x', y], z\rangle$ is a proper subloop of Q, and so $[[x, y][x', y], z] = [[x, y], z][[x', y], z]$ by Proposition 3.1 and $c\ell(H) \le 2$. Using analogous arguments, we establish the linearity of $[[x, y], z]$ and $[[x, y, z], u]$ in all variables, and also the linearity of $[x, y, [z, u]]$ and $[x, y, [z, u, v]]$ in the variables z, u, v.

Let us prove the linearity of $[x, y, [z, u]]$ in x. We have $xL(z, y) = x[x, y, z]^{-1}$ in any Moufang loop. In particular, $xL([z, u], y) = x[x, y, [z, u]]^{-1}$. Moreover, the companion of the pseudo-automorphism $L([z, u], y)$ is the central element $[y, [z, u]]$, which means that $L([z, u], y)$ is an automorphism. It is in fact a center automorphism because $x\backslash(xL([z, u], y)) = [x, y, [z, u]]^{-1}$ is central. By Lemma 2.1, the mapping $x \mapsto x\backslash(xL([z, u], y)) = [x, y, [z, u]]^{-1}$ is a homomorphism, and this shows that $[x, y, [z, u]]$ is linear in x.

The linearity of $[x, y, [z, u, v]]$ in x follows analogously, using the left inner mapping $L([z, u, v], y)$ and the fact that $[y, [z, u, v]]$, $[x, y, [z, u, v]] \in Z(Q)$.

Finally, the subloops $\langle x, y, [z, u]\rangle$, $\langle x, y, [z, u, v]\rangle$ are proper in Q, the associator mapping is therefore alternating on them by Proposition 3.1, and so $(x, y, z, u) \mapsto [x, y, [z, u]]$ and $(x, y, z, u, v) \mapsto [x, y, [z, u, v]]$ are linear in y, too. □

4. Moufang loops of odd order with commuting inner mappings

In this section we prove our first main result:

Theorem 4.1. *Let Q be a Moufang loop of odd order with $\operatorname{Inn} Q$ abelian. Then Q has nilpotency class at most 2.*

Recall that a finite Moufang loop is of odd order if and only if it is (uniquely) 2-divisible.

Call Q a *minimal counterexample* to Theorem 4.1 if Q is a uniquely 2-divisible Moufang loop, $\operatorname{Inn} Q$ is abelian, $c\ell(Q) = 3$, $c\ell(H) < 3$ for every proper subloop H of Q, and $rank(Q) \geq 3$.

If Theorem 4.1 does not hold, then there is indeed a minimal counterexample to it, as defined above. To see this, consider any counterexample Q to Theorem 4.1, a 2-divisible Moufang loop with $\operatorname{Inn} Q$ abelian such that $c\ell(Q) > 2$. Then Q is nilpotent by [13] (we need $|Q| < \infty$ here), and upon replacing Q with a suitable factor loop, we can assume that $c\ell(Q) = 3$. Every strictly descending chain of subloops of Q of nilpotency class three has a minimal element, and upon replacing Q with that minimal element, we can assume that $c\ell(H) < 3$ for every proper subloop H of Q. Finally, Theorem 4.1 holds for groups, so we must have $rank(Q) \geq 3$ by diassociativity.

Lemma 4.2. *Let Q be a Moufang loop and let x, y, $z \in Q$ be such that $T(x)L(y, z) = L(y, z)T(x)$. Then $[[x, y, z], x] = 1$.*

Proof. Once again, $xL(z, y) = x[x, y, z]^{-1}$ in any Moufang loop. Thus $[x, y, z]^{-1}x = (x[x, y, z]^{-1})T(x) = xL(z, y)T(x) = xT(x)L(z, y) = xL(z, y) = x[x, y, z]^{-1}$. □

The identity $[[x, y, z], x] = 1$ plays a prominent role in the theory of Moufang loops, as the following result of Bruck shows.

Lemma 4.3. *Let Q be a Moufang loop. Then Q satisfies all or none of the following identities:*

(i) $[[x, y, z], x] = 1$,
(ii) $[x, y, [y, z]] = 1$,
(iii) $[x, y, z]^{-1} = [x^{-1}, y, z]$,
(iv) $[x, y, z]^{-1} = [x^{-1}, y^{-1}, z^{-1}]$,
(v) $[x, y, z] = [x, zy, z]$,
(vi) $[x, y, z] = [x, z, y^{-1}]$,
(vii) $[x, y, z] = [x, xy, z]$.

When these identities hold, the associator $[x, y, z]$ lies in the center of $\langle x, y, z\rangle$, and the following identities hold for all integers n:

$$[x, y, z] = [y, z, x] = [y, x, z]^{-1},$$
$$[x^n, y, z] = [x, y, z]^n,$$
$$[xy, z] = [x, z][[x, z], y][y, z][x, y, z]^3,$$
$$xL(y, z) = xR(y, z) = x[x, y, z].$$

Proof. The only observation not proved in [2, Lemma VII.5.5] is $xL(y,z) = xR(y,z) = x[x,y,z]$. But $xL(y,z) = x[x,z,y]^{-1} = x[x,y,z] = x[x,y^{-1},z^{-1}] = xL(y^{-1},z^{-1}) = xR(y,z)$. □

Lemma 4.4. *Let Q be a minimal counterexample to Theorem 4.1. Then*

$$[[x,y,z],u] = [[x,y],z,u] = 1$$

for every x, y, z, $u \in Q$.

Proof. By Lemma 4.2, $[[x,y,z],x] = 1$. By Lemma 4.3, the associator mapping is alternating. By Proposition 3.3, the mappings $(x,y,z,u) \mapsto [[x,y,z],u]$, $(x,y,z,u) \mapsto [[x,y],z,u]$ are linear. Parts (i) and (ii) of Lemma 4.3 then imply that the two mappings are alternating.

If $rank(Q) = 3$, we conclude that $[[x,y,z],u] = [[x,y],z,u] = 1$ for every x, y, z, $u \in Q$, since the expressions are trivial on any 3 elements. For the rest of the proof, we can therefore assume that $rank(Q) \geq 4$.

Then any three elements generate a proper subloop of nilpotency class at most 2. Consequently, by Proposition 3.1, $(x,y) \mapsto [x,y]$ is linear, and we have

$$\begin{aligned}[[x,y,z],u] &= [(x(yz))^{-1} \cdot (xy)z, u] = [x(yz),u]^{-1}[(xy)z,u] \\ &= ([x,u] \cdot [y,u][z,u])^{-1}([x,u][y,u] \cdot [z,u]) = [[x,u],[y,u],[z,u]].\end{aligned}$$

Now, $\langle x,u,[y,u],[z,u]\rangle$ is a proper subloop of Q, so $[[x,y,z],u] = [[x,u],[y,u],[z,u]] = 1$.

Another proper subloop of Q is $H = \langle x^{-1}, xL(z,u), y\rangle$. The restriction of $T(y)$ to H is then an automorphism of H, since the companion y^3 of $T(y)$ is nuclear by Proposition 3.1. Using $xL(y,z) = x[x,z,y]^{-1}$, we calculate

$$\begin{aligned}[xT(y),z,u]^{-1} &= (xT(y))^{-1} \cdot xT(y)L(u,z) = x^{-1}T(y) \cdot xL(u,z)T(y) \\ &= (x^{-1} \cdot xL(u,z))T(y) = [x,z,u]^{-1}T(y),\end{aligned}$$

so $[xT(y),z,u] = [x,z,u]T(y)$.

Yet another proper subloop of Q is $\langle x^{-1},[x,y],z,u\rangle = \langle x^{-1},xT(y),z,u\rangle$. By Proposition 3.1 and the above calculation, we have

$$\begin{aligned}[[x,y],z,u] &= [x^{-1} \cdot xT(y),z,u] = [x^{-1},z,u][xT(y),z,u] \\ &= [x,z,u]^{-1} \cdot [x,z,u]T(y) = [[x,z,u],y] = 1.\end{aligned}$$

□

Lemma 4.5. *Let Q be a minimal counterexample to Theorem 4.1. Then*

$$[[x,y],z] = 1$$

for every x, y, $z \in Q$.

Proof. Let $Q(1/2) = (Q,*)$ be the associated Bruck loop. By Lemma 2.2, $(Q,*)$ is a commutative Moufang loop. Let $K = \langle [x,y],z,u\rangle$. By Lemma 4.4, $[[x,y],z,u] = 1$, and the Moufang theorem implies that K is an associative subloop of Q, necessarily with $c\ell(K) \leq 2$. Then Corollary 2.4 shows that $K(1/2)$ is an abelian group. This means that $[x,y] \in Z(Q(1/2))$ for every x, $y \in Q$.

Calculating in $\langle x,y\rangle$, we have $xT(y) = x*[x,y]$, by Lemma 2.5(i), and thus

$$xT(y)T(z) = x*[x,y]*[x*[x,y],z].$$

The commutator mapping is linear in the group $\langle [x,y],x,z\rangle$, so

$$\begin{aligned}[x*[x,y],z] &= [x^{1/2}[x,y]x^{1/2},z] = [x^{1/2},z][[x,y],z][x^{1/2},z]\\ &= [x,z]^{1/2}[[x,y],z][x,z]^{1/2} = [x,z]*[[x,y],z],\end{aligned}$$

where we have used Lemma 2.5(ii). Altogether,

$$xT(y)T(z) = x*[x,y]*[x,z]*[[x,y],z].$$

Since $T(y)T(z) = T(z)T(y)$, we deduce that $[[x,y],z] = [[x,z],y]$.

On the other hand, recall that the mapping $(x,y,z)\mapsto [[x,y],z]$ is alternating by Proposition 3.3, hence $[[x,y],z] = [[x,z],y] = [[x,y],z]^{-1}$, or $[[x,y],z]^2 = 1$. Then $[[x,y],z] = 1$ follows by unique 2-divisibility. $\square$

Lemma 4.6. *Let Q be a minimal counterexample to Theorem 4.1. Then*

$$[[x,y,z],u,v] = 1$$

for every x, y, z, u, $v\in Q$.

Proof. By Lemmas 4.2, 4.3 and Proposition 3.3, $(x,y,z,u,v)\mapsto [[x,y,z],u,v]$ is a linear mapping. Assume for a while that $rank(Q)=3$. Then any expression $[[x,y,z],u,v]$ using only the 3 generators of Q will vanish (we can use Lemma 4.3 to cover the case $[[x,y,z],x,y]=1$). For the rest of the proof, we can therefore assume that $rank(Q)>3$.

Then $c\ell(\langle x,y,z\rangle)\le 2$, which means that the associators in Q and $Q(1/2)$ agree, by Lemma 3.2. Let $L^*(y,z)$ denote the left inner mapping in $Q(1/2)$. Then $xL^*(y,z) = x*[x,y,z]^{-1} = x^{1/2}[x,y,z]^{-1}x^{1/2} = x[x,y,z]^{-1} = xL(y,z)$, because $[x^{1/2},[x,y,z]] = 1$. Since the left inner mappings in Q and in the commutative Moufang loop $Q(1/2)$ agree, $\operatorname{Inn} Q(1/2)$ is abelian, and so $c\ell(Q(1/2))\le 2$ by [2, Theorem VII.11.5]. It follows that $[[x,y,z],u,v]=1$. $\square$

Theorem 4.1 now follows from Lemmas 4.4, 4.5 and 4.6.

Problem 4.7. *Let Q be an infinite uniquely 2-divisible Moufang with* $\operatorname{Inn} Q$ *abelian. Is $c\ell(Q)\le 2$?*

5. Uniquely 6-divisible Moufang loops with commuting inner mappings

The finiteness assumption can be removed from Theorem 4.1 in case of 6-divisible Moufang loops, cf. Theorem 5.3.

Lemma 5.1. *Let Q be a Moufang loop in which $[[x,y,z],x]=1$ holds for every x, y, $z\in Q$. Then*

$$zT(y)T(x) = zT(yx)\cdot [x,y,z]^{-3}$$

holds for every x, y, $z\in Q$.

Proof. By Lemma 4.3, $[x,y,z]$ is central in $\langle x,y,z\rangle$. Note that the following calculation takes place in $\langle x,y,z\rangle$, and that every associator $[u,v,w]$ that appears below satisfies

$\langle u, v, w\rangle = \langle x, y, z\rangle$. Hence

$$\begin{aligned} zT(y)T(x) &= x^{-1}(y^{-1}zy)x \\ &= x^{-1}(y^{-1}z\cdot yx)[y^{-1}z, y, x] \\ &= x^{-1}(y^{-1}\cdot z(yx))[y^{-1}z, y, x][y^{-1}, z, yx] \\ &= (x^{-1}y^{-1})z(yx)\cdot[y^{-1}z, y, x][y^{-1}, z, yx][x^{-1}, y^{-1}, z(yx)]^{-1} \\ &= zT(yx)\cdot[y^{-1}z, y, x][y^{-1}, z, yx][x^{-1}, y^{-1}, z(yx)]^{-1}. \end{aligned}$$

Moreover, by Lemma 4.3,

$$\begin{aligned} [y^{-1}z, y, x] &= [y^{-1}, y^{-1}z, x] = [y^{-1}, z, x] = [x, y, z]^{-1}, \\ [y^{-1}, z, yx] &= [y, yx, z] = [y, x, z] = [x, y, z]^{-1}, \\ [x^{-1}, y^{-1}, z(yx)] &= [x^{-1}, y^{-1}, (zy)x[z, y, x]^{-1}] = [x^{-1}, y^{-1}, (zy)x] \\ &= [x^{-1}, x^{-1}(y^{-1}z^{-1}), y^{-1}] = [x^{-1}, y^{-1}z^{-1}, y^{-1}] \\ &= [y^{-1}, y^{-1}z^{-1}, x^{-1}]^{-1} = [y^{-1}, z^{-1}, x^{-1}]^{-1} = [x, y, z], \end{aligned}$$

and the result follows. □

Lemma 5.2. *Let Q be a Moufang loop in which $[[x, y, z], x] = 1$, all commutators are nuclear, and $[[x, y], z] = [x, y, z]^2$ holds. Then*

$$zT(y)T(x) = zT(x)T(y)\cdot[x, y, z]^{-4}$$

for every x, y, $z\in Q$.

Proof. By Lemma 5.1, and since all commutators are in the nucleus, we have

$$\begin{aligned} zT(yx) &= zT([y^{-1}, x^{-1}](xy)) \\ &= zT([y^{-1}, x^{-1}])T(xy)\cdot[xy, [y^{-1}, x^{-1}], z]^3 = zT([y^{-1}, x^{-1}])T(xy). \end{aligned}$$

By Lemma 4.3,

$$zT([y^{-1}, x^{-1}]) = z[z, [y^{-1}, x^{-1}]] = z[[y^{-1}, x^{-1}], z]^{-1} = z[y^{-1}, x^{-1}, z]^{-2} = z[x, y, z]^2.$$

Since $[x, y, z]$ is central in $\langle x, y, z\rangle$ by Lemma 4.3, we calculate

$$zT(yx) = zT([y^{-1}, x^{-1}])T(xy) = (z[x, y, z]^2)T(xy) = zT(xy)\cdot[x, y, z]^2.$$

Finally, using the last equality and Lemmas 4.3, 5.1, we have

$$\begin{aligned} zT(y)T(x) &= zT(yx)[x, y, z]^{-3} = zT(xy)[x, y, z]^{-1} = zT(xy)[x, y, z]^3\cdot[x, y, z]^{-4} \\ &= zT(xy)[y, x, z]^{-3}\cdot[x, y, z]^{-4} = zT(x)T(y)\cdot[x, y, z]^{-4}, \end{aligned}$$

as desired. □

Theorem 5.3. *Let Q be a uniquely 6-divisible Moufang loop with $\mathrm{Inn}\, Q$ abelian. Then Q has nilpotency class at most 2.*

Proof. Every subloop H of Q with $\mathit{rank}(H) = 2$ is a group and hence satisfies $c\ell(H)\le 2$. By Lemma 2.2, $Q(1/2)$ is a commutative Moufang loop. By [2, Lemma VII.5.7], $x^3 = 1$ for every element x in the derived subloop of a commutative Moufang loop. Since $Q(1/2)$ contains no elements of order 3, we conclude that $Q(1/2)$ is an abelian group. By Lemma 2.3, commutators are nuclear, associators are central, and $[[x, y], z] = [x, y, z]^2$ in Q. Then Lemmas 4.2 and 5.2 yield $[x, y, z]^4 = 1$. Since Q contains no elements of even order, we deduce that Q is a group and $c\ell(Q)\le 2$. □

6. Moufang 2-loop of Csörgő type

In this section, we show that Theorem 4.1 cannot be extended to Moufang loops of even order.

Throughout the section, let X denote an associative ring and consider X as a natural $\mathbb{Z}$-module, that is, $nx = x + \cdots + x$ is well defined for all $n \in \mathbb{Z}$ and $x \in X$.

Our construction is a generalization of a construction by R. H. Bruck. Indeed, if X is an algebra over field of characteristic $\neq 2$, then the construction is precisely [2, Example 3, p.128]. We mention that Bruck did not establish any properties pertaining to the commutativity of the inner mapping group.

Proposition 6.1. *Let X_1 be an additive subgroup of X such that $uu = 0$ and $uv + vu = 0$ holds for every u, $v \in X_1$. For $n \geq 1$, denote by X_n the additive subgroup of X generated by $u_1 \cdots u_n$, where $u_i \in X_1$. Define multiplication on the subspace $Q = X_1 \times X_2 \times X_3$ by*

$$(a, b, c)(a', b', c') = (a + a', b + b' + aa', c + c' + ba'). \tag{6.1}$$

Then Q is a Moufang loop with neutral element $(0, 0, 0)$ and inverse $(a, b, c)^{-1} = (-a, -b, -c + ba)$. Moreover,

$$\begin{aligned}
[(a, b, c), (a', b', c')] &= (0, 2aa', ba' - b'a), \\
[[(a, b, c), (a', b', c')], (a'', b'', c'')] &= (0, 0, 2aa'a''), \\
[(a, b, c), (a', b', c'), (a'', b'', c'')] &= (0, 0, aa'a''), \\
(a, b, c)L((a', b', c'), (a'', b'', c'')) &= (a, b, c + aa'a''), \\
(a, b, c)R((a', b', c'), (a'', b'', c'')) &= (a, b, c + aa'a''), \\
(a, b, c)T((a', b', c')) &= (a, b + 2aa', c + ab' + ba')
\end{aligned}$$

hold for every (a, b, c), (a', b', c'), $(a'', b'', c'') \in Q$.

Proof. We have $(a, b, c)(0, 0, 0) = (0, 0, 0)(a, b, c) = (a, b, c)$ for every $(a, b, c) \in Q$, so $(0, 0, 0)$ is the neutral element of Q. Solving $(a, b, c)(a', b', c') = (a'', b'', c'')$ for (a, b, c) or (a', b', c') leads to

$$\begin{aligned}
(a, b, c) &= (a'' - a', b'' - b' - a''a', c'' - c' - (b'' - c')a'), \\
(a', b', c') &= (a'' - a, b'' - b - aa'', c'' - c'b(a'' - a)),
\end{aligned}$$

respectively, where we have used the properties of X_1. Consequently,

$$(a, b, c)^{-1} = (-a, -b, -c + ba).$$

By straightforward calculation,

$$\begin{aligned}
(a, b, c)T((a', b', c')) &= (a', b', c')^{-1}((a, b, c)(a', b', c')) \\
&= (-a', -b', -c' + b'a')(a + a', b + b' + aa', c + c' + ba') \\
&= (a, b + 2aa', c + ab' + ba').
\end{aligned}$$

Let $x = (a, b, c)$, $y = (a', b', c')$, $z = (a'', b'', c'')$. Upon solving the equation $xy = yx \cdot [x, y]$ for $[x, y]$, we get

$$[(a, b, c), (a', b', c')] = (0, 2aa', ba' - b'a),$$

and upon solving the equation $(xy)z = x(yz) \cdot [x, y, z]$ for $[x, y, z]$, we get

$$[(a, b, c), (a', b', c'), (a'', b'', c'')] = (0, 0, aa'a'').$$

It follows that $(0,0,c) \in Z(Q)$ for every $c \in X_3$, and $(0,b,c) \in N(Q)$ for every $(b,c) \in X_2 \times X_3$. In particular, all commutators are in the nucleus and all associators are in the center of Q.

We have $((xy)x)z = (xy)(xz) \cdot [xy,x,z] = (x(y(xz)) \cdot [x,y,xz])[xy,x,z] = x(y(xz)) \cdot [x,y,xz][xy,x,z]$. Hence the Moufang identity $((xy)x)z = x(y(xz))$ holds if and only if $[x,y,xz][xy,x,z]$ vanishes. With x, y, z as above, we have $[x,y,xz][xy,x,z] = (0,0,aa'(a+a''))(0,0,(a+a')aa'') = (0,0,aa'a'')(0,0,a'aa'') = (0,0,aa'a''+a'aa'') = (0,0,0)$.

Lemma 4.3 yields $xL(y,z) = xR(y,z) = x[x,y,z]$, finishing the proof. □

Proposition 6.2. *Let $Q = X_1 \times X_2 \times X_3$ be the Moufang loop constructed in Proposition 6.1, with the same assumptions on X. Let H be defined on $X_1 \times X_2$ with multiplication*

$$(u,v)(u',v') = (u+u', v+v'+2uu').$$

Then H is a group and $\operatorname{Inn} Q = \{T_a;\ a \in Q\} \cong H$. Moreover, if $4X = \{4x;\ x \in X\} = 0$ then $\operatorname{Inn} Q$ is an abelian group of exponent 4.

Proof. It is easy to see that H is associative, has neutral element $(0,0)$, and $(u,v)^{-1} = (-u,-v+2uu) = (-u,-v)$. Moreover,

$$\begin{aligned}
[(u,v),(u',v')] &= (u,v)^{-1}(u',v')^{-1}(u,v)(u',v') \\
&= (-u-u', -v-v'+2uu')(u+u', v+v'+2uu') \\
&= (0, 4uu' - 2(u+u')(u+u')) = (0, 4uu'),
\end{aligned}$$

and $(u,v)^4 = (2u,2v)^2 = (4u,4v)$. Hence H is an abelian group of exponent 4 when $4X = 0$.

For $(u,v) \in H$ define the map $S(u,v) : Q \to Q$ by

$$(a,b,c)S(u,v) = (a, b+2au, c+av+bu).$$

As

$$\begin{aligned}
(a,b,c)S(u,v)S(u',v') &= (a, b+2au, c+av+bu)S(u',v') \\
&= (a, b+2au+2au', c+av+bu+av'+(b+2au)u') \\
&= (a, b+2a(u+u'), c+a(v+v'+2uu')+b(u+u')) \\
&= (a,b,c)S(u+u', v+v'+2uu'),
\end{aligned}$$

the set $\{S(u,v);\ (u,v) \in X_1 \times X_2\}$ is isomorphic to H. By Proposition 6.1, we have

$$L((a',b',c'),(a'',b'',c'')) = R((a',b',c'),(a'',b'',c'')) = S(0, a'a'')$$

and

$$T((a',b',c')) = S(a',b').$$

This shows at once that $\operatorname{Inn} Q$ is isomorphic to H, and that it consists of conjugations of Q. □

We now construct a ring X for which Proposition 6.1 yields a Moufang loop Q of Csörgő type.

Put $R = \mathbb{Z}_4$, $X = R^7$ and let $\{e_1, \ldots, e_7\}$ be a set of free generators of the R-module X. Define the multiplication on the generators according to

$$\begin{array}{c|ccccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
\hline
e_1 & 0 & e_4 & e_5 & 0 & 0 & e_7 & 0 \\
e_2 & -e_4 & 0 & e_6 & 0 & -e_7 & 0 & 0 \\
e_3 & -e_5 & -e_6 & 0 & e_7 & 0 & 0 & 0 \\
e_4 & 0 & 0 & e_7 & 0 & 0 & 0 & 0 \\
e_5 & 0 & -e_7 & 0 & 0 & 0 & 0 & 0 \\
e_6 & e_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

and extend it to X by R-linearity. This turns X into an associative ring satisfying $4X = 0$. In order to verify associativity, we observe that all products $e_i(e_je_k)$ vanish except when $\{i,j,k\} = \{1,2,3\}$, and then

$$\begin{aligned}
e_1(e_2e_3) &= e_1e_6 = e_7 = e_4e_3 = (e_1e_2)e_3, & e_1(e_3e_2) &= -e_1e_6 = -e_7 = e_5e_2 = (e_1e_3)e_2, \\
e_2(e_1e_3) &= e_2e_5 = -e_7 = -e_4e_3 = (e_2e_1)e_3, & e_2(e_3e_1) &= -e_2e_5 = e_7 = e_6e_1 = (e_2e_3)e_1, \\
e_3(e_1e_2) &= e_3e_4 = e_7 = -e_5e_2 = (e_3e_1)e_2, & e_3(e_2e_1) &= e_3e_4 = -e_7 = -e_6e_1 = (e_3e_2)e_1.
\end{aligned}$$

Let $X_1 = Re_1 + Re_2 + Re_3$. Then X_1 is an additive subgroup of X satisfying $uu = 0$ and $uv + vu = 0$ for every u, $v \in X_1$. Moreover, we have $X_2 = Re_4 + Re_5 + Re_6$ and $X_3 = Re_7$.

Let Q be the Moufang loop of order 4^7 constructed from X as in Proposition 6.1. By Proposition 6.2, $\mathrm{Inn}(Q)$ is an abelian group of order 4^6 and exponent 4. By Proposition 6.1, we have

$$[[(e_1, 0, 0), (e_2, 0, 0)], (e_3, 0, 0)] = (0, 0, 2e_7) \neq (0, 0, 0),$$

which implies that the nilpotency class of Q is at least 3 (in fact, it is equal to 3). Hence Q is a Moufang loop of Csörgő type.

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