THE POINCARÉ GROUP IN A DEMISEMIDIRECT PRODUCT WITH A NON-ASSOCIATIVE ALGEBRA WITH REPRESENTATIONS THAT INCLUDE PARTICLES AND QUARKS

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ABSTRACT. The quarks have always been a puzzle, as have the particles' mass and mass/spin relations as they seemed to have no coordinates in configuration space and/or momentum space. The solution to this seems to lie in the marriage of ordinary Poincaré group representations with a non-associative algebra made through a demisemidirect product. Then, the work of G. Dixon applies; so, we may obtain all the relations between masses, mass and spin, and the attribution of position and momentum to quarks - this in spite of the old restriction that the Poincaré group cannot be extended to a larger group by any means (including the (semi)direct product) to get even the mass relations. Finally, we will briefly discuss a possible connection between the phase space representations of the Poincaré group and the phase space representations of the object we will obtain. This will take us into Leibniz (co)homology.

1. INTRODUCTION

When I was a graduate student at the University of Rochester in 1964-1970, I heard a colloquium given by L. O'Raifeartaigh[1] with the message that there was nothing one could do to the Poincaré group to obtain a larger group which would have particles as well as any relation between the various particles' masses, (Okubo theory) [6]. Moreover, there was no way to obtain a relation between the masses and spins of the particles, (Regge theory) [11]. Furthermore, there was nothing you could do about quarks which seemed to have neither positions nor momenta. In the intervening years, the work on Leibniz Algebras by M. K. Kinyon [5], the book by G. M. Dixon [2], and the work by J-L. Loday and T. Pirashvili [7] on Leibniz algebras and cohomology has caused a rethinking of this problem.

We look to *non-asssociative* algebras as a way to enlarge the Lie algebra of the Poincaré group. We will obtain a "Leibniz algebra", a "Lie rack", and a "Lie digroup", and the relation between the last two (Kinyon). What are the appropriate non-associative things with which to extend the Lie algebra of the Poincaré group? Well, the octonions and algebras we can make from them (Dixon). Question: How does cohomology enter the game? Answer, time permitting: You may algebraically

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[1]L. O'Raifeartaigh produced a paper in 1965 that claimed the result. The proof war erroneous and R. Jost and I. Segal provided mathematical proofs. Then in O'Raifeartaigh and with A. Bohm extended the theorem.

characterize the phase spaces (symplectic spaces) on which the algebra operates by the "kernel of the coboundary operator on the set of two-forms" - according to Guillemin and Sternberg [3], at least in the case of a group. The generalization of this to the non-associative case is in Loday and Pirashvili and extended by Lodder [8] to obtain the symplectic cases. In this way, we will attribute the coordinates of position, momentum (and perhaps spin) to the quarks, and not just to the particles.

Here in section 2, we will give an outline of what a Leibniz algebra, a Lie rack, and a Lie digroup are, as well as defining the demisemidirect product. In section 3 we will look at vector spaces that are made from certain non-associative algebras that include the octonions and indicate how we should introduce the Poincaré Lie algebra into them. Then in section 4, we will make some comments on how and why the cohomology gets into the game.

2. Leibniz algebras, Lie racks, and Lie digroups

In this section, we will take M. Kinyon's definition of a Leibniz algebra:

Definition 1. [5] *A Leibniz algebra* $(\mathfrak{g}, [\cdot,\cdot])$ *is a vector space* $\mathfrak{g}$ *together with a bilinear mapping* $[\cdot,\cdot\} : \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ *satisfying the Jacobi identity*

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]] \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{g}$.*,*

Loday, Lodder, and Pirashvili take $[Y,[X,Z]]$ replaced by $-[[X,Z],Y]$.

If, in addition to (1), we have $[X,Y] = -[Y,X]$, then $\mathfrak{g}$ becomes a Lie algebra and the Jacobi identity becomes the usual one.

Take [5]

$$S \equiv \{[X,X]; X \in \mathfrak{g}\}. \tag{2.2}$$

Then S is an ideal in $\mathfrak{g}$ and $\mathfrak{h} \equiv \mathfrak{g}/S$ is a Lie algebra.

Definition 2. [5] *We define*

$$ad_X(Y) \equiv [X,Y]\ \forall X, Y \in \mathfrak{g}; \tag{2.3}$$

so,

$$\ker(ad) = \{X \in \mathfrak{g} : [X,\cdot] = 0\}. \tag{2.4}$$

Definition 3. [5] *A Leibniz algebra* $\mathfrak{g}$ *is said to* <u>*split*</u> *over* $\mathcal{E} \subset \mathfrak{g}$ *if* $\mathcal{E}$ *is an ideal in* $\mathfrak{g}$ *such that* $S \subset \mathcal{E} \subset \ker(ad)$ *and there is a Lie subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *such that* $\mathfrak{g} = \mathcal{E} \oplus \mathfrak{h}$*, a direct sum of vector spaces. Then* $\forall u, v \in \mathcal{E}$*, and* $X, Y \in \mathfrak{h}$*, it follows that*

$$[u+X, v+Y] = Xv + [X,Y]. \tag{2.5}$$

Conversely, given a Lie algebra $\mathfrak{h}$ *and an* $\mathfrak{h}$*-module* V*, form* $\mathfrak{g} \equiv V \oplus \mathfrak{h}$ *and define a bracket on* $\mathfrak{g}$ *by (2.5). Then* $(\mathfrak{g}, [\cdot,\cdot])$ *is a Leibniz algebra called the* <u>*demisemidirect product*</u> *of* V *and* $\mathfrak{h}$*. Then* $S \simeq \mathfrak{h}V$ *and* $\ker(ad) = V \oplus \{X \in \mathfrak{h} : Xv = 0\ \forall v \in V\} \cap Z(\mathfrak{h})$*, where* $Z(\mathfrak{h})$ *is the center of* $\mathfrak{h}$*. Furthermore,* $V \simeq V \oplus \{0\}$ *is an ideal such that* $S \subset V \subset \ker(ad)$ *and* $\mathfrak{g}/V \simeq \mathfrak{h}$*; so,* $\mathfrak{g}$ *splits over* V*.*

We will use this definition where we take $\mathfrak{h}$ equal to the Poincaré Lie algebra $\mathfrak{p}$, and V to be an appropriate $\mathfrak{p}$-module.

Definition 4. [5] *A Lie rack* $(Q, \circ, 1)$ *is a smooth manifold* Q *with bilinear operation* $\circ$ *and a distinguished element* $1 \in Q$ *such that 1) for all* $x, y, z \in Q$, $x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$, *2.* $\forall a, b \in Q$, *there exists a unique* $x \in Q$ *such that* $a \circ x = b$, *3.* $1 \circ x = x$, $x \circ 1 = 1$ *for all* $x \in Q$, *and such that* $\circ : Q \times Q \to Q$ *is a smooth mapping.*

Example 1. [5] *Let* H *be a Lie group and* V *an* H*-module. On* $Q = V \times H$, *define binary operation* $\circ$ *by*

$$(u, A) \circ (v, B) \equiv (Av, ABA^{-1}) \tag{2.6}$$

for all $u, v \in V$ *and* $A, B \in H$. *Setting* $\mathbf{1} = (0, 1)$, *then* $(Q, \circ, \mathbf{1})$ *is a (linear) Lie rack.*

We have in mind H equal to the Lie group $\mathcal{P}$, the Poincaré group and V an appropriate $\mathcal{P}$-module.

Definition 5. [5] *A dialgebra* $(\mathcal{A}, \vdash, \dashv)$ *is a vector space* $\mathcal{A}$ *together with bilinear mappings* $\vdash, \dashv : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ *such that* $\forall x, y, z \in \mathcal{A}$,

$$x \vdash (y \dashv z) = (x \vdash y) \dashv z, \tag{2.7}$$

$$x \dashv (y \vdash z) = x \dashv (y \dashv z) \tag{2.8}$$

$$(x \dashv y) \vdash z = (x \vdash y) \vdash z. \tag{2.9}$$

Lemma 1. [5] *Given a dialgebra* $(\mathcal{A}, \vdash, \dashv)$ *and defining a bracket by*

$$[x, y] \equiv x \vdash y - y \dashv x, \tag{2.10}$$

then $(\mathcal{A}, [\cdot, \cdot])$ *is a Leibniz algebra.*

Example 2. *If* $\mathcal{A} \equiv V \oplus \mathfrak{g}$, $\mathfrak{g} \subseteq End(V)$, $\mathfrak{g}$ *a Lie algebra, then*

$$(u, X) \vdash (v, Y) \equiv (Xv, XY), \tag{2.11}$$

$$(u, X) \dashv (v, Y) \equiv (0, XY). \tag{2.12}$$

Hence $(\mathcal{A}, [\cdot, \cdot])$ *is the demisemidirect product of* V *with the Lie algebra* $\mathfrak{g}$. *Again, take* $\mathfrak{g}$ *to be* $\mathfrak{p}$.

Definition 6. [5] *A disemigroup* $(G, \vdash, \dashv)$ *is a set* G *with two binary operations* $\vdash$ *and* $\dashv$ *satisfying* $(G, \vdash)$ *and* $(G, \dashv)$ *are semigroups and* $(G, \vdash, \dashv)$ *is a dialgebra. A disemigroup* $(G, \vdash, \dashv)$ *satisfying 1. there exists* $1 \in G$ *such that* $1 \vdash x = x \dashv 1 = x$ *for all* $x \in G$ *and 2.* $\forall x \in G$, *there exists* $x^{-1} \in G$ *such that* $x \vdash x^{-1} = x^{-1} \dashv x = 1$ *is called a digroup. A Lie digroup* $(G, \vdash, \dashv)$ *is a smooth manifold* G *with* $(G, \vdash, \dashv)$ *a digroup such that the digroup operations* $\vdash, \dashv : G \times G \to G$ *and the inversion* $(\cdot)^{-1} : G \to G$ *are smooth mappings.*

Example 3. [5] *Let* H *be a group, and* M *a set on which* H *acts on the left. Suppose there exists a point* $e \in M$ *such that* $he = e$ *for all* $h \in H$ *and suppose* H *acts transitively on* $M \backslash \{e\}$. *Then on* $G \equiv M \times H$, *define*

$$(u, h) \vdash (v, k) = (hv, hk) \tag{2.13}$$

$$(u, h) \dashv (v, k) = (u, hk) \tag{2.14}$$

for all $u, v \in M$, $h, k \in H$. *Then* $(G, \vdash, \dashv)$ *is a digroup and* $(u, h)^{-1} = (e, h^{-1})$.

Now we make a definition that will get us the connection between Lie digroups and Lie racks:

Definition 7. [5] *For x in digroup G, define $\circ$ on G by*

$$x \circ y \equiv x \vdash y \dashv y^{-1}. \tag{2.15}$$

Then we have the

Lemma 2. [5] *Let $(G, \vdash, \dashv)$ be a Lie digroup. Consequently, $(G, \circ, 1)$ is a Lie rack and thus the tangent space T_1G has the structure of a Leibniz algebra.*

Example 4. [5] *Let H be a Lie group, V an H-module, and set $G = V \times H$. As in the previous example, define $\vdash, \dashv$ by (2.13), (2.14). Then $(G, \vdash, \dashv)$ is a (linear) Lie digroup. The distinguished unit is $(0, 1)$ and the inverse of (u, A) is $(0, A^{-1})$.*

We summarize with the theorem:

Theorem 1. [5] *Let $G = V \times H$ with H a Lie group and V an H-module. If $(G, \vdash, \dashv)$ is the (linear) Lie digroup defined by (2.13), (2.14), then the induced (linear) Lie rack is $(G, \circ, 1)$ defined by (2.6). Conversely, every (linear) Lie rack is induced from a (linear) Lie digroup*

Proof. $(u, A) \circ (v, B) = (u, A) \vdash (v, B) \dashv (0, A^{-1}) = (Av, ABA^{-1})$ for all u,v$\in V$ and A,B$\in H$. □

We will take H equal to the Poincaré Lie group $\mathcal{P}$ and $\mathfrak{h}$ the Poincaré Lie algebra $\mathfrak{p}$. V is a $\mathcal{P}$-module to be determined in the next section.

3. The structures $V = \mathbb{Q}$, $\mathbb{O}$, $\mathbb{C} \times \mathbb{Q} \times \mathbb{O}$, etc.

We will obtain the Poincaré Lie algebra $\mathfrak{p}$ in terms of the Pauli spin algebra and then discuss the various choices for V as being a $\mathfrak{p}$-module.

First, the Poincaré group is $\mathbb{R}^4 \rtimes \mathcal{L}$, where $\mathcal{L}$ is the group of Lorentz transformations and $\mathbb{R}^4$ has the metric $diag(1, -1, -1, -1)$. $\mathbb{R}^4$ is the Minkowski space-time $\mathfrak{M}^4$. With the Cayley transform of $\mathbb{R}^4$, we identify this $\mathbb{R}^4$ with the set of real linear combinations of the Pauli spin matrices by

$$(t, x, y, z) \in \mathbb{R}^4 \mapsto t\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3 \tag{3.1}$$

where σ_0 is the 2×2 identity matrix and $\sigma_1, \sigma_2, \sigma_3$ satisfy the general Pauli conditions $\sigma_j = \overline{\sigma_j}^T$, $\sigma_j\sigma_k = i\sigma_l$, (j, k, l) a cyclic permutation of $(1, 2, 3)$. Then we have $\det(t\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3) = t^2 - x^2 - y^2 - z^2 = \parallel (t, x, y, z) \parallel$. This may be realized with the standard basis for the Pauli spin algebra, but we shall not need that here.

Now, we may obtain the action of the double cover of $\mathcal{L}$ to be $SL(2, \mathbb{C})$ where, for $A \in SL(2, \mathbb{C})$ and $p = t\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3$,

$$A : p \mapsto A \cdot p = Ap\overline{A}^T. \tag{3.2}$$

In this way, we have a semidirect product $\mathfrak{M} \rtimes SL(2, \mathbb{C})$ which henceforth we shall call $\mathcal{P}$. But the Pauli spin matrices are a basis for the 2×2 complex matrices as well. Hence, every element of $\mathcal{P}$ is in the complex span of the σ_js.

Let $\mathbb{Q}$ be the quaternions; i.e., the set $span\{1, q_1, q_2, q_3\}$ where $q_j^2 = -1$, and $q_jq_{j+1} = q_{j+2}$, $j = 1, 2, 3 \bmod 3$. The multiplication table for the qs is

$$\begin{matrix} 1 & q_1 & q_2 & q_3 \\ q_1 & -1 & q_3 & -q_2 \\ q_2 & -q_3 & -1 & q_1 \\ q_3 & q_2 & -q_1 & -1 \end{matrix}. \tag{3.3}$$

Just as the reals may be embedded into the complex numbers, the complex numbers may be embedded into the quaternions, but in a non-unique way: $\mathbb{C} \hookrightarrow \mathbb{Q}$, $x + iy \mapsto x1 + yq_j$ for any q_j. In a similar fashion, we may map the Pauli spin algebra into $\mathbb{Q}$ by

$$\sigma_0 \mapsto 1,\ i\sigma_j \mapsto -q_j,\ \text{for } j = 1, 2, 3 \tag{3.4}$$

for any given choice of the basis for the Pauli spin operators and the basis for the quaternions. Through this map, we have an action of $\mathfrak{p}$ and hence $\mathcal{P}$ on $\mathbb{Q}$.

From [2], the set of octonions, $\mathbb{O}$, is the span of the set $\{1, e_j, j = 1, \cdots, 7\}$ satisfying the multiplication table

$$\begin{array}{cccccccc} 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ e_1 & -1 & e_6 & e_4 & -e_3 & e_7 & -e_2 & -e_5 \\ e_2 & -e_6 & -1 & e_7 & e_5 & -e_4 & e_1 & -e_3 \\ e_3 & -e_4 & -e_7 & -1 & e_1 & e_6 & -e_5 & e_2 \\ e_4 & e_3 & -e_5 & -e_1 & -1 & e_2 & e_7 & -e_6 \\ e_5 & -e_7 & e_4 & -e_6 & -e_2 & -1 & e_3 & e_1 \\ e_6 & e_2 & -e_1 & e_5 & -e_7 & -e_3 & -1 & e_4 \\ e_7 & e_5 & e_3 & -e_2 & e_6 & -e_1 & -e_4 & -1 \end{array}; \tag{3.5}$$

i.e., $e_a e_{a+1} = e_{a+5} = e_{a-2}$, $a = 1, \cdots, 7 \bmod 7$. You may check that the octonion multiplication is not (always) associative. Check, for example, the multiplication of e_1, e_3 and e_5.

Now

$$\{1 \mapsto 1, q_1 \mapsto e_a, e_2 \mapsto e_{a+1}, q_3 \mapsto e_{a+5}\} \tag{3.6}$$

for any $a \in \{1, \cdots, 7\} \bmod 7$ defines an inclusion $\mathbb{Q} \hookrightarrow \mathbb{O}$. Without loss off generality, choose $a = 1$. By this inclusion and (3.6), we have a natural action on the left of $\mathfrak{p}$ and hence $\mathcal{P}$ on $\mathbb{O}$.

We have, in fact, a natural action of $\mathfrak{p}$ and $\mathcal{P}$ on $\mathbb{C} \otimes \mathbb{Q}$, on $\mathbb{C} \otimes \mathbb{Q} \otimes \mathbb{O} \equiv \mathbb{T}$, on $\mathbb{T}^2$, etc. Let V of the previous section equal any of these vector spaces. Let V_L (V_R) equal V acting on the left (right) of V. This also hosts a natural action of $\mathfrak{p}$ and $\mathcal{P}$. Dixon [2] made the discovery that we may write the isospin, the hypercharges, and the charge operators in terms of the actions of $SU(2)$ ($\subset \mathcal{P}$) on $\mathbb{T}^2$. Moreover, Dixon shows that the relations among masses and mass/spins may be obtained in this fashion. Then in addition, working in the demisemidirect product, you have in addition the position and momentum for any of these objects.

What remains to be checked is that these various operatorss are Poincaré invariants. This comes about (I conjecture) because of two properties. The first of these is the fact that the action of $\mathfrak{p}$ (i.e., of $\mathbb{Q}$) on V, V_L, V_R decomposes into two parts: the part containing only (the injection of) $\mathbb{Q}$, and the part containing $\mathbb{T}\backslash\mathbb{Q}$. The other fact is that there is a symmetry about the (arbitrary) "third" axis for the Poincaré group in the case of massive, spinning particles at the origin of phase space. Specifically, one has the following: Representations (resp. phase space representations) of the massive spinning case of $\mathcal{P} = \mathbb{R}^4 \rtimes \mathcal{L} = \mathbb{R}^4 \rtimes SL(2, \mathbb{C})$ are formed [12] by taking the subgroup $H = \mathcal{L}_{p,s} = [SL(2, C)]_{p_*, s_*}$ (resp. $H' = \mathbb{R}p_* \rtimes \mathcal{L}_{p,s} = \mathbb{R}p_* \rtimes [SL(2, C)]_{p_*, s_*}$) where $p_* = m(1, 0, 0, 0) \in \mathbb{R}^4$ is the momentum at rest, $s_* = S(0, 0, 0, 1)$ is the spin at rest, m is the mass, S is the spin, and where $\mathcal{L}_{p,s}$ is the part of $\mathcal{L}$ that fixes p and s. The representation spaces are then $\mathcal{P}/H$ (resp. $\mathcal{P}/H'$). We remark that we have $\mathcal{P}$-invariance of these spaces and that

$\mathcal{L}/\mathcal{L}_{p,s} = SU(2)$! All the operators that Nixon and his forrunners have used in classifying the particles and quarks are just at p_*, s_* only! The Poincaré transformations then may be used to express these operators at general (q, p, s) where we have q representing the $\mathfrak{M}^4$-part.

We have achieved models in which we may attribute momentum and position to the quarks, as well as the particles, and in which we may discuss the $\mathcal{P}$-invariance of the operators which are inputs for the mass formulas and the mass/spin relations.

4. Leibniz (Co)homology

How does the space $\mathcal{P}/H'$ arise, and does that derivation have an equivalent in the Leibniz algebra set up? Why bother?

Phase spaces (or symplectic spaces in mathematical terminology) are an essential ingredient in the discussion of classical mechanics; the entire structure of Hamiltonian dynamics is based on it. Guillemin and Sternberg [3] have shown that for any Lie algebra/Lie group, the phase spaces (symplectic spaces) on which the group may act are all found in a certain set determined by the coboundry operator. The coboundary operaor is something entering into Lie (co)homology. $\mathcal{P}/H'$ is just one example of one such phase space [12].

Now

$$d(g_1 \otimes \cdots \otimes g_n) = \sum_{1 \le i < j \le n} (-1)^j g_1 \otimes \cdots \otimes g_{i-1} \otimes [g_i, g_j] \otimes g_{i+1} \otimes \cdots \otimes \widehat{g_j} \otimes \cdots \otimes g_n, \tag{4.1}$$

where the $g_i \in$ Lie algebra $\mathfrak{g}$, defines the (Lie) boundary map. One may use the analog in the Leibniz algebra setting! [7] If all the (co)homology theory and the identification of the symplectic spaces remains the same in analog, then the picture for the demisemidirect product $V + \mathcal{P}$ would be complete. Although specific examples have been worked out, there is as yet no general theory that will do the trick although there has been some progress [7][8]. This is a point at which we will just say that it is under consideration.

References

[1] Bohm, A. and O'Raifeartaigh, L., Phys. Rev. 171 (1968), 1698 - 1701.
[2] Dixon, G. M., *Division Algebras: Octonions, Quaternions, Complex Numbers and the Algrbraic Design of Physics*, Kluwer Acad. Pubs., Dordrecht, 1994.
[3] Guillemin, V. and Sternberg, S., *Symplectic Techniques in Physics*, Cambridge Univ. Press, New York, 1991.
[4] Jost, R., Helv. Phys. Acta, 39 (1966), 369-?.
[5] Kinyon, M. K., J. Lie Theory, 17 (2007), 99 - 114.
[6] Kuriyan, J. and Schecter, J. "Lectures on Unitary Symmetry" by S. Okubo, chapter 4, 1964, The University of Rochester, Department of Physiiics and Astronomy, Rochester, New York.
[7] Loday, J-L and Pirashvili, T., Math. Ann. 296 (1993), 139 - 158.
[8] Lodder, J. M., Ann. Inst. Fourier 48 (1998), 73 - 95.
[9] O'Raifeartaigh, L., Phys. Rev. Letters 14 (1965), 575.
[10] O'Raifeartaigh, L., Phys. Rev. 161 (1967), 1571 - 1575; Phys. Rev. 164 (1967), 2000 - 2002.
[11] Regge, T., Nuovo Cimento, 14 (1959), 951-?; 18 (1960), 947-956.
[12] Schroeck, F. E., Jr., *Quantum Mechanics on Phase Space*, Kluwer Acad. Pubs., Dordrecht, 1996, Section III.4.C.iii.
[13] Segal, I., J. Funct. Analysis, 1 (1967), 1-21.

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