TRANSITION EFFECT MATRICES AND QUANTUM MARKOV CHAINS

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Abstract

A transition effect matrix (TEM) is a quantum generalization of a classical stochastic matrix. By employing a TEM we obtain a quantum generalization of a classical Markov chain. We first discuss state and operator dynamics for a quantum Markov chain. We then consider various types of TEMs and vector states. In particular, we study invariant, equilibrium and singular vector states and investigate projective, bistochastic, invertible and unitary TEMs.

1 Introduction

This article is dedicated to Pekka Lahti on the occasion of his sixtieth birthday. Lahti is an important pioneer in the field of foundations of quantum mechanics. His main speciality is quantum measurement theory [5, 6, 7, 9, 10, 24, 25, 26, 27, 30, 31]. The simplest general type of quantum measurement is a two-valued (yes-no or 1-0) quantum measurement called an **effect** [5, 8, 11, 20, 21, 29, 32]. An effect is represented by a bounded operator A on a Hilbert space H satisfying $0 \le A \le I$. The effect A corresponds to the measurement outcome 1 and it's complement $A' = I - A$ corresponds to the measurement outcome 0. In general, this measurement may be imprecise (unsharp, fuzzy). If the measurement is precise (sharp) then the effect corresponds to a projection operator on H. We thus call projection operators **sharp effects**. This is consistent with the fact that effects can have a spectrum of any closed subset of $[0,1]$ while the spectrum of a projection is contained in the two element set $\{0,1\}$. Thus, a general effect is analogous to a fuzzy set while a projection is analogous to a set.

Of course, measurements may have more than two values. A general measurement is represented by a normalized positive operator-valued measure (POVM) on H and these objects have been extensively studied by Lahti and many others [5, 8, 20, 21, 29, 32]. The special case of a sharp measurement is represented by a projection-valued measure (PVM) on H. In the present work we shall only consider discrete measurements. Such measurements are represented by a finite or infinite sequence $(A_1, A_2, \ldots)$ of effects such that $\sum A_i = I$ where in the infinite case, convergence is in the strong operator topology. The effect A_i corresponds to the measurement resulting in the i-th outcome. If the quantum system is initially in the state ρ, where ρ is a density operator on H, and the measurement is performed but the result is not observed (registered) then we shall assume that the post-measurement state is given by the generalized Lüders formula [16, 19]

$$\mathcal{L}(\rho) = \sum A_i^{1/2}\rho A_i^{1/2}$$

Moreover, if the i-th outcome is observed, then the post-measurement state of the system becomes

$$A_i^{1/2}\rho A_i^{1/2}/\mathrm{tr}(\rho A_i)$$

if $\mathrm{tr}(\rho A_i) \neq 0$. We interpret $A_i^{1/2}\rho A_i^{1/2}$ as the partial (unnormalized) state ρ conditioned on the occurrence of the outcome i. A standard axiom of quantum mechanics says that the probability that the i-th outcome is observed in the state ρ is given by

$$P_\rho(A_i) = \mathrm{tr}(\rho A_i)$$

In this article it is useful to extend our notation from states, which are special cases of effects, to arbitrary effects. We define the **sequential product** of two effects A and B by $A \circ B = A^{1/2}BA^{1/2}$. It is easy to check that $A \circ B$ is again an effect and we interpret $A \circ B$ to be the effect obtained by first performing measurement A and then performing measurement B. There is considerable literature on sequential products [14, 15, 16, 18, 19] and there are physically motivated reasons for this being the correct definition.

One of the purposes of this article is to show how (discrete) measurements can be employed to construct transition effect matrices and quantum Markov chains. A **stochastic matrix** is a square matrix $M = [p_{ij}]$ where $0 \leq p_{ij} \leq 1$ and $\sum_i p_{ij} = 1$ for every j. Thus, each column of M is a discrete probability distribution. If $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a probability distribution, it is easy to check that

$$M[\lambda_1, \lambda_2, \ldots, \lambda_n]^T \tag{1.1}$$

is again a probability distribution where $[\lambda_1, \lambda_2, \ldots, \lambda_n]^T$ is the column vector obtained by transposing the row vector $[\lambda_1, \lambda_2, \ldots, \lambda_n]$. If $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the initial distribution for a classical Markov chain, then (1.1) is the distribution of the chain at one time step. We now generalize these concepts to the quantum mechanical realm. Quantum Markov chains and the closely related concepts of quantum Markov processes and quantum random walks have been studied for many years [1, 2, 3, 12, 13, 22, 28]. More recently, there have been applications of quantum random walks to quantum computation and information theory [22, 23, 33]. We shall present a different approach which we believe is closer to the classical Markov chains and is easier to visualize than some of the other approaches.

A natural quantum generalization of a stochastic matrix is a transition effect matrix (TEM). Specifically, a TEM is a matrix $\mathcal{A} = [A_{ij}]$ where A_{ij} are effects satisfying $\sum_i A_{ij} = I$ for every j. Thus, $\mathcal{A}$ is a matrix of effects for which each column is a discrete POVM or measurement. This is a generalization because if $[p_{ij}]$ is a stochastic matrix then $[p_{ij}I]$ is a TEM. A **quantum Markov chain** is a pair $(G, \mathcal{A}_k)$ where G is a directed graph and $\mathcal{A}_k = [A_{ij}^k]$, $k = 1, 2, \ldots$, is a TEM whose entry A_{ij}^k labels the edge from vertex j to vertex i (if there is no edge from vertex j to vertex i, then $A_{ij}^k = 0$). We think of the vertices of G as sites that a quantum system can occupy and A_{ij}^k is the effect (or quantum event) that the system undergoes a transition from site j to site i during the discrete time step from time $k-1$ to time k. For example, the sites may be points of location for a quantum particle or possible configurations of a quantum computer. In the former case, A_{ij}^k may be the dynamical operators for the particle and in the latter case, A_{ij}^k may be given by a program instructing the quantum computer to evolve from configuration j to configuration i during the k-th time step. If the system occupies site j at time $k-1$, then it must evolve to site $i = 1, 2, \ldots$ in one time step. Hence, $\sum_i A_{ij}^k = I$ which justifies $\mathcal{A}_k$ being a TEM. We may think of the j-th column A_{ij}^k, $i = 1, 2, \ldots$, as a measurement of the j-th site to determine where the system will evolve from this site during the k-th time step.

The chain $(G, \mathcal{A}_k)$ is initially described by a **vector state** $S = (S_1, S_2, \ldots)$

where S_i are positive trace class operators such that $\sum \operatorname{tr}(S_i) = 1$. We call the entries S_i of S **partial states**. Then $\operatorname{tr}(S_i)$ is the probability that the system is initially at site i. We use the notation

$$\operatorname{tr}(S) = (\operatorname{tr}(S_1), \operatorname{tr}(S_2), \dots)$$

so $\operatorname{tr}(S)$ is the initial distribution for the chain. The expression $A_{ij}^1 \circ S_j$ can be interpreted as the partial state for the system occupying site i at the first time step conditioned on it being in the partial state Sj at site j initially. It follows that $\sum_j A_{ij}^1 \circ S_j$ is the partial state that the system occupies site i at the first time step. This motivates our definition that

$$\mathcal{A}_1 \circ S = \left[A_{ij}^1\right] \circ [S_1, S_2, \dots]^T = [U_1, U_2, \dots]^T$$

where $U_i = \sum_j A_{ij}^1 \circ S_j$. Of course, this is a generalization of matrix multiplication. We shall later show that $\mathcal{A}_1(S) = \mathcal{A}_1 \circ S$ is again a vector state.

If the initial vector state in S, then $\mathcal{A}_1(S)$ is the vector state for the system and $\operatorname{tr}[\mathcal{A}_1(S)]$ is the distribution at one time step. Continuing this process

$$\mathcal{A}_{(k)}(S) = \mathcal{A}_k \circ (\mathcal{A}_{k-1} \circ (\cdots \mathcal{A}_1 \circ S)) \tag{1.2}$$

is the vector state for the system and $\operatorname{tr}\left[\mathcal{A}_{(k)}(S)\right]$ is the distribution at the k-th time step. We call $\mathcal{A}_{(k)}$ the **state dynamics** for the system. As we shall see, there is a dual dynamics called the operator dynamics. Although these two types of dynamics do not coincide, we shall show they are statistically equivalent. We say that the chain $(G, \mathcal{A}_k)$ is **homogeneous** (**stationary**) if $\mathcal{A}_k = A$, $k = 1, 2, \dots$.In this case the one step transitions are time independent and the dynamics is given by iterations

$$\mathcal{A}_{(k)}(S) = A \circ (A \circ (\cdots A \circ S))$$

of the single TEM A. Most of the results of this paper involve homogeneous chains which we denote by (G, A).

Besides quantum Markov chains, this article considers various types of vector states and TEMs. We shall study invariant, equilibrium and singular vector states. We also investigate projective, bistochastic, invertible and unitary TEMs. This work is a continuation of the author's investigations in [17].

2 Vector States and TEMs

Let H be a complex Hilbert space and let $\mathcal{B}(H)$ be the set of bounded linear operators on H. We denote the set of positive operators on H by $\mathcal{B}^+(H)$ and the set of positive trace class operators on H by $\mathcal{T}^+(H)$. The following sets of operators are also useful:

$\mathcal{E}(H) = \{A \in \mathcal{B}(H)\colon 0 \le A \le I\}$
$\mathcal{P}(H) = \{P \in \mathcal{B}(H)\colon P = P^* = P^2\}$
$\mathcal{D}(H) = \{D \in \mathcal{T}^+(H)\colon \operatorname{tr}(D) = 1\}$

Of course, $\mathcal{D}(H), \mathcal{P}(H) \subseteq \mathcal{E}(H) \subseteq \mathcal{B}^+(H)$. We call the elements of $\mathcal{E}(H)$ **effects**, the elements of $\mathcal{P}(H)$ **projections** or **sharp effects** and the elements of $\mathcal{D}(H)$ **density operators** or **states**. A **vector state** is a finite or infinite sequence $S = (S_1, S_2, \dots)$ where $S_i \in \mathcal{T}^+(H)$ and $\sum \operatorname{tr}(S_i) = 1$. A vector state S is **singular** if $S_i = \delta_{ij} T$ where δ_{ij} is the Kronecker delta and $T \in \mathcal{D}(H)$. We denote the one-dimensional projection onto the span of a unit vector $x \in H$ by $\widehat{x}$. The set $\mathcal{S}(H)$ of all vector states on H is clearly a convex set. For $S = (S_1, S_2, \dots) \in \mathcal{S}(H)$ we write

$$\operatorname{tr}(S) = (\operatorname{tr}(S_1), \operatorname{tr}(S_2), \dots)$$

Then $\operatorname{tr}\colon \mathcal{S}(H) \to \ell_1(\mathbb{R})$ is an affine map.

Theorem 2.1. *The extreme points of $\mathcal{S}(H)$ are precisely the singular elements of the form $(0, \dots, 0, \widehat{x}, 0, \dots)$.*

Proof. Suppose $0 < \lambda < 1$, $S = (S_1, S_2, \dots), T = (T_1, T_2, \dots) \in \mathcal{S}(H)$ and

$$(0, \dots, 0, \widehat{x}, 0, \dots) = \lambda S + (1 - \lambda) T$$

where $\widehat{x}$ is the i-th entry. Then $S_j = T_j = 0$ for $j \ne i$ and $\widehat{x} = \lambda S_i + (1 - \lambda) T_i$. Since it is well-known that $\mathcal{P}(H)$ is the extremal set of $\mathcal{E}(H)$, we have that $S_i = T_i = \widehat{x}$. Hence, $S = T$ so $(0, \dots, 0, \widehat{x}, 0, \dots)$ is extremal. Conversely, suppose $S = (S_1, S_2, \dots) \in \mathcal{S}(H)$ is extremal. Assume that $S_i, S_j \ne 0$. Without loss of generality, we can take $i = 1$, $j = 2$. Letting $\lambda = \operatorname{tr}(S_1)$, $1 - \lambda = \sum_{i=2}^{\infty} \operatorname{tr}(S_i)$ we have

$$S = \lambda \left(\frac{S_1}{\lambda}, 0, 0, \dots \right) + (1 - \lambda) \left(0, \frac{S_2}{1 - \lambda}, \frac{S_3}{1 - \lambda}, \cdots \right)$$

Since $0 < \lambda < 1$, this gives a contradiction. Hence, $S_i \neq 0$ for some i and $S_j = 0$, $j \neq i$. Letting $S_i = T$ we conclude that S is the singular state with $S_j = \delta_{ij}T$. It is well-known that we can represent T as $T = \sum \lambda_i \widehat{x}_i$ where $\lambda_i > 0$, $\sum \lambda_i = 1$. Hence

$$S = (0, \ldots, 0, T, 0, \ldots) = \sum \lambda_i (0, \ldots, 0, \widehat{x}_i, 0, \ldots)$$

Since S is extremal we have that $\lambda_j = 1$ for some j and
$S = (0, \ldots, 0, \widehat{x}_j, 0, \ldots)$. □

Since $\mathcal{S}(H)$ is the closed convex hull of its extreme points in the appropriate topology, we can gain a lot of information by studying the dynamics of a quantum Markov chain on its extreme points. Theorem 2.1 shows that these extreme points have a very simple form.

An **effect matrix** is a finite or infinite square matrix $A = [A_{ij}]$ where $A_{ij} \in \mathcal{E}(H)$ for all i, j. A **transition effect matrix** (TEM) is an effect matrix $A = [A_{ij}]$ where $\sum_i A_{ij} = I$ for every j and the convergence is in the strong operator topology in the infinite case. We also call a TEM a **stochastic effect matrix** or a **quantum stochastic matrix**. For a TEM A and a vector state $S \in \mathcal{S}(H)$ we define $A \circ S$ by $(A \circ S)_i = \sum_j A_{ij} \circ S_j$. It is easy to check that $A \circ S \in \mathcal{S}(H)$ [17]. If A, B are effect matrices of the same size on H, we say that $A \circ B$ is **defined** if $\sum_k A_{ik} \circ A_{kj}$ converges in the strong operator topology to an effect C_{ij} for all i, j. We then define $(A \circ B)_{ij} = C_{ij}$ for all i, j and write $C = A \circ B$. An effect matrix A is called a **dual** TEM if its transpose A^T is a TEM. Thus, A is a dual TEM if and only if $\sum_j A_{ij} = I$ for all i. Of course, the transpose of a TEM is a dual TEM. It is shown in [17] that if A, B are dual TEMs of the same size on H, then $A \circ B$ exists and $A \circ B$ is again a dual TEM. The following result was proved in [17] (in general $(A \circ B)^T \neq B^T \circ A^T$).

Lemma 2.2. *If A and B are TEMs of the same size on H, then*

$$\operatorname{tr}\left[(A^T \circ B^T)^T \circ S\right] = \operatorname{tr}\left[B \circ (A \circ S)\right]$$

for all $S \in \mathcal{S}(H)$.

As discussed in Section 1, a quantum Markov chain is a pair $(G, \mathcal{A}_k)$ where $\mathcal{A}_k = [A_{ij}^k]$ is a TEM, $k = 1, 2, \ldots$. For $S \in \mathcal{S}(H)$ we obtain the vector states $\mathcal{A}_k(S)$ given by (1.2) and call $\mathcal{A}_{(k)}$, $k = 1, 2, \ldots$, the **state dynamics** of the system. We now discuss a dual to the state dynamics.

Care must be taken because the matrix product is nonassociative and when we write $\mathcal{A}_n \circ \cdots \circ \mathcal{A}_2 \circ \mathcal{A}_1$ we mean

$$\mathcal{A}_n \circ \cdots \circ \{\mathcal{A}_4 \circ [\mathcal{A}_3 \circ (\mathcal{A}_2 \circ \mathcal{A}_1)]\}$$

Corresponding to the quantum Markov chain $(G, \mathcal{A}_k)$ we form the matrices

$$\mathcal{A}^{(k)} = \left(\mathcal{A}_1^T \circ \mathcal{A}_2^T \circ \cdots \circ \mathcal{A}_k^T\right)^T$$

Notice that $\mathcal{A}^{(k)}$, $k = 1, 2, \ldots$, is a TEM and we call $\mathcal{A}^{(k)}$ the **operator dynamics**. The analogy between the state and operator dynamics and the Schrödinger and Heisenberg pictures in usual quantum mechanics is discussed in [17]. Examples are given in [17] which show that $\mathcal{A}^{(k)} \circ S \neq \mathcal{A}_{(k)}(S)$ in general. However, the next result shows that the two types of dynamics are statistically equivalent.

Theorem 2.3. *If $(G, \mathcal{A}_k)$ is a quantum Markov chain, then for every $S \in \mathcal{S}(H)$ we have*

$$\operatorname{tr}\left[\mathcal{A}^{(k)} \circ S\right] = \operatorname{tr}\left[\mathcal{A}_{(k)}(S)\right]$$

Proof. We prove the result by induction on k. The result clearly holds for $k = 1$. Suppose the result holds for $k \in \mathbb{N}$. Applying Lemma 2.2 gives

$$\begin{aligned}\operatorname{tr}\left[\mathcal{A}^{(k+1)} \circ S\right] &= \operatorname{tr}\left[\left(\mathcal{A}_1^T \circ (\mathcal{A}_2^T \circ \cdots \circ \mathcal{A}_{k+1}^T)\right)^T \circ S\right] \\ &= \operatorname{tr}\left[(\mathcal{A}_2^T \circ \cdots \circ \mathcal{A}_{k+1}^T)^T \circ (\mathcal{A}_1(S))\right]\end{aligned}$$

Now $(G, \mathcal{B}_k)$ is a Markov chain where $\mathcal{B}_i = \mathcal{A}_{i+1}$, $i = 1, 2, \ldots$. Hence, by the induction hypothesis we have

$$\begin{aligned}\operatorname{tr}\left[\mathcal{A}^{(k+1)} \circ S\right] &= \operatorname{tr}\left[\mathcal{B}^{(k)} \circ \mathcal{A}_1(S)\right] = \operatorname{tr}\left[\mathcal{B}_{(k)}\left(\mathcal{A}_1(S)\right)\right] \\ &= \operatorname{tr}\left[\mathcal{A}_{(k+1)}(S)\right]\end{aligned}$$

The result follows by induction. □

3 Types of TEMs

A TEM is **projective** if all its entries are projections. If $A = [A_{ij}]$ is projective, it follows that $A_{ik}A_{jk} = 0$ for $i \neq j$ and that each column of A is a discrete PVM.

Theorem 3.1. (a) *If $A \circ A^T$ exists for a TEM A, then A is projective if and only if $A \circ A^T$ is diagonal.* (b) *If $A^T \circ A$ exists for a TEM A, then A is projective if and only if the diagonal elements of $A^T \circ A$ are I.*

Proof. (a) If A is projective and $i \neq j$, then

$$(A \circ A^T)_{ij} = \sum_k A_{ik} \circ A_{jk} = 0$$

Hence, $A \circ A^T$ is diagonal. Conversely, suppose $A \circ A^T$ is diagonal. Then for $i \neq j$ we have

$$\sum_k A_{ik} \circ A_{jk} = (A \circ A^T)_{ij} = 0$$

It follows that $A_{ik} \circ A_{jk} = 0$ for $i \neq j$. Since $\sum_j A_{jk} = I$ we have

$$A_{ik}^2 = A_{ik} \circ A_{ik} = A_{ik}$$

for every i, k. Hence, A is projective.
(b) If A is projective, then the diagonal element

$$(A^T \circ A)_{ii} = \sum_k A_{ki} \circ A_{ki} = \sum_k A_{ki} = I$$

Conversely, if $(A^T \circ A)_{ii} = I$ for every i, then

$$\sum_k A_{ki}^2 = (A^T \circ A)_{ii} = I$$

Suppose $A_{ki}^2 \neq A_{ki}$ for some k and i. Since $A_{ki}^2 \leq A_{ki}$ there exists an $x \in H$ with $\|x\| = 1$ such that $\langle A_{ki}^2 x, x\rangle < \langle A_{ki}, x, x\rangle$. Since $\sum_k A_{ki} = I$ we have

$$1 = \sum_k \left\langle A_{ki}^2 x, x\right\rangle < \sum_k \langle A_{ki} x, x\rangle = \|x\|^2 = 1$$

This is a contradiction. Hence, $A_{ki}^2 = A_{ki}$ for every i and k so A is projective. □

We denote the diagonal effect matrix whose diagonal elements are I again by I. We say that an effect matrix A is **bistochastic** if A and A^T are TEMs, **invertible** if there is an effect matrix B such that $A \circ B = B \circ A = I$ and **unitary** if $A \circ A^T = A^T \circ A = I$.

Theorem 3.2. *The following statements are equivalent for an effect matrix A.* (a) *A is invertible.* (b) *A is projective and bistochastic.* (c) *A is unitary.* (d) *A is a projective TEM and $A\circ A^T=A^T\circ A$.*

Proof. To show that (a) implies (b), suppose that $A\circ B=B\circ A=I$. Then for $i\ne j$ we have $\sum_k A_{ik}\circ B_{kj}=0$. Hence, $A_{ik}\circ B_{kj}=0$ for $i\ne j$ and it follows that [19]

$$A_{ik}B_{kj}=B_{kj}A_{ik}=0$$

for every k and $i\ne j$. Moreover, $\sum_k A_{ik}B_{ki}=I$. Hence, for every $x\in H$ and every i we have that

$$\langle x,x\rangle=\sum_k\langle A_{ik}\circ B_{ki}x,x\rangle\le\sum_k\langle A_{ik}x,x\rangle$$

By symmetry, we that $\langle x,x\rangle\le\sum_k\langle B_{ik}x,x\rangle$ for every i. Since $A_{ik}\circ B_{kj}=0$ for every k and $i\ne j$ we have

$$\begin{aligned}\langle A_{ik}\circ B'_{ki}x,x\rangle&=\langle(A_{ik}-A_{ik}\circ B_{ki})\,x,x\rangle\\&=\left\langle A_{ik}^{1/2}x,A_{ik}^{1/2}x\right\rangle-\langle A_{ik}\circ B_{ki}x,x\rangle\\&\le\sum_j\left\langle B_{kj}A_{ik}^{1/2}x,A_{ik}^{1/2}x\right\rangle-\langle A_{ik}\circ B_{ki}x,x\rangle\\&=\sum_j\langle A_{ik}\circ B_{kj}x,x\rangle-\langle A_{ik}\circ B_{ki}x,x\rangle\\&=\sum_{j\ne i}\langle A_{ik}\circ B_{kj}x,x\rangle=0\end{aligned}$$

Hence, $\langle A_{ik}\circ B'_{ki}x,x\rangle=0$ for all $x\in H$ so $A_{ik}\circ B'_{ki}=0$. It follows that [19]

$$A_{ik}B'_{ki}=B'_{ki}A_{ik}=0$$

Hence,

$$A_{ik}=B_{ki}\circ A_{ik}\le B_{ki}$$

By symmetry, $B_{ki}\le A_{ik}$ so $B_{ki}=A_{ik}$ and $B=A^T$. We also have that

$$A_{ik}=B_{ki}\circ A_{ik}=A_{ik}\circ A_{ik}=A_{ik}^2$$

so $A_{ik}\in\mathcal{P}(H)$ for every i,k. Hence, A is projective. Finally,

$$\sum_k A_{ik}=\sum_k A_{ik}\circ A_{ik}=\sum_k A_{ik}\circ B_{ki}=(A\circ B)_{ii}=I$$

and by symmetry $\sum_k A_{ki} = I$ so A is bistochastic. To show that (b) implies (c), suppose A is projective and bistochastic. We then have

$$(A \circ A^T)_{ij} = \sum_k A_{ik} \circ A_{jk} = \delta_{ij} \sum_k A_{ik} = \delta_{ij} I$$

and similarly

$$(A^T \circ A)_{ij} = \sum_k A_{ki} \circ A_{kj} = \delta_{ij} I$$

Hence, $A \circ A^T = A^T \circ A = I$ so A is unitary. Since (c) implies (a) is trivial we have that (a), (b), (c) are equivalent. Now (b) together with (c) imply (d). Also, (d) implies (b) because if (d) holds, then

$$\begin{aligned} \sum_k A_{jk} &= \sum_k A_{jk} \circ A_{jk} = (A \circ A^T)_{jj} = (A^T \circ A)_{jj} = \sum_i A_{ij} \circ A_{ij} \\ &= \sum_i A_{ij} = I \end{aligned}$$

so A is bistochastic. Hence, (a), (b), (c) and (d) are equivalent. □

A TEM A is **symmetric** if $A = A^T$, is **pure projective** if its entries are one-dimensional projections and is **commutative** if all its entries commute. If A is pure projective, then its columns correspond to orthonormal bases for H. Any 2×2 bistochastic matrix has the form

$$A = \begin{bmatrix} B & B' \\ B' & B \end{bmatrix}$$

so the rows (columns) are permutations of each other and A is commutative and symmetric. These properties need not be satisfied for 3×3 bistochastic matrices. For example, let $A, B, C \in \mathcal{E}(H)$ with $A + B + C = I$ and $AB \neq BA$. Then the following bistochastic matrix is not symmetric or commutative and its rows (columns) are not permutations of each other.

$$\begin{bmatrix} A & B & C \\ C & 0 & C' \\ B & B' & 0 \end{bmatrix}$$

Also, the following TEM is symmetric (and hence, bistochastic) but not commutative

$$\begin{bmatrix} A & B & C \\ B & B' & 0 \\ C & 0 & C' \end{bmatrix}$$

Nevertheless, we have the following result.

Theorem 3.3. *If A is a 3×3 pure projective bistochastic matrix, then the rows (columns) of A are permutations of each other (and hence, is commutative).*

Proof. Let A have the form $A=[\widehat{x}_{ij}]_{3\times 3}$. Then

$$\widehat{x}_{12}+\widehat{x}_{13}=\widehat{x}_{11}'=\widehat{x}_{21}+\widehat{x}_{31}$$

Hence, $\widehat{x}_{13}\circ\widehat{x}_{22}=\widehat{x}_{31}\circ\widehat{x}_{22}$ and it follows that

$$\left|\langle x_{22},x_{13}\rangle\right|^2\widehat{x}_{13}=\left|\langle x_{22},x_{31}\rangle\right|^2\widehat{x}_{31}$$

If $\langle x_{22},x_{13}\rangle\neq 0$, then $\widehat{x}_{13}=\widehat{x}_{31}$. If $\langle x_{22},x_{13}\rangle=0$, then $\widehat{x}_{22}=\widehat{x}_{11}$. Continuing this process, we find that any entry of a row is an entry of some other row. The result now follows. □

The following example shows that Theorem 3.3 does not hold for 4×4 pure projective bistochastic matrices and that such matrices need not be commutative. Let $\{x_1,x_2,x_3,x_4\}$ be an orthonormal bases for $\mathbb{C}^4$ and let $v=2^{-1/2}(x_3+x_4)$, $w=2^{-1/2}(x_3-x_4)$. Then $\widehat{x}_3$ and $\widehat{v}$ do not commute. The following matrix is pure projective and bistochastic.

$$A=\begin{bmatrix} \widehat{x}_1 & \widehat{x}_2 & \widehat{x}_3 & \widehat{x}_4 \\ \widehat{v} & \widehat{w} & \widehat{x}_1 & \widehat{x}_2 \\ \widehat{x}_2 & \widehat{x}_1 & \widehat{x}_4 & \widehat{x}_3 \\ \widehat{w} & \widehat{v} & \widehat{x}_2 & \widehat{x}_1 \end{bmatrix}$$

Computing the square of A we have

$$A\circ A=\begin{bmatrix} \widehat{x}_1+\frac{1}{2}\,\widehat{x}_4 & \frac{1}{2}\,\widehat{x}_4 & 0 & \widehat{x}_2+\widehat{x}_3 \\ 0 & \widehat{w}+\widehat{x}_1 & \frac{1}{2}\,\widehat{v}+\widehat{x}_2 & \frac{1}{2}\,\widehat{v} \\ \frac{1}{2}\widehat{x}_3 & \widehat{x}_2+\frac{1}{2}\,\widehat{x}_3 & \widehat{x}_1+\widehat{x}_4 & 0 \\ \widehat{v}+\widehat{x}_2 & 0 & \frac{1}{2}\,\widehat{w} & \frac{1}{2}\,\widehat{w}+\widehat{x}_1 \end{bmatrix}$$

Since A is unitary and $A \circ A$ is not unitary (it is not bistochastic), this illustrates the fact that if A and B are unitary, then $A \circ B$ need not be unitary. Hence, the unitary matrices do not form a group. This also shows that the sequential product of bistochastic matrices need not be bistochastic.

We should also point out that if A is unitary, then

$$A^T \circ (A \circ B) \neq (A^T \circ A) \circ B = B$$

in general. For example, using our previous matrix A we have

$$A^T \circ (A \circ A) = \begin{bmatrix} \widehat{x}_1 & \widehat{x}_2 & \frac{1}{2}\,\widehat{v} + \frac{1}{2}\,\widehat{w} & \frac{1}{2}\,\widehat{v} + \frac{1}{2}\,\widehat{w} \\ \widehat{v} & \widehat{w} & \widehat{x}_1 & \widehat{x}_2 \\ \widehat{x}_2 & \widehat{x}_1 & \widehat{x}_4 & \widehat{x}_3 \\ \frac{1}{2}\,\widehat{x}_4 + \frac{1}{2}\,\widehat{x}_3 & \frac{1}{2}\,\widehat{x}_4 + \frac{1}{2}\,\widehat{x}_3 & \widehat{x}_2 & \widehat{x}_1 \end{bmatrix} \neq A$$

Using the same vectors as before, the following is a symmetric, pure projective TEM that is not commutative

$$B = \begin{bmatrix} \widehat{x}_3 & \widehat{x}_2 & \widehat{x}_1 & \widehat{x}_4 \\ \widehat{x}_2 & \widehat{v} & \widehat{w} & \widehat{x}_1 \\ \widehat{x}_1 & \widehat{w} & \widehat{v} & \widehat{x}_2 \\ \widehat{x}_4 & \widehat{x}_1 & \widehat{x}_2 & \widehat{x}_3 \end{bmatrix}$$

Since B is unitary, we have $B \circ B = I$.

We say that a TEM A is **invariant** if $A^T \circ A^T = A^T$. The reason for this terminology is that the operator dynamics then satisfies

$$A^{(k)} = (A^T \circ A^T \circ \cdots \circ A^T)^T = A$$

for $k = 1, 2, \ldots$. Hence, after the first time step the distribution of the corresponding homogeneous quantum Markov chain remains invariant, independent of the initial distribution. We now give some examples of invariant TEMs. If $\{x_1, x_2\}$ is an orthonormal basis for $\mathbb{C}^2$, then

$$\begin{bmatrix} \widehat{x}_1 & \widehat{x}_1 \\ \widehat{x}_2 & \widehat{x}_2 \end{bmatrix}$$

is invariant. For a less trivial example, let $\{y_1, y_2\}$ be another orthonormal basis for $\mathbb{C}^2$ and form the TEM

$$A = \begin{bmatrix} \widehat{x}_1 + a\widehat{x}_2 & (1-a)\widehat{y}_1 \\ (1-a)\widehat{x}_2 & a\widehat{y}_1 + \widehat{y}_2 \end{bmatrix}$$

where $a = |\langle x_2, y_1\rangle|^2 / \left(1 + |\langle x_2, y_1\rangle|^2\right)$. Letting $b = |\langle x_2, y_1\rangle|^2$ we have

$$\begin{aligned} A^T \circ A^T &= \begin{bmatrix} \widehat{x}_1 + [a^2 + (1-a)^2 b]\,\widehat{x}_2 & [a(1-a)(1+b) + (1-a)b]\,\widehat{x}_2 \\ [a(1-a)(1+b) + (1-a)b]\,\widehat{y}_1 & [a^2 + (1-a)^2 b]\,\widehat{y}_1 + \widehat{y}_2 \end{bmatrix} \\ &= A^T \end{aligned}$$

Thus A, is invariant.

Theorem 3.4. *A projective TEM $A = [P_{ij}]$ is invariant if and only if $P_{ji} \leq P_{jj}$ for all i, j and $P_{ki} \circ P_{jk} = 0$ for all i, j, k with $j \neq k$.*

Proof. Suppose the conditions hold. We then have

$$\begin{aligned} (A^T \circ A^T)_{ij} &= \sum_k P_{ki} \circ P_{jk} = P_{ji} \circ P_{jj} + \sum_{k \neq j} P_{ki} \circ P_{jk} \\ &= P_{ji} \circ P_{jj} = P_{ji} = (A^T)_{jj} \end{aligned}$$

Hence, $A^T \circ A^T = A^T$ so A is invariant. Conversely, suppose A is invariant. Then

$$P_{ji} = (A^T)_{ij} = (A^T \circ A^T)_{ij} = \sum_k P_{ki} \circ P_{jk}$$

Hence,

$$\begin{aligned} P_{ji} = P_{ji} \circ P_{ji} = P_{ji} \circ \left(\sum_k P_{ki} \circ P_{jk} \right) &= P_{ji} \circ P_{jj} + \sum_{k \neq j} (P_{ji} \circ P_{ki}) \circ P_{jk} \\ = P_{ji} \circ P_{jj} \end{aligned}$$

This implies that $P_{ji} \leq P_{jj}$ [19]. It follows that $\sum_{k \neq j} P_{ki} \circ P_{jk} = 0$ so that $P_{ki} \circ P_{jk} = 0$ for $j \neq k$. $\square$

In the 2×2 case, it follows from Theorem 3.4 that every projective invariant TEM is commutative. However, this does not hold for the 3×3 case.

Let $P, Q \in \mathcal{P}(H)$ with $PQ \neq QP$. Then the following TEM is projective and invariant but not commutative.

$$\begin{bmatrix} I & P & Q \\ 0 & P' & 0 \\ 0 & 0 & Q' \end{bmatrix}$$

The next result characterizes invariance in the 2×2 case.

Theorem 3.5. *The following TEM is invariant if and only if* $B^2 = B \circ A$ *and* $A' \circ B = AA'$

$$\begin{bmatrix} A & B \\ A' & B' \end{bmatrix}$$

Proof. To check invariance we have

$$\begin{bmatrix} A & A' \\ B & B' \end{bmatrix} \circ \begin{bmatrix} A & A' \\ B & B' \end{bmatrix} = \begin{bmatrix} A^2 + A' \circ B & AA' + A' \circ B' \\ B \circ A + B'B & B \circ A' + B'^2 \end{bmatrix}$$

If $A' \circ B = AA'$, then $A^2 + A' \circ B = A^2 + AA' = A$ and

$$AA' + A' \circ B' = AA' + A' - A' \circ B = AA' + A' - AA' = A'$$

If $B \circ A = B^2$, then $B \circ A + B'B = B^2 + B'B = B$ and

$$B \circ A' + B'^2 = B - B \circ A + B'^2 = B - B^2 + B'^2 = B'B + B'^2 = B'$$

Therefore, the two conditions imply invariance. Conversely if the given TEM is invariant, then by the previous calculations we have that $A = A^2 + A' \circ B$ so

$$A' \circ B = A - A^2 = AA'$$

Moreover,

$$B = B \circ A + B'B = B \circ A + B - B^2$$

so that $B \circ A = B^2$. □

It is clear that the set of TEMs of a fixed size form a convex set. Characterizing the set of extreme points for this convex set is an open problem. For the related problem of extreme points of normalized POVM we refer the reader to [20, 21]. However, we have the following result.

Theorem 3.6. (a) *If A is projective, then A is an extremal.* (b) *If A is a 2×2 TEM, then A is extremal if and only if A is projective.*

Proof. (a) Suppose A is projective and $A = \lambda B + (1-\lambda)C$ where $0 < \lambda < 1$ and B, C are TEMs. Then $A_{ij} = \lambda B_{ij} + (1-\lambda)C_{ij}$ and since A_{ij} is extremal in $\mathcal{E}(H)$ [8, 32] we have that $B_{ij} = C_{ij} = A_{ij}$. Hence, A is extremal. (b) Suppose A is a 2×2 TEM. If A is projective, it follows from (a) that A is extremal. Conversely, suppose A is not projective. We can assume without loss of generality that A_{11} and A_{21} are not projections. Then A_{11} is not extremal in $\mathcal{E}(H)$ so $A_{11} = \lambda B + (1-\lambda)C$ $0 < \lambda < 1$, $B, C \in \mathcal{E}(H)$ with $B \neq C$. Hence,

$$A_{21} = I - A_{11} = \lambda(I - B) + (1-\lambda)(I - C)$$

and we have

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \lambda \begin{bmatrix} B & A_{12} \\ I - B & A_{22} \end{bmatrix} + (1-\lambda) \begin{bmatrix} C & A_{12} \\ I - C & A_{22} \end{bmatrix}$$

Since $B \neq C$, A is not extremal. □

4 Invariant Vector States

A vector state $S \in \mathcal{S}(H)$ is **invariant** under a TEM A if $A \circ S = S$. Of course, if S is invariant under A, then the distribution for a corresponding quantum Markov chain in the vector state S is constant.

Theorem 4.1. *Let $A = [\widehat{x}_{ij}]$ be a pure projection TEM and let $S = (S_1, S_2, \ldots)$ be a vector state. If $S_i = \lambda_i \widehat{x}_{ii}$ for all i where $\lambda_i \geq 0$, $\sum \lambda_i = 1$, then S is invariant under A. Conversely, if $\langle x_{ik}, x_{ii} \rangle \neq 0$ for all i, k with $i \neq k$ and S is invariant under A, then $S_i = \lambda_i \widehat{x}_{ii}$ for all i, where $\lambda_i \geq 0$, $\sum \lambda_i = 1$.*

Proof. Suppose $S_i = \lambda_i \widehat{x}_{ii}$ for all i, where $\lambda_i \geq 0$, $\sum \lambda_i = 1$. Then

$$(A \circ S)_i = \sum_j \widehat{x}_{ij} S_j \widehat{x}_{ij} = \sum_j \lambda_j \widehat{x}_{ij} \widehat{x}_{jj} \widehat{x}_{ij} = \lambda_i \widehat{x}_{ii} = S_i$$

Hence, $A \circ S = S$ and S is invariant under A. Conversely, suppose S is invariant under A. Then for all i we have

$$S_i = (A \circ S)_i = \sum_j \widehat{x}_{ij} S_j \widehat{x}_{ij} = \sum_j \langle S_j x_{ij}, x_{ij} \rangle \widehat{x}_{ij} \tag{4.1}$$

Moreover

$$S_i = [A \circ (A \circ S)]_i = \sum_j \widehat{x}_{ij} (A \circ S)_j \widehat{x}_{ij} = \sum_j \widehat{x}_{ij} \left(\sum_k \widehat{x}_{jk} S_k \widehat{x}_{jk} \right) \widehat{x}_{ij}$$

$$= \sum_j \widehat{x}_{ij} \left(\sum_k \langle S_k x_{jk}, x_{jk} \rangle \widehat{x}_{jk} \right) \widehat{x}_{ij}$$

$$= \langle S_i x_{ii}, x_{ii} \rangle \widehat{x}_{ii} + \sum_j \sum_{k \neq j} \langle S_k x_{ik}, x_{ik} \rangle \left| \langle x_{jk}, x_{ij} \rangle \right|^2 \widehat{x}_{ij} \tag{4.2}$$

Applying (4.1), Equation (4.2) becomes

$$\sum_{j \neq i} \langle S_j x_{ij}, x_{ij} \rangle \widehat{x}_{ij} = \sum_{k \neq i} \langle S_k x_{ik}, x_{ik} \rangle \left| \langle x_{ik}, x_{ii} \rangle \right|^2 \widehat{x}_{ii} + \sum_{j \neq i} \sum_{k \neq j} \langle S_k x_{jk}, x_{jk} \rangle \left| \langle x_{jk}, x_{ij} \rangle \right|^2 \widehat{x}_{ij} \tag{4.3}$$

From (4.1) we obtain

$$S_j = \sum_k \langle S_k x_{jk}, x_{jk} \rangle \widehat{x}_{jk}$$

Hence, for $i \neq j$ we have

$$\langle S_j x_{ij}, x_{ij} \rangle = \sum_{k \neq j} \langle S_k x_{jk}, x_{jk} \rangle \left| \langle x_{jk}, x_{ij} \rangle \right|^2 \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$\sum_{k \neq i} \langle S_k x_{ik}, x_{ik} \rangle \left| \langle x_{ik}, x_{ii} \rangle \right|^2 = 0$$

Assuming that $\langle x_{ik}, x_{ii} \rangle \neq 0$ for $i \neq k$, we conclude that $\langle S_k x_{ik}, x_{ik} \rangle = 0$ for all i, k with $i \neq k$. It follows from (4.1) that

$$S_i = \langle S_i x_{ii}, x_{ii} \rangle \widehat{x}_{ii}$$

Letting $\lambda_i = \langle S_i x_{ii}, x_{ii} \rangle$, we have that $\lambda_i \geq 0$ and

$$\sum \lambda_i = \sum_i \langle S_i x_{ii} x_{ii} \rangle = \sum_i \operatorname{tr}(S_i) = 1$$

The result now follows □

From Theorem 4.1 we have that if $\langle x_{ik}, x_{ii}\rangle \neq 0$ for $i \neq k$, then $S = (S_1, S_2, \ldots)$ is invariant under $A = [\widehat{x}_{ij}]$ if and only if $S_i = \lambda_i \widehat{x}_{ii}$ for all i, where $\lambda_i \geq 0$, $\sum \lambda_i = 1$. The next example shows that the condition $\langle x_{ik}, x_{ii}\rangle \neq 0$ for $i \neq k$ is necessary for the theorem to hold. In the exceptional case in which this inequality does not hold, there are other invariant states. Let $\{x_1, x_2\}$ be an orthonormal basis for $\mathbb{C}^2$ and form the pure projective TEM

$$A = \begin{bmatrix} \widehat{x}_1 & \widehat{x}_2 \\ \widehat{x}_2 & \widehat{x}_1 \end{bmatrix} \tag{4.5}$$

and the vector state $S = \left(\frac{1}{4}I, \frac{1}{4}I\right)$. Then $A \circ S = S$ so S is invariant under A but S does not have the form $(\lambda_1 \widehat{x}_1, (1-\lambda)\widehat{x}_1)$. In fact we have the following result.

Theorem 4.2. *A vector state $S = (S_1, S_2)$ is invariant under the TEM A given by (4.5) if and only if $S_1 = \alpha\widehat{x}_1 + \beta\widehat{x}_2$ and $S_2 = (1-\alpha-2\beta)\widehat{x}_1 + \beta\widehat{x}_2$ where $\alpha, \beta \geq 0$ and $\alpha + 2\beta \leq 1$.*

Proof. If $S = (S_1, S_2)$ where S_1 and S_2 have the given form, it is easy to check that $A \circ S = S$. Conversely, suppose $A \circ S = S$. Then

$$S_1 = \widehat{x}_1 S_1 \widehat{x}_1 + \widehat{x}_2 S_2 \widehat{x}_2, \qquad S_2 = \widehat{x}_2 S_1 \widehat{x}_2 + \widehat{x}_1 S_2 \widehat{x}_1$$

Hence, $\widehat{x}_1 S_1 = \widehat{x}_1 S_1 \widehat{x}_1$ so $\widehat{x}_1 S_1 = S_1 \widehat{x}_1$ and similarly, $\widehat{x}_1 S_2 = S_2 \widehat{x}_1$. It follows that $S_1 = \alpha_1\widehat{x}_1 + \alpha_2\widehat{x}_2$, $S_2 = \beta_1\widehat{x}_1 + \beta_2\widehat{x}_2$ where $\alpha_i, \beta_i \geq 0$, $i = 1, 2$ and $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 1$. Since $S_2 = S_1\widehat{x}_2 + S_2\widehat{x}_1$ we have that

$$S_2 = \alpha_2\widehat{x}_2 + \beta_1\widehat{x}_1$$

Hence, $\alpha_2 = \beta_2$ and $\beta_1 = 1 - \alpha_1 - 2\alpha_2$. Letting $\alpha = \alpha_1$ and $\beta = \alpha_2$ gives the result. $\square$

The next result characterizes invariant vector states for projective TEMs.

Theorem 4.3. *A vector state $S = (S_1, S_2, \ldots)$ is invariant under a projective TEM $A = [P_{ij}]$ if and only if $S_i P_{ii} = P_{ii} S_i$ and $\sum_{j \neq i} P_{ij} \circ S_j = P'_{ii} S_i$ for every i.*

Proof. If the conditions hold, then for every i we have

$$(A \circ S)_i = \sum_j P_{ij} \circ S_j = P_{ii} \circ S_i + \sum_{j \neq i} P_{ij} \circ S_j = P_{ii} S_i + P'_{ii} S_i = S_i$$

Hence, $A \circ S = S$ so S is invariant under A. Conversely, suppose S is invariant under A. Then for every i we have

$$S_i = (A \circ S)_i = P_{ii} \circ S_i + \sum_{j \neq i} P_{ij} \circ S_j \tag{4.6}$$

Taking the sequential product of both sides with P_{ii} on the left gives

$$\sum_{j \neq i} P_{ii} \circ (P_{ij} \circ S_j) = 0$$

Hence, $P_{ii} \circ (P_{ij} \circ S_j) = 0$ for every i, j with $j \neq i$. It follows that $P_{ii}(P_{ij} \circ S_j) = 0$ [19]. Applying (4.6) we have that $P_{ii}S_i = P_{ii} \circ S_i$ so $P_{ii}S_i = S_iP_{ii}$ for every i. Hence, by (4.6) we have

$$S_i = P_{ii}S_i + \sum_{j \neq i} P_{ij} \circ S_j$$

Therefore, for every i we have

$$P'{}_{ii}S_i = S_i - P_{ii}S_i = \textstyle\sum_{j \neq i} P_{ij} \circ S_j \qquad \square$$

We next consider invariant states under unitary effect matrices and invariant singular states

Theorem 4.4. *If A is unitary, then $S = (S_1, S_2, \ldots)$ is invariant under A if and only if $A_{ij} \circ S_j = A_{ij}S_i$ for all i, j.*

Proof. Suppose $A \circ S = S$. Then for every i we have $\sum_j A_{ij} \circ S_j = S_i$. For $k \neq i$ this gives

$$A_{ik}S_i = \sum_j A_{ik}(A_{ij}S_jA_{ij}) = A_{ik} \circ S_k$$

Hence, $A_{ik} \circ S_i = S_iA_{ik}$ for $k \neq i$ so that

$$A_{ii}S_i = S_iA_{ii} = A_{ii} \circ S_i$$

Thus, $A_{ij} \circ S_j = A_{ij}S_i$ for all i, j. Conversely, suppose that $A_{ij} \circ S_j = A_{ij}S_i$ for every i, j. We then have

$$(A \circ S)_i = \sum_j A_{ij} \circ S_j = \sum_j A_{ij}S_i = S_i$$

Hence, $A \circ S = S$. $\square$

A 0–I TEM is a TEM whose entries are all 0 or I. Thus, a 0–I TEM has exactly one I in each column.

Theorem 4.5. (a) *Let T be the singular vector state $T_j = \delta_{jk}S$. Then T is invariant under the TEM $S = [A_{ij}]$ if and only if $A_{kk} \circ S = S$.* (b) *A TEM A takes every singular state to a singular state if and only if A is a 0–I TEM.*

Proof. (a) If T is invariant under A, then we have

$$\delta_{ik}S = T_i = (A \circ T)_i = \sum_j A_{ij} \circ T_j = A_{ik} \circ S$$

Hence, $A_{kk} \circ S = S$. Conversely, if $A_{kk} \circ S = S$ we have that [19]

$$A_{kk}S = SA_{kk} = S$$

Therefore,

$$S = \sum_j S \circ A_{jk} = S + \sum_{j \neq k} S \circ A_{jk}$$

But this implies that $S \circ A_{jk} = 0$ for $j \neq k$. Hence, $SA_{jk} = A_{jk}S = 0$ for $j \neq k$. We conclude that

$$(A \circ T)_i = \sum_j A_{ij} \circ T_j = A_{ik} \circ S = \delta_{ik}S = T_i$$

Thus, T is invariant under A.
(b) Let $T_j = \delta_{rj}S_1$ and $W_j = \delta_{sj}S_2$ be singular vector states and suppose $A \circ T = W$. Then for every i we have

$$(A \circ T)_i = \sum_k A_{ij} \circ T_k = A_{ir} \circ S_1 = W_i = \delta_{si}S_2$$

Hence, $A_{ir} \circ S_1 = 0$ if $i \neq s$. Since this holds for every $S_1 \in \mathcal{D}(H)$, $A_{ir} = 0$ for every $i \neq s$. Therefore, $A_{ir} = \delta_{ir}I$. Since r is arbitrary A is a 0–I TEM. Conversely, suppose A is a 0–I TEM. If $T_j = \delta_{rj}S$ is a singular vector state, then we have

$$(A \circ T)_i = A_{ir} \circ S = S$$

for some i and $(A \circ T)_j = 0$ for $j \neq i$. Hence, $(A \circ T)_j = \delta_{ij}S$. We conclude that $A \circ T = W$ where $W_j = \delta_{ij}S$. Hence, A takes singular vector states to singular vector states. □

5 Equilibrium Vector States

A vector state S is an **equilibrium vector state** for a TEM A if there exists a $T \in \mathcal{S}(H)$ such that $\lim_{n\to\infty} A_{(n)}(S) = T$ in the strong operator topology. We then call T an **attractive vector state** for A. Clearly, an invariant vector state under A is an equilibrium and an attractive vector state for A. Conversely, if T is attractive for A with $\lim A_{(n)}(S) = T$, then T is invariant under A because

$$A \circ T = A \circ \lim_{n\to\infty} A_{(n)}(S) = \lim_{n\to\infty} A \circ A_{(n)}(S) = \lim_{n\to\infty} A_{(n+1)}(S) = T$$

Hence, a vector state is attractive for A if and only if it is invariant under A. It appears to be quite difficult to characterize the equilibrium vector states for a given TEM. We can nevertheless solve this problem completely for 2×2 pure projective TEMs. This is a special case of the 2×2 projective TEMs considered in [17]. However, we shall employ a different technique here which may be useful for more general cases.

Consider a pure projective TEM

$$A = \begin{bmatrix} \widehat{x}_1 & \widehat{y}_1 \\ \widehat{x}_2 & \widehat{y}_2 \end{bmatrix}$$

We shall derive the general form for the operator dynamics $A^{(n)}$. Using the notation $(x,y) = |\langle x,y\rangle|^2$ for $x,y \in \mathbb{C}^2$ we obtain

$$(A^T)^2 = A^T \circ A^T = \begin{bmatrix} \widehat{x}_1 + (y_1,x_2)\widehat{x}_2 & (y_2,x_2)\widehat{x}_2 \\ (x_1,y_1)\widehat{y}_1 & (x_2,y_1)\widehat{y}_1 + \widehat{y}_2 \end{bmatrix}$$

$$(A^T)^3 = \begin{bmatrix} \widehat{x}_1 + (x_1,y_1)(y_1,x_2)\widehat{x}_2 & \left[(y_2,x_2) + (y_1,x_2)^2\right]\widehat{x}_2 \\ \left[(x_1,y_1) + (x_2,y_1)^2\right]\widehat{y}_1 & (y_2,x_2)(x_2,y_1)\widehat{y}_1 + \widehat{y}_2 \end{bmatrix}$$

$$(A^T)^4$$

$$= \begin{bmatrix} \widehat{x}_1 + \left[(x_1,y_1) + (x_2,y_1)^2\right](y_1,x_2)\widehat{x}_2 & \left[(y_2,x_2)(x_2,y_1)^2 + (y_2,x_2)\right]\widehat{x}_2 \\ \left[(x_1,y_1) + (x_1,y_1)(y_1,x_2)^2\right]\widehat{y}_1 & \left[(y_2,x_2) + (y_1,x_2)^2\right](y_1,x_2)\widehat{y}_1 + \widehat{y}_2 \end{bmatrix}$$

It is now clear that $(A^T)^n$ has the form

$$(A^T)^n = \begin{bmatrix} \widehat{x}_1 + a_n\widehat{x}_2 & b_n\widehat{x}_2 \\ c_n\widehat{y}_1 & d_n\widehat{y}_1 + \widehat{y}_2 \end{bmatrix}$$

We then have

$$(A^T)^{n+1} = \begin{bmatrix} \widehat{x}_1 + (y_1, x_2)c_n\widehat{x}_2 & [(x_2, y_1)d_n + (x_2, y_2)]\,\widehat{x}_2 \\ [(x_1, y_1) + a_n(y_1, x_2)]\,\widehat{y}_1 & (y_1, x_2)b_n\,\widehat{y}_1 + \widehat{y}_2 \end{bmatrix}$$

$$(A^T)^{n+2}$$
$$= \begin{bmatrix} \widehat{x}_1 + (y_1, y_2)\,[(x_1, y_1) + a_n(y_1, x_2)]\,\widehat{x}_2 & [(y_1, x_2)^2 b_n + (x_2, y_2)]\,\widehat{x}_2 \\ [(x_1, y_1) + (y_1, x_2)^2 c_n]\,\widehat{y}_1 & (y_1, x_2)\,[(x_2, y_1)d_n + (x_2, y_2)]\,\widehat{y}_1 + \widehat{y}_2 \end{bmatrix}$$

We now obtain the recurrence relation:

$$b_{n+2} = (y_1, x_2)^2 b_n + (x_2, y_2)$$

Solving this recurrence relation with the initial conditions $b_1 = 1$, $b_2 = (x_2, y_2)$ we obtain:

$$\begin{aligned} b_{2n+1} &= (x_2, y_2)\left[1 + (y_1, x_2)^2 + (y_1, x_2)^4 + \cdots + (y_1, x_2)^{2n-2}\right] + (y_1, x_2)^{2n} \\ b_{2n} &= (x_2, y_2)\left[1 + (y_1, x_2)^2 + (y_1, x_2)^4 + \cdots + (y_1, x_2)^{2n-2}\right] \end{aligned}$$

If $(y_1, x_2) \neq 1$, then $0 \leq (y_1, x_2) < 1$ and we conclude that

$$\lim_{n\to\infty} b_n = \frac{(x_2, y_2)}{1 - (y_1, x_2)^2} = \frac{1 - (y_1, x_2)}{1 - (y_1, x_2)^2} = \frac{1}{1 + (y_1, x_2)}$$

If $(y_1, x_2) = 0$, then $(x_1, y_2) = 1$ and we again obtain

$$\lim_{n\to\infty} b_n = 1 = \frac{1}{1 + (y_1, x_2)}$$

Since $a_n = 1 - b_n$ we conclude that

$$\lim_{n\to\infty} a_n = 1 - \frac{1}{1 + (y_1, x_2)} = \frac{(y_1, x_2)}{1 + (y_1, x_2)}$$

Similar analyses apply to c_n and d_n so that

$$\lim_{n\to\infty} c_n = \frac{1}{1+(y_1,x_2)}, \qquad \lim_{n\to\infty} d_n = \frac{(y_1,x_2)}{1+(y_1,x_2)}$$

Letting $a=(y_1,x_2)=|\langle y_1,x_2\rangle|^2$ we conclude that in the strong operator topology we have

$$\lim_{n\to\infty}(A^T)^n = \begin{bmatrix} \widehat{x}_1 + \frac{a}{1+a}\,\widehat{x}_2 & \frac{1}{1+a}\,\widehat{x}_2 \\ \frac{1}{1+a}\,\widehat{y}_1 & \frac{a}{1+a}\,\widehat{y}_1 + \widehat{y}_2 \end{bmatrix}$$

It follows from Theorem 2.3 that every vector state $S=(S_1,S_2)$ is an equilibrium state for A and that

$$\begin{aligned}\lim_{n\to\infty} A_{(n)}(S) &= \begin{bmatrix} \widehat{x}_1 + \frac{a}{1+a}\,\widehat{x}_2 & \frac{1}{1+a}\,\widehat{y}_1 \\ \frac{1}{1+a}\,\widehat{x}_2 & \frac{a}{1+a}\,\widehat{y}_1 + \widehat{y}_2 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \\ &= \begin{bmatrix} \langle S_1x_1,x_1\rangle\widehat{x}_1 + \frac{a\langle S_1x_2,x_2\rangle\widehat{x}_2+\langle S_2,y_1,y_1\rangle\widehat{y}_1}{1+a} \\ \langle S_2y_2,y_2\rangle\widehat{y}_2 + \frac{\langle S_1x_2,x_2\rangle\widehat{x}_2+a\langle S_2y_1,y_1\rangle\widehat{y}_1}{1+a} \end{bmatrix} \end{aligned} \tag{5.1}$$

Moreover, all the attractive states have the form (5.1). Finally, the equilibrium distribution is given by

$$\lim_{n\to\infty} \operatorname{tr}\left[A_{(n)}(S)\right] = \frac{1}{1+a}\begin{bmatrix} \langle S_1x_1,x_1\rangle + \langle S_2y_1,y_1\rangle + a\operatorname{tr}(S_1) \\ \langle S_1x_2,x_2\rangle + \langle S_2y_2,y_2\rangle + a\operatorname{tr}(S_2) \end{bmatrix}$$

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