A THEOREM AND A QUESTION ABOUT EPICOMPLETE ARCHIMEDEAN LATTICE-ORDERED GROUPS

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Dedicated to the memory of Paul Conrad and to his work.

Abstract. G always denotes an archimedean ℓ-group. We call G **epicomplete** if G is divisible, σ-complete, and laterally σ-complete. $\mathcal{X}$ always denotes a Tychonoff space. $D(\mathcal{X})$ is the set of continuous $f : \mathcal{X} \longrightarrow [-\infty, +\infty]$ with $f^{-1}((-\infty, +\infty))$ dense. It is familiar and useful that every G has representations in various $D(\mathcal{X})$. It is known that (**A**) G is epicomplete with weak unit e if and only if there is a compact basically disconnected space $\mathcal{K}$ and an ℓ-group isomorphism $G \approx D(\mathcal{K})$ with $e \longmapsto \mathbf{1}_{\mathcal{K}}$; thus G admits a compatible f-ring multiplication with e as identity.

This paper concerns the situation without weak unit and involves this particular kind of representation: $H \overset{J}{\leq} D(\mathcal{X})$ means H is an ℓ-group in $D(\mathcal{X})$ situated as $[x \notin \text{closed } \mathcal{F} \implies \exists h \in H$ with $0 < h(x) < +\infty$ and $h(\mathcal{F}) = \{0\}]$ and $[p \in \beta\mathcal{X} \setminus \mathcal{X}$ and $h \in H \implies \beta h(p) = 0$ or $\pm\infty]$. A **J-representation** of G is an ℓ-group isomorphism $G \approx H \overset{J}{\leq} D(\mathcal{X})$. Our Theorem is the following analogue of (A) with $D(\mathcal{K})$ replaced by $D(\mathcal{K}, p) = \{f \in D(\mathcal{K}) : f(p) = 0\}$. (**B**) The following are equivalent for G: ι) G is epicomplete with no weak units and has a J-representation; $\iota\iota$) there is compact basically disconnected $\mathcal{K}$ with non-isolated P-point p and an ℓ-isomorphism $G \approx D(\mathcal{K}, p)$; $\iota\iota\iota$) G is epicomplete with no weak units and has a compatible reduced f-ring multiplication. Our Question is: (**C**) Are these conditions satisfied by every epicomplete G with no weak units? Or, more generally, does *every* G have a J-representation? (We conjecture: "No.")

1. Introduction

We continue the discussion in the Abstract, including various definitions and explanatory remarks. A general reference is [14]; other references are [1], [6], [10], [23] for ordered algebras, and [12] for topology.

The notation $G \leq H$ means G is a sub-ℓ-group of H. A **weak unit** of an ℓ-group H is a positive $u \in H$ with $[u \wedge h = 0 \implies h = 0]$. For example, in $C(\mathcal{X})$, a positive function is a weak unit if and only if its cozero set is dense. (The **cozero set** of f is $\operatorname{coz} f = \{x \in \mathcal{X} : f(x) \neq 0\}$. The **zero set** of f is $\mathfrak{z}f = \mathcal{X} \setminus \operatorname{coz} f$.)

G is called **σ-complete** (resp., **laterally σ-complete**) if, whenever $\{g_n\}$ is a countable subset of G with an upper bound in G (resp., which is pairwise disjoint, i. e., $m \neq n \implies |g_m| \wedge |g_n| = 0$), then $\{g_n\}$ has a supremum in G (denoted $\vee_n^G g_n$).

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The term "epicomplete" above is used because of Theorem 4.9 of [2]: in the category of archimedean ℓ-groups with ℓ-group homomorphisms, G has the property $[G \xrightarrow{\varphi} H$ categorically epic and monic $\implies \varphi$ is an isomorphism] (which defines **epicomplete**) if and only if G is divisible, σ-complete and laterally σ-complete. (Such G are called "sequentially inextensible" in [13].) Some further remarks about epicompleteness appear below (§6).

The space $\mathcal{X}$ is called **basically disconnected** (BD) if the closure of each cozero set is open. The connection of BD with σ-completeness originates in the Nakano-Stone Theorem (see [14]): $C(\mathcal{X})$ (or $C^*(\mathcal{X})$) is σ-complete if and only if $\mathcal{X}$ is BD.

Provide $[-\infty, +\infty]$ with the obvious order and compact topology. Under the pointwise order ($f \leq g$ means $f(x) \leq g(x)$ for each $x \in \mathcal{X}$), $D(\mathcal{X})$ is a lattice. Addition and multiplication in $D(\mathcal{X})$ are partially defined pointwise: $f_1 + f_2 = f_3$ means $f_1(x) + f_2(x) = f_3(x)$ for each $x \in \bigcap_{i=1,2,3} f_i^{-1}(-\infty, +\infty)$; likewise $f_1 \cdot f_2 = f_3$. Given f_1 and f_2, there may be no f_3.

Any sublattice G of $D(\mathcal{X})$ for which $[g \in G \implies -g \in G$ and $g_1, g_2 \in G \implies g_1 + g_2 \in D(\mathcal{X})$ and $g_1 + g_2 \in G]$ is an archimedean ℓ-group, and we say "G is an ℓ-group in $D(\mathcal{X})$" and write $G \leq D(\mathcal{X})$. If also $[g_,, g_2 \in G \implies g_1 \cdot g_2 \in D(\mathcal{X})$ and $g_1 \cdot g_2 \in G]$, then G is a reduced archimedean f-ring: we say "G is an f-ring in $D(\mathcal{X})$"; if, also, the constant function $1_{\mathcal{X}} \in G$, then it is the ring identity.

For any $F \subseteq D(\mathcal{X})$, we let F^* denote the set of bounded functions in F.

By Proposition 2.2 of [17], $D(\mathcal{X})$ is closed under either addition or multiplication precisely when $\mathcal{X}$ satisfies the condition [each dense cozero set, $\mathcal{S}$, in $\mathcal{X}$ is C^*-embedded in $\mathcal{X}$] (which means $\mathcal{S} \subseteq \mathcal{X} \subseteq \beta\mathcal{S}$, where $\beta\mathcal{S}$ denotes the Čech - Stone compactification of $\mathcal{S}$; see [14], [12]). That property is called **quasi-F**. It is easy to see that every basically disconnected space is quasi-F ([11]).

There are many theorems of the general form $[\forall G \exists G \approx \widehat{G} \leq D(\mathcal{X})]$, and frequently, if G has further properties then so does $\widehat{G}$, *inter alia* [31]; [17]; [19] *and* [20]; [24]; [7]; [21]; [25]; [5]. The first three of these have the virtue of $\mathcal{X}$ being as small as possible in some sense; see the remarks in §1.1 below. Our focus is the following extract from [19] (as explained in Remark 1.1(f) below)).

Definition 1. *$H \overset{J}{\leq} D(\mathcal{X})$ means H is an ℓ-group in $D(\mathcal{X})$ and both of the following conditions hold.*

($J_{\mathcal{X}}$) If $\mathcal{F}$ is closed in $\mathcal{X}$ and $x \in \mathcal{X} \setminus \mathcal{F}$, then there is $h \in H$ with $0 < h(x) < +\infty$ and $h(\mathcal{F}) = \{0\}$.

(J_∞) For each $p \in \beta\mathcal{X} \setminus \mathcal{X}$ and each $h \in H$, $\beta h(p) \in \{0, \pm\infty\}$.

*A **J-representation** of G is an ℓ-group isomorphism $G \approx \widehat{G} \overset{J}{\leq} D(\mathcal{X})$.*

In (J_∞), $\beta\mathcal{X}$ denotes the Čech - Stone compactification of $\mathcal{X}$ and $\beta h \in D(\beta\mathcal{X})$ is the unique continuous extension of h ($\in D(\mathcal{X})$). (The J in "J-representation" stands for Johnson. This name and the notation were *not* coined by the third author.)

We will, when expedient, use language such as "(J) holds in $H \leq D(\mathcal{X})$."

1.1. **Remarks.** (a) If $H \overset{J_{\mathcal{X}}}{\leq} D(\mathcal{X})$, then whenever $\mathcal{U}$ is open and $x \in \mathcal{U}$, there is a <u>bounded</u> $h \in H$ with $0 < h(x) < +\infty$ and $\operatorname{coz} h \subseteq \mathcal{U}$. (By ($J_{\mathcal{X}}$), there is $h_1 \in H$

with $0 < h_1(x) < +\infty$ and $\operatorname{coz} h_1 \subseteq \mathcal{U}$, so there is $h_2 \in H$ with $0 < h_2(x) < +\infty$ and $\operatorname{coz} h_2 \subseteq h_1^{-1}\left(\left(\frac{h_1(x)}{2}, 2h_1(x)\right)\right)$. Take $h = h_1 \wedge h_2$.)

(b) $H \overset{J}{\leq} D(\mathcal{X})$ implies $\mathcal{X} = \bigcup_{h \in H} \beta h^{-1}(\mathbb{R} \setminus \{0\})$, a union of open sets in the compact space $\beta\mathcal{X}$. Thus, $\mathcal{X}$ is open in $\beta\mathcal{X}$, so is locally compact.

(c) If G has a weak unit e_G, then there is $G \approx \widehat{G} \leq D(\mathcal{X})$ with $\widehat{e_G} = 1_{\mathcal{X}}$, $\widehat{G}$ satisfies $(J_{\mathcal{X}})$ and $\mathcal{X}$ is compact, so (J_∞) holds vacuously. This is the Yosida representation, [31]; such representation is essentially unique, [16]. The space $\mathcal{X}$ is the space of "values of G;" it is called the **Yosida space** of G and denoted $\mathcal{Y}(G, e_G)$ (or $\mathcal{Y}G$ when no confusion is possible).

(d) If $G \overset{J}{\leq} D(\mathcal{X})$ with $\mathcal{X}$ compact, then there is $u \in G^*$ with $u \geq 1_{\mathcal{X}}$. (For each $x \in \mathcal{X}$, there is $g_x \in G$ with $0 < g_x(x) < +\infty$; cover $\mathcal{X}$ by sets of the form $g_x^{-1}\left(\frac{1}{2}g_x(x), \frac{3}{2}g_x(x)\right)$, etc..) u is a weak unit, and $G \approx \frac{1}{u}G \overset{J}{\leq} D(\mathcal{X})$ is "the" representation of Remark (c).)

(e) If A is an archimedean ℓ-ring with identity e_A, then e_A is a weak unit, and the Yosida representation of Remark (c), $A \approx \widehat{A} \overset{J}{\leq} D(\mathcal{X})$, has $\widehat{A}$ as an f-ring in $D(\mathcal{X})$ and the representation is a ring isomorphism. This is the Henriksen-Johnson representation ([17]); $\mathcal{X}$ is (naturally homeomorphic to) the space of maximal ℓ-ideals of A. (This description follows [16].)

(f) To the present point, if A is a reduced archimedean f-ring, then there is $A \approx \widehat{A} \overset{J}{\leq} D(\mathcal{X})$, which is a ring isomorphism having $\widehat{A}$ an f-ring in $D(\mathcal{X})$. Such representation is essentially unique. This is the Johnson representation ([19], [20]); the $\mathcal{X}$ is a space of "certain" maximal ℓ-ideals of A. If A has an identity, this representation is that of Remark (e).

(g) Each of the Y-, HJ-, J-representations is appropriately functorial. This shan't concern us here, except for a few comments in §§6.2 and 6.3 below.

1.2. **The way ahead.** In proving Theorem B of the Abstract, we shall pay attention to inferring as much as possible from various weakenings of the hypotheses; e.g., $(J_{\mathcal{X}})$, rather than (J), relatively uniformly (ru-) complete (§2) rather than σ-complete, etc. .The statement that ends this section (Theorem 1) illustrates, by compiling the high points.

Let $\mathcal{X}$ be any (Tychonoff) space. We let

$$D_0(\mathcal{X}) = \{f \in D(\mathcal{X}) : \beta f(\beta\mathcal{X} \setminus \mathcal{X}) = \{0\}\}$$

(the functions that "vanish at ∞") and

$$D_K(\mathcal{X}) = \{f \in D(\mathcal{X}) : cl_{\mathcal{X}}(\operatorname{coz} f) \text{ is compact}\}$$

(the functions of compact support); these are convex sublattices of $D(\mathcal{X})$. Set $C_0(\mathcal{X}) = D_0(\mathcal{X}) \cap C(\mathcal{X})$ and $C_K(\mathcal{X}) = D_K(\mathcal{X}) \cap C(\mathcal{X})$; the former is a convex subring of $C(\mathcal{X})$, the latter is an ℓ-ideal. For $p \in \mathcal{X}$, set

$$D(\mathcal{X}, p) = \{f \in D(\mathcal{X}) : f(x) = 0\};$$

this is a convex sublattice of $D(\mathcal{X})$.

Suppose $D(\mathcal{X})$ is an ℓ-group, thus also an f-ring. Then $D_0(\mathcal{X})$, $D_K(\mathcal{X})$, and $D(\mathcal{X}, p)$ are ℓ-group ideals (convex ℓ-subgroups) and $D_K(\mathcal{X})$ is a ring ideal while $D_0(\mathcal{X})$ and $D(\mathcal{X}, p)$ need not be. If p is a **P-point** (meaning that the intersection of countably many neighborhoods of p is a neighborhood of p), then $D(\mathcal{X}, p)$ is a ring ideal (since then $f(p) = 0 \implies \mathfrak{z}f$ is a neighborhood of $p \implies \forall g \in D(\mathcal{X})$ $((fg)(p) = 0)$. (See [14] about P-points.)

For $\mathcal{X}$ locally compact, there is the one-point compactification $\alpha\mathcal{X} = \mathcal{X} \cup \{\alpha\}$ ([12]). Then, $f \in D_0(\mathcal{X}) \implies \exists \widehat{f} \in D(\alpha\mathcal{X}, \alpha)$ with $\widehat{f}\mid_{\mathcal{X}} = f$; this defines a natural lattice isomorphism $D_0(\mathcal{X}) \approx D(\alpha\mathcal{X}, \alpha)$ which is a group isomorphism if either $D_0(\mathcal{X})$ or $D(\alpha\mathcal{X}, \alpha)$ is a group (and, hence, both are).

For $G \leq H$, $\langle G \rangle_H \equiv \{h \in H : \exists g \in G \text{ with } |h| \leq g\}$, the ℓ-group ideal in H generated by G. When $\langle G \rangle_H = G$, i. e., when G is an ℓ-group ideal in H, we write $G \underset{\bullet}{\leq} H$.

Theorem 1. *Suppose G is divisible and $G \leq D(\mathcal{X})$.*

(Theorem 2) If G is ru-complete and $(J_{\mathcal{X}})$ holds, then $C_K(\mathcal{X}) \underset{\bullet}{\leq} G$.

(Theorem 4) G is σ-complete and (J) holds if and only if $\mathcal{X}$ is locally compact and BD, and $C_K(\mathcal{X}) \underset{\bullet}{\leq} G \underset{\bullet}{\leq} D_0(\mathcal{X})$.

(Theorem 6) If G is laterally σ-complete, $\mathcal{X}$ is not compact, and (J) holds, then $\mathcal{X}$ is locally compact, $\mathcal{X}$ and $\alpha\mathcal{X}$ are BD with α as P-point in $\alpha\mathcal{X}$, and $G \leq D_K(\mathcal{X}) = D_0(\mathcal{X}) = \langle G \rangle_{D_0\langle\mathcal{X}\rangle}$.

In §5, we sum up the information obtained thus far, then add what is required to obtain (more than) Theorem B.

This paper owes a large debt to [19], which is about f-algebras: in Theorem 1, Theorem 4 and Theorem 6 are distinctly related to [19] 5.10 and 5.13(i), respectively; one might say that the former de-ringify the latter.

2. Uniform completeness

Consider a sequence (a_n) in G, $a \in G$ and $b \in G^+$. Then (a_n) is said to be **ru-Cauchy** (b) (relatively uniformly Cauchy, regulated by b) if $[\forall k \in \mathbb{N}\ \exists\, n(k) \in \mathbb{N}$ $(n \geq n(k) \implies k\left|a_n - a_{n(k)}\right| \leq b)]$. If $[\forall k \in \mathbb{N}\ \exists\, n(k) \in \mathbb{N}$ $(n \geq n(k) \implies$ $k|a_n - a| \leq b)]$, one says that (a_n) **ru-converges to** a (b), written $a_n \longrightarrow a\,(b)$. (One thinks of an inequality "$kh \leq b$" as "$h \leq \varepsilon b$, for $\varepsilon = \frac{1}{k}$.") G is **ru-complete** if for each $b \in G^+$ every sequence that is ru-Cauchy(b) is ru-convergent (b) to some element of G (see [23]).

The result of this section is the following theorem of Stone-Weierstrass type. ($\mathcal{X}$ need not be locally compact here.)

Theorem 2. *Suppose $G \overset{J_{\mathcal{X}}}{\leq} D(\mathcal{X})$. If G is divisible and ru-complete, then $C_K(\mathcal{X}) \underset{\bullet}{\leq} G$.*

For the proof of this result, we shall: state some approximation lemmas, use these to prove the theorem, then indicate proofs of the lemmas.

Lemma 1. *Suppose G is divisible and $G \overset{J_{\mathcal{X}}}{\leq} D(\mathcal{X})$.*

1. *Suppose $\mathcal{F}$ is closed, $x \in \mathcal{X} \setminus \mathcal{F}$, and that $a \in G^+$ has $0 < a(x) < +\infty$ and $\mathcal{F} \subseteq \mathfrak{z}a$. Then, for all $\delta \in \mathbb{R}$ with $0 < \delta < a(x)$ there is $0 \leq b \in G^*$ with [$b(x) > a(x) - \delta$; $\mathcal{F} \subseteq \mathfrak{z}b$; $\forall y \in \mathcal{X}\,(b(y) < a(x) + \delta)$].*
2. *Suppose $\mathcal{F}$ is closed, $x \in \mathcal{X} \setminus \mathcal{F}$, and $0 < 2\varepsilon < 1$. Then, there is $b \in G$ with [$0 \leq b(y) < 1$ for all $y \in \mathcal{X}$; $\mathcal{F} \subseteq \mathfrak{z}b$; $1 - \varepsilon < b(x)$].*

(3) *Suppose $\mathcal{F}$ is closed, $\mathcal{K}$ is compact with $\mathcal{K}\cap\mathcal{F}=\varnothing$, and $0<2\varepsilon<1$. Then, there is $b\in G$ with $[0\le b(y)\le 1$ for all $y\in\mathcal{X}$; $\mathcal{F}\subseteq \mathfrak{z}b$; $1-\varepsilon\le b(y)$ for all $y\in\mathcal{K}]$.*

Corollary 1. *Suppose G is divisible and $G\overset{J_{\mathcal{X}}}{\le} D(\mathcal{X})$.*

(1) *If $f\in C_K(\mathcal{X})^+$ and $\varepsilon>0$, then there is $a\in G^+\cap C_K(\mathcal{X})$ with $f(x)-\frac{3}{2}\varepsilon\le a(x)\le f(x)$ for all $x\in\mathcal{X}$.*
(2) *If $f\in C_K(\mathcal{X})^+$, then there is a sequence (a_n) in $G^+\cap C_K(\mathcal{X})$ with $a_n\uparrow f$ uniformly (in the usual sense) on $\mathcal{X}$.*

Note that (2) is a version of the Stone-Weierstrass theorem.

The following result converts the uniform convergence in Corollary 1 to ru-convergence.

Lemma 2. *Suppose $G\le D(\mathcal{X})$, that $a,b\in G$ with $b\ge 0$ and that (a_n) is a sequence in G.*

(1) *If $a_n\longrightarrow a(b)$, then, for all $\mathcal{Y}\subseteq\mathcal{X}$ with $b\mid_{\mathcal{Y}}$ bounded, $a_n\longrightarrow a$ uniformly on $\mathcal{Y}$.*
(2) *Suppose b is bounded away from 0 on $\bigcup_{n\in\mathbb{N}}\operatorname{coz}a_n$. If (a_n) is uniformly Cauchy on $\mathcal{X}$, then (a_n) is ru-Cauchy (b).*

Proof of Theorem 2. It suffices to show that $C_K(\mathcal{X})^+\subseteq G$. Let $f\in C_K(\mathcal{X})^+$. By Corollary 1(2), there is a sequence (a_n) in G with $a_n\uparrow f$ uniformly on $\mathcal{X}$. Thus, $\bigcup_{n\in\mathbb{N}}\operatorname{coz}a_n\subseteq\operatorname{coz}f$ and (a_n) is uniformly Cauchy on $\mathcal{X}$. Apply Lemma 1(3) with $\mathcal{K}=\overline{\operatorname{coz}f}$, $\mathcal{F}=\varnothing$, $\varepsilon=\frac{1}{2}$: there is $b\in G^*$ with $\frac{1}{2}\le b(y)$ for each $y\in\mathcal{K}$. By Lemma 2(2), (a_n) is ru-Cauchy (b). Since G is ru-complete, there is $a\in G$ with $a_n\longrightarrow a(b)$. By Lemma 2(1), $a_n\longrightarrow a$ uniformly on $\mathcal{X}$, so $f=a\in G$. □

Proof of Lemma 1. (1) Let $\mathcal{F}$ and a be as stated, and suppose $0<\delta<a(x)$. Set $\mathcal{U}=a^{-1}(a(x)-\delta,a(x)+\delta)$. Since $(J_{\mathcal{X}})$ holds, there is $c\in G^+$ with $a(x)-\delta<c(x)<+\infty$ and $\mathcal{X}\setminus\mathcal{U}\subseteq\mathfrak{z}c$. Let $b=c\wedge a$.

(2) Let $\mathcal{F}$, x and ε be as stated. By $(J_{\mathcal{X}})$ and divisibility, there is $a\in G^+$ with $1-\frac{3}{4}\varepsilon<a(x)<1-\frac{1}{4}\varepsilon$ and $\mathcal{F}\subseteq\mathfrak{z}a$. Now apply (1) with $\delta=\frac{1}{4}\varepsilon$ to obtain b.

(3) Let $\mathcal{F}$, $\mathcal{K}$ and ε be as stated. For each $x\in\mathcal{K}$, use (2) to obtain $b_x\in G^+$ with $0\le b_x<1$, $\mathcal{F}\subseteq\mathfrak{z}b_x$ and $1-\frac{\varepsilon}{2}\le b_x(x)$. $\left\{b_x^{-1}((1-\varepsilon,1)):x\in\mathcal{K}\right\}$ forms an open cover of compact $\mathcal{K}$, so there is a finite subcover $\left\{b_{x_i}^{-1}((1-\varepsilon,1)):i=1,2,\cdots,n\right\}$. Set $b=\bigvee_{i=1}^{n}b_{x_i}$. □

Proof of Corollary 1. (1) Let $f\in C_K(\mathcal{X})^+$ and $\varepsilon>0$. Take $B\in\mathbb{R}$ with $B\ge f(y)$ for each $y\in\mathcal{X}$, and let $\mathcal{K}=f^{-1}\left(\left[\frac{\varepsilon}{2},B\right]\right)$. Since $\overline{\operatorname{coz}f}$ is compact, $\mathcal{K}$ is also. For each $x\in\mathcal{K}$, set $\mathcal{V}_x=f^{-1}\left(\left(f(x)-\frac{\varepsilon}{2},f(x)+\frac{\varepsilon}{2}\right)\right)$: $\mathcal{V}_x$ is an open set containing x; we may choose open $\mathcal{U}_x\subseteq\mathcal{V}_x$ with $x\in\mathcal{U}_x\subseteq\overline{\mathcal{U}_x}\subseteq\mathcal{V}_x$. Note that in Lemma 1(3), the number "1" can be replaced by any positive real number, by divisibility. Now use Lemma 1(3) to produce $b_x\in G$ satisfying $[\forall y\in\mathcal{X},\ 0\le b_x(y)\le f(x)-\frac{\varepsilon}{2}$;

$\mathcal{X}\setminus\mathcal{V}_x \subseteq \mathfrak{z}b_x$; $\forall y\in\overline{\mathcal{U}_x}$, $f(x)-\varepsilon \leq b_x(y)$]. The $\mathcal{U}_x$ cover $\mathcal{K}$, so there are $x_1, x_2, \cdots, x_n$ with $\mathcal{K}\subseteq \bigcup_{i=1}^{n}\mathcal{U}_{x_i}$. Set $a=\bigvee_{i=1}^{n} b_{x_i}$.

(2) This follows directly from (1). □

Proof of Lemma 2. This is completely routine. □

3. σ-completeness

The main result here is Theorem 4 below. Its proof requires some variations on Nakano-Stone, Theorem 3(1).

Theorem 3. (1) *(Nakano-Stone) $C(\mathcal{X})$, or $C^*(\mathcal{X})$, is σ-complete if and only if $\mathcal{X}$ is BD.*

(2) *$D(\mathcal{X})$ is σ-complete if and only if $\mathcal{X}$ is BD; then $D(\mathcal{X})$ is an ℓ-group and $D_K(\mathcal{X}) \underset{\bullet}{\leq} D_0(\mathcal{X}) \underset{\bullet}{\leq} D(\mathcal{X})$, with all three σ-complete.*

(3) *If $\mathcal{X}$ is locally compact, the following are equivalent.*
   - (a) *$C_K(\mathcal{X})$ is σ-complete.*
   - (b) *$D_K(\mathcal{X})$ is σ-complete.*
   - (c) *If $\mathcal{U}$ is a cozero set in $\mathcal{X}$ with compact closure, $\overline{\mathcal{U}}$, then $\overline{\mathcal{U}}$ is BD.*
   - (d) *$\mathcal{X}$ is BD.*
   - (e) *$C_0(\mathcal{X})$ is σ-complete.*
   - (f) *$D_0(\mathcal{X})$ is σ-complete.*

Theorem 4. *$G \overset{J}{\leq} D(\mathcal{X})$ and G is both divisible and σ-complete if and only if $\mathcal{X}$ is locally compact and BD and $C_K(\mathcal{X}) \underset{\bullet}{\leq} G \underset{\bullet}{\leq} D_0(\mathcal{X})$.*

The proofs require some lemmas.

Lemma 3. *If $I \underset{\bullet}{\leq} G$ and G is either divisible or σ-complete, then I is the same.*

The easy proof is omitted.

Lemma 4. (1) *Suppose $G \overset{J_{\mathcal{X}}}{\leq} D(\mathcal{X})$. If $\{f_\gamma : \gamma\in\Gamma\}\cup\{g\}\subseteq G$, then $g=\bigvee_{\gamma\in\Gamma} f_\gamma$ in G if and only if this is so in $D(\mathcal{X})$.*

(2) *Suppose $g=\bigvee_{\gamma\in\Gamma} f_\gamma$ in $D(\mathcal{X})$ or in $G \overset{J_{\mathcal{X}}}{\leq} D(\mathcal{X})$. If $\mathcal{W}$ is open and $\mathfrak{z}f_\gamma \supseteq \mathcal{W}$ for each $\gamma\in\Gamma$, then $\mathfrak{z}g\supseteq\mathcal{W}$.*

(3) *Suppose that $\mathcal{X}$ is a locally compact space and that $G \overset{J_{\mathcal{X}}}{\leq} D(\mathcal{X})$. If $\{f_\gamma : \gamma\in\Gamma\}\cup\{f\}\subseteq G$, then $f=\bigvee_{\gamma\in\Gamma} f_\gamma$ in G if and only if $\left\{x\in\mathcal{X} : f(x)=\bigvee_{\gamma\in\Gamma} f_\gamma(x)\right\}$ is dense in $\mathcal{X}$. In particular, this is true in $C(\mathcal{X})$ and $D(\mathcal{X})$ for locally compact $\mathcal{X}$.*

Proof. (1) Suppose $f\in D(\mathcal{X})$ with $g\geq f$. If $g\neq f$, then there is $0\neq h\in G^+$ with $\operatorname{coz} h\subseteq (g-f)^{-1}\left(\left(\frac{1}{n},+\infty\right]\right)$ for some $n\in\mathbb{N}$, by ($J_{\mathcal{X}}$). As noted in Remark

1.1(a), one can choose such h bounded; say $h(x) \leq m \in \mathbb{N}$ for all $x \in \mathcal{X}$. Then,

$$f(x) \leq g(x) - \frac{h(x)}{mn}, \text{ for each } x \in \mathcal{X}.$$

Thus,

$$mnf \leq mng - h;$$

the function on the right is in G.

If $g = \bigvee_{\gamma\in\Gamma} f_\gamma$ in G then g is an upper bound for $\{f_\gamma : \gamma \in \Gamma\}$ in $D(\mathcal{X})$. Also, $mng = \bigvee_{\gamma} mnf_\gamma$ in G, so $mng - h$ is not an upper bound for $\{mnf_\gamma : \gamma \in \Gamma\}$ for any $0 \lneq h \in G$. If $f \in D(\mathcal{X})$ with $f \lneq g$, then the argument above shows that f is not an upper bound for $\{f_\gamma : \gamma \in \Gamma\}$.

If $f = \bigvee_{\gamma\in\Gamma} f_\gamma$ in $D(\mathcal{X})$ and $f \in G$, then it is clear that $f = \bigvee_{\gamma\in\Gamma} f_\gamma$ in G.

(2) We show: within $D(\mathcal{X})$ or $G \overset{J_\mathcal{X}}{\leq} D(\mathcal{X})$, for any $\{f_\gamma : \gamma \in \Gamma\}$ and open $\mathcal{W}$ with $\mathfrak{z}f_\gamma \supseteq \mathcal{W}$ for each γ, if $h \geq f_\gamma$ for each γ and $h(p) > 0$ for some $p \in \mathcal{W}$, then there is $h' \in G$ (or $D(\mathcal{X})$) with $h \gneq h' \geq f_\gamma$ for each $\gamma \in \Gamma$. For, we may choose v in G (or in $D(\mathcal{X})$) with $v(p) > 0$ and $\operatorname{coz} v \subseteq \mathcal{W}$, then set $u = |h| \wedge v$. Take $h' = h - u$.

(3) This, in the case $G \in \mathbf{W}$ and $\mathcal{X} = \mathcal{Y}G$ (the Yosida space of G), is Lemma 4.1(a) in [3]. The proof there, which uses the Baire category theorem, works in this setting, where the space $(\mathcal{X})$ is locally compact.. □

Lemma 5. *If $\mathcal{X}$ has the property [for each $p \in \mathcal{X}$, there is an open set $\mathcal{U}$ containing p with $\overline{\mathcal{U}}$ BD], then $\mathcal{X}$ is BD.*

Proof. Suppose $f \in C(\mathcal{X})$. To show that $\overline{\operatorname{coz} f}$ is open, let $p \in \overline{\operatorname{coz} f}$. Choose $\mathcal{U}$ open with $p \in \mathcal{U}$ and $\overline{\mathcal{U}}$ BD. Then $cl_{\overline{\mathcal{U}}}(\operatorname{coz}(f|_{\overline{\mathcal{U}}}))$ is open in $\overline{\mathcal{U}}$ and contains p, so there is an open set $\mathcal{V}$ in $\mathcal{X}$ with $p \in \mathcal{V} \cap \overline{\mathcal{U}} \subseteq cl_{\overline{\mathcal{U}}}(\operatorname{coz}(f|_{\overline{\mathcal{U}}})) \subseteq \overline{\operatorname{coz} f}$. But $p \in \mathcal{V} \cap \mathcal{U} \subseteq \overline{\operatorname{coz} f}$. □

Proof of Theorem 3. (2) Suppose $\mathcal{X}$ is BD. Recall that to show that $D(\mathcal{X})$ is σ-complete it suffices to show that whenever $\{f_n : n \in \mathbb{N}\} \subseteq D(\mathcal{X})^+$ has upper bound $f \in D(\mathcal{X})$ it follows that there is $g = \bigvee_{n\in\mathbb{N}} f_n \in D(\mathcal{X})$. Set $\mathcal{Y} = f^{-1}(\mathbb{R})$. Since $f \in D(\mathcal{X})$, $\mathcal{Y}$ is dense in $\mathcal{X}$; since $\mathcal{X}$ is BD and $\mathcal{Y}$ is a cozero set, $\mathcal{Y}$ is C^*-embedded in $\mathcal{X}$, so $\mathcal{Y}$ is BD. By Nakano-Stone, there is $h \in C(\mathcal{Y})$ with $h = \bigvee_{n\in\mathbb{N}} (f_n|_\mathcal{Y})$, since each $f_n|_\mathcal{Y} \leq f|_\mathcal{Y}$. Since $\mathcal{Y}$ is C^*-embedded in $\mathcal{X}$, there is $g \in D(\mathcal{X})$ with $h = g|_\mathcal{Y}$. Clearly, $g = \bigvee_{n\in\mathbb{N}} f_n$. Conversely, when $D(\mathcal{X})$ is σ-complete, so is its ideal $C(\mathcal{X})$, by Lemma 3: $\mathcal{X}$ is BD. By Lemma 4, $D_K(\mathcal{X})$ and $D_0(\mathcal{X})$ are σ-complete.

(3) We show (a) $\implies$ (c) $\implies$ (d), which shows that $D(\mathcal{X})$ is a σ-complete ℓ-group, by (2), so $D_K(\mathcal{X})$ and $D_0(\mathcal{X})$ are, also (i.e., (b) and (f) hold). But, then, (b) $\implies$ (a) and (f) $\implies$ (e) $\implies$ (a), by Lemma 3.

Assume (a): $C_K(\mathcal{X})$ is σ-complete. Now suppose that $\mathcal{U}$ is a cozero set in $\mathcal{X}$ with $\mathcal{E} \equiv \overline{\mathcal{U}}$ compact. We show $C(\mathcal{E})$ is σ-complete and apply Nakano-Stone to conclude that (c) holds. To that end, suppose $\{f_n : n \in \mathbb{N}\} \cup \{f\} \subseteq C(\mathcal{E})^+$ with $f_n \leq f$ for each $n \in \mathbb{N}$. By local compactness, $\mathcal{E}$ has a neighborhood $\mathcal{V}$ with compact closure; then f and each of the f_n have extensions, $\overline{f}$ and $\overline{f_n}$, in $C(\mathcal{X})$ satisfying [for each $n \in \mathbb{N}$, $f_n \le f$ and $\operatorname{coz}\overline{f}$, $\operatorname{coz}\overline{f_n} \subseteq \mathcal{V}$ (§3.11 in [14])]. Thus, $\overline{f}$ and the $\overline{f_n}$ are in $C_K(\mathcal{X})$, so there is $h \in C_K(\mathcal{X})$ with $h = \bigvee_{n\in\mathbb{N}} \overline{f_n}$. The embedding $C_K(\mathcal{X}) \le C(\mathcal{X})$ preserves all existing joins, so $h = \bigvee_{n\in\mathbb{N}} \overline{f_n}$ in $C(\mathcal{X})$, and thus $\mathcal{P} = \left\{x \in \mathcal{X} : h(x) = \bigvee_{n\in\mathbb{N}} \overline{f_n}(x)\right\}$ is dense in $\mathcal{X}$ (Lemma 4). Since $\mathcal{E}$ is regular closed, $\mathcal{P} \cap \mathcal{E}$ is dense in $\mathcal{E}$. But $\mathcal{P} \cap \mathcal{E} = \left\{x \in \mathcal{E} : h(x) = \bigvee_{n\in\mathbb{N}} \overline{f_n}(x)\right\}$; so, by Lemma 4, $h|_{\mathcal{E}} = \bigvee_{n\in\mathbb{N}} \left(\overline{f_n}|_{\mathcal{E}}\right) = \bigvee_{n\in\mathbb{N}} f_n$ in $C(\mathcal{E})$.

Lemma 5 shows that (c) $\Longrightarrow$ (d). □

Lemma 6. (1) *If G is divisible and σ-complete, then G is a vector lattice.*

(2) *(Lemma 39.2 and Theorem 39.4 in* [23]*) A σ-complete vector lattice is ru-complete.*

(3) *If G is divisible and σ-complete, then G is ru-complete.*

Proof. The scalar multiplication in (1) is defined: for $\alpha \in \mathbb{R}$ and $g \in G$, take rationals $\alpha_n \uparrow \alpha$ and set $\alpha g = \bigvee_{n\in\mathbb{N}} \alpha_n g$. We omit the details. (3) follows from (1) and (2). □

Corollary 2. *If $G \overset{J_{\mathcal{X}}}{\le} D(\mathcal{X})$ and G is both divisible and σ-complete, then $C_K(\mathcal{X}) \underset{\bullet}{\le} G$.*

Proof. Apply Lemma 6 and Theorem 2. □

Much of the further calculation required for the proof of Theorem 4 is provided by the following.

Proposition 1. *Suppose $\mathcal{X}$ is locally compact, $G \le D(\mathcal{X})$, and G is divisible and σ-complete.*

(1) *If $(J_{\mathcal{X}})$ holds, then $\{f \in D_0(\mathcal{X}) : \exists a \in G \text{ with } |f| \le a\} \subseteq G$.*

(2) *If $(J_{\mathcal{X}})$ and (J_∞) both hold, then $G \subseteq D_0(\mathcal{X})$.*

Proof. (1) Suppose $f \in D_0(\mathcal{X})$ and $a \in G$ with $0 \le f \le a$. Since $f \in D_0(\mathcal{X})$, $\mathcal{U} = \{x \in \mathcal{X} : 0 < f(x) < +\infty\}$ is a σ-compact open subset of $\mathcal{X}$.

Fix $n \in \mathbb{N}$. By local compactness, for each $x \in \mathcal{U}$, there are open sets $\mathcal{U}_x$ and $\mathcal{V}_x$ with

$$x \in \mathcal{U}_x \subseteq \overline{\mathcal{U}_x} \subseteq \mathcal{V}_x = f^{-1}\left(\left(f(x) - \frac{1}{2n}, f(x) + \frac{1}{2n}\right)\right) \subseteq \mathcal{U}$$

with $\overline{\mathcal{U}_x}$ compact. There is $a_x \in C_K(\mathcal{X})$ with $0 \le a_x \le f(x) - \frac{1}{2n}$, $a_x|_{\overline{\mathcal{U}_x}} = \left(f(x) - \frac{1}{2n}\right) \vee 0$ and $a_x(\mathcal{X} \setminus \mathcal{V}_x) = \{0\}$; by Corollary 2, $a_x \in G$. $\{\mathcal{U}_x : x \in \mathcal{X}\}$ covers $\mathcal{U}$, so there is a countable subcover, $\{\mathcal{U}_{x_i} : i \in \mathbb{N}\}$. Set $a_n = \bigvee_{i\in\mathbb{N}} (a \wedge a_{x_i})$ in G. Since $0 \le a \wedge a_{x_i} \le f$ for each i, we have $a_n(x) = \bigvee_{i\in\mathbb{N}} (a \wedge a_{x_i})(x) \le f(x)$ on a dense subset of $\mathcal{X}$, by Lemma 4. Hence, $0 \le a_n \le f$. If $f(y) \in \mathbb{R}$, then $y \in \mathcal{U}_{x_i}$ for some i, and a straightforward calculation shows that $f(y) - \frac{1}{n} \le a_n(y)$ Thus, $f - \frac{1}{n} \le a_n \le f$.

Now set $b = \bigvee_{n\in\mathbb{N}} a_n$ in G. As before, $b \leq f$ by Lemma 4; [$f - \frac{1}{n} \leq b \leq f$ for each $n \in \mathbb{N}$] $\implies f = b \in G$.

(2) Suppose $a \in G^+$. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = (\beta a)^{-1}([1,n])$. We show that the hypotheses imply $\beta a(\beta\mathcal{X} \setminus \mathcal{X}) \subseteq \{0\}$. Each $\mathcal{U}_n$ is compact and, since (J_∞) holds, contained in $\mathcal{X}$: $\mathcal{U}_n = a^{-1}([1,n])$. By local compactness, there are, for each $n \in \mathbb{N}$, $a_n \in C_K(\mathcal{X})$ with $0 \leq a_n \leq 1$ and $a_n(\mathcal{U}_n) = \{1\}$. By Corollary 2, each $a_n \in G$. Set $b = \bigvee_{n\in\mathbb{N}} (a_n \wedge a)$ in G. Since $b(\mathcal{U}_n) = \{1\}$ for each n, we have $\beta b\left(\overline{\bigcup_{n\in\mathbb{N}} \mathcal{U}_n}^{\beta\mathcal{X}}\right) = \{1\}$. Hence, $\beta a(x) \geq 1 \implies x \in \overline{\bigcup_{n\in\mathbb{N}} \mathcal{U}_n}^{\beta\mathcal{X}} \implies \beta b(x) = 1 \implies x \in \mathcal{X}$, by (J_∞). Since $\beta a(\beta\mathcal{X} \setminus \mathcal{X}) \subseteq \{0, \pm\infty\}$, the result follows. □

Proof of Theorem 4. If $G \overset{J}{\leq} D(\mathcal{X})$ and G is divisible and σ-complete, then $C_K(\mathcal{X}) \underset{\bullet}{\leq} G$, by Corollary 2, and $C_K(\mathcal{X})$ is σ-complete, by Lemma 3. By Remark 1.1(b), $\mathcal{X}$ is locally compact, so $\mathcal{X}$ is BD (by Theorem 3, where one sees that $D(\mathcal{X})$ is an ℓ-group and $D_0(\mathcal{X}) \underset{\bullet}{\leq} D(\mathcal{X})$). Now apply Proposition 1 to conclude that $G \underset{\bullet}{\leq} D_0(\mathcal{X})$.

The converse results from the following three observations. First, if $\mathcal{X}$ is BD and $G \underset{\bullet}{\leq} D_0(\mathcal{X})$, then G is divisible and σ-complete, by Theorem 3 and Lemma 3. Second, if $\mathcal{X}$ is locally compact and $C_K(\mathcal{X}) \leq G \leq D(\mathcal{X})$, then $G \overset{J_{\mathcal{X}}}{\leq} D(\mathcal{X})$ (because $C_K(\mathcal{X}) \overset{J_{\mathcal{X}}}{\leq} D(\mathcal{X})$ and $(J_{\mathcal{X}})$ is "inherited up"). Finally, if $G \subseteq D_0(\mathcal{X})$, then $G \overset{J_\infty}{\leq} D(\mathcal{X})$ (because $D_0(\mathcal{X}) \subseteq D(\mathcal{X})$ satisfies (J_∞) and (J_∞) is "inherited down"). □

4. Lateral σ-completeness

The main result here is Theorem 6. We begin with two clarifying results, the second of which is a variation on Nakano-Stone for lateral σ-completeness.

Recall that $\alpha\mathcal{X}$ denotes the one-point compactification of the locally compact space $\mathcal{X}$: $\alpha\mathcal{X} = \mathcal{X} \cup \{\alpha\}$.

Proposition 2. *Suppose $\mathcal{X}$ is locally compact, not compact.*

(1) *If $\alpha\mathcal{X}$ is BD, then $\mathcal{X}$ is BD.*
(2) *If $\mathcal{X}$ is BD and α is a P-point in $\alpha\mathcal{X}$, then $\alpha\mathcal{X}$ is BD.*

In what follows, we extend the definition of lateral σ-completeness to a lattice L with bottom 0: L is laterally σ-complete if whenever $\{f_n : n \in \mathbb{N}\} \subseteq L$ satisfies $[n \neq m \implies f_n \wedge f_m = 0$ (f_n and f_m are "disjoint")], $\bigvee_{n\in\mathbb{N}} f_n$ exists in L.

Theorem 5. (1) *$D(\mathcal{X})^+$ is laterally σ-complete if and only if $\mathcal{X}$ is BD.*
(2) *If $\mathcal{X}$ is locally compact and not compact, then $D_0(\mathcal{X})^+$ is laterally σ-complete if and only if $\alpha\mathcal{X}$ is BD with α a P-point.*

Regarding Theorem 5(2) vs. Theorem 3(3), see Proposition 5 below. Also, regarding the ubiquity of the situation [$\alpha\mathcal{X}$ is BD with α a P-point], see §6.2 below.

Theorem 6. *Suppose* $G \overset{J}{\leq} D(\mathcal{X})$ *with* $\mathcal{X}$ *not compact. If* G *is divisible and laterally* σ*-complete, then:*

(1) $\mathcal{X}$ *is (locally compact and) BD with* α *a P-point in* $\alpha\mathcal{X}$*, and*
(2) $G \leq D_K(\mathcal{X}) = D_0(\mathcal{X}) = \langle G \rangle_{D_0(\mathcal{X})}$.

Note: Conclusion (2) makes sense, since conclusion (1) and Proposition 2 imply $\mathcal{X}$ is BD, so $D(\mathcal{X})$ is an ℓ-group and $D_K(\mathcal{X}) \underset{\bullet}{\leq} D_0(\mathcal{X}) \underset{\bullet}{\leq} D(\mathcal{X})$, by Theorem 3.

We proceed to the proofs, with occasional interjection of lemmata.

Lemma 7. *If* $\mathcal{Y}$ *has a point* p *with the property [* $\overline{\operatorname{coz} g}$ *is open whenever* $g \in C(\mathcal{Y})$ *has* $g(p) = 0$ *], then* $\mathcal{Y}$ *is BD.*

Proof. Suppose $0 \leq f \in C(\mathcal{Y})$. We wish to show that $\overline{\operatorname{coz} f}$ is open; this is true, by hypothesis when $f(p) = 0$. There is, thus, no loss of generality in assuming that $0 \leq f \leq 1$ and $f(p) = 1$. Suppose $y \in \overline{\operatorname{coz} f}$; we need to identify an open set $\mathcal{U}$ with $y \in \mathcal{U} \subseteq \overline{\operatorname{coz} f}$. When $y \in \operatorname{coz} f$, take $\mathcal{U} = \operatorname{coz} f$, so suppose $y \in \overline{\operatorname{coz} f} \setminus \operatorname{coz} f$. Choose $0 \leq h \in C(\mathcal{Y})$ with $h(p) = 0$, $h(y) = 1$, and with $\operatorname{coz} h \subseteq f^{-1}\left(\left[0, \frac{1}{2}\right]\right)$. Then $(h \wedge f)(p) = 0$ and $y \in \overline{\operatorname{coz} hf} \subseteq \overline{\operatorname{coz} f}$, and $\overline{\operatorname{coz}(h \wedge f)}$ is open, by hypothesis. Take $\mathcal{U} = \overline{\operatorname{coz}(h \wedge f)\, yt}$. $\square$

Proof of Proposition 2. (1) If $\alpha\mathcal{X}$ is BD, then $C(\alpha\mathcal{X})$ is σ-complete, by Theorem 3(1). Since $C_K(\mathcal{X}) \underset{\bullet}{\leq} C_0(\mathcal{X}) \approx C(\alpha\mathcal{X}, \alpha) \underset{\bullet}{\leq} C(\alpha\mathcal{X})$, it follows that $C_K(\mathcal{X})$ is σ-complete (Lemma 3), so $\mathcal{X}$ is BD, By Theorem 3(3).

(2) When $\mathcal{X}$ is BD, with α a P-point in $\alpha\mathcal{X}$, we apply Lemma 7. If $0 \leq f \in C(\alpha\mathcal{X})$ with $f(\alpha) = 0$, then $\alpha \in \operatorname{int}(\mathfrak{z}f)$, since α is a P-point of $\alpha\mathcal{X}$, so $\overline{\operatorname{coz} f}^{\alpha\mathcal{X}} = \overline{\operatorname{coz} f}^{\mathcal{X}}$; since $\mathcal{X}$ is BD, the latter set is open. $\square$

In the following Proposition, we describe the basic method (called **counting by threes)** of inferring from lateral σ-completeness. This process will be employed several times in the proofs of Theorems 5 and 6. A picture will make the assertions clear (the last one following from Lemma 4(2)).

Proposition 3. (1) *In* $D(\mathcal{X})$*, say that* $\{f_1, f_2, f_3, \cdots\}$ *is* ***linear*** *if* $|m - n| \geq 3 \implies f_m \wedge f_n = 0$.

(2) *In* $\mathcal{X}$*, let* $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3 \cdots$ *be closed sets with* $\mathcal{U}_n \subseteq \operatorname{int}\mathcal{U}_{n+1}$ *for each* n*. For each* $n > 1$*, set* $\mathcal{K}_n = \mathcal{U}_n \setminus \operatorname{int}\mathcal{U}_{n-1}$*, and for each* $n > 2$*, set* $\mathcal{F}_n = \mathcal{U}_{n-2} \cup (\mathcal{X} \setminus \operatorname{int}\mathcal{U}_{n+1})$*; also, set* $\mathcal{K}_1 = \mathcal{U}_1$ *and* $\mathcal{F}_i = \mathcal{X} \setminus \operatorname{int}\mathcal{U}_{i+1}$ *for* $i = 1, 2$*. These are closed sets,* $\bigcup_{n=1}^{\infty} \mathcal{K}_n = \bigcup_{n=1}^{\infty} \mathcal{U}_n$*, and* $\mathcal{K}_n \cap \mathcal{F}_n = \varnothing$ *for each* n*.*

(3) *Suppose* $\{\mathcal{U}_n, \mathcal{K}_n, \mathcal{F}_n\}$ *is as in (2), and* $\{f_n : n \in \mathbb{N}\} \subseteq D(\mathcal{X})^+$ *has* $\mathfrak{z}f_n \supseteq \mathcal{F}_n$ *for each* $n \geq 3$*. Then* $\{f_1, f_2, \cdots\}$ *is linear.*

(4) *Suppose* $\{\mathcal{U}_n, \mathcal{K}_n \mathcal{F}_n, f_n\}$ *is as in (3). Then each of the families* $\{f_{3n-2}\}$*,* $\{f_{3n-1}\}, \{f_{3n}\}$ *is a pairwise disjoint family. There may exist in* $D(\mathcal{X})$*, or in a* $G \leq D(\mathcal{X})$

$$f^{(2)} = \bigvee_{n=1}^{\infty} f_{3n-2};\; f^{(1)} = \bigvee_{n=1}^{\infty} f_{3n-1};\; f^{(0)} = \bigvee_{n=1}^{\infty} f_{3n};$$

if so, then

$$f = f^{(2)} \vee f^{(1)} \vee f^{(0)} = \bigvee_{n=1}^{\infty} f_n,$$

in $D(\mathcal{X})$, *or* G. *Moreover,* $\operatorname{int}\left(\mathcal{X} \setminus \bigcup_{n\in\mathbb{N}} \mathcal{U}_n\right) \subseteq \mathfrak{z}f$ *in* $D(\mathcal{X})$, *and in* G, *provided* $G \overset{J_{\mathcal{X}}}{\leq} D(\mathcal{X})$.

Proof of Theorem 5. (1) Suppose $\mathcal{X}$ is BD and $\{f_n : n \in \mathbb{N}\}$ is a pairwise disjoint family in $D(\mathcal{X})^+$, so that $\operatorname{coz} f_n \cap \operatorname{coz} f_m = \varnothing$ when $n \neq m$. Then $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \operatorname{coz} f_n$ is a cozero set, so $\overline{\mathcal{U}}$ is open. Define $f \in D(\mathcal{U})$ as $f(x) = f_n(x)$ when $x \in \operatorname{coz} f_n$. Then let $f_1 \in D(\overline{\mathcal{U}})$ extend f ($\mathcal{U}$ is C^*-embedded in $\mathcal{X}$, thus in $\overline{\mathcal{U}}$), and define $f_2 \in D(\mathcal{X})$ by [$f_2 = f_1$ on $\overline{\mathcal{U}}$: $f_2 |_{\mathcal{X}\setminus\overline{\mathcal{U}}} = 0$]. Then $f_2 = \bigvee_{n\in\mathbb{N}} f_n$.

Now suppose that $D(\mathcal{X})^+$ is laterally σ-complete, and that $g \in C(\mathcal{X})$ with $0 \leq g \leq 1$. For each $n \in \mathbb{N}$, set $\mathcal{U}_n = g^{-1}\left(\left[\frac{1}{n}, 1\right]\right)$ and $\mathcal{K}_n$, $\mathcal{F}_n$ as in Proposition 3:

$$\begin{aligned}
\mathcal{K}_1 &= g^{-1}(\{1\}) \\
\mathcal{K}_n &\subseteq g^{-1}\left(\left[\frac{1}{n}, \frac{1}{n-1}\right]\right), \text{ for } n > 1; \\
\mathcal{F}_1 &= g^{-1}\left(\left[0, \frac{1}{2}\right)\right); \\
\mathcal{F}_2 &= g^{-1}\left(\left[0, \frac{1}{3}\right)\right); \\
\mathcal{F}_n &= g^{-1}\left(\left[0, \frac{1}{n+1}\right) \cup \left[\frac{1}{n-2}, 1\right]\right).
\end{aligned}$$

For each n, $\operatorname{int}(\mathfrak{z}g) \subseteq \mathcal{F}_n$, and $\mathcal{K}_n$ and $\mathcal{F}_n$ are completely separated: there are $f_n \in C(\mathcal{X})$ with [$0 \leq f_n \leq 1$; $\mathcal{F}_n \subseteq \mathfrak{z}f_n$; $f_n(\mathcal{K}_n) = \{1\}$]. Thus, the f_n form a linear set, and we have $f^{(2)}, f^{(1)}, f^{(0)} \in D(\mathcal{X})$, and hence $f = f^{(2)} \vee f^{(1)} \vee f^{(0)} \in D(\mathcal{X})$ as in Proposition 3. Since $\operatorname{coz} g = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$, Proposition 3 implies that $f(\operatorname{coz} g) = \{1\}$ and that $f(\operatorname{int} \mathfrak{z}g) = \{0\}$: i.e., that $\overline{\operatorname{coz} g}$ is open.

(2) We suppose $\mathcal{X}$ is locally compact and not compact. If $\alpha\mathcal{X}$ is BD; then $\mathcal{X}$ is BD, by Proposition 2. If $\{f_n : n \in \mathbb{N}\}$ is a pairwise disjoint family in $D_0(\mathcal{X})^+$, then $f = \bigvee_{n\in\mathbb{N}} f_n \in D(\mathcal{X})$, by part (1) of this theorem. If, also, α is a P-point in $\alpha\mathcal{X}$, then each $\mathfrak{z}f_n$ is a neighborhood of α, as is $\bigcap_{n\in\mathbb{N}} \mathfrak{z}f_n$. This implies that $f(\alpha) = 0$; i.e., $f \in D_0(\mathcal{X})$.

Now suppose $D_0(\mathcal{X})$ is laterally σ-complete. We show, first, that α is a P-point. To that end, let $\mathcal{U}$ be a G_δ containing α; then $\mathcal{U}$ contains a set $\bigcap_{n\in\mathbb{N}} (\mathcal{X} \setminus \mathcal{U}_n)$, where each $\mathcal{U}_n$ is a compact subset of $\mathcal{X}$ and $\mathcal{U}_n \subseteq \operatorname{int}\mathcal{U}_{n+1}$ for each n. Follow the procedure described in Proposition 3: define the sets $\mathcal{K}_n$ and $\mathcal{F}_n$ as there, and then choose functions $f_n \in C_K(\mathcal{X}) \subseteq D_0(\mathcal{X})$ with [$0 \leq f_n \leq 1$; $\mathcal{F}_n \subseteq \mathfrak{z}f_n$; $f_n(\mathcal{K}_n) = \{1\}$] Now form $f = f^{(2)} \vee f^{(1)} \vee f^{(0)} = \bigvee_{n\in\mathbb{N}} f_n$ in $D_0(\mathcal{X})^+$. f has an extension $\overline{f} \in D(\alpha\mathcal{X})$, and $\left\{x \in \alpha\mathcal{X} : \overline{f}(x) < 1\right\}$ is a neighborhood of α contained in $\mathcal{U}$.

To see that $\alpha\mathcal{X}$ is BD, we can use Lemma 7. Suppose $g \in C(\alpha\mathcal{X})$ has $g(\alpha) = 0$; we wish to show that $\overline{\operatorname{coz} g}$ is open. We may assume $0 \le g \le 1$; for each $n \in \mathbb{N}$ set $\mathcal{U}_n = g^{-1}\left(\left[\frac{1}{n}, 1\right]\right)$ and proceed exactly as in the second part of the proof of part (1) of this theorem to reach the desired conclusion. □

The following result represents some of the information in Theorem 6, assuming only that $(J_{\mathcal{X}})$ holds.

Theorem 7. *Suppose $G \overset{J_{\mathcal{X}}}{\le} D(\mathcal{X})$ and that $\mathcal{X}$ is locally compact. If G is divisible and laterally σ-complete, then:*

(1) *$\mathcal{X}$ is BD (so $D(\mathcal{X})$ is an ℓ-group), and*
(2) *$D_0(\mathcal{X}) \le \langle G \rangle_{D(\mathcal{X})}$.*

Proof. (1) By Theorem 3(3), it suffices to show that [$\mathcal{U}$ is cozero with $\overline{\mathcal{U}}$ compact $\implies$ $\overline{\mathcal{U}}$ BD] holds. So, suppose $f \in C(\mathcal{X})$ has $0 \le f \le 1$ and $\overline{\operatorname{coz} f}$ compact. We apply the process described in Proposition 3, setting $\mathcal{U}_n = f^{-1}\left(\left[\frac{1}{n}, 1\right]\right)$ for each $n \in \mathbb{N}$. Note that each $\mathcal{U}_n$ is compact; $(J_{\mathcal{X}})$ holds, so we may apply Lemma 1(3): there are $g_n \in G$ with [$0 \le g_n \le 1$; $\mathfrak{z}g_n \supseteq \mathcal{F}_n$; $g_n(\mathcal{K}_n) \subseteq \left[\frac{1}{2}, 1\right]$]. Then $\{g_n : n \in \mathbb{N}\}$ is a linear set; by lateral σ-completeness, obtain functions $g^{(2)}, g^{(1)}, g^{(0)}$ and then get $g = g^{(2)} \vee g^{(1)} \vee g^{(0)} \in G$, which satisfies [$g\mid_{\operatorname{coz} f} \ge \frac{1}{2}$; $\operatorname{int} \mathfrak{z}f \subseteq \mathfrak{z}g$]. Hence, $\overline{\operatorname{coz} f}$ is open.

(2) Let $f \in D_0(\mathcal{X})^+$. Again following Proposition 3, for each $n \in \mathbb{N}$ set $\mathcal{U}_n = f^{-1}\left(\left[\frac{1}{n}, n\right]\right)$ to obtain $\mathcal{F}_n$, $\mathcal{K}_n$; note that, since $f \in D_0(\mathcal{X})$, $\mathcal{U}_n$ is compact, so $\mathcal{K}_n$ is, also. Now use $(J_{\mathcal{X}})$ and the compactness of the $\mathcal{K}_n$ to apply Lemma 1(3) to produce $g_n \in G$ for each n with [$0 \le g_n \le 1$; $\mathcal{F}_n \subseteq \mathfrak{z}g_n$; $g_n\mid_{\mathcal{K}_n} \ge \frac{1}{2}$].

Then $\{g_n : n \in \mathbb{N}\}$ is a linear set, as is $\{2ng_n : n \in \mathbb{N}\}$. (Note that $2ng_n\mid_{\mathcal{K}_n} \ge f\mid_{\mathcal{K}_n}$.) Now complete the process from Proposition 3 to obtain $g = g^{(2)} \vee g^{(1)} \vee g^{(0)} \in G$ with $g \ge f$. □

The next result represents more of the information in Theorem 6.

Theorem 8. *Suppose $G \overset{J}{\le} D(\mathcal{X})$, and that $\mathcal{X}$ is not compact. If G is divisible and laterally σ-complete, then:*

(1) *$\mathcal{X}$ is locally compact and BD, with α a P-point in $\alpha\mathcal{X}$, and*
(2) *$G \le D_0(\mathcal{X}) = \langle G \rangle_{D(\mathcal{X})}$.*

Proof. Recall that $\mathcal{X}$ is locally compact, since (J) holds (Remark 1.1(b)), so we have the conclusions of Theorem 7. Hence, we need only prove: (2′) $G \subseteq D_0(\mathcal{X})$, and (1′) α is a P-point in $\alpha\mathcal{X}$.

(2′) Let $g \in G^+$. By (J_∞), $\beta g(\beta\mathcal{X} \setminus \mathcal{X}) \subseteq \{0, \pm\infty\}$, so it suffices to show that [$\beta g(x) = +\infty \implies x \in \mathcal{X}$]. So, suppose $\beta g(x) = +\infty$. We again apply Proposition 3, setting, for each $n \in \mathbb{N}$, $\mathcal{U}_n = g^{-1}([1, n])$. Note that each $\mathcal{U}_n$ is a compact subset of $\mathcal{X}$ and that $x \in \operatorname{cl}_{\beta\mathcal{X}}\left(\bigcup_{n\in\mathbb{N}} \mathcal{U}_n\right)$. Following Proposition 3, get the sets $\mathcal{K}_n$ and $\mathcal{F}_n$ for each $n \in \mathbb{N}$. Since each $\mathcal{K}_n$ is compact and $(J_{\mathcal{X}})$ holds, we may apply Lemma 1(3): for each $n \in \mathbb{N}$ there is $f_n \in G$ with [$0 \le f_n \le 2$; $\mathcal{F}_n \subseteq \mathfrak{z}f_n$; $f_n\mid_{\mathcal{K}_n} \ge 1$]. Then $\{f_n : n \in \mathbb{N}\}$ is a linear set in the laterally σ-complete G, so the process of Proposition 3 yields $f^{(2)}, f^{(1)}, f^{(0)}, f \in G$ with [$f^{(i)} = \bigvee_{k\in\mathbb{N}} f_{3k-2}$, for

$i = 0, 1, 2$; $f = f^{(2)} \vee f^{(1)} \vee f^{(0)}]$. One sees that $0 \leq f \leq 2$ and that $\beta f(x) \geq 1$, so $x \in \mathcal{X}$, by (J_∞).

(1′) Suppose $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \cdots$ are neighborhoods of α in $\alpha\mathcal{X}$; we wish to show that $\bigcap_{n\in\mathbb{N}} \mathcal{V}_n$ is a neighborhood of α in $\alpha\mathcal{X}$. There are compact sets $\mathcal{U}_n \subseteq \mathcal{X}$ with $\mathcal{U}_n \subseteq \operatorname{int}\mathcal{U}_{n+1}$ for each $n \in \mathbb{N}$ with $\bigcup_{n\in\mathbb{N}} (\mathcal{X} \setminus \mathcal{V}_n) \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$. Now follow the process in Proposition 3. First, get sets $\mathcal{K}_n$ and $\mathcal{F}_n$ for each $n \in \mathbb{N}$. Note that the $\mathcal{K}_n$ are compact: using $(J_\mathcal{X})$, apply Lemma 1(3) to produce $f_n \in G$ with $[\, 0 \leq f_n \leq 2$; $\mathcal{F}_n \subseteq \mathfrak{z} f_n$; $1 \leq f_n \mid_{\mathcal{K}_n}]$. The linear set $\{f_n : n \in \mathbb{N}\}$ leads, in the familiar way, to $f \in G$ with $0 \leq f \leq 2$ and $f\left(\bigcup_{n\in\mathbb{N}} \mathcal{K}_n\right) \subseteq [1, 2]$. Since $f \in G \leq D_0(\mathcal{X})$, by (2′), f has an extension $\overline{f} \in D(\alpha\mathcal{X})$ with $\overline{f}(\alpha) = 0$. Thus $\mathcal{V} = \overline{f}^{-1}([0, 1))$ is a neighborhood of α with $\mathcal{V} \cap \left(\bigcup_{n\in\mathbb{N}} \mathcal{K}_n\right) = \varnothing$. Since $\bigcup \mathcal{K}_n = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$, this means that $\mathcal{V} \subseteq \bigcap_{n\in\mathbb{N}} \mathcal{V}_n$. □

Proposition 4. *Let $\mathcal{X}$ be locally compact and not compact. Then, $D_K(\mathcal{X}) = D_0(\mathcal{X})$ if and only if α is a P-point in $\alpha\mathcal{X}$.*

Proof. Suppose α is a P-point in $\alpha\mathcal{X}$ and $f \in D_0(\mathcal{X})$. Extend f to $g \in D(\alpha\mathcal{X})$, so $g^{-1}(0)$ is a neighborhood of α. Then $\alpha \notin \overline{\operatorname{coz} g}^{\alpha\mathcal{X}}$, so $\overline{\operatorname{coz} f}^{\mathcal{X}} = \overline{\operatorname{coz} g}^{\alpha\mathcal{X}}$, which is compact: $f \in D_K(\mathcal{X})$.

Now suppose $D_K(\mathcal{X}) = D_0(\mathcal{X})$. If $\mathcal{V}$ is a G_δ in $\alpha\mathcal{X}$ with $\alpha \in \mathcal{V}$, then there is $f \in C(\alpha\mathcal{X})$ with $\alpha \in \mathfrak{z} f \subseteq \mathcal{V}$. Thus, $f \mid_\mathcal{X} \in D_0(\mathcal{X}) = D_K(\mathcal{X})$, so $\overline{\operatorname{coz} f}^{\mathcal{X}}$ is compact: $\mathcal{V}$ is a neighborhood of α in $\alpha\mathcal{X}$. □

Proof of Theorem 6. Assume the hypotheses: $G \overset{J}{\leq} D(\mathcal{X})$, with $\mathcal{X}$ locally compact and not compact, and with G divisible and laterally σ-complete. Then the conclusions of Theorems 7 and 8 hold: $\mathcal{X}$ is locally compact and BD, α is a P-point in $\alpha\mathcal{X}$, and $G \leq D_0(\mathcal{X}) = \langle G \rangle_{D(\mathcal{X})}$. By Proposition 4, $D_K(\mathcal{X}) = D_0(\mathcal{X})$. □

A final remark: inevitable comparison of Theorem 5(2) with Theorem 3(3) produces the following variant, (whose proof we omit).

Proposition 5. (1) *$C(\mathcal{X})$ is laterally σ-complete if and only if $\mathcal{X}$ is a P-space.*

(2) *Suppose $\mathcal{X}$ is locally compact. $C_0(\mathcal{X})$ is laterally σ-complete if and only if $\mathcal{X}$ is discrete.*

(3) *Suppose $\mathcal{X}$ is locally compact. $C_K(\mathcal{X})$ is laterally σ-complete if and only if $\mathcal{X}$ is finite.*

5. Epicompleteness

We now combine §§ 2, 3, 4 and a little further argument to get Theorem 9 below, then expand that to the main result of this paper, Theorem 11 (= Theorem B of the abstract). We then add one more characterization in Theorem 12.

Theorem 9. $G \overset{J}{\leq} D(\mathcal{X})$ *and G is epicomplete with no weak units if and only if*

(a) *$\mathcal{X}$ is BD and locally compact and not compact, with α a P-point in $\alpha\mathcal{X}$, and*

(b) $G = D_0(\mathcal{X})$.

Part of this is contained in the following result.

Theorem 10. *Suppose* $G \overset{J_{\mathcal{X}}}{\leq} \mathcal{X}$ *with* $\mathcal{X}$ *locally compact. If* G *is epicomplete, then*

(1) $\mathcal{X}$ *is BD, and*
(2) $D_0(\mathcal{X}) \leq G$.

Part of the hypothesis in Theorems 9 and 10 is that G be σ-complete. This can be "weakened" to ru-complete because, for G divisible and laterally σ-complete, ru-completeness implies σ-completeness; see [30] and Theorem 5.4 of [15]. (The converse is Lemma 6.)

Proof of Theorem 10. Suppose $G \overset{J_{\mathcal{X}}}{\leq} D(\mathcal{X})$ and G is divisible.

If G is laterally σ-complete and $\mathcal{X}$ is locally compact, then $\mathcal{X}$ is BD and $D_0(\mathcal{X}) \leq \langle G \rangle_{D(\mathcal{X})}$, by Theorem 7. If G is σ-complete, then G is ru-complete (Lemma 6), so $C_K(\mathcal{X}) \leq G$, by Theorem 2.

So, when $\mathcal{X}$ is locally compact and G is epicomplete, $\mathcal{X}$ is BD and both $C_K(\mathcal{X}) \leq G$ and $D_0(\mathcal{X}) \leq \langle G \rangle_{D(\mathcal{X})}$ hold. If $f \in D_0(\mathcal{X})^+$, there is $g \in G$ with $f \leq g$. For each $n \in \mathbb{N}$, set $\mathcal{U}_n = f^{-1}\left(\left[\frac{1}{n}, n\right]\right)$. For each n, $\mathcal{U}_n$ is a compact subset of $\mathcal{X}$ and $\mathcal{U}_n \subseteq \operatorname{int}\mathcal{U}_{n+1}$, so we may choose $h_n \in C_K(\mathcal{X}) \leq G$ with [$h_n(\mathcal{U}_n) = \{1\}$; $h_n\left(\mathcal{X} \setminus \mathcal{U}_{n+1}\right) = \{0\}$; $0 \leq h_n \leq 1$]. Now take $g_n = fh_n$ for each n. Each $g_n \in C_K(\mathcal{X}) \subseteq G$, and each $g_n \leq f \leq g$. Hence, $\bigvee_{n \in \mathbb{N}} g_n = f \in G : D_0(\mathcal{X}) \leq G$. □

Proof of Theorem 9. Suppose (a) holds. Then $\alpha\mathcal{X}$ is BD, by Proposition 2, so $D_0(\mathcal{X})$ is laterally σ-complete, by Theorem 5. Since $\mathcal{X}$ is BD, $D_0(\mathcal{X})$ is σ-complete, by Theorem 3, so $D_0(\mathcal{X})$ is epicomplete, since it is clearly divisible. Since $\mathcal{X}$ is locally compact, $D_0(\mathcal{X}) \overset{J_{\mathcal{X}}}{\leq} D(\mathcal{X})$; since α is a P-point of $\alpha\mathcal{X}$, $D_0(\mathcal{X}) \overset{J_{\infty}}{\leq} D(\mathcal{X})$ and $D_0(\mathcal{X})$ contains no weak units.

Conversely, suppose $G \overset{J}{\leq} D(\mathcal{X})$ and G is epicomplete with no weak units. Since (J) holds and G contains no weak units, $\mathcal{X}$ is locally compact and not compact. By Theorem 10, $\mathcal{X}$ is BD and $D_0(\mathcal{X}) \leq G$. By Theorem 8, α is a P-point in $\alpha\mathcal{X}$ and $G \leq D_0(\mathcal{X})$. □

Recall, from 1.2, that when $\mathcal{X}$ is locally compact and not compact there is a lattice isomorphism $D_0(\mathcal{X}) \approx D(\alpha\mathcal{X}, \alpha)$. If $\mathcal{X}$ is also BD (so that $D_0(\mathcal{X})$ is also a group), this is an ℓ-group isomorphism. Note that if $\mathcal{K}$ is compact and $p \in \mathcal{K}$, then $\mathcal{X} = \mathcal{K} \setminus \{p\}$ has $\alpha\mathcal{X} = \mathcal{K} = \mathcal{X} \cup \{p\}$, and $\mathcal{X}$ is BD if $\mathcal{K}$ is BD, by Proposition 2.

Theorem 11. *The following are equivalent for* $G \in |\mathbf{Arch}|$.

(1) G *is epicomplete with no weak units and has a* J*-representation.*
(2) *There is a compact BD space* $\mathcal{K}$ *with a non-isolated P-point* p *for which* $G \approx D(\mathcal{K}, p)$.
(3) G *is epicomplete with no weak units and there is on* G *a compatible reduced* f*-ring multiplication.*

Proof. (1)$\Longleftrightarrow$(2) by Theorem 9 and the remarks that preceed this Theorem.

(2) $\Longrightarrow$ (3), where the multiplication is that of $D(\mathcal{K},p)$ (which is inherited from $D(\mathcal{K})$).

(3) $\Longrightarrow$ (1), by Johnson's representation theorem, 1.1 (f) above. □

We now proceed to another characterization of the groups G of Theorem 11. $I \subseteq G$ is called a **complete ideal** (respectively, a **$\boldsymbol{\sigma}$-ideal**) in G if $I \underset{\bullet}{\leq} G$ and $[B \subseteq I$ (resp., B is countable and $B \subseteq I$) and $f = \vee B \in G \implies f \in I\,]$. A **P-set** in a compact space $\mathcal{Y}$ is a closed subset $\mathcal{E}$ for which each G_δ-set containing $\mathcal{E}$ is a neighborhood of $\mathcal{E}$ (so p is a P-point if and only if $\{p\}$ is a P-set).

If $\mathcal{M}$ is compact and BD (so that $D(\mathcal{M})$ is an ℓ-group), and $\mathcal{E}$ is a closed subset of $\mathcal{M}$, set $D(\mathcal{M},\mathcal{E}) = \{f \in D(\mathcal{M}) : \mathfrak{z}f \supseteq \mathcal{E}\}$ (generalizing the previous $D(\mathcal{M},p)$). For any $S \subseteq D(\mathcal{M})$, set $\mathfrak{z}S = \bigcap\{\mathfrak{z}s : s \in S\}$ and $\operatorname{coz} S = \bigcup\{\operatorname{coz} s : s \in S\}$; note that $\mathfrak{z}(D(\mathcal{M},\mathcal{E})) = \mathcal{E}$. For any clopen $\mathcal{U} \subseteq \mathcal{M}$, let $\chi_{\mathcal{U}}$ denote the characteristic function of $\mathcal{U}$; note that $\chi_{\mathcal{U}} \in D(\mathcal{M})$.

Proposition 6. *Suppose $\mathcal{M}$ and $\mathcal{E}$ are as in the preceeding paragraph.*

1. *$D(\mathcal{M},\mathcal{E}) \underset{\bullet}{\leq} D(\mathcal{M})$. If I is an ℓ-group ideal in $D(\mathcal{M})$, then $I \subseteq D(\mathcal{M},\mathfrak{z}I)$; for any clopen $\mathcal{U} \supseteq \mathfrak{z}I$, $1-\chi_{\mathcal{U}} \in I$.*
2. *If $\mathcal{E}$ is a P-set in $\mathcal{M}$, then $D(\mathcal{M},\mathcal{E})$ is a σ-ideal in $D(\mathcal{M})$. If I is a σ-ideal in $D(\mathcal{M})$, then $\mathfrak{z}I$ is a P-set in $\mathcal{M}$ and $I = D(\mathcal{M},\mathfrak{z}I)$.*
3. *If $\mathcal{E}$ is a regular closed set in $\mathcal{M}$, then $D(\mathcal{M},\mathcal{E})$ is a complete ideal in $D(\mathcal{M})$. If I is a complete ideal in $D(\mathcal{M})$, then $\mathfrak{z}I$ is a regular closed set in $\mathcal{M}$ and $I = D(\mathcal{M},\mathfrak{z}I)$.*
4. *$D(\mathcal{M},\mathcal{E})$ is a σ-ideal in $D(\mathcal{M})$ with a weak unit if and only if $\mathcal{E}$ is clopen, in which case $D(\mathcal{M},\mathcal{E})$ is a complete ideal in which $1-\chi_{\mathcal{E}}$ is a weak unit.*

Proof. (1) The first two assertions are obvious. Suppose $\mathcal{U}$ is clopen with $\mathcal{U} \supseteq \mathfrak{z}I$. For each $x \in \mathcal{M}\setminus\mathcal{U}$, there is $f_x \in I$ with $f_x(x) > 1$ and $0 \leq f_x \leq 2$. $\left\{f_x^{-1}((1,2)) : x \in \mathcal{M}\setminus\mathcal{U}\right\}$ forms an open cover of the compact set $\mathcal{M}\setminus\mathcal{U}$: there is a finite subcover, say $\left\{f_{x_i}^{-1}((1,2)) : 1 \leq i \leq n\right\}$. Then $f = \bigvee_{i=1}^{n} f_{x_i} \in I$ and $f \geq \chi_{\mathcal{M}\setminus\mathcal{U}} = 1-\chi_{\mathcal{U}}$, so the latter function is in I.

(2) Suppose $\mathcal{E}$ is a P-set in $\mathcal{M}$ and $\{f_n : n \in \mathbb{N}\} \subseteq D(\mathcal{M},\mathcal{E})$ has $f = \bigvee_{n\in\mathbb{N}} f_n \in D(\mathcal{M})$. Since $\mathcal{E}$ is a P-set and $\bigcap \mathfrak{z}f_n$ is a G_δ containing $\mathcal{E}$, $f \in D(\mathcal{M},\mathcal{E})$. Thus, $D(\mathcal{M},\mathcal{E})$ is a σ-ideal.

Now suppose I is a σ-ideal. If $\mathcal{F}$ is any G_δ containing $\mathfrak{z}I$, then, since $\mathcal{M}$ is BD, there are clopen sets $\mathcal{U}_n$, for $n \in \mathbb{N}$, with $\mathcal{F} \supseteq \bigcap_{n\in\mathbb{N}} \mathcal{U}_n \supseteq \mathfrak{z}I$. By (1), each $1-\chi_{\mathcal{U}_n} \in I$, and $s = \bigvee_{n\in\mathbb{N}} \left(1-\chi_{\mathcal{U}_n}\right)$ exists in $D(\mathcal{M})$, since $D(\mathcal{M})$ is σ-complete. Since I is a σ-ideal, $s \in I$; so $\mathcal{V} = \mathfrak{z}s \supseteq \mathfrak{z}I$. Since each $1-\chi_{\mathcal{U}_n}$ takes only the values 0 and 1, it is clear that s does also, so $\mathcal{V}$ is clopen, with $\mathfrak{z}I \subseteq \mathcal{V} \subseteq \bigcap_{n\in\mathbb{N}} \mathcal{U}_n \subseteq \mathcal{F}$. Hence, $\mathfrak{z}I$ is a P-set.

Let $f \in D(\mathcal{M},\mathfrak{z}I)^+$, where I is a σ-ideal in $D(\mathcal{M})$. Since $\mathcal{M}$ is BD, $\overline{\operatorname{coz} f}$ is clopen; but $\mathcal{M}\setminus\operatorname{coz} f$ is a G_δ containing $\mathfrak{z}I$ and $\mathfrak{z}I$ is a P-set, so $\mathfrak{z}I \subseteq \mathcal{M}\setminus\overline{\operatorname{coz} f}$. Thus, $\chi_{\overline{\operatorname{coz} f}} \in I$, so $f = \bigvee_{n\in\mathbb{N}} \left(f \wedge n\cdot\chi_{\overline{\operatorname{coz} f}}\right) \in I$.

(3) If $\mathcal{E}$ is a regular closed subet of $\mathcal{M}$, let $0 \leq f \notin I = D(\mathcal{M},\mathcal{E})$. Then, $f(p) > 0$ for some $p \in \mathcal{E}$. Since $\mathcal{E}$ is regular closed, it follows that $f(x) > 0$ for some $x \in \operatorname{int}\mathcal{E} = \operatorname{int}(\mathfrak{z}I)$. Hence, f cannot be the supremum of any collection of members of I, so I is a complete ideal.

If $\mathfrak{z}I$ is not regular closed, then there is $p \in \mathfrak{z}I \setminus \overline{\operatorname{int}\mathfrak{z}I}$, so $p \in \overline{\operatorname{coz} I}$. Choose an open set $\mathcal{V}$ with $p \in \mathcal{V}$ and $\mathcal{V} \cap \operatorname{int}\mathfrak{z}I = \varnothing$; there is $f \in D(\mathcal{M})$ with $0 \leq f \leq 1$, $f(p) = 1$ and $f(\mathcal{M} \setminus \mathcal{V}) = \{0\}$. For each $x \in \mathcal{V} \cap \operatorname{coz} I$, there is $f_x \in I$ with $0 \leq f_x \leq 1$ and $f_x(x) = 1$; for each such x, $0 \leq f_x \wedge f \leq f_x$ and $f_x \wedge f \in I$. Since $\mathcal{V} \cap \operatorname{coz} I$ is dense in $\mathcal{V}$, $f = \bigvee \{f_x \wedge f : x \in \mathcal{V} \cap \operatorname{coz} I\}$; but $f \notin I$, so I is not a complete ideal in $D(\mathcal{M})$.

The last assertion here is a special case of the similar assertion in (2).

(4) Suppose $D(\mathcal{M},\mathcal{E})$ is a σ-ideal with weak unit e. Since e is a weak unit, $\operatorname{coz} e$ is dense in $\mathcal{M} \setminus \mathcal{E}$. This means that $\overline{\operatorname{coz} e} = \overline{\mathcal{M} \setminus \mathcal{E}}$, with the former clopen, since $\mathcal{M}$ is BD. Since $\mathcal{E}$ is a P-set, $\mathfrak{z}e$ is a neighborhood of $\mathcal{E}$, and this implies that $\overline{\operatorname{coz} e} = \mathcal{M} \setminus \mathcal{E}$. Thus, $\mathcal{E}$ is clopen.

If $\mathcal{E}$ is clopen, then $D(\mathcal{M},\mathcal{E})$ is a complete ideal, by (3), so is a σ-ideal. By (1), $1 - \chi_{\mathcal{E}} \in D(\mathcal{M},\mathcal{E})$ and is clearly a weak unit in $D(\mathcal{M},\mathcal{E})$. □

Remark 1. *(a) Part (2) of this Proposition can also be obtained by combining Theorem 0 and Theorem 1 of* [29].

(b) In much of the proof of Proposition 6, the only role played by the assumption that $\mathcal{M}$ be BD is to assure that $D(\mathcal{M})$ is a group. Indeed, parts (1) and (3) and the first statement of (2) are, mutatis mutandis, valid for any ℓ-group $G \overset{J_{\mathcal{X}}}{\leq} D(\mathcal{M})$ for compact $\mathcal{M}$.

Theorem 12. *For an archimedean ℓ-group G, the equivalent conditions of Theorem 11 are equivalent to:*

(4) There is a compact BD space $\mathcal{K}$ and a σ-ideal I in $D(\mathcal{M})$ which is not a complete ideal and for which $G \approx I$.

Proof. If condition (2) of Theorem 11 holds, then $G \approx D(\mathcal{K},p)$ for some non-isolated P-point p in a compact BD space $\mathcal{K}$. Thus, $\{p\}$ is not a regular closed set in $\mathcal{K}$, so $D(\mathcal{K},p)$ is not a complete ideal in $D(\mathcal{K})$, by Proposition 6(3).

Now suppose I is a non-complete σ-ideal in $D(M)$, where $\mathcal{M}$ is compact BD. Since $D(\mathcal{M})$ is epicomplete, I is, also. By part (2) of Proposition 6, $I = D(\mathcal{M},\mathfrak{z}I)$, with $\mathfrak{z}I$ a P-set that is not regular closed. Hence, $\mathfrak{z}I$ is not clopen, so I has no weak unit, by part (4) of Proposition 6. Set $\mathcal{X} = \mathcal{M} \setminus \mathfrak{z}I$; one readily sees that $I \ni f \longmapsto f|_{\mathcal{X}} \in D(\mathcal{X})$ is a J-representation of I, so condition (1) of Theorem 11 is satisfied. □

6. Concluding remarks

6.1. **$\alpha\mathcal{X}$ BD with P-point α.** (a) Let $\mathcal{K}$ be compact with $\mathcal{E}$ a closed subset of $\mathcal{K}$. By $\mathcal{K}/\mathcal{E}$, we mean the quotient of $\mathcal{K}$ obtained by collapsing $\mathcal{E}$ to a point, $p_{\mathcal{E}}$. Then $\mathcal{K}/\mathcal{E} = (\mathcal{K} \setminus \mathcal{E}) \cup \{p_{\mathcal{E}}\}$, with the quotient topology. If $\mathcal{K}$ is a BD space and $\mathcal{E}$ is a non-open P-set, then $\mathcal{K}/\mathcal{E} = (\mathcal{K} \setminus \mathcal{E}) \cup \{p_{\mathcal{E}}\}$ is BD with $p_{\mathcal{E}}$ a non-isolated p-point. So, $\mathcal{X} = \mathcal{K} \setminus \mathcal{E}$ is BD, locally compact but not compact, with α a p-point in $\alpha\mathcal{X}$, by Proposition 2.

(This construct can be used to fashion a direct proof of [Theorem 12(4) $\Longrightarrow$ Theorem 11(2)], as follows. By Proposition 6, I is of the form $D(\mathcal{K}, \mathcal{E})$ for $\mathcal{K}$ compact BD and $\mathcal{E}$ a non-open P-set in $\mathcal{K}$. Then $I = D(\mathcal{K}, \mathcal{E}) \approx D(\mathcal{K}/\mathcal{E}, p_{\mathcal{E}}) = D(\alpha\mathcal{X}, \alpha)$.)

(b) The well-known Stone duality theorem between Boolean algebras A and Boolean spaces $\mathcal{S}\mathsf{A}$ has $\mathsf{A} \approx clop\mathcal{S}\mathsf{A}$, and then Boolean homomorphisms $\mathsf{A} \xrightarrow{\varphi} \mathsf{B}$ correspond to continuous maps $\mathcal{S}\mathsf{A} \xleftarrow{\mathcal{S}\varphi} \mathcal{S}\mathsf{B}$. See [26], especially §§20-22, for various details and many examples.

The Boolean algebra A is σ-complete (is a "σ-algebra") if and only if $\mathcal{S}\mathsf{A}$ is BD. For an ideal I in a Boolean algebra A, with quotient $\mathsf{A} \xrightarrow{\varphi} \mathsf{A}/\mathsf{I}$, I is a complete ideal (respectively, a σ-ideal) if and only if the inclusion $\mathcal{S}\mathsf{A} \xleftarrow{\mathcal{S}\varphi} \mathcal{S}(\mathsf{A}/\mathsf{I})$ is as a clopen set (resp'ly, a P-set). This produces many (indeed, all) examples as in (a) above.

(c) Starting with a compact BD space $\mathcal{K}$ (of course, $clop\mathcal{K}$ is a σ-algebra with $\mathcal{K} \approx \mathcal{S}(clop\mathcal{K})$), P-sets, $\mathcal{T}$, are produced exactly as $\mathcal{T} = \bigcap \mathcal{W}$ with $\mathcal{W} \subseteq clop\mathcal{K}$ satisfying [$\{\mathcal{V}_n : n \in \mathbb{N}\} \subseteq \mathcal{W} \implies \exists \mathcal{V} \in \mathcal{W}$ with $\mathcal{V} \subseteq \bigcap_{n\in\mathbb{N}} \mathcal{V}_n$]. It is not hard to see that: first, [each P-set in $\mathcal{K}$ is clopen if and only if $\mathcal{K}$ satisfies the **countable chain condition** (ccc): it has no uncountable pairwise disjoint family of non-void open sets]; and, second, [$\mathcal{X}$ locally compact BD with ccc $\implies$ α is not a P-point in $\alpha\mathcal{X}$]. (See [27], [28] and [29].)

(d) An immediate example.

Let $\mathcal{D}$ be an uncountable discrete space, set $\lambda\mathcal{D} = \mathcal{D} \cup \{\lambda\}$, where neighborhoods of λ have countable complements, and let $\mathcal{K} = \beta(\lambda\mathcal{D})$ (the Čech-Stone compactification). Then $\mathcal{K}$ is BD with λ a non-isolated P-point.

This generalizes. Take uncountable I and compact BD spaces $\{\mathcal{K}_i : i \in I\}$; to the topological sum $\sum_{i\in I} \mathcal{K}_i$, adjoin λ as above (neighborhoods of λ contain all but countably many of the $\mathcal{K}_i$). Then $\mathcal{K} = \beta\left(\lambda \sum_{i\in I} \mathcal{K}_i\right)$ is BD with λ as non-isolated P-point.

6.2. **Epicomplete monoreflections.** It is a theorem in [2] that, in the category of archimedean ℓ-groups [$\forall G \; \exists \; G \overset{\beta_G}{\leq} \beta G$ with the properties: βG is epicomplete; β_G is epic; $\forall$ morphism $G \xrightarrow{\varphi} H$ with H epicomplete, $\exists! \; \beta G \xrightarrow{\overline{\varphi}} H$ with $\overline{\varphi}\beta_G = \varphi$]. "Most of the time," βG has no weak units, and we are in the situation of this paper. Regarding when βG might have a weak unit, see [22] and the remarks in [4], 3.3 (especially (b) and (g)).

The only general calculation of βG from G is in [4] which shows, at some labor, that for $G = C_K(\mathcal{Y})$ or $G = C_0(\mathcal{Y})$ (where $\mathcal{Y}$ is locally compact and not compact) $\beta G = B_L(\mathcal{Y})$, the ℓ-group of Baire functions with Lindelöf cozero set. We note, without proof, the equivalence of: i) $B_L(\mathcal{Y})$ has weak units; ii) $B_L(\mathcal{Y}) = B(\mathcal{Y})$ (all Baire functions); iii) $\mathcal{Y}$ is Lindelöf. It is pointed out in [4], §6.3, that when G has no weak unit, $B_L(\mathcal{Y})$ is of the form specified in Theorem 11(2) (neglecting to mention "assuming $\mathcal{Y}$ is not Lindelöf"). Also, Theorem 11(3) is visible: pointwise multiplication.

One notes here that $G = C_K(\mathcal{Y})$ and $G = C_0(\mathcal{Y})$ are, as presented, in a J-representation, and we pointed out above how βG has, in these cases, a J-representation. We do not know any general relation between [G has a J-representation] and [βG has a J-representation].

6.3. **Compatible multiplication.** It is a theorem from [9] and [16] that, in the category of ℓ-groups with distinguished weak unit (and unit-preserving morphisms), [$\forall H \; \exists \; H \overset{r_H}{\leq} rH$ with the properties: rH is an f-ring with identity which is a weak unit; r_H is epic; $\forall$ morphism $H \xrightarrow{\varphi} A$ with A an f-ring with identity, $\exists! \; rH \xrightarrow{\overline{\varphi}} A$ (which is even an f-ring morphism) with $\overline{\varphi} r_H = \varphi$].

It is not known if a similar theorem is true *sans* weak units and identities, but it seems doubtful.

Consider an epicomplete monoreflection $G \leq \beta G$ as in 6.2. If βG satisfies the conditions of Theorem 11(1) (it contains no weak units and has a J-representation), then βG has a compatible ring multiplication, as in Theorem 12(4). Define

$$\varrho G = \bigcap \{A : G \subseteq A; A \text{ is a sub-}f\text{-ring of } \beta G\}$$

as a candidate for the "similar theorem" of the previous paragraph. Since we view that as doubtful, it would seem to follow that "all βG's satisfy the conditions of Theorem 11" is doubtful.

For the record, we note that the (central) question "Does every archimedean ℓ-group have a J-representation?" was inserted by Hager into Charles Holland's Black Swamp Problem Book, [18], on 03/08/03.

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