FINITE DUALITIES, IN PARTICULAR IN FULL HOMOMORPHISMS

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ABSTRACT. This paper is a survey of several results concerning finite dualities, a special case of the famous Constraint Satisfaction Problem (CSP). In CSP, the point is to characterize a class $\mathcal{C}$ of objects X determined by constraints represented by the requirement of the *existence* of structure preserving mappings *from* X into special objects. In a finite duality, such a class $\mathcal{C}$ is characterized by the *non-existence* of special maps *into* X from a finite system of objects.

In the first third of the article we recall some well-known facts concerning constraints represented by classical homomorphisms of relational systems. In the second part we present several results, not yet published but mostly already submitted, concerning the variant of full homomorphisms. The third part contains a few results on hypergraphs and complexes in this context. These form part of an investigation recently undertaken, and appear here first.

In the Constraint Satisfaction Problem, one is concerned with objects X endowed with a given type of structure subjected to *constraints*, usually represented by a system of special objects $\mathcal{B}$, in our case always finite, and the requirement that there exist a mapping $X \to B \in \mathcal{B}$ suitably linked with the structures. (For a more precise formulation see Section 1 below.)

One endeavours to find a characterization, as transparent as possible, of the resulting class. This can sometimes be done by requiring the non-existence of special maps $A_i \to X$ from a finite list of objects $A_1, \dots, A_n$, or by requiring the non-existence of subobjects isomorphic to any of the A_is. Then we speak of a *finite duality.*

Most of this paper is a survey of already known (but in the second and third part not yet published) results. After a very concise report on some general facts (Sections 1–3) we discuss the relation between

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prohibiting morphisms and prohibiting subobjects. This is a very easy matter, but since it is usually treated as folklore, if at all, we feel it should be formulated explicitly; hence we devote to it a special section (Section 4).

Our main intent is to inform the reader on finite dualitioes concerning full homomorphisms (and therir variants; for instance, in graphs these are the maps $f : (X, R) \to (Y, S)$ such that xRy iff $f(x)Sf(y)$). For various reasons, one of them beeing a closer immediate tie with the subobject prohibiting condition, they gained in recent years a growing interest (see e.g. the extensive treatment of graphs in this respect in [6], or the characteristics of Gallai monochromes in [5]; see also [9], [10]). Recently the authors proved a general theorem on the existence of finite dualities for any (finite) constraint system in any category of relational structures of finite type (in [4], submitted for publication). This fact is quoted here as Proposition 5.2 and Theorem 5.3. From [4] we also present a few resulting concrete facts concerning Ramsey lists (Sections 6–7).

The fact in Proposition 5.2 naturally lead to the question whether a similar statement holds for "unbounded" finitary structures. More specifically, does one have finite dualities in (variants of) full homomorphisms in the case of hypergraphs? In Sections 8 and 9 we present a few (not yet published) results of an investigation that has only begun. It turns out that, typically, there are no non-trivial dualities; certain special hypergraphs (complexes), however, do behave somewhat differently.

The paper is concluded by several remarks and problems.

1. The Constraint Satisfaction Problem

To illustrate the type of problems and facts to be presented, let us start with the simple example of a finite binary relation. Given such relations, R on a set X and R' on a set X', a mapping $f : X \to X'$ is a *homomorphism* $G = (X, R) \to G' = (X', R')$ if

(hom) $$(x, y) \in R \Rightarrow (f(x), f(y)) \in R'.$$

Homomorphisms capture many combinatorial properties of relations; for a detailed treatment of graph homomorphisms see [10]. Of a particular interest will be the case where there is a fixed target B and we ask whether there is a homomorphism $G \to B$. That is, we are interested in the class

$$\{G \mid \text{there is a homomorphism } f : G \to B\}.$$

The target B, or a system of targets $\mathcal{B}$, is what we speak of as a *constraint.* When we ask whether a given graph G satisfies the constraint, we are interested only in whether a homomorphism from G into B exists, and not in the homomorphism itself.

Speaking of binary relations as a *simple* example is not quite correct. Note that, already in the case of symmetric graphs and $B = K_n$ (the complete graph with n vertices), checking whether a given graph satisfies the constraint is equivalent to checking whether it is n-colorable, an extremely difficult task.

1.1. More generally, consider a category $\mathfrak{C}$, that is,

- a specification of *objects* of interest (relations, relational systems, hypergraphs etc.; in our case they will always be finite), and
- a specification of *morphisms*, that is maps which in one way or another repect the structure. For instance, for relational systems $(R_i)_{i\in J}$ resp. $(R'_i)_{i\in J}$ on X resp. X', they will be the *homomorphisms* $f : (X, (R_i)) \to (X', (R'_i))$ satisfying

$$\forall i \in J,\ (x_1, \ldots, x_{n_i}) \in R_i \quad \Longrightarrow \quad (f(x_1), \ldots, f(x_{n_i})) \in R_i,$$

or the *full homomorphisms* $f : (X, (R_i)) \to (X', (R'_i))$ satisfying

$$\forall i \in J,\ (x_1, \ldots, x_{n_i}) \in R_i \quad \Longleftrightarrow \quad (f(x_1), \ldots, f(x_{n_i})) \in R_i.$$

The *Constraint Satisfaction Problem* (briefly, *CSP*) is that of determining, for a (typically finite) system $\mathcal{B}$ of objects, the class

$$\mathbf{CSP}(\mathcal{B}) = \{X \text{ object of } \mathfrak{C} \mid \exists B \in \mathcal{B},\ \exists \text{ morphism } X \to B \text{ in } \mathfrak{C}\}.$$

2. Forbidding (homo)morphisms

2.1. In a complementary way, the class $\mathbf{CSP}(\mathcal{B})$ can be represented by *forbidding* (instead of *requiring*) homomorphisms, namely as

$$\mathbf{Forb}(\mathcal{A}) = \{X \mid \text{there is no } f : A \to X \text{ with } A \in \mathcal{A}\}$$

Indeed we can take

$$\mathcal{A} = \{A \mid \text{there is no } f : A \to B \text{ with } B \in \mathcal{B}\}. \tag{2.1}$$

(If $X \in \mathbf{CSP}(\mathcal{B})$ then $X \to B$ for some $B \in \mathcal{B}$ and if we had $A \to X$ we would have $A \to B$; if $X \notin \mathbf{CSP}(\mathcal{B})$ then $X \notin \mathbf{Forb}(\mathcal{A})$ because of the identity $X \to X$.)

This is, of course, trivial. The less trivial question is whether we can find, for a finite $\mathcal{B}$, a *finite* $\mathcal{A}$ such that

$$\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\mathcal{B}).$$

Then we speak of a *finite duality*. First defined in [17], finite dualities have been intensively studied from the conbinatorial and logical point of view, and also in the optimization context.

Note that if one has a finite duality as above then the class $\mathbf{CSP}(\mathcal{B})$ is obviously decidable in polynomial time. A more general (and very interesting) problem, into which we will not go here, is that of an equality $\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\mathcal{B})$ with at least "transparently described" $\mathcal{A}$, in which case we obtain a so called good characterization of $\mathbf{CSP}(\mathcal{B})$, that is, at least a deterministic decision procedure for both the positive and negative membership questions.

3. Finite dualities for relations with standard homomorphisms

The following theorem has recently been proven, as a combination of results of [3] and [19]:

3.1. Theorem. *In the case of finite binary relations, there is a finite duality* $\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\{B\})$ *if and only if the class* $\mathbf{CSP}(\{B\})$ *is first order definable.*

Moreover, if the finite duality exists then the $\mathcal{A}$ can be chosen in a surprisingly special way. Namely, one has ([12])

3.2. Theorem. *If* $\mathcal{B} = \{B\}$ *admits a finite duality then it admits a dualiy* $\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\{B\})$ *with* $\mathcal{A}$ *a finite set of finite trees.*

For classical graphs (symmetric antireflexive relations), there are no non-trivial finite dualities. But for oriented graphs they abound ([11]), although such did not appear to be the case at the outset of the investigations. Furthermore, theorems similar to 3.1 above can be proven for relational structures of finite types; the dualities are well characterized and abundant ([**?**]).

3.3. Encouraged by these results, one might expect something similar for finite algebras. But the facts there are entirely different. It has been recently shown ([13]) that there are no such dualities at all. One has

Theorem. *Let* Δ *be a finite type. Then for every finite set* $\mathcal{A}$ *of finite algebras of a type* Δ *and every finite algebra* B *of this type there exists a finite algebra* A *such that* $A \in \mathbf{Forb}(\mathcal{A})$ *and* $A \notin \mathbf{CSP}(\{B\})$.

This not only excludes a duality, but even the existence of a finite $\mathcal{A}$ such that

$$\mathbf{Forb}(\mathcal{A}) \subseteq \mathbf{CSP}(B).$$

Among bounded structures this is a special feature of algebras. Similar inclusions in more general relational structures may yield non-trivial classes even when there is no non-trivial duality. For instance, it can be shown that the existence of an inclusion $\mathbf{Forb}(\mathcal{A}) \subseteq \mathbf{CSP}(\mathcal{B})$ in graphs amounts to the boundedness of the chromatic numbers of the graphs in $\mathbf{Forb}(\mathcal{A})$; such $\mathcal{A}$ were characterized in [17].

For hypergraphs there is, however, a fact reminiscent of the Theorem above; see 8.3 below, and the non-existence of non-trivial inclusions $\mathbf{Forb}(\mathcal{A}) \subseteq \mathbf{CSP}(\mathcal{B})$ for complexes in 9.3.

Homomorphisms of algebras have special properties distinguishing them from general homomorphisms of relational structures. Thus for instance, for a one-one homomorphism of algebras one has

$$x = \alpha_i(x_1, \dots, x_{n_i}) \Longleftrightarrow f(x) = \alpha'_i(f(x_1), \dots, f(x_{n_i}))).$$

equivalent with the formally weaker condition with $\Longrightarrow$. This makes them, in a sense, structurally close to full homomorphisms (recall 1.1.). But the lack of dualities is in the specific nature of objects (algebras). As we will see below, for general relational objects the fullness condition is no obstacle to finite dualities. In fact it even helps and the dualities are much more frequent than in the standard homomorphism cases.

For more see e.g. [9]

4. Intermezzo: Forbidden homomorphisms and forbidden subobjects

4.1. It is often the case that important classes of objects are characterized by prohibiting a *system of subobjects* (that is, subsets endowed with induced structures, like for instance induced subgraphs – in general one can consider *extremal* or *strong* monomorphisms, [1]) rather than *homomorphisms from a system of objects.* (In fact, the idea of prohibiting subobjects emerged prior to that of prohibiting morphisms.) For example, planar graphs are characterized by the absence of two specific configurations ([?]), and similarly one can characterize distributive lattices ([?]). In the cases we have in mind it can always be done in this, perhaps more transparent, way.

Let us introduce the following notation ($X \nrightarrow Y$ stands for "there is no morphism from X to Y").

$$\begin{array}{lll} X \to \mathcal{A} & \text{for} & \exists A \in \mathcal{A},\ X \to A, \\ \mathcal{A} \to X & \text{for} & \exists A \in \mathcal{A},\ A \to X, \\ X \nrightarrow \mathcal{A} & \text{for} & \forall A \in \mathcal{A},\ X \nrightarrow A, \\ \mathcal{A} \nrightarrow X & \text{for} & \forall A \in \mathcal{A},\ A \nrightarrow X. \end{array}$$

Thus, $\mathbf{Forb}(\mathcal{A}) = \mathcal{A} \nrightarrow\ = \{X \mid \mathcal{A} \nrightarrow X\}$, $\mathbf{CSP}(\mathcal{B}) =\ \to \mathcal{B}\ = \{X \mid X \to \mathcal{B}\}$. If we further set

$$\mathcal{N}(\mathcal{B}) =\ \nrightarrow \mathcal{B} = \{X \mid X \nrightarrow \mathcal{B}\} \tag{4.1}$$

we see that the trivial fact from the first paragraph of 2.1 can be expressed as

$$\mathcal{N}(\mathcal{B}) \nrightarrow X \quad \text{iff} \quad X \to \mathcal{B} \tag{4.2}$$

and the dualities $\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\mathcal{B})$ we are discussing can be rewritten as

$$P \setminus (\mathcal{A} \to) =\ \to \mathcal{B}. \tag{4.3}$$

4.2. The categories we discuss in this paper have the following properties:

(a) for an object A there are, up to isomorphism, only finitely many objects C such that there exists an *onto* morphism $A \to C$,

(b) each morphism $f : A \to B$ can be written as a composition of an onto one and an injection (that is an embedding of a subobject), symbolically

$$f = (A \twoheadrightarrow C \hookrightarrow B)$$

where we use $\twoheadrightarrow$ to indicate morphisms onto and $\hookrightarrow$ to indicate injections

Consequently, the duality (4.2) gives rise to the equality

$$P \setminus ((\mathcal{A}\twoheadrightarrow)\hookrightarrow) =\ \to\mathcal{B}.$$

and finally, setting $\mathcal{A}_1 = \mathcal{A}\twoheadrightarrow$, to

$$P \setminus (\mathcal{A}_1\hookrightarrow) =\ \to\mathcal{B},$$

Hence if we write

$$\begin{aligned} \mathbf{Forb}_{\text{sub}}(\mathcal{A}_1) &= \mathcal{A}_1 \not\hookrightarrow = \\ &= \{X \mid X \text{ has no subobject isomorphic with an } A \in \mathcal{A}_1\} \end{aligned}$$

we have our original finite duality $\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\mathcal{B})$ replaced by a finite "subobject duality"

$$\mathbf{Forb}_{\mathrm{sub}}(\mathcal{A}_1) = \mathbf{CSP}(\mathcal{B}).$$

These types of situations were part of the pre-CSP motivation for studying finite dualities in [17].

5. CSP in relational systems with full homomorphisms

The categories of relational systems, and similarly the categories of hypergraphs and complexes we will be discussing later, have properties (a) and (b) from 4.2. They also obviously have the property that

(a∗) every object has only finitely many subobjects.

A property that these categories do not have, but that holds true in the variant with full homomorphisms, is that

(c) each onto morphism $f : A \to B$ is a *retract*, that is, there is a $g : B \to A$ such that fg is the identity.

An object A is said to be *reduced* if it has no non-trivial (≡ non-isomorphic) retract $r : A \to B$. Obviously, each object has a reduced retract (that is, a retract $r : A \to B$ with reduced B) and consequently each finite duality can be replaced with one in which all the objects in $\mathcal{A}$ and $\mathcal{B}$ are reduced.

An object A is *critical* with respect to a system of objects $\mathcal{B}$ if

- it is reduced,
- $A \not\to \mathcal{B}$, and
- if $A' \to A \not\to A'$ then $A' \to \mathcal{B}$.

Set

$$\mathcal{N}_0(\mathcal{B}) = \{X \in \mathcal{N}(\mathcal{B}) \mid X \text{ critical w.r.t. } \mathcal{B}\}.$$

The following simple lemma plays a crucial role in the theorem below. What it does is reduce the $\mathcal{N}(\mathcal{B})$ from (4.1) to its essential part. The proof of the theorem will not be presented – it is in [4] – but the lemma will be useful in a variant that does not immediately follow and that we will prove later.

5.1. Lemma. *In a category satisfying* (a), (a∗), (b) *and* (c) *one has*

$$\mathcal{N}_0(\mathcal{B}) \not\to X \quad \textit{iff} \quad X \to \mathcal{B}.$$

Proof. Take the $\mathcal{N}(\mathcal{B})$ from (4.1) and then consider just its reduced elements, forming a set $\mathcal{A}$. We still have $\mathcal{A} \not\to X$ iff $X \to \mathcal{B}$. It is easy to see that for a reduced A every morphism $A \to X$ is one-to-one.

Thus, if $A \in \mathcal{A}$ is not critical we have a proper subobject $A' \hookrightarrow A$ such that still $A' \not\to X$. Hence we can restrict ourselves to the smallest A of $\mathcal{A}$ (smallest in the order of "being a subobject": recall that our objects are finite) and these constitute precisely the $\mathcal{N}_0(\mathcal{B})$. $\square$

Let Δ be any finite type. Denote by $\mathbf{Rel}_{\mathsf{full}}(\Delta)$ the category of relational systems of this type with full homomorphisms.

5.2. Proposition. ([4]) *Let $\Delta = (n_t)_{t\in T}$ and let $\mathcal{B}$ be a finite set of objects of $\mathbf{Rel}_{\mathsf{full}}(\Delta)$. Let $m > \max_{t\in T} n_t$. Then, with possibly finitely many exceptions, every A critical with respect to $\mathcal{B}$ can be embedded into an object of $\mathbf{Rel}_{\mathsf{full}}(\Delta)$ carried by the power X^m where*

$$X = X_B \cup \{\omega\}$$

for some $B \in \mathcal{B}$ and $\omega \notin X_B$.

As an immediate consequence one obtains

5.3. Theorem. *In $\mathbf{Rel}_{\mathsf{full}}(\Delta)$ there exists for every finite set of objects $\mathcal{B}$ a finite system of objects $\mathcal{A}$ and a finite duality*

$$\mathcal{A} \not\to X \quad \textit{iff} \quad X \to \mathcal{B}.$$

For graphs and one element $\mathcal{B} = \{B\}$ this was proved (among other results) independently in [6]. Moreover, there is proved an interesting fact that one can find a duality with $|A| \leq |B|+1$ for all the $A \in \mathcal{A}$. In this result it is essential that the graphs are not necessarily connected. For connected graphs the situation is unclear – see 7.2 below.

6. Ramsey lists

In contrast with the general fact about $\mathcal{B}$ from 5.3, it is seldom possible to complete a finite $\mathcal{A}$ to a finite duality with $\mathcal{A}$ on the left hand side. For instance, in the category of graphs (with full homomorphisms), there are only the following four systems $\mathcal{A}$ with fewer than three elements:

$$\{K_1\}, \quad \{K_2\}, \quad \{K_3, P_3\} \quad \text{and} \quad \{K_3, P_4\}$$

(The K_ns are complete graphs, the P_ns are paths, see below). There are infinitely many such systems with three elements, however (see 7.1 below).

It is no wonder such systems are relatively rare, for they have a very strong combinatorial property.

A collection of reduced objects $\mathcal{A} = \{A_1, \dots, A_n\}$ is said to be a *Ramsey list*, or, briefly, to be *Ramsey*, if there is a finite system of objects $\mathcal{F}$ in the given category such that each reduced object that is not isomorphic to an object of $\mathcal{F}$ has a subobject isomorphic to one of the A_is. (The reader may wish to consult [15] and [8] for a general background on Ramsey theory.)

6.1. Proposition. *Let $\mathcal{C}$ be a category satisfying* (a), (a∗), (b) *and* (c), *and let $\mathcal{A}$ be a finite collection of reduced objects of $\mathcal{C}$. Then $\mathcal{A}$ is Ramsey iff there is a finite duality*

$$\mathcal{A} \not\to X \quad \textit{iff} \quad X \to \mathcal{B}.$$

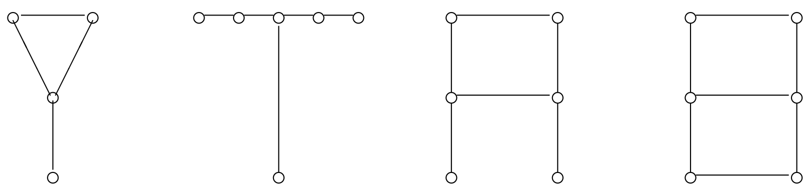
Proof. If there is such a duality then it suffices to take for $\mathcal{F}$ the set of all subobjects of the elements of $\mathcal{B}$.

On the other hand, if $\mathcal{A}$ is Ramsey set $\mathcal{B} = \{F \mid \mathcal{A} \not\to F \text{ and } F \in \mathcal{F}\}$. □

7. Examples of concrete dualities in $\mathbf{ConnGraph}_{\mathsf{full}}$

7.1. Some particular graphs. We will use the following symbols for particular graphs. Here ij indicates that both (i, j) and (j, i) are in the relation.

- $K_n = (\{0, 1, \dots, n-1\}, \{ij \mid i \neq j\})$ is the complete graph with n vertices,
- P_n is the n-path $(\{0, 1, \dots, n\}, \{01, 12, \dots, (n-1)n\})$,
- C_n is the n-cycle $(\{0, 1, \dots, n-1\}, \{01, 12, \dots, (n-1)0\})$,
- $Y = (\{0, 1, 2, 3\}, \{01, 12, 23, 13\})$,
- $T = (\{0, 1, 2, 3, 4, 5\}, \{01, 12, 23, 34, 25\})$,
- $A = (\{0, 1, 2, 3, 4, 5\}, \{01, 12, 23, 34, 45, 14\})$,
- and $B = (\{0, 1, 2, 3, 4, 5\}, \{01, 12, 23, 34, 45, 14, 05\})$.



Y T A B

All the examples in this section are in the category of *connected* symmetric graphs.

7.2. Several dualities following from 5.2.

1. For complete graphs we have

$$\{K_{n+1}, P_3, Y\} \not\to X \quad \textit{iff} \quad X \to K_n.$$

2. For paths:

$$\{P_4, C_3, A, C_5\} \not\to X \quad \textit{iff} \quad X \to P_3,$$

and for $n \geq 4$*,*

$$\{P_{n+1}, T, C_3, A, B, C_5, \ldots, C_{n+2}\} \not\to X \quad \textit{iff} \quad X \to P_n.$$

3. For cycles:

$$\{P_4, C_3, A\} \not\to X \quad \textit{iff} \quad X \to C_5,$$

and for $n \geq 6$*,*

$$\{P_{n-1}, T, C_3, A, B, C_5, \ldots, C_{n-1}\} \not\to X \quad \textit{iff} \quad X \to C_n.$$

Remarks. 1. Note the similarities of the left duals of the paths and the cycles. Compare for instance the dualities

$$\{P_5, T, C_3, A, B, C_5, C_6\} \not\to X \quad \text{iff} \quad X \to P_4$$

and

$$\{P_6, T, C_3, A, B, C_5, C_6\} \not\to X \quad \text{iff} \quad X \to C_7.$$

2. The duality $\{P_4, C_3, A\} \not\to X$ iff $X \to C_5$ from 3 above is a characteristics of Gallai monochromes proved in [5].

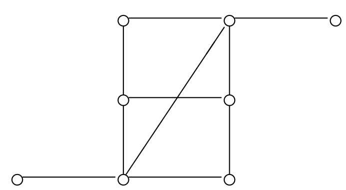
7.3. A special example, and problem. By tedious checking we obtain the duality, for the A from 7.1,

$$\{P_4, C_3, C_5, E\} \not\to X \quad \text{iff} \quad X \to A$$

with

$$E = (\{0,1,2,3,4,5,6,7\}, \{01, 12, 23, 34, 45, 14, 17, 26, 46, 67\}),$$

a relatively complex graph (in this context).



E

This contrasts with the result of [6], as $|E| = |A| + 2$, and it can be shown by tedious checking that the duality cannot be achieved with smaller graphs. It should not be forgotten, however, that our examples concern the category $\mathbf{ConnGraph}_{\mathsf{full}}$, while the mentioned result speaks of general, not just connected, obstruction graphs. As far as we know, the problem of the bound on the sizes of the $A \in \mathcal{A}$ in the connected case is open.

7.4. Effectiveness of determining the left hand side.
All the examples have been established using variants of the construction from 5.2. In the case of a symmetric binary relation, the starting object does not need to be as big as X^m; it suffices to take, roughly speaking, B redoubled with a point added, then filling in a suitable structure between the two B-layers. The search for the structure went, more or less, by brute force. Can the search be done more effectively, if not for any graph then at least for some interesting class of graphs?

8. Hypergraphs

The question naturally arises whether the general theorem 5.3 has to do with the boundedness of the type. What happens if the arity of the structure increases with the size? It turns out that, already in the simplest unbounded structure, namely in the case of hypergraphs, there are no non-trivial dualities, whether we consider the standard homomorphisms or the full ones.

8.1. A *hypergraph* is a couple $H = (V_H, \mathcal{E}_H)$ with $\mathcal{E}_H \subseteq \exp(X_H)$.
The complete hypergraph, that is any of the $S = (V_S, \mathfrak{P}(V_S))$, will be referred to as a *simplex*, for reasons that will become apparent in the next section.

A natural extension of the notion of a graph homomorphism is that of a *hypergraph homomorphism* (further, briefly, just *homomorphism*) $f : G \to H$, a mapping $f : V_G \to V_H$ such that

$$\forall X \subseteq V_G, \quad X \in \mathcal{E}_G \implies f[X] \in \mathcal{E}_H.$$

The resulting category will be denoted by

$$\mathbf{Hypgraph}.$$

Extending the concept of full homomorphisms from graphs to relational structures, we obtain the *full homomorphisms* between hypergraphs satisfying

$$\forall X \subseteq V_G, \quad X \in \mathcal{E}_G \iff f[X] \in \mathcal{E}_H;$$

the resulting category will be designated

$$\mathbf{Hypgraph}_{\mathsf{full}}.$$

To avoid a messy discussion (caused by the fact that if $f : H \to G$ then G has a void hyperedge only when H does), we will prove the facts concerning the subcategory

$$\mathbf{Hypergraph}^{\circ}$$

generated by the hypergraphs H such that $\emptyset \in \mathcal{E}_H$. The corresponding subcategory restricted to full homomorphisms is $\mathbf{Hypgraph}^{\circ}_{\mathsf{full}}$.

8.2. Proposition. *There is no non-trivial duality in* $\mathbf{Hypergraph}^{\circ}$.

Proof. If each $B \in \mathcal{B}$ has the feature that $\mathcal{E}_B$ contains no nonvoid edges, then each $X \in \mathbf{CSP}(\mathcal{B})$ has this feature as well. And, if $\mathcal{B}$ were to participate in a duality with a finite set $\mathcal{A}$, then no $A \in \mathcal{A}$ could have this feature. But a hypergraph X such that $\mathcal{E}_X$ contained nonvoid edges, but such that its nonvoid edges were bigger than any of those of the $\mathcal{E}_A$s, $A \in \mathcal{A}$, would violate the duality.

Now if there is a one-element edge in some $B \in \mathcal{B}$ then $X \to \mathcal{B}$ for any X. In this case, $\mathcal{B}$ participates in a trivial duality with $\mathcal{A}$ taken to be ϕ.

Finally, let there be non-void edges and let all of them have at least two points. Choose a set X such that $|X| > \max_{B \in \mathcal{B}} |V_B| \cdot \max_{A \in \mathcal{A}} |V_A|$, and set $C = (X, \mathcal{E})$, where

$$\mathcal{E} = \{M \subseteq X \mid |M| > \max_{A \in \mathcal{A}} |V_A|\}.$$

Then $C \not\to \mathcal{B}$ and $\mathcal{A} \not\to C$. $\square$

8.3. Recall 3.3. In $\mathbf{Hypgraph}_{\mathsf{full}}$ we have a similar situation. (A complex is a hypergraph such that all subsets of hyperedges are hyperedges; see the definition at the beginning of Section 9 below.)

Lemma. *For every system* $A_1, \ldots, A_k$, $B_1 \ldots, B_l$ *of hypergraphs there exists a complex* C *such that there is no full homomorphism* $C \to B_i$ *and no full homomorphism* $A_j \hookrightarrow C$ *for any of the* A_j *unless it is a simplex.*

Proof. Set

$$b = \max_i |B_i|, \quad a = \max_i |A_i|.$$

Choose a set V_C with cardinality $(a+2)b$ and set

$$\mathcal{E}_C = \{E \subseteq V_C \mid |E| \leq a+1\}.$$

If f is a homomorphism into some $B \in \mathcal{B}$, choose an $x \in V_B$ such that $|f^{-1}[\{x\}]| \geq a+2$. Pick $E, E' \subseteq f^{-1}[\{x\}]$ such that $|E| = a+1$ and $|E'| = a+2$. Then $f[E] = f[E'] = \{x\}$. As $E \in \mathcal{E}_C$, $\{x\}$ is in $\mathcal{E}_{B_i}$. But then f is not full, since $E' \notin \mathcal{E}_C$.

Now let there be a full homomorphism $f : A_j \to C$ for some j. Let $E \subseteq V_{A_j}$ be arbitrary. Since $f[E]$ is in $\mathcal{E}_C$, E is in $\mathcal{E}_{A_j}$. □

Corollary. *There is no non-trivial duality in* $\mathbf{Hypgraph}^{\circ}_{\mathsf{full}}$.

Proof. Confront the A_i and B_j with the C from the lemma, and in addition the hypergraph D with $V_D = V_C$ and

$$\mathcal{E}_D = \{E \subseteq V_D \mid |E| = a+1 \text{ or } 0\}.$$

For the same reasons as above, $D \nrightarrow \mathcal{B}$. There must be an $A_j \hookrightarrow D$, which makes A_j discrete, and, since $A_j \nrightarrow B \in B$, all the one-element subsets of the B's are hyperedges. But then again the simplex among the A_is can be mapped to any such B. □

9. Complexes

9.1. A *complex* is a hypergraph H such that

(1) $\forall x \in V_H,\ \{x\} \in \mathcal{E}_H$,
(2) $\forall E \in V_H, \forall E' \subseteq E,\ E' \in \mathcal{E}_H$.

(This is a well-known concept from combinatorial topology – often also referred to as *abstract complex*, see e.g. [**?**]; in accordance with this we have called complete hypergraphs *simplices* – recall 8.1.)

The category of complexes with standard resp. full homomorphisms will be denoted by

$$\mathbf{Compl} \quad \text{resp.} \quad \mathbf{Compl}_{\mathsf{full}}.$$

We will also be interested in the subcategory of $\mathbf{Compl}_{\mathsf{full}}$ constituted by the complexes of dimension at most k, in other words, with the size of the hyperedges bounded by k. It will be denoted by

$$\mathbf{Compl}^k_{\mathsf{full}}.$$

A full homomorphism in this context is a mapping $f : G \to H$ such that

$$\forall X \subseteq V_G, |X| \leq k, \quad X \in \mathcal{E}_G \iff f[X] \in \mathcal{E}_H.$$

9.2. Proposition. *There is no non-trivial duality in* **Compl**.

Proof. Take a $B \in \mathcal{B}$ and choose an $x_0 \in V_B$. Now construct C as follows. First, let $\overline{A}$ be the disjoint sum of all the $A \in \mathcal{A}$. Set

$$V_C = (V_B \setminus \{x_0\}) \cup V_{\overline{A}},$$

supposing the union disjoint, and define $\mathcal{E}_C$ by

$$U \in \mathcal{E}_C \quad \text{if} \quad \begin{cases} \text{either } U \in \mathcal{E}_B \text{ and } x_0 \notin U, \\ \text{or } U = (W \setminus \{x_0\}) \cup V, \ x_0 \in W \in \mathcal{E}_B, \ \emptyset \neq V \in \mathcal{E}_{\overline{A}} \\ \text{or } U \in \mathcal{E}_{\overline{A}}. \end{cases}$$

Then C is a complex, $C \to B$, and $A \hookrightarrow C$ for all $A \in \mathcal{A}$. □

9.3. Proposition. *In* $\mathbf{Compl}_{\mathsf{full}}$ *there is no non-trivial finite duality. Moreover, there are no non-trivial finite sets* $\mathcal{A}$ *and* $\mathcal{B}$ *such that*

$$\mathbf{Forb}(\mathcal{A}) \subseteq \mathbf{CSP}(\mathcal{B}).$$

Proof. Let us specify what we understand by trivial. There is, of course, the trivial duality

$$A \not\to X \quad \text{iff} \quad X \to B$$

with B empty and A a one-point simplex; this will be excluded.

Now let $\mathcal{B}$ contain a non-empty complex B_0. Suppose

$$\mathbf{Forb}(\mathcal{A}) \subseteq \mathbf{CSP}(\mathcal{B}).$$

We can apply Lemma 8.3 since the C there is a complex. One of the A_j is a simplex, but then $A_j \to B_0$, which is a contradiction. □

9.4. The proof of the following proposition needs only a very small modification of that of 5.2. But since 5.2 cannot be applied directly, we will present it in some detail. Also it is an opportunity to illustrate the principle of the proof that was omitted.

Proposition. *Let* $B_1, \ldots, B_r$ *be objects of* $\mathbf{Compl}^k_{\mathsf{full}}$. *Let* $m > k$. *Then, with possibly finitely many exceptions, every* A *critical with respect to* $B_1, \ldots, B_r$ *can be embedded into an object of* $\mathbf{Compl}^k_{\mathsf{full}}$ *carried by* X^m *where*

$$X = V_B \cup \{\omega\}$$

for some $B \in \mathcal{B}$ *and* $\omega \notin V_B$.

Consequently, for every finite set of objects $B_1, \ldots, B_r$ *there is a finite set of objects* $A_1, \ldots, A_n$ *and a duality*

$$\{A_1, \ldots, A_n\} \not\to X \quad \textit{iff} \quad X \to \{B_1, \ldots, B_r\}.$$

Proof. Since A is reduced, it suffices to find a full homomorphism from A into an object as stated.

Consider a critical A. For every $a \in A$ there is a full homomorphism $i(a) : A \setminus \{a\} \to B_{i(a)}$. Assume that A is sufficiently large so as to contain distinct $a_1, \ldots, a_m$ such that the $B_{i(a_j)}$s coincide. Designate the common value B, and choose full homomorphisms

$$f_i : A \setminus \{a_i\} \to B.$$

Set $X = V_B \cup \{\omega\}$ and define mappings

$$f_i^+ : V_A \to X$$

by setting

$$f_i^+(x) = \begin{cases} f_i(x) \text{ if } x \neq a_i, \\ \omega \text{ if } x = a_i. \end{cases}$$

Now define

$$B^+ = (X, \mathcal{E}^+) \text{ with } \mathcal{E}^+ = \{U \subseteq X \mid |U| \leq k,\ U \setminus \{\omega\} \in \mathcal{E}_B\}.$$

If $U \in \mathcal{E}_A$ then $f_i^+[U] \setminus \{\omega\} = f_i[U \setminus a_i] \in \mathcal{E}_B$ and hence all the f_i^+ are homomorphisms $A \to B^+$, though not necessarily full. Consider the map

$$f : A \to X^m \quad \text{defined by} \quad p_i f = f_i^+,$$

where $p_i : X^m \to X$ is the i^{th} projection. Let

$$\mathcal{E} = \{f(U) \mid U \in \mathcal{E}_A\} \cup \{U \mid |U| = 1\}$$

Thus defined, $(X^m, \mathcal{E})$ is an object of $\mathbf{Compl}^k_{\mathsf{full}}$, and f and the p_is are homomorphisms. We claim that f is full. For if $f[U] \in \mathcal{E}$ for some $U \subseteq V_A$ with $|U| \leq k$ then, since $m > k$, there is an i such that $a_i \notin U$, hence $f_i^+[U] = f_i[U]$. Therefore

$$\omega \notin f_i[U] = f_i^+[U] = p_i f[U] \in \mathcal{E}^+,$$

hence $U \in \mathcal{E}_B$, and since f_i is full, $U \in \mathcal{E}_A$. $\square$

9.5. It is a trivial observation that for a complex X one has

$$X \in \mathbf{Compl}^k_{\mathsf{full}} \quad \text{iff} \quad S_{k+1} \not\to X$$

where S_{k+1} is the simplex with $k+1$ vertices. This, together with 9.4, yields the following

Corollary. *In contrast with the negative fact of 9.3, in* $\mathbf{Compl}_{\mathsf{full}}$ *there exists for each finite system of objects* $\mathcal{B}$ *a finite system* $\mathcal{A}$ *such that*

$$\mathbf{Forb}_{sub}(\mathcal{A}) \subseteq \mathbf{CSP}(\mathcal{B}).$$

10. A few concluding remarks and open problems

10.1. Finite dualities constitute only a small part of the CSP problem. A very important question is that of $\mathbf{Forb}(\mathcal{A})$ with $\mathcal{A}$ not necessarily finite, but given by a criterion which is transparent. A good example is the characterization of bipartite graphs as being those into which no odd cycle embeds. Another such criterion is algorithmically generated $\mathcal{A}$. For such results concerning bounded tree width dualities, see [10].

10.2. In Proposition 5.2 (and similar results), the search for the elements of $\mathcal{A}$ is restricted to subobjects of a well defined object. In some cases this suffices to present a satisfactory list, but in general the brute force search is too hard. Is there an effective search algorithm?

10.3. The existence of a non-trivial duality $\mathbf{Forb}(\mathcal{A}) = \mathbf{CSP}(\mathcal{B})$ implies the existence of a non-trivial subobject duality $\mathbf{Forb}_{\mathrm{sub}}(\mathcal{A}) = \mathbf{CSP}(\mathcal{B})$. It would be useful to study the situations in which the latter exists and the former is absent. Note that for the "inclusion characterization" we have such a phenomenon in complexes: compare 9.3 with 9.5.

10.4. In the hypergraph case there are other natural choices of morphisms to be analyzed. A set of subsets can be viewed as a generalized topology, and the open continuous maps constitute one of the fullness type choices. In this particular case one has negative results similar to those in Section 8, but there are some special features of interest.

10.5. Let us recall, once again, the problem of the size of the $A \in \mathcal{A}$ bounded by the $|B|$ in the case of connected symmetric graphs with full homomorphisms (see 7.2). Does $|B|+2$ suffice? A very easy bound is $2|B|$, obviously too big.

An analogous question for standard homomorphisms is highly non-trivial and was studied in [20].

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