RESIDUATED FRAMES WITH APPLICATIONS TO DECIDABILITY

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Abstract. Residuated frames provide relational semantics for substructural logics and are a natural generalization of Kripke frames in intuitionistic and modal logic, and of phase spaces in linear logic. We explore the connection between Gentzen systems and residuated frames and illustrate how frames provide a uniform treatment for semantic proofs of cut-elimination, the finite model property and the finite embeddability property. We use our results to prove the decidability of the equational and/or universal theory of several varieties of residuated lattice-ordered groupoids, including the variety of involutive FL-algebras.

Substructural logics and their algebraic formulation as varieties of residuated lattices and FL-algebras provide a general framework for a wide range of logical and algebraic systems, such as

Classical propositional logic	$\leftrightarrow$	Boolean algebras
Intuitionistic logic	$\leftrightarrow$	Heyting algebras
Łukasiewicz logic	$\leftrightarrow$	MV-algebras
Abelian logic	$\leftrightarrow$	abelian lattice-ordered groups
Basic fuzzy logic	$\leftrightarrow$	BL-algebras
Monoidal t-norm logic	$\leftrightarrow$	MTL-algebras
Intuitionistic linear logic	$\leftrightarrow$	ILL-algebras
Full Lambek calculus	$\leftrightarrow$	FL-algebras

as well as lattice-ordered groups, symmetric relation algebras and many other systems.

In this paper we introduce residuated frames and show that they provide relational semantics for substructural logics and representations for residuated structures. Our approach is driven by the applications of the theory. As is the case with Kripke frames for modal logics, residuated frames provide a valuable tool for solving both algebraic and logical problems. Moreover we show that there is a direct link between Gentzen-style sequent calculi and our residuated frames, which gives insight into the connection between a cut-free proof system and the finite embeddability property for the corresponding variety of algebras.

We begin with an overview of residuated structures and certain types of closure operators called nuclei. This leads to the definition of residuated frames (Section 3) and Gentzen frames (Section 4), illustrated by several examples. We then prove a general homomorphism theorem in the setting of Gentzen frames (Thm. 4.2) and

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apply it to the particular examples. Results that we obtain in Section 5 include the cut-elimination property for several logical systems, the decidability of logics and of varieties of residuated structures, the finite model property, and the finite embeddability property. The homomorphism theorem generalizes and simplifies ideas found in a variety of papers [2–4, 16, 18, 22, 24, 25] that address the above types of problems in otherwise seemingly unrelated ways. Thus the notion of residuated frame provides a unifying framework for the analysis of various logical and algebraic properties and for their proof in a general setting.

For example in a paper by Ono and Komori [18], and later by Okada and Terui [16] cut-elimination and decidability for the full Lambek calculus (and other systems) are proved using monoid semantics and phase spaces. In [3, 4] Blok and van Alten prove the finite embeddability property for several classes of residuated structures using a combinatorial argument, and Belardinelli, Jipsen, Ono [2] and Wille [24] give algebraic proofs of cut-elimination and decidability for FL-algebras and involutive residuated lattices.

In the current paper we present a common generalization of these results, and use it to prove several new results. In particular, we consider all subvarieties of residuated lattice-ordered unital groupoids ($r\ell u$-groupoids) defined by an equation using the symbols $\{\vee, \cdot, 1\}$ and prove cut elimination for sequent calculi associated with these equational classes (Cor. 5.16). We apply this result to obtain the finite model property for many of these classes (essentially those where the defining equation corresponds to a sequent rule whose premises are no more complex than its conclusion, see Thm. 5.18). For integral $r\ell u$-groupoids we are able to prove the stronger finite embeddability property for *all* subvarieties defined by $\{\vee, \cdot, 1\}$-equations (Thm. 5.21), which implies that the universal theory of each of these classes is decidable.

In Sections 6 and 7 we adapt our techniques to involutive residuated structures, and prove similar results about them, including the decidability of the equational theory of involutive FL-algebras and its corresponding generalizations (Cor. 7.7, 7.9). Section 8 concludes with some further results about one-sided and perfect involutive frames.

We note that similar generalized Kripke frames have been introduced and applied to residuated structures independently in [13] and in algebraic form in [7]. The correspondence of this approach to ours is discussed in more detail after the definition of residuated frames in Section 3. Our somewhat more general perspective is required to establish the fundamental link between Gentzen sequent calculi and residuated frames.

1. Residuated structures

In this section we define the residuated structures that will be used in the paper. The definition of a residuated lattice is given in the general setting, following [6]. For more on the properties of residuated lattices and related structures the reader is referred to [14] and [8].

A *residuated lattice* is of the form $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and the following *residuation* property holds for all $x, y, z \in A$

$$\text{(res)} \qquad xy \leq z \quad \text{iff} \quad x \leq z/y \quad \text{iff} \quad y \leq x\backslash z.$$

Here $\leq$ denotes the lattice order and xy stands for $x \cdot y$. We refer to the operations of $\mathbf{A}$ as meet, join, multiplication, left and right division, and multiplicative unit, respectively. The residuation property (res) can be reformulated in equational form [6], hence the class RL of residuated lattices forms a variety. We recall some basic results that follow from the (res) property.

Lemma 1.1. *Multiplication preserves all existing joins, and divisions preserve all existing meets in the numerator and convert all existing joins in the denominator to meets, i.e. if $\bigvee X$ and $\bigwedge Y$ exist then*

$$(\bigvee X)y = \bigvee_{x \in X} xy \qquad (\bigvee X)\backslash z = \bigwedge_{x \in X} x\backslash z \qquad z/(\bigvee X) = \bigwedge_{x \in X} z/x$$

$$y(\bigvee X) = \bigvee_{x \in X} yx \qquad x\backslash(\bigwedge Y) = \bigwedge_{y \in Y} x\backslash y \qquad (\bigwedge Y)/x = \bigwedge_{y \in Y} y/x$$

Lattice-ordered groups (ℓ-groups for short) are defined as lattices with an order-preserving group operation and a unit element. They can be considered as residuated lattices, where $x\backslash y = x^{-1}y$ and $y/x = yx^{-1}$. Relative to RL, they are axiomatized by the equation $x(x\backslash 1) = 1$. We also mention another important example. If $\mathbf{M} = (M, \cdot, 1)$ is a monoid, then the multiplication can be extended to the powerset $\mathcal{P}(M)$, by $X \cdot Y = \{xy : x \in X, y \in Y\}$, and divisions are defined by $X\backslash Y = \{z \in M : X \cdot \{z\} \subseteq Y\}$ and $Y/X = \{z \in M : \{z\} \cdot X \subseteq Y\}$. The structure $(\mathcal{P}(M), \cap, \cup, \cdot, \backslash, /, \{1\})$ is a residuated lattice.

An *FL-algebra* is an algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ such that $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ is a residuated lattice and 0 is an arbitrary element of A. We denote the variety of FL-algebras by FL.

Of special importance are residuated lattices (and FL-algebras) that satisfy the equations $xy = yx$, $x \leq 1$ and $x \leq x^2$. They are called *commutative*, *integral* and *contractive*, respectively. Note that commutativity implies $x\backslash y = y/x$, and in this case $x \to y$ is used for the common value.

The constant 0 allows for the definition of two negation operations ${\sim}x = x\backslash 0$ and $-x = 0/x$. An FL-algebra is called *involutive* (*InFL-algebra*) if it satisfies the equations ${\sim}{-x} = x = -{\sim}x$. In an InFL-algebra we can define operations dual to multiplication and the divisions by

$$x + y = {\sim}[(-y) \cdot (-x)],\ x \mathbin{\dot{-}} y = {\sim}[(-y)\backslash(-x)] \text{ and } y \mathbin{\dot{\leftharpoondown}} x = -[({\sim}x)/({\sim}y)].$$

It then follows that $x + y = -[({\sim}y) \cdot ({\sim}x)] = {\sim}x\backslash y = x/{\sim}y$, $x \mathbin{\dot{-}} y = x({\sim}y)$, $y \mathbin{\dot{\leftharpoondown}} x = ({\sim}y)x$ and that $(A, \vee, \wedge, +, \mathbin{\dot{\leftharpoondown}}, \mathbin{\dot{-}}, 0, 1)$ is an FL-algebra, called the *dual* of the original one (note the order of the operations). The variety of FL-algebras is denoted by InFL. A *cyclic* FL-algebra is one that satisfies ${\sim}x = -x$. Note that cyclicity is a consequence of commutativity.

Boolean algebras are term equivalent to InFL-algebras that satisfy $x \cdot y = x \wedge y$. Also, ℓ-groups are term equivalent to cyclic InFL-algebras that satisfy $x + y = x \cdot y$. In this case we have ${\sim}x = x^{-1}$ and $0 = 1$. Finally, we mention that symmetric relation algebras and MV-algebras are examples of cyclic involutive FL-algebras.

Many subsequent results apply to more general residuated structures that are not assumed to be lattice-ordered, associative or have a unit element. Hence we end this section with the following definitions.

A *pogroupoid* is a structure $\mathbf{G} = (G, \leq, \cdot)$ where $\leq$ is a partial order on G and the binary operation $\cdot$ is order preserving. A *residuated pogroupoid*, or *rpo-groupoid*, is a structure $\mathbf{G} = (G, \leq, \cdot, \backslash, /)$ where $\leq$ is a partial order on G and the residuation property (res) holds. It follows that multiplication is order preserving. If $\leq$ is a lattice order then $(G, \wedge, \vee, \cdot, \backslash, /)$ is said to be a *$r\ell$-groupoid*, and if this algebra is extended with a constant 1 that is a multiplicative unit, or with an arbitrary constant 0 then it is said to be a *$r\ell u$-groupoid* or a *$r\ell z$-groupoid* respectively. Note that a residuated lattice is an associative $r\ell u$-groupoid, and an FL-algebra is an associative $r\ell uz$-groupoid. *Involutive $r\ell u$-groupoids* are defined like InFL-algebras, but without assuming associativity. The varieties of $r\ell u(z)$-groupoids and involutive $r\ell u$-groupoids are denoted by $\mathsf{RLU(Z)G}$ and InGL, respectively.

2. Nuclei

Galois connections. Given two posets $\mathbf{P}$ and $\mathbf{Q}$, we say that the maps ${}^{\rhd} : P \to Q$ and ${}^{\lhd} : Q \to P$ form a *Galois connection* if for all $p \in P$ and $q \in Q$,

$$q \leq p^{\rhd} \text{ iff } p \leq q^{\lhd}.$$

Recall that a *closure operator* γ on a poset $\mathbf{P}$ is a map that is increasing, monotone and idempotent, i.e., $x \leq \gamma(x)$, $x \leq y$ implies $\gamma(x) \leq \gamma(y)$, and $\gamma(\gamma(x)) = \gamma(x)$, for all $x, y \in P$. The image $\gamma[P] = \{\gamma(p) : p \in P\}$ of *γ-closed* elements is denoted by P_γ, and $\mathbf{P}_\gamma$ denotes the associated poset.

The following results are folklore in the theory of Galois connections.

Lemma 2.1. *Assume that the maps ${}^{\rhd} : P \to Q$ and ${}^{\lhd} : Q \to P$ form a Galois connection between the posets $\mathbf{P}$ and $\mathbf{Q}$. Then the following properties hold.*

(i) *The maps ${}^{\rhd}$ and ${}^{\lhd}$ are both order reversing. Moreover, they convert existing joins into meets, i.e., if $\bigvee X$ exists in P for some $X \subseteq P$, then $\bigwedge\{x^{\rhd} : x \in X\}$ exists in Q and $(\bigvee X)^{\rhd} = \bigwedge\{x^{\rhd} : x \in X\}$, and likewise for ${}^{\lhd}$.*

(ii) *The maps ${}^{\rhd\lhd} : P \to P$ and ${}^{\lhd\rhd} : Q \to Q$ are both closure operators.*

(iii) ${}^{\rhd\lhd\rhd} = {}^{\rhd}$ *and* ${}^{\lhd\rhd\lhd} = {}^{\lhd}$

(iv) *For all $q \in Q$, $q^{\lhd} = \max\{p \in P : q \leq p^{\rhd}\}$ and for all $p \in P$, $p^{\rhd} = \max\{q \in Q : p \leq q^{\lhd}\}$.*

Given a relation $R \subseteq A \times B$ between two sets A and B, for $X \subseteq A$ and $Y \subseteq B$ we define

$$\begin{array}{lll} X\ R\ Y & \text{iff} & x\ R\ y \text{ for all } x \in X,\, y \in Y \\ x\ R\ Y & \text{iff} & \{x\}\ R\ Y \\ X\ R\ y & \text{iff} & X\ R\ \{y\} \\ XR = \{y \in B : X\ R\ y\} & & xR = \{x\}R \\ RY = \{x \in A : x\ R\ Y\} & & Ry = R\{y\} \end{array}$$

So, XR contains the elements of B that are related to *all* elements of X. Note that $XR = \bigcap_{x \in X} xR$ and similarly for RY. We define the maps ${}^{\rhd} : \mathcal{P}(A) \to \mathcal{P}(B)$ and ${}^{\lhd} : \mathcal{P}(B) \to \mathcal{P}(A)$, by $X^{\rhd} = XR$ and $Y^{\lhd} = RY$.

Lemma 2.2. *A pair of maps ${}^{\rhd} : \mathcal{P}(A) \to \mathcal{P}(B)$ and ${}^{\lhd} : \mathcal{P}(B) \to \mathcal{P}(A)$ form a Galois connection iff $X^{\rhd} = XR$ and $Y^{\lhd} = RY$, for some relation $R \subseteq A \times B$. In this case the relation is given by $x\ R\ y$ iff $x \in \{y\}^{\lhd}$ (iff $y \in \{x\}^{\rhd}$).*

The pair $({}^{\rhd}, {}^{\lhd})$ is called the Galois connection *induced* by R. We denote by γ_R the closure operator $\gamma_R : \mathcal{P}(A) \to \mathcal{P}(A)$ associated with R, where $\gamma_R(X) = X^{\rhd\lhd}$ (see Lemma 2.1(ii)).

Given a closure operator γ on a complete lattice $\mathbf{P}$, we say that a subset D of P is a *basis* for γ, if the elements in $\gamma[P]$ are exactly the meets of elements of D. In particular we have $D \subseteq \gamma[P]$. Note that the interior operator in a topological space is a closure operator under the dual order, hence this notion of basis is equivalent to the usual one in topology.

Lemma 2.3. *Let A and B be sets.*

(i) *If R is a relation between A and B, then γ_R is a closure operator on $\mathcal{P}(A)$.*
(ii) *If γ is a closure operator on $\mathcal{P}(A)$, then $\gamma = \gamma_R$ for some relation R with domain A.*
(iii) *If $(^{\rhd}, ^{\lhd})$ is the Galois connection induced by R, then the collection of the sets $\{b\}^{\lhd}$, where $b \in B$, forms a basis for γ_R.*

Proof. Statement (i) follows from Lemma 2.2 and Lemma 2.1(ii).

For (ii), let $B = \gamma[\mathcal{P}(A)]$ and define $R \subseteq A \times B$ with $a \mathrel{R} C$ iff $a \in C$. Then for $X \subseteq A$ and $Y \subseteq B$, we have $X^{\rhd} = \{C \in B : X \subseteq C\}$ and $Y^{\lhd} = \bigcap Y$. Therefore, $\gamma_R(X) = \bigcap\{C \in B : X \subseteq C\} = \gamma(X)$, for all $X \subseteq A$.

For (iii) note that γ-closed elements are of the form $Y^{\lhd}$, where $Y \subseteq B$, and that $Y^{\lhd} = RY = \bigcap_{y \in Y} Ry = \bigcap_{y \in Y}\{y\}^{\lhd}$. □

Part (iii) of the above result will be used repeatedly in the following form.

Corollary 2.4. *For a closed set $X \subseteq A$ and $a \in A$ we have*

$$a \in X \qquad \textit{iff} \qquad \forall b \in B\ [X \subseteq \{b\}^{\lhd} \Rightarrow a \in \{b\}^{\lhd}].$$

Nuclei. A *nucleus* on a pogroupoid $\mathbf{G}$ is a closure operator γ on (the poset reduct of) $\mathbf{G}$ such that $\gamma(a)\gamma(b) \leq \gamma(ab)$ for all $a, b \in G$. The concept of a nucleus was originally defined in the context of Brouwerian algebras (e.g. [20]) and quantales (e.g. [19]).

Lemma 2.5. [11] *If γ is a closure operator on a pogroupoid $\mathbf{G}$, then the following statements are equivalent:*

(i) *γ is a nucleus.*
(ii) *$\gamma(\gamma(x)\gamma(y)) = \gamma(xy)$ for all $x, y \in G$.*

If $\mathbf{G}$ is residuated, then the above conditions are also equivalent to:

(iii) *$x/y,\ y\backslash x \in G_\gamma$ for all $x \in G_\gamma$, $y \in G$.*

Let $\mathbf{G} = (G, \leq, \cdot)$ be a residuated groupoid, let γ be a nucleus on $\mathbf{G}$, and for all $x, y \in G$ define $x \circ_\gamma y = \gamma(x \circ y)$. The structure $\mathbf{G}_\gamma = (G_\gamma, \leq, \circ_\gamma)$, is called the γ-*image* of $\mathbf{G}$. If $\mathbf{G}$ has a unit, is lattice ordered and/or is residuated, then the γ-*retraction* is defined to have the operations $\gamma(1)$, $\wedge$, $\vee_\gamma$ (where $x \vee_\gamma y = \gamma(x \vee y)$), and $\backslash$, $/$, respectively. So for example if $\mathbf{G} = (G, \wedge, \vee, \cdot, \backslash, /, 1)$ is a residuated lattice and γ is a nucleus on $\mathbf{G}$, then the γ-retraction of $\mathbf{G}$ is the algebra $\mathbf{G}_\gamma = (G_\gamma, \wedge, \vee_\gamma, \circ_\gamma, \backslash, /, \gamma(1))$, where $x \circ_\gamma y = \gamma(x \cdot y)$ and $x \vee_\gamma y = \gamma(x \vee y)$.

Lemma 2.6. [8, 10]

(i) *The nucleus retraction $\mathbf{G}_\gamma$ of a pogroupoid $\mathbf{G}$ is a pogroupoid and the properties of lattice-ordering, being residuated and having a unit are preserved.*
(ii) *In the above cases, the nucleus γ is a $\{\cdot, \vee, 1\}$-homomorphism from $\mathbf{G}$ to $\mathbf{G}_\gamma$ (if $\vee$ and 1 exist). In particular, if t is a $\{\cdot, \vee, 1\}$-formula, then $\gamma(t^{\mathbf{G}}(\bar{x})) = t^{\mathbf{G}_\gamma}(\gamma(\bar{x}))$, for all sequences $\bar{x}$ of elements in G.*

(iii) *All equations and inequations involving* $\{\cdot, \vee, 1\}$ *are preserved. For example, if* $\mathbf{G}$ *is associative, commutative, integral or contracting, then so is* $\mathbf{G}_\gamma$.
(iv) *In particular, if* $\mathbf{G}$ *is a residuated lattice and* γ *is a nucleus on it, then the* γ*-retraction* $\mathbf{G}_\gamma$ *of* $\mathbf{G}$ *is a residuated lattice.*

Nuclei on powersets. A ternary relational structure is a pair $\mathbf{W} = (W, \circ)$, where W is a set and $\circ \subseteq W^3$. On the powerset $\mathcal{P}(W)$ of W we define the operation $X \circ Y = \{z \in W : (x, y, z) \in \circ \text{ for some } x \in X, y \in Y\}$ and we write $x \circ y$ for the set $\{x\} \circ \{y\}$ and $x \circ Y$ for $\{x\} \circ Y$. Also, we define the sets $X/Y = \{z \mid \{z\} \circ Y \subseteq X\}$ and $Y \backslash X = \{z \mid Y \circ \{z\} \subseteq X\}$. It is easy to see that the algebra $\mathcal{P}(\mathbf{W}) = (\mathcal{P}(W), \cap, \cup, \cdot, \backslash, /)$ is a residuated groupoid.

A ternary relation structure $\mathbf{W} = (W, \circ)$ is said to be *associative* if it satisfies $(x \circ y) \circ z = x \circ (y \circ z)$, i.e., if it satisfies the following equivalence

$$\exists u[(x, y, u) \in \circ \text{ and } (u, z, w) \in \circ] \quad \text{iff} \quad \exists v[(x, v, w) \in \circ \text{ and } (y, z, v) \in \circ].$$

It is said to *have a unit* $E \subseteq W$ if $x \circ E = \{x\} = E \circ x$, i.e., if

$$\exists e \in E[(x, e, y) \in \circ] \quad \text{iff} \quad x = y \quad \text{iff} \quad \exists e \in E[(e, x, y) \in \circ]$$

Lemma 2.7. *If* γ *is a closure operator over a ternary relation structure* $(W, \circ)$ *and* $\mathcal{D}$ *is a basis for* γ*, then the following statements are equivalent:*
(i) γ *is a nucleus on* $\mathcal{P}(\mathbf{W})$.
(ii) $C/\{w\}$, $\{w\}\backslash C \in \mathcal{P}(W)_\gamma$*, for all* $C \in \mathcal{D}$ *and* $w \in W$.

Proof. We will use the equivalent condition for a nucleus given in Lemma 2.5 (iii). Obviously, this condition implies (ii). For the converse, assume (ii) holds, let $\mathbf{L} = \mathcal{P}(\mathbf{W})$, and consider $X \in L_\gamma$ and $Y \in L$. Since $\mathcal{D}$ is a basis for γ, there exists $\mathcal{X} \subseteq \mathcal{D}$ such that $X = \bigcap \mathcal{X}$. By Lemma 1.1 $X/Y = (\bigcap \mathcal{X})/(\bigcup_{y \in Y}\{y\}) = \bigcap_{C \in \mathcal{X}} \bigcap_{y \in Y} C/\{y\}$. By assumption, $C/\{y\}$ is a γ-closed element, hence X/Y is γ-closed since the intersection of closed elements of a closure operator on a complete lattice is also closed. □

3. Frames

After giving the definition of a residuated frame, we discuss a range of examples, most of which will play a role in the subsequent applications.

As noted earlier, if $(W, \circ)$ is a ternary relational structure, then $\mathcal{P}(W, \circ)$ is a residuated groupoid, and if $(W, \circ)$ is associative or has a unit, the same holds for $\mathcal{P}(W, \circ)$. Also, we have seen that if $R \subseteq W \times W'$, then γ_R is a closure operator on $\mathcal{P}(W)$. We would like to know for which relations $R \subseteq W \times W'$, γ_R is a nucleus on $\mathcal{P}(W, \circ)$. The following condition characterizes the relations that give rise to nuclei.

A relation $N \subseteq W \times W'$ is called *nuclear* on $(W, \circ)$ if for every $u, v \in W, w \in W'$, there exist subsets $u \backslash\!\!\backslash w$ and $w /\!\!/ v$ of W' such that

$$u \circ v \; N \; w \quad \text{iff} \quad v \; N \; u \backslash\!\!\backslash w \quad \text{iff} \quad u \; N \; w /\!\!/ v \quad .$$

Note that the statement above makes use of our notation $x \; N \; Y$ for $\{x\} \; N \; Y$ and $X \; N \; y$ for $X \; N \; \{y\}$, where $X \; N \; Y$ if $x \; N \; y$, for all $x \in X$ and $y \in Y$.

Lemma 3.1. *If* $(W, \circ)$ *is a ternary relation structure and* $N \subseteq W \times W'$*, then* γ_N *is a nucleus on* $\mathcal{P}(W, \circ)$ *iff* N *is a nuclear relation.*

Proof. By Lemma 2.3(iii), the collection $\mathcal{D} = \{\{w\}^{\lhd} : w \in W'\}$ forms a basis for γ_R. So, by Lemma 2.5(iv) γ_N is a nucleus iff $\{w\}^{\lhd}/\{u\}$ and $\{u\}\backslash\{w\}^{\lhd}$ are γ-closed, for all $u \in W$ and $w \in W'$ (here $\backslash$ and $/$ are calculated in $\mathcal{P}(W, \circ)$).

Since $\mathcal{D}$ is a basis, $\{u\}\backslash\{w\}^{\lhd}$ is closed iff $\{u\}\backslash\{w\}^{\lhd} = \bigcap \mathcal{X}$, for some $\mathcal{X} \subseteq \mathcal{D}$ iff $\{u\}\backslash\{w\}^{\lhd} = \bigcap_{c \in u \mathbin{\backslash\!\!\backslash} w} \{c\}^{\lhd}$, for some $u \mathbin{\backslash\!\!\backslash} w \subseteq W'$. This is equivalent to the statement that for all $v \in W$,

$$v \in \{u\}\backslash\{w\}^{\lhd} \quad \text{iff} \quad v \in \textstyle\bigcap_{c \in u \mathbin{\backslash\!\!\backslash} w} \{c\}^{\lhd}.$$

Transforming this statement further we obtain for all $v \in W$,

$$u \circ v \subseteq \{w\}^{\lhd} \quad \text{iff} \quad v \in \{c\}^{\lhd} \text{ for all } c \in u \mathbin{\backslash\!\!\backslash} w.$$

or, equivalently, $u \circ v \ N \ w$ iff $v \ N \ c$ for all $c \in u \mathbin{\backslash\!\!\backslash} w$. So, $\{u\}\backslash\{w\}^{\lhd}$ is closed iff there exists $u \mathbin{\backslash\!\!\backslash} w \subseteq W'$ such that $u \circ v \ N \ w$ iff $v \ N \ u \mathbin{\backslash\!\!\backslash} w$. Likewise, we obtain the second equivalence of a nuclear relation. □

A *residuated frame* (or r-frame for short) is a structure of the form $\mathbf{W} = (W, W', N, \circ, \mathbin{\backslash\!\!\backslash}, \mathbin{/\!\!/})$, where $(W, \circ)$ is a ternary relational structure and $N \subseteq W \times W'$ is a nuclear relation on $(W, \circ)$ with respect to $\mathbin{\backslash\!\!\backslash}, \mathbin{/\!\!/}$. Concretely, this means

- W and W' are sets,
- N is a binary relation from W to W', called the *Galois relation*,
- $\circ \subseteq W^3$, $\quad \mathbin{\backslash\!\!\backslash} \subseteq W \times W' \times W$, $\quad \mathbin{/\!\!/} \subseteq W' \times W \times W$, and
- for all $u, v \in W$ and $w \in W'$

$$(u \circ v) \ N \ w \quad \text{iff} \quad v \ N \ (u \mathbin{\backslash\!\!\backslash} w) \quad \text{iff} \quad u \ N \ (w \mathbin{/\!\!/} v)$$

  where $x \star y = \{z : (x, y, z) \in \star\}$ for $\star \in \{\circ, \mathbin{\backslash\!\!\backslash}, \mathbin{/\!\!/}\}$.

It follows from Lemma 2.6 and Lemma 3.1 that $\mathcal{P}(W, \circ)_{\gamma_N}$ is a $r\ell$-groupoid, called the *Galois algebra of* $\mathbf{W}$, and denoted by $\mathbf{W}^+$. In detail,

$$\begin{aligned}
\mathbf{W}^+ &= (\gamma_N[\mathcal{P}(W)], \cap, \cup_{\gamma_N}, \circ_{\gamma_N}, \backslash, /), \text{ where} \\
X \cup_{\gamma_N} Y &= \gamma_N(X \cup Y) \\
X \circ_{\gamma_N} Y &= \gamma_N(\{z \in W : (x, y, z) \in \circ\}) \\
X \backslash Y &= \{z : X \circ z \subseteq Y\} \\
Y / X &= \{z : z \circ X \subseteq Y\}.
\end{aligned}$$

A unital r-frame (or ru-frame) $\mathbf{W} = (W, W', N, \circ, \mathbin{\backslash\!\!\backslash}, \mathbin{/\!\!/}, E)$ is an r-frame with a set $E \subseteq W$ such that $\gamma_N(E)$ is a unit, and a ruz-frame $\mathbf{W} = (W, W', N, \circ, \mathbin{\backslash\!\!\backslash}, \mathbin{/\!\!/}, E, D)$ is a ru-frame with a distinguished subset $D \subseteq W$. In either case $\mathbf{W}^+$ has a unit $1 = \gamma_N(E)$, and in the latter case also has $\gamma_N(D)$ as interpretation for the constant 0. In particular the Galois algebra of an associative ru-frame is a residuated lattice, and the Galois algebra of an associative ruz-algebra is an FL-algebra. Actually the weaker conditions

- $[(x \circ y) \circ z]^{\rhd} = [x \circ (y \circ z)]^{\rhd}$ (weak associativity)
- $(x \circ E)^{\rhd} = x^{\rhd} = (E \circ x)^{\rhd}$, for all $x \in W$ (weak unit)

suffice to show that the Galois algebra is an FL-algebra and would allow for a slightly more general definition of an associative ru-frame.

The residuated frames we consider extend the *formal contexts* of formal concept analysis [12]. These structures consist of triples (W, W', N), where $N \subseteq W \times W'$, so they capture the lattice operations, but not the multiplication, residuals and unit element.

Furthermore, residuated frames are related to, but more general than, the reduced separated frames (or RS frames) of [13]. The latter assume that the map $w \mapsto \{w\}^{\rhd\lhd}$ is a bijection (i.e., the frame is separated) and that the copy of W in W^+ consists of completely join irreducible elements (i.e., the frame is reduced). These conditions make the relation N behave like a partial order – the restriction of the order of W^+ on the copy, in W^+, of the union $W \cup W'$, and set up a duality between perfect lattices and RS frames. The analogue of the ternary relation $\circ$ on W in the context of RS-frames is given by a ternary relation $R \subseteq W \times W \times W'$ that satisfies a compatibility condition. For RS frames, the relations $\circ$ and R are interdefinable by means of the nuclear relation N. The particular, the residuated frames that are needed for applications in this paper are rarely RS-frames.

To illustrate the generality of residuated frames, we now consider a series of examples.

The Dedekind-MacNeille completion. Given a poset $\mathbf{P} = (P, \leq)$, we can define the residuated frame $\mathbf{W_P} = (P, P, \leq, \circ, \backslash\!\backslash, /\!/)$, where $\circ, \backslash\!\backslash, /\!/$ are the empty set. The nuclear property for $\leq$ is vacuously true.

The poset $\mathcal{P}(P)_{\gamma_\leq}$ (the poset reduct of $\mathbf{W}^+_\mathbf{P}$) of closed sets is called the *Dedekind-MacNeille completion* of $\mathbf{P}$. The maps on $\mathcal{P}(L)$ of the Galois connection involved in the Dedekind-MacNeille completion are usually denoted by ^{u} and ^{l}, where $A^\rhd = A^u = \{x \in L \mid x \leq a \text{ for all } a \in A\}$ and $A^\lhd = A^l = \{x \in L \mid x \geq a \text{ for all } a \in A\}$, for all $A \subseteq L$.

It is well known that the map $x \mapsto \{x\}^\lhd$ is an embedding of $\mathbf{P}$ into $\mathbf{W}^+_\mathbf{P}$.

Residuated pogroupoids. Let $\mathbf{G} = (G, \leq, \cdot, \backslash, /)$ be a residuated pogroupoid and define $x \circ y = \{xy\}$, $x \backslash\!\backslash y = \{x \backslash y\}$ and $x /\!/ y = \{x/y\}$. Then $\mathbf{W_G} = (G, G, \leq, \cdot, \backslash\!\backslash, /\!/)$ is a residuated frame since the nuclear property for $\leq$ is just the residuation property for $\mathbf{G}$. If $\mathbf{G}$ is associative or has a unit, then $\mathbf{W}^+_\mathbf{G}$ has the same properties. In particular, if $\mathbf{G}$ is a residuated lattice, then $\mathbf{W}^+_\mathbf{G}$ is a complete residuated lattice, and the map $x \mapsto \{x\}^\lhd$ is an embedding of $\mathbf{G}$ into $\mathbf{W}^+_\mathbf{G}$ (Corollary 4.5). Hence $\mathbf{W}^+_\mathbf{G}$ is called the Dedekind-MacNeille completion of $\mathbf{G}$.

Partial subalgebras. Let $\mathbf{A}$ be a residuated lattice and $\mathbf{B}$ a partial subalgebra of $\mathbf{A}$, i.e., B is any subset of A, and each operation $f^\mathbf{A}$ on A induces a partial operation $f^\mathbf{B}$ on B by $f^\mathbf{B}(b_1, \ldots, b_n) = f^\mathbf{A}(b_1, \ldots, b_n)$ if this latter value is in B, and undefined otherwise. Define $(W, \circ, 1)$ to be the submonoid of $\mathbf{A}$ generated by B. A *unary linear polynomial* of $(W, \circ, 1)$ is a map u on W of the form $u(x) = v \circ x \circ w$, where v, w are elements of W. Such polynomials are also known as *sections* and we denote the set of all sections by S_W. Let $W' = S_W \times B$, and define $x \; N \; (u, b)$ by $u(x) \leq^\mathbf{A} b$. Given $x, y \in W$ and $u \in S_W$, one can define sections $u'(x) = u(x \circ y)$ and $u''(y) = u(x \circ y)$. We will also use the notation $u' = u(\_ \circ y)$ and $u'' = u(x \circ \_)$. Now define $x \backslash\!\backslash (u, b) = (u(x \circ \_), b)$ and $(u, b) /\!/ y = (u(\_ \circ y), b)$. Then it is easy to see that $\mathbf{W_{A,B}} = (W, W', N, \circ, \backslash\!\backslash, /\!/)$ is a residuated frame. The same holds for the case where $\mathbf{A}$ is a *rpo*-groupoid.

Corollary 5.19 below shows that the map $x \mapsto \{x\}^\lhd$ is an embedding of the partial subalgebra $\mathbf{B}$ of $\mathbf{A}$ into the $r\ell u$-groupoid $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$.

Phase spaces. An (intuitionistic) phase space is a pair $(\mathbf{M}, D)$, where $\mathbf{M} = (M, \cdot, 1)$ is a monoid and D is an intersection closed family of subsets of M with the property that for all sets X, Y

$$X, Y \subseteq M \text{ and } Y \in D \quad \text{implies} \quad X\backslash Y, Y/X \in D.$$

Since intersection-closed families and closure operators are interdefinable on powersets, in view of Lemma 2.5 a phase space can be defined as a pair $(\mathbf{M}, \gamma)$, where $\mathbf{M}$ is a commutative monoid and γ is a nucleus on $\mathcal{P}(\mathbf{M})$. Let N_γ be the binary relation on M defined by $x\ N_\gamma\ y$ iff $x \in \gamma(\{y\})$; also let $x \mathbin{\backslash\!\backslash} z = \{x\}\backslash\gamma(\{z\})$ and $z \mathbin{/\!/} y = \gamma(\{z\})/\{y\}$. It is easy to verify that $(M, M, N_\gamma, \cdot, \mathbin{\backslash\!\backslash}, \mathbin{/\!/}, \{1\})$ is a residuated frame. Indeed, $x \cdot y\ N_\gamma\ z$ iff $x \cdot y \in \gamma(\{z\})$ iff $y \in \{x\}\backslash\gamma(\{z\})$, and likewise $x \in \gamma(\{z\})/\{y\}$.

Perfect (residuated) lattices. According to [7] a lattice $\mathbf{L}$ is called perfect if every element of L is a join of completely join-irreducible elements of $\mathbf{L}$ and a meet of completely meet-irreducible elements of $\mathbf{L}$. This example generalizes the Kripke frame (also called atom structure) of an atomic Boolean algebra with operators, since a Boolean algebra is perfect if and only if it is atomic. The set of all completely join-irreducible (meet-irreducible) elements is denoted by $J^\infty(\mathbf{L})$ ($M^\infty(\mathbf{L})$).

Let $\mathbf{L}$ be a $r\ell$-groupoid with a perfect lattice reduct, and for $x, y \in J^\infty(\mathbf{L})$, $w \in M^\infty(\mathbf{L})$ define

$$\begin{aligned} x \circ y &= \{z \in J^\infty(\mathbf{L}) : z \le xy\} \\ x \mathbin{\backslash\!\backslash} w &= \{v \in M^\infty(\mathbf{L}) : x\backslash w \le v\} \\ w \mathbin{/\!/} y &= \{v \in M^\infty(\mathbf{L}) : w/y \le v\}. \end{aligned}$$

Since $\mathbf{L}$ is perfect, we have $xy = \bigvee(x \circ y)$, $x\backslash w = \bigwedge(x \mathbin{\backslash\!\backslash} w)$, and $w/y = \bigwedge(w \mathbin{/\!/} y)$, hence the nucleus property for $\le$ follows from the residuation property. Therefore $\mathbf{W}_{\mathbf{L}}^\infty = (J^\infty(\mathbf{L}), M^\infty(\mathbf{L}), \le, \circ, \mathbin{\backslash\!\backslash}, \mathbin{/\!/})$ is a residuated frame.

It is shown in [7] that the perfect lattice reduct of $\mathbf{L}$ is embedded in the lattice reduct of $(\mathbf{W}_{\mathbf{L}}^\infty)^+$. Since every element of $\mathbf{L}$ is a join of elements of $J^\infty(\mathbf{L})$ and multiplication distributes over existing joins, it follows that the map $a \mapsto \downarrow(a) \cap J^\infty(\mathbf{L})$ is an embedding of $\mathbf{L}$ into $(\mathbf{W}_{\mathbf{L}}^\infty)^+$. If L is complete (e.g. in the finite case) then this map is an isomorphism.

The system GL. Let $\mathcal{L} = \{\wedge, \vee, \cdot, \backslash, /, 1, 0\}$ be the language of FL-algebras. Terms in this language correspond to propositional formulas in substructural logic, hence the set of all terms (over some fixed countable set of variables) is denoted by Fm. Let $\circ$ be a binary symbol, ε a constant symbol, and define $(W, \circ, \varepsilon)$ to be the free groupoid with unit ε generated by the set Fm. As in the partial subalgebra example, S_W denotes the set of unary linear polynomials of $(W, \circ, \varepsilon)$. (However we do not assume associativity of $\circ$, hence $u(x)$ cannot in general be written in the form $v \circ x \circ w$.) A (single-conclusion) *sequent* is a pair $(x, b) \in W \times Fm$, which is traditionally written $x \Rightarrow b$, and the symbol $\Rightarrow$ is called the *sequent separator*. A *sequent rule* is a pair $(\{s_1, \ldots, s_n\}, s_0)$ where $s_0, \ldots, s_n$ are sequents. Such rules are usually presented in the form

$$\frac{s_1 \quad s_2 \quad \ldots \quad s_n}{s_0} \qquad \text{or} \qquad \frac{}{s_0}$$

with rules of the latter form referred to as *axioms*. Finally, a *Gentzen system* is a set of sequent rules.

Consider the Gentzen system **GL** for the non-associative full Lambek calculus, given by the rules in Figure 1 and all their uniform substitution instances (i.e., a, b, c range over Fm, x, y range over W and u ranges over S_W). The system is essentially obtained from Gentzen's system **LJ** for intuitionitic logic, by removing all the implicit *structural rules*.

$$\frac{x \Rightarrow a \quad u(a) \Rightarrow c}{u(x) \Rightarrow c}\ (\mathrm{CUT}) \qquad \frac{}{a \Rightarrow a}\ (\mathrm{Id})$$

$$\frac{x \Rightarrow a \quad u(b) \Rightarrow c}{u(x \circ (a\backslash b)) \Rightarrow c}\ (\backslash\mathrm{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a\backslash b}\ (\backslash\mathrm{R})$$

$$\frac{x \Rightarrow a \quad u(b) \Rightarrow c}{u((b/a) \circ x) \Rightarrow c}\ (/\mathrm{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\mathrm{R})$$

$$\frac{u(a \circ b) \Rightarrow c}{u(a \cdot b) \Rightarrow c}\ (\cdot\mathrm{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot\mathrm{R})$$

$$\frac{u(a) \Rightarrow c}{u(a \wedge b) \Rightarrow c}\ (\wedge\mathrm{L}\ell) \qquad \frac{u(b) \Rightarrow c}{u(a \wedge b) \Rightarrow c}\ (\wedge\mathrm{L}r) \qquad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge\mathrm{R})$$

$$\frac{u(a) \Rightarrow c \quad u(b) \Rightarrow c}{u(a \vee b) \Rightarrow c}\ (\vee\mathrm{L}) \qquad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee\mathrm{R}\ell) \qquad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee\mathrm{R}r)$$

$$\frac{u(\varepsilon) \Rightarrow a}{u(1) \Rightarrow a}\ (1\mathrm{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathrm{R})$$

FIGURE 1. The system **GL**.

A *proof* in **GL** is defined inductively in the usual way as a labeled rooted tree (where the order of the branches does not matter). Formally, every sequent (S, s) is considered as a proof with *assumption* S and *conclusion* s. Moreover, if $\mathsf{P}_1, \ldots, \mathsf{P}_n$ are proofs with sets of assumptions $S_1, \ldots, S_n$ and conclusions $s_1, \ldots, s_n$, respectively, and if $(\{s_1, \ldots s_n\}, s_0)$ is an instance of a rule in **GL**, then the tree (denoted by)

$$\frac{\mathsf{P}_1 \quad \ldots \quad \mathsf{P}_n}{s_0}$$

is a proof with set of assumptions $S_1 \cup \cdots \cup S_n$ and conclusion s_0. If there is a proof of a sequent s in **GL** from assumptions S, then we write $S \vdash_{\mathbf{GL}} s$ and say that s is *provable in* **GL** *from* S. If S is empty we simply write $\vdash_{\mathbf{GL}} s$ and say that s is provable in **GL**. For more on **GL**, see [10].

We take $W' = S_W \times Fm$, where S_W is the set of all unary linear polynomials in W and define the relation N by

$$x\ N\ (u, a) \qquad \text{iff} \qquad \vdash_{\mathbf{GL}} (u(x) \Rightarrow a).$$

Then

$$x \circ y\ N\ (u, a) \ \text{ iff } \vdash_{\mathbf{GL}} u(x \circ y) \Rightarrow a \ \text{ iff } \ x\ N\ (u(\_ \circ y), a) \ \text{ iff } \ y\ N\ (u(x \circ \_), a).$$

Hence N is a nuclear relation where the appropriate subsets of W' are given by

$$(u, a) /\!\!/ x = \{(u(\_ \circ x), a)\} \text{ and } x \backslash\!\!\backslash (u, a) = \{(u(x \circ \_), a)\}.$$

We denote the resulting residuated frame by $\mathbf{W_{GL}}$.

Let $\mathbf{Fm}$ be the countably generated absolutely free algebra over the language of FL-algebras. Unlike some of the previous examples we cannot expect the map $x \mapsto \{x\}^{\lhd}$ to be an embedding of $\mathbf{Fm}$ into $\mathbf{W}^{+}_{\mathbf{GL}}$. However in Section 4 we show that this map has some weak properties of a homomorphism, referred to as a *quasi-homomorphism* in [2].

We will use this quasi-homomorphism to prove the cut-elimination property for $\mathbf{GL}$, namely that the system obtained from $\mathbf{GL}$ by removing (all instances of) the cut rule has exactly the same provable sequents as $\mathbf{GL}$.

We say that an $r\ell u$-groupoid $\mathbf{G}$ *satisfies* the sequent $x \Rightarrow a$ (also that the sequent *holds* or is *valid* in $\mathbf{G}$) if for every homomorphism $f : \mathbf{Fm} \to \mathbf{G}$, $f(x^{\mathbf{Fm}}) \leq f(a)$. Here $x^{\mathbf{Fm}}$ denotes the formula obtained from x by replacing $\circ$ with $\cdot$. The following theorem states that $\mathbf{GL}$ is sound with respect to the variety of $r\ell u$-groupoids. The proof proceeds by induction on the rules (and axioms) of $\mathbf{GL}$.

Lemma 3.2. (*Soundness*) [10] *Every sequent that is provable in* $\mathbf{GL}$ *is valid in all* $r\ell u$*-groupoids.*

We will show in Section 4 that the converse is also true, i.e., $r\ell u$-groupoids provide a complete semantics.

The system $\mathbf{GL_a}$ is defined to be $\mathbf{GL}$ augmented by the structural rule of associativity

$$\frac{u((x \circ y) \circ z) \Rightarrow c}{\overline{u(x \circ (y \circ z)) \Rightarrow c}}\ (\mathrm{a})$$

The double line means that we assume two rules, the one stated (read downwards) and its inverse (read upward). Other structural rules can be added to obtain further *basic* substructural logic systems, *exchange*, *contraction*, *left weakening* (or *integrality*) and *right weakening*

$$\frac{u(x \circ y) \Rightarrow c}{u(y \circ x) \Rightarrow c}\ (\mathrm{e}) \qquad \frac{u(x \circ x) \Rightarrow c}{u(x) \Rightarrow c}\ (\mathrm{c}) \qquad \frac{u(\varepsilon) \Rightarrow c}{u(x) \Rightarrow c}\ (\mathrm{i}) \qquad \frac{x \Rightarrow \varepsilon}{x \Rightarrow a}\ (\mathrm{o})$$

The (o) rule only applies to an extension $\mathbf{GL}^0$ of $\mathbf{GL}$ with the rules

$$\frac{x \Rightarrow \varepsilon}{x \Rightarrow 0}\ (0\mathrm{R}) \qquad \frac{}{0 \Rightarrow \varepsilon}\ (0\mathrm{L})$$

Note that a residuated frame for the system $\mathbf{GL}^0$ uses $W' = S_W \times (Fm \cup \{\varepsilon\})$. It is easy to see that the sequent $x \Rightarrow \varepsilon$ is provable in $\mathbf{GL}^0$ iff the sequent $x \Rightarrow 0$ is provable. Also, the systems $\mathbf{GL}$ and $\mathbf{GL}^0$ prove the same sequents with non-empty right-hand side. In that sense the two systems are essentially equivalent, but $\mathbf{GL}^0$ supports the addition of further structural rules, like (o). We extend the subscript notation, so for example $\mathbf{GL_{ae}}$ is $\mathbf{GL}$ plus associativity and exchange. Furthermore we abbreviate the combination of (i) and (o) to *weakening* (w). The system $\mathbf{GL^0_{aecw}}$ is equivalent to Gentzen's original system $\mathbf{LJ}$ for intuitionistic logic.

The system FL. The Gentzen system $\mathbf{FL}$ is an associative version of $\mathbf{GL}^0$. The only difference is that now W (containing the left-hand sides of sequents) is defined as the free monoid over the set Fm of formulas. Consequently, x, y, z range over sequences of formulas. Note that $\mathbf{FL}$ has the same rules as $\mathbf{GL}$, but different rule instances. Traditionally sequents like $u(a) \Rightarrow d$ are denoted by $\Gamma, A, \Delta \Rightarrow \delta$. The system $\mathbf{FL}$ was introduced by H. Ono in [17] and is called *full Lambek calculus*. In

$$\frac{a \le b \quad u(b) \le c}{u(a) \le c}\ (\text{cut}) \qquad \frac{}{a \le a}\ (\text{id}) \qquad \frac{a \le b \quad c \le d}{ac \le bd}\ (\cdot\text{r})$$

$$\frac{a \le b \quad u(c) \le d}{u(a(b\backslash c)) \le d}\ (\backslash\text{l}) \qquad \frac{ab \le c}{b \le a\backslash c}\ (\backslash\text{r})$$

$$\frac{a \le b \quad u(c) \le d}{u((c/b)a) \le d}\ (/\text{l}) \qquad \frac{ab \le c}{a \le c/b}\ (/\text{r})$$

$$\frac{u(a) \le c}{u(a \wedge b) \le c}\ (\wedge\text{l}\ell) \qquad \frac{u(b) \le c}{u(a \wedge b) \le c}\ (\wedge\text{l}r) \qquad \frac{a \le b \quad a \le c}{a \le b \wedge c}\ (\wedge\text{r})$$

$$\frac{u(a) \le c \quad u(b) \le c}{u(a \vee b) \le c}\ (\vee\text{l}) \qquad \frac{a \le b}{a \le b \vee c}\ (\vee\text{r}\ell) \qquad \frac{a \le c}{a \le b \vee c}\ (\vee\text{r}r)$$

$$\frac{|u| \le a}{u(1) \le a}\ (1\text{l}) \qquad \frac{a \le b \quad 1 \le c}{a \le bc}\ (1\text{r}\ell) \qquad \frac{1 \le b \quad a \le c}{a \le bc}\ (1\text{r}r)$$

FIGURE 2. The system **PL**.

contrast, *Lambek calculus* is a system without connectives and rules for $\wedge$, $\vee$, 1 and 0.

Since $\circ$ is an associative operation, we omit any parentheses. In fact $\circ$ is traditionally denoted by comma, and the elements of W are concretely realized as finite sequences of formulas. On the other hand the operation $\cdot$ on **Fm** is not associative. Note that if a, b, c, d are formulas from the sequent $a \circ b \circ c \Rightarrow d$ by using two applications of $(\cdot\text{L})$ we can prove the distinct sequents $(a \cdot b) \cdot c \Rightarrow d$ and $a \cdot (b \cdot c) \Rightarrow d$.

Note that **FL** is equivalent to the system $\mathbf{GL_a}$. Likewise, $\mathbf{FL_e}$ is equivalent to $\mathbf{GL_{ae}}$. As shown by the next result, the naming similarity between **FL** and FL-algebras is not a coincidence.

Lemma 3.3. [9] *Every sequent that is provable in* **FL** *is valid in all FL-algebras.*

It turns out that FL is an *equivalent algebraic semantics* for **FL**. For more on **FL**, see for example [8]. The corresponding residuated frame $\mathbf{W_{FL}}$ is associative.

The systems PL and $\mathbf{PL'_a}$. The *algebraic Gentzen system* **PL** is given in Figure 2. Apart from the superficial replacement of the separator $\Rightarrow$ by the symbol $\le$, in **PL** a sequent is identified with a pair of formulas (or terms) from Fm. Therefore a, b, c range over formulas and u ranges over unary linear polynomials of the groupoid $(Fm, \cdot)$, where $\cdot$ is the operation corresponding to the connective of Fm. The phrase "*algebraic* Gentzen system" refers to the fact that sequents can be considered inequations, since both sides are algebraic terms. In (1l) $|u|$ denotes the formula resulting from $u(1)$ by reducing the formula as if this occurrence of 1 were an identity for "$\cdot$".

We define $W = Fm$, and $\circ$ on W by $a \circ b = a \cdot b$, if neither of a, b is 1, and by $a \circ 1 = a = 1 \circ a$. Clearly $(W, \circ, 1)$ is a groupoid with unit. We define $W' = S_W \times Fm$ and the relation N by

$$x \; N \; (v, a) \text{ iff } \vdash_{\mathbf{PL}} v(x) \Rightarrow a.$$

Note that v is different than the u used in the definition of **PL**, as it involves $\circ$ (which in turn is evaluated as $\cdot$ most of the times). The relation N is nuclear for the same reasons as in **GL**. We denote the associated residuated frame by $\mathbf{W_{PL}}$.

The subscript notation referring to the basic structural rules (e), (c), (i) and (o) is defined also for **PL**. So $\mathbf{PL_a}$ is obtained by adding the associativity rule (a) to **PL**.

Lemma 3.4. *Every sequent that is provable in* $\mathbf{PL_a}$ *is valid in all FL-algebras. The same holds for* **PL** *and* $r\ell u$*-groupoids.*

In [14] another associative version $\mathbf{PL'_a}$ of $\mathbf{PL_a}$ is defined that is more of an algebraic rendering of **FL** than of $\mathbf{GL_a}$. Unlike in the system $\mathbf{PL_a}$, where the two sides of the sequents (i.e., elements of W) are formulas, in $\mathbf{PL'_a}$ they are equivalence classes of formulas. Let $\equiv_m$ be the congruence on the algebra $\mathbf{Fm}$ that makes the identifications $1 \cdot a \equiv_m a \equiv_m a \cdot 1$ and $(a \cdot b) \cdot c \equiv_m a \cdot (b \cdot c)$, for all $a, b, c \in Fm$. The relation $\equiv_m$ is the least congruence such that the quotient $(Fm/{\equiv_m}, \cdot, 1)$ is a monoid. We define $(W, \circ, 1)$ to be this monoid and sequents to be pairs (a, b) of elements of W (but use the notation $a \leq b$). Apart from this modification, the system $\mathbf{PL_a}$ is defined by the same set of rules as **PL**. In other words, the two systems have the same rules, but they have different rule instances, very much like in the case of **GL** and **FL**.

Similar considerations allow the definition of systems like $\mathbf{PL'_{ae}}$. Also, frames like $\mathbf{W_{PL_a}}$ and $\mathbf{W_{PL'_a}}$ are defined in the obvious way.

Lemma 3.5. [14] *Every sequent that is provable in* $\mathbf{PL'_a}$ *is valid in all FL-algebras.*

The system ML. The algebraic Gentzen system **ML** given in Figure 3 was defined in [15], and is based on the same sequents as **PL**.

One can observe that the rules do not mention unary linear polynomials, but instead the rules (\res) and (/res) are bidirectional (their inverses are also assumed in the system). We mention that the use of the *context* in the form of sections u was necessary in **PL** in order to be able to access deep inside a formula and apply the left rules. The same effect is accomplished in **ML** by the use of the two bidirectional rules: we use the rules downward to isolate in the left-hand side of $\leq$ the part we want to access, apply the desired left-hand side rule and use the bidirectional rules applied upward to move back the shifted context. This is also seen in the proof of the fact that N is nuclear, which holds for different reasons than in **PL**.

Lemma 3.6. [15] *Every sequent that is provable in* **ML** *is valid in all* $r\ell u$*-groupoids.*

We take $W = W' = Fm$, $\circ$ is defined as in $\mathbf{W_{PL}}$ and N is defined by

$$x \; N \; y \text{ iff } \vdash_{\mathbf{ML}} x \leq y.$$

We have

$$x \circ y \; N \; z \text{ iff } \vdash_{\mathbf{ML}} (x \circ y \leq z) \text{ iff } \vdash_{\mathbf{ML}} (x \leq z/y) \text{ iff } x \; N \; z/y \text{ iff } y \; N \; x\backslash z.$$

So, $y /\!\!/ x = \{y/x\}$ and $x \backslash\!\!\backslash y = \{x\backslash y\}$. The resulting residuated frame is denoted by $\mathbf{W_{ML}}$.

$$\frac{a \le b \quad b \le c}{a \le c}\ (\mathrm{tr}) \qquad \frac{}{a \le a}\ (\mathrm{id}) \qquad \frac{a \le b \quad c \le d}{ac \le bd}\ (\cdot)$$

$$\frac{a \le b \quad c \le d}{b\backslash c \le a\backslash d}\ (\backslash\mathrm{o}) \qquad \frac{ab \le c}{\overline{b \le a\backslash c}}\ (\backslash\mathrm{res})$$

$$\frac{a \le b \quad c \le d}{c/b \le d/a}\ (/\mathrm{o}) \qquad \frac{ab \le c}{\overline{a \le c/b}}\ (/\mathrm{res})$$

$$\frac{a \le c}{a \wedge b \le c}\ (\wedge\mathrm{lt}\ell) \qquad \frac{b \le c}{a \wedge b \le c}\ (\wedge\mathrm{lt}r) \qquad \frac{a \le b \quad a \le c}{a \le b \wedge c}\ (\wedge\mathrm{rt})$$

$$\frac{a \le c \quad b \le c}{a \vee b \le c}\ (\vee\mathrm{lt}) \qquad \frac{a \le b}{a \le b \vee c}\ (\vee\mathrm{rt}\ell) \qquad \frac{a \le c}{a \le b \vee c}\ (\vee\mathrm{rt}r)$$

$$\frac{a \le c \quad b \le 1}{ab \le c}\ (1\mathrm{r}\ell) \quad \frac{a \le 1 \quad b \le c}{ab \le c}\ (1\mathrm{r}r) \quad \frac{a \le b \quad 1 \le c}{a \le bc}\ (1\mathrm{r}\ell) \quad \frac{1 \le b \quad a \le c}{a \le bc}\ (1\mathrm{r}r)$$

Figure 3. The system **ML**.

4. Gentzen frames

After a short discussion of the similarities of the previous examples, we define a common abstraction called a Gentzen frame. This allows us to prove a quasi-homomorphism result that yields simultaneously the three embeddings claimed in the last section and will be instrumental in obtaining the new results in the paper.

Note that in $\mathbf{W_{GL}}$, if $a\ N\ (u, c)$ and $b\ N\ (u, c)$, then $\vdash_{\mathbf{GL}} u(a) \Rightarrow c$ and $\vdash_{\mathbf{GL}} u(b) \Rightarrow c$. In view of the rule ($\vee$L), we obtain $\vdash_{\mathbf{GL}} u(a \vee b) \Rightarrow c$, namely $a \vee b\ N\ (u, c)$. Therefore, in $\mathbf{W_{GL}}$ we have the implication

$$\text{if } a\ N\ z \text{ and } b\ N\ z, \text{ then } a \vee b\ N\ z.$$

The same argument works for $\mathbf{W_{PL}}$, $\mathbf{W_{PL_a}}$, $\mathbf{W_{PL'_a}}$ and $\mathbf{W_{FL}}$. Note that the implication also holds in $\mathbf{W_{ML}}$ for similar, but slightly different reasons, since we do not have a context u.

Interestingly enough, the same implication holds for $\mathbf{W_G}$, where $\mathbf{G}$ is a GL-algebra. Indeed, if $a\ N\ c$ and $b\ N\ c$, then $a \le c$ and $b \le c$, so $a \vee b \le c$ and $a \vee b\ N\ c$. Furthermore, if $\mathbf{A}$ is a $r\ell u$-groupoid, $\mathbf{B}$ a partial subalgebra of $\mathbf{A}$ and $a, b, a \vee b \in B$, then $a\ N\ c$ and $b\ N\ c$, namely $a \le c$ and $b \le c$ implies $a \vee b \le c$ and $a \vee b\ N\ c$.

In other words, the above implication, which we also write in the form

$$\frac{a\ N\ z \quad b\ N\ z}{a \vee b\ N\ z}\ (\vee\mathrm{L})$$

holds in the residuated frames $\mathbf{W_{GL}}$, $\mathbf{W_{FL}}$, $\mathbf{W_{PL}}$, $\mathbf{W_{ML}}$, $\mathbf{W_{PL'_a}}$, $\mathbf{W_G}$, $\mathbf{W_{A,B}}$, for all $z \in W'$ and all a, b that are elements of Fm for the first four frames, elements of $Fm/{\equiv_m}$ for $\mathbf{W_{PL'_a}}$, elements of G for $\mathbf{W_G}$, and elements of B for $\mathbf{W_{A,B}}$. Note that the sets Fm, $Fm/{\equiv_m}$, G and B are subsets of W, in the corresponding frames, and they actually generate it as a groupoid under the operation $\circ$. Moreover, they are all (partial) $\mathcal{L}$-algebras. Furthermore, these sets can be identified with a subset of W' and their elements are exactly the right-hand sides of the sequents in each case. In the case of $\mathbf{W_{GL}}$, $\mathbf{W_{FL}}$, $\mathbf{W_{PL}}$, $\mathbf{W_{PL'_a}}$, every $b \in Fm$ can be identified with the element (id, b) of W', where id is the identity polynomial. The same identification works for elements of B for $\mathbf{W_{A,B}}$. Finally, in the case of $\mathbf{W_{ML}}$ and $\mathbf{W_G}$ the corresponding set Fm or G is the set W' itself.

The above considerations lead to the following definition about a pair of a residuated frame and a special partial $\mathcal{L}$-algebra.

Definition and quasi-homomorphism. A *Gentzen ru-frame* is a pair $(\mathbf{W}, \mathbf{B})$ where

(i) $\mathbf{W} = (W, W', N, \circ, \backslash\!\!\backslash, /\!\!/, \{\varepsilon\})$ is a ru-frame with $\circ$ a binary *operation*,
(ii) $\mathbf{B}$ is a partial $\mathcal{L}$-algebra,
(iii) $(W, \circ, \varepsilon)$ is a groupoid with unit generated by $B \subseteq W$,
(iv) there is an injection of B into W' (under which we will identify B with a subset of W') and
(v) N satisfies the rules of **GN** (Figure 4) for all $a, b \in B$, $x, y \in W$ and $z \in W'$.

A rule is understood to hold only in case all the expressions in it make sense. For example, $(\wedge\mathrm{L}\ell)$ is read as, if $a, b, a \wedge b \in B$, $z \in W'$ and $a\ N\ z$, then $a \wedge b\ N\ z$.

$$\frac{x\ N\ a \quad a\ N\ z}{x\ N\ z}\ (\mathrm{CUT}) \qquad \frac{}{a\ N\ a}\ (\mathrm{Id})$$

$$\frac{x\ N\ a \quad b\ N\ z}{x \circ (a\backslash b)\ N\ z}\ (\backslash\mathrm{L}) \qquad \frac{a \circ x\ N\ b}{x\ N\ a\backslash b}\ (\backslash\mathrm{R})$$

$$\frac{x\ N\ a \quad b\ N\ z}{(b/a) \circ x\ N\ z}\ (/\mathrm{L}) \qquad \frac{x \circ a\ N\ b}{x\ N\ b/a}\ (/\mathrm{R})$$

$$\frac{a \circ b\ N\ z}{a \cdot b\ N\ z}\ (\cdot\mathrm{L}) \qquad \frac{x\ N\ a \quad y\ N\ b}{x \circ y\ N\ a \cdot b}\ (\cdot\mathrm{R})$$

$$\frac{a\ N\ z}{a \wedge b\ N\ z}\ (\wedge\mathrm{L}\ell) \qquad \frac{b\ N\ z}{a \wedge b\ N\ z}\ (\wedge\mathrm{L}r) \qquad \frac{x\ N\ a \quad x\ N\ b}{x\ N\ a \wedge b}\ (\wedge\mathrm{R})$$

$$\frac{a\ N\ z \quad b\ N\ z}{a \vee b\ N\ z}\ (\vee\mathrm{L}) \qquad \frac{x\ N\ a}{x\ N\ a \vee b}\ (\vee\mathrm{R}\ell) \qquad \frac{x\ N\ b}{x\ N\ a \vee b}\ (\vee\mathrm{R}r)$$

$$\frac{\varepsilon\ N\ z}{1\ N\ z}\ (1\mathrm{L}) \qquad \frac{}{\varepsilon\ N\ 1}\ (1\mathrm{R})$$

FIGURE 4. The system **GN**.

A *Gentzen ruz-frame* is a Gentzen ru-frame extended with the set $\{\varepsilon\}^{\lhd}$, and (iv),(v) are modified as follows:

(iv') there is an injection of $B \cup \{\varepsilon\}$ into W' (under which we will identify $B \cup \{\varepsilon\}$ with a subset of W') and

(v') N satisfies the rules of **GN** (Figure 4) for all $a, b \in B$, $x, y \in W$ and $z \in W'$ as well as

$$\frac{x \ N \ \varepsilon}{x \ N \ 0} \ (0R) \qquad \frac{}{0 \ N \ \varepsilon} \ (0L)$$

A *cut-free Gentzen frame* is defined in the same way, but it is not stipulated to satisfy the (CUT) rule. A *sequent* in the (possibly cut-free) Gentzen frame $(\mathbf{W}, \mathbf{B})$ is an element of $W \times B$. We use the notation $x \Rightarrow a$, or $x \leq a$ for such a pair.

Lemma 4.1. *Keeping the notation of the previous section,* $(\mathbf{W_G}, \mathbf{G})$, $(\mathbf{W_{A,B}}, \mathbf{B})$, $(\mathbf{W_{GL}}, \mathbf{Fm})$, $(\mathbf{W_{FL}}, \mathbf{Fm})$, $(\mathbf{W_{PL}}, \mathbf{Fm})$, $(\mathbf{W_{PL'_a}}, \mathbf{Fm}/{\equiv_m})$ *and* $(\mathbf{W_{ML}}, \mathbf{Fm})$ *are Gentzen frames.*

Proof. To check that $\mathbf{W_{PL}}$ satisfies **GN**, we focus on the rules for product and 1, as the other are immediate. The rule $(\cdot L)$ of **GN** reads as

$$\frac{v(a \circ b) \leq c}{v(a \cdot b) \leq c}$$

for $\mathbf{W_{PL}}$. If neither of a, b are 1, then the two inequalities in the rule are identical. If one of them is 1, then the rule is a consequence of the rule (1l) of **PL**. Likewise, the rule

$$\frac{a \leq b \quad c \leq d}{a \circ c \leq bd} \ (\cdot R)$$

follows from $(\cdot r)$, $(1r\ell)$ and $(1rr)$ of **PL**. The rule (1L) of **GN** is immediate and (1R) are follows from (id), as $\varepsilon_W = 1$.

The argument for $\mathbf{W_{ML}}$ is similar. For $(\cdot R)$, we now use the rules

$$\frac{a \leq c}{a1 \leq c} \qquad \frac{b \leq c}{1b \leq c}$$

which follow from $(1r\ell)$ and $(1rr)$. This also shows that the last two rules of **ML** can be simplified slightly.

The verification for the remaining frames is straight forward. ☐

We will see more Gentzen frames later. The following theorem will yield a common generalization of the promised embeddings of the previous section.

Theorem 4.2. *Let* $(\mathbf{W}, \mathbf{B})$ *be a cut-free Gentzen ru-frame. For all* $a, b \in B$, $X, Y \in \mathbf{W}^+$ *and for every connective* $\bullet$*, if* $a \bullet^{\mathbf{B}} b$ *is defined, then*

(i) $1^{\mathbf{B}} \in \gamma_N(\varepsilon) \subseteq \{1_{\mathbf{B}}\}^{\lhd}$.

(ii) *If* $a \in X \subseteq \{a\}^{\lhd}$ *and* $b \in Y \subseteq \{b\}^{\lhd}$*, then* $a \bullet^{\mathbf{B}} b \in X \bullet^{\mathbf{W}^+} Y \subseteq \{a \bullet^{\mathbf{B}} b\}^{\lhd}$.

(iii) *In particular,* $a \bullet^{\mathbf{B}} b \in \{a\}^{\lhd} \bullet^{\mathbf{W}^+} \{b\}^{\lhd} \subseteq \{a \bullet^{\mathbf{B}} b\}^{\lhd}$.

(iv) *If, additionally,* N *satisfies* (CUT) *then* $\{a\}^{\lhd} \bullet^{\mathbf{W}^+} \{b\}^{\lhd} = \{a \bullet^{\mathbf{B}} b\}^{\lhd}$.

Furthermore, if $(\mathbf{W}, \mathbf{B})$ *is a cut-free ruz-frame we have*

(v) $0^{\mathbf{B}} \in \{\varepsilon\}^{\lhd} \subseteq \{0^{\mathbf{B}}\}^{\lhd}$.

Proof. (i) Here $\bullet = 1$, so by assumption $1^{\mathbf{B}}$ is defined. By (1R), we have $\varepsilon \in \{1^{\mathbf{B}}\}^{\lhd}$, so $\gamma_N(\varepsilon) \subseteq \{1^{\mathbf{B}}\}^{\lhd}$.

On the other hand, if $\gamma_N(\varepsilon) \subseteq \{z\}^{\lhd}$, then $\varepsilon \in \{z\}^{\lhd}$ and $\varepsilon \ N \ z$. Therefore $1^{\mathbf{B}} \ N \ z$ by (1L), and hence $1^{\mathbf{B}} \in \{z\}^{\lhd}$. Thus, $1^{\mathbf{B}} \in \gamma_N(\varepsilon)$.

(ii) We will give the proof for the connectives $\vee$, $\cdot$ and $\backslash$. The proof for the remaining two connectives follows the same ideas.

Let $\bullet = \vee$. If $x \in X$, then $x \in \{a\}^{\lhd}$, or equivalently $x\ N\ a$. By ($\vee$Rℓ), $x\ N\ a \vee b$, hence $x \in \{a \vee b\}^{\lhd}$. Consequently $X \subseteq \{a \vee b\}^{\lhd}$, and similarly, we obtain $Y \subseteq \{a \vee b\}^{\lhd}$ using ($\vee$Rr). Therefore $X \cup Y \subseteq \{a \vee b\}^{\lhd}$ and hence $X \vee Y = \gamma_N(X \cup Y) \subseteq \{a \vee b\}^{\lhd}$.

On the other hand, let $z \in W'$ and assume $X \vee Y \subseteq \{z\}^{\lhd}$. Then $a \in X \subseteq X \vee Y \subseteq \{z\}^{\lhd}$, so $a\ N\ z$. Similarly, $b\ N\ z$, so $a \vee b\ N\ z$ by ($\vee$L), hence $a \vee b \in \{z\}^{\lhd}$. Thus, $a \vee b \in X \vee Y$, by Corollary 2.4.

Let $\bullet = \cdot$. If $x \in X$ and $y \in Y$, then $x \in \{a\}^{\lhd}$ and $y \in \{b\}^{\lhd}$, i.e., $x\ N\ a$ and $x\ N\ b$. It follows from ($\cdot$R) that $x \circ y\ N\ a \cdot b$, hence $x \circ y \in \{a \cdot b\}^{\lhd}$. Consequently, $X \circ Y \subseteq \{a \cdot b\}^{\lhd}$ and therefore $X \cdot^{\mathbf{W}^+} Y = \gamma_N(X \circ Y) \subseteq \{a \cdot b\}^{\lhd}$.

On the other hand, let $z \in W'$ and assume $X \cdot^{\mathbf{W}^+} Y \subseteq \{z\}^{\lhd}$. Since $a \circ b \in X \circ Y \subseteq \gamma_N(X \circ Y) = X \cdot^{\mathbf{W}^+} Y$, we have $a \circ b \in \{z\}^{\lhd}$, so $a \circ b\ N\ z$. Consequently, $a \cdot b\ N\ z$, by ($\cdot$L), hence $a \cdot b \in \{z\}^{\lhd}$. Thus, $a \cdot b \in X \cdot Y$.

Let $\bullet = \backslash$. If $x \in X \backslash^{\mathbf{W}^+} Y$ then $X \circ \{x\} \subseteq Y$. Since $a \in X$ and $Y \subseteq \{b\}^{\lhd}$, we have $a \circ x \in \{b\}^{\lhd}$, i.e., $a \circ x\ N\ b$. By ($\backslash$R) we obtain $x\ N\ a \backslash b$, hence $x \in \{a \backslash b\}^{\lhd}$.

On the other hand, if $Y \subseteq \{z\}^{\lhd}$, then $b \in \{z\}^{\lhd}$, so $b\ N\ z$. For all $x \in \{a\}^{\lhd}$, $x\ N\ a$, so $x \circ (a \backslash b)\ N\ z$, by ($\backslash$L), i.e., $x \circ (a \backslash b) \in \{z\}^{\lhd}$, for all $x \in \{a\}^{\lhd}$. Consequently, $\{a\}^{\lhd} \circ \{a \backslash b\} \subseteq \{z\}^{\lhd}$, for all $\{z\}^{\lhd}$ that contain Y, so $\{a\}^{\lhd} \circ \{a \backslash b\} \subseteq Y$. Since $X \subseteq \{a\}^{\lhd}$, we have $X \circ \{a \backslash b\} \in Y$, so $a \backslash b \in X \backslash^{\mathbf{W}^+} Y$.

Statement (iii) is a direct consequence of (ii) for $X = \{a\}^{\lhd}$ and $Y = \{b\}^{\lhd}$.

(iv) We first show that if (CUT) holds and $c \in Z \subseteq \{c\}^{\lhd}$ then $\{c\}^{\lhd} = Z$. If $x \in \{c\}^{\lhd}$, then $x\ N\ c$. To show that $x \in Z$, let $Z \subseteq \{z\}^{\lhd}$, for some $z \in W'$. Since $c \in Z$ by assumption, we get $c \in \{z\}^{\lhd}$, or equivalently $c\ N\ z$. By (CUT) we obtain $x\ N\ z$, namely $x \in \{z\}^{\lhd}$. Consequently, $x \in Z$ by Corollary 2.4.

Taking $c = a \bullet b$ and $Z = \{a\}^{\lhd} \bullet \{b\}^{\lhd}$, we obtain $\{a\}^{\lhd} \bullet \{b\}^{\lhd} = \{a \bullet b\}^{\lhd}$ from (ii).

(v) The (0L) rule immediately implies that $0 \in \{\varepsilon\}^{\lhd}$, and it follows from (0R) that $x \in \{\varepsilon\}^{\lhd}$ implies $x \in \{0^{\mathbf{B}}\}^{\lhd}$ for any $x \in W$. Hence $\{\varepsilon\}^{\lhd} \subseteq x \in \{0^{\mathbf{B}}\}^{\lhd}$. $\square$

Note that the conditions in **GN** are not only sufficient, but also necessary for condition (ii). We mention that there is a generalization of the above theorem that refers to a notion of a Gentzen frame where W is not necessarily a groupoid and B is a relational structure, but we do not have any applications at this point and we do not state or prove it.

Corollary 4.3. *If* $(\mathbf{W}, \mathbf{B})$ *is a Gentzen frame, the map* $x \mapsto \{x\}^{\lhd}$ *from* $\mathbf{B}$ *to* $\mathbf{W}^+$ *is a homomorphism from the partial algebra* $\mathbf{B}$ *into the* $r\ell u(z)$*-groupoid* $\mathbf{W}^+$*.*

If $(\mathbf{W}, \mathbf{B})$ is a cut-free Gentzen frame, then $x \mapsto \{x\}^{\lhd}$ comes close to being a homomorphism, hence it is called a *quasi-homomorphism*.

In a cut-free Gentzen frame $(\mathbf{W}, \mathbf{B})$ the relation N is called *antisymmetric on* B if, for all $a, b \in B$, $a\ N\ b$ and $b\ N\ a$ implies $a = b$.

Corollary 4.4. *If* $(\mathbf{W}, \mathbf{B})$ *is a (cut-free) Gentzen frame and* N *is antisymmetric on* B*, then the map* $x \mapsto \{x\}^{\lhd}$ *from* $\mathbf{B}$ *to* $\mathbf{W}^+$ *is a (quasi-)embedding.*

Proof. We only need to show that the map in injective. Assume $\{a\}^{\lhd} = \{b\}^{\lhd}$, for $a, b \in B$. Recall that $a \in \{a\}^{\lhd}$ and $b \in \{b\}^{\lhd}$, so $a\ N\ b$ and $b\ N\ a$. Consequently, $a = b$. $\square$

Embedding into the DM-completion and representation. Clearly if $\mathbf{G}$ is an $r\ell u$-groupoid, the Galois relation $\leq$ of $\mathbf{W_G}$ is antisymmetric on G.

Corollary 4.5. *Let* $\mathbf{G}$ *be an* $r\ell u$*-groupoid. The map* $x \mapsto \{x\}^{\lhd}$ *from* $\mathbf{G}$ *to* $\mathbf{W}^+_{\mathbf{G}}$ *is an embedding.*

Therefore every residuated lattice can be embedded into a complete one.

Corollary 4.6. [5] *Every residuated lattice is a subalgebra of the nucleus image of the power set of a monoid.*

It follows from the proof of this result that commutative residuated lattices are represented by commutative monoids. More generally, since all monoid identities are preserved by nuclei, residuated lattices defined by monoid equations that are preserved under the powerset construction are represented by the class of monoids that satisfy these identities. This includes all linear and balanced monoid identities (where an identity is linear if each variable appears at most once on each side, and balanced if each variable appears on both sides).

5. Cut elimination

The *cut elimination property* for a Gentzen system states that the set of provable sequents does not change if the cut rule is removed from the system. Since the effect of the cut rule cannot be simulated by composing the other rules (the cut rule is not *derivable*) traditional arguments of cut elimination try to replace the cut rule locally, by exploring the particulars of each occurrence of the rule. Such arguments consist of an algorithm for rewriting proofs and in each rewriting step the resulting proof is simpler (according to some complexity measure). The idea is that the proof is rewritten locally so that the cut rules are pushed upward in the proof until they finally disappear. The argument usually proceeds by double induction and consists of a tedious case analysis.

A semantical proof of the cut elimination for $\mathbf{FL}$ is given in [2]. This method was then extended in [10] to systems including $\mathbf{GL}$, $\mathbf{FL}$, $\mathbf{FL_{ec}}$ and various other extensions. We will obtain the cut elimination for all the above systems, as well as for $\mathbf{PL}$, $\mathbf{PL'_a}$ and $\mathbf{ML}$ among others, as corollaries of a general theorem. Note that the notion of a sequent differs in these systems. To account for this we will prove the theorem in a general setting and then instantiate it to the particular cases.

Cut elimination for GN. Let $(\mathbf{W}, \mathbf{B})$ be a cut-free Gentzen $ru(z)$-frame. From now on we assume that B is a total $\mathcal{L}$-algebra. For every homomorphism $f : \mathbf{Fm} \to \mathbf{B}$, we let $\bar{f} : \mathbf{Fm} \to \mathbf{W}^+$ be the $\mathcal{L}$-homomorphism that extends the assignment $p \mapsto \{f(p)\}^{\lhd}$, for all variables p of $\mathbf{Fm}$. [More generally, we may define the assignment by $p \mapsto Q_p$, where Q_p is any set such that $\{f(p)\}^{\rhd\lhd} \subseteq Q_p \subseteq \{f(p)\}^{\lhd}$.]

Lemma 5.1. *If* $(\mathbf{W}, \mathbf{B})$ *is a cut-free Gentzen frame and* $\mathbf{B}$ *a total algebra, then for every homomorphism* $f : \mathbf{Fm} \to \mathbf{B}$*, we have* $f(a) \in \bar{f}(a) \subseteq \{f(a)\}^{\lhd}$*, for all* $a \in Fm$*. If* $(\mathbf{W}, \mathbf{B})$ *is a Gentzen frame, then* $\bar{f}(a) = \{f(a)\}^{\lhd}$*, for all* $a \in Fm$*.*

Proof. Let $f : \mathbf{Fm} \to \mathbf{B}$ be a homomorphism. By definition of $\bar{f}$ and the axiom (id), the statement holds for the propositional variables. For $a = 1$, by Theorem 4.2(i), we have $f(1) = 1^{\mathbf{B}} \in \gamma_N(\varepsilon) = 1^{\mathbf{W}^+} \subseteq \{1^{\mathbf{B}}\}^{\lhd} = \{f(1)\}^{\lhd}$. We proceed by induction. Assume that $f(a) \in \bar{f}(a) \subseteq \{f(a)\}^{\lhd}$ and $f(b) \in \bar{f}(b) \subseteq \{f(b)\}^{\lhd}$. By Theorem 4.2(iii), for each connective $\bullet$, we have $f(a) \bullet_B f(b) \in \bar{f}(a) \bullet^{\mathbf{W}^+} \bar{f}(b) \subseteq$

$\{f(a) \bullet^{\mathbf{B}} f(b)\}^{\lhd}$. Since f and $\bar{f}$ are homomorphisms, we have $f(a \bullet b) \in \bar{f}(a \bullet b) \subseteq \{f(a \bullet b)\}^{\lhd}$. Finally, if $(\mathbf{W}, \mathbf{B})$ is a Gentzen frame, then $\bar{f}(a) = \{f(a)\}^{\lhd}$, by Theorem 4.2(iv). $\square$

To account for the different types of sequents in the applications we will use the most general type. For this section an *(intuitionistic) sequent* is an element of $Fm^{\circ} \times Fm$, where $\mathbf{Fm}^{\circ} = (Fm^{\circ}, \circ, \varepsilon)$ denotes the absolutely free algebra in the signature $\{\circ, \varepsilon\}$ over the set Fm. We use the notation $x \Rightarrow a$ for sequents.

Let $(\mathbf{W}, \mathbf{B})$ be a Gentzen frame. Note that every map $f : Fm \to B$ extends inductively to a map $f^{\circ} : Fm^{\circ} \to W$ by $f^{\circ}(x \circ^{\mathbf{Fm}^{\circ}} y) = f^{\circ}(x) \circ^{W} f^{\circ}(y)$. Likewise, every homomorphism $f : \mathbf{Fm} \to \mathbf{G}$ into an $\mathcal{L}$-algebra $\mathbf{G}$ extends to a homomorphism $f^{\circ} : \mathbf{Fm}^{\circ} \to \mathbf{G}$. A sequent $x \Rightarrow a$ is said to be *valid* in $(\mathbf{W}, \mathbf{B})$, if for every homomorphism $f : \mathbf{Fm} \to \mathbf{B}$, we have $f^{\circ}(x)\ N\ f(a)$. A sequent $x \Rightarrow a$ is said to be *valid* in a residuated ℓ-groupoid $\mathbf{G}$, if it is valid in the Gentzen frame $(\mathbf{W_G}, \mathbf{G})$, namely if for all homomorphisms $f : \mathbf{Fm} \to \mathbf{G}$, we have $f^{\circ}(x) \leq f(a)$.

Theorem 5.2. *If* $(\mathbf{W}, \mathbf{B})$ *is a cut free Gentzen* $r(u)(z)$*-frame, then every sequent that is valid in* $\mathbf{W}^{+}$ *is also valid in* $(\mathbf{W}, \mathbf{B})$.

Proof. Assume that $x \Rightarrow a$ is valid in $\mathbf{W}^{+}$ and let $f : \mathbf{Fm} \to \mathbf{B}$ be a homomorphism. We will show that $f^{\circ}(x)\ N\ f(a)$. Since $x \Rightarrow a$ is valid in $\mathbf{W}^{+}$, for the homomorphism $\bar{f} : \mathbf{Fm} \to \mathbf{W}^{+}$, we have $\bar{f}(x) \subseteq \bar{f}(a)$. If $x = t^{\mathbf{Fm}^{\circ}}(b_1, \ldots, b_n)$, for $b_1, \ldots, b_n \in Fm$, then $\bar{f}(x) = \bar{f}(x^{\mathbf{Fm}^{\circ}}(b_1, \ldots, b_n)) = x^{\mathbf{W}^{+}}(\bar{f}(b_1), \ldots, \bar{f}(b_n))$. By Lemma 5.1, $\bar{f}(a) \subseteq \{f(a)\}^{\lhd}$ and $f(b) \in \bar{f}(b)$, for all formulas b of t. Hence

$$
\begin{array}{lll}
f^{\circ}(x) & = f^{\circ}(t^{\mathbf{Fm}^{\circ}}(b_1, \ldots, b_n)) & \\
 & = t^{(W,\circ)}(f(b_1), \ldots, f(b_n)) & (f^{\circ} \text{ is a homomorphism extending } f) \\
 & \in t^{\mathcal{P}(W)}(\bar{f}(b_1), \ldots, \bar{f}(b_n)) & (\circ \text{ in } \mathcal{P}(W) \text{ is element-wise}) \\
 & \subseteq t^{\mathbf{W}^{+}}(\bar{f}(b_1), \ldots, \bar{f}(b_n)) & (\gamma_N \text{ is a closure operator}) \\
 & = \bar{f}^{\circ}(x) &
\end{array}
$$

It follows that $f^{\circ}(x) \in \bar{f}^{\circ}(x) \subseteq \bar{f}(a) \subseteq \{f(a)\}^{\lhd}$, and therefore $f^{\circ}(x) \in \{f(a)\}^{\lhd}$, i.e., $f^{\circ}(x)\ N\ f(a)$. $\square$

Remark 5.3. Theorem 5.2 is valid even in case $\mathbf{B}$ is a partial algebra. The necessary modifications in various parts of the section are easy. Given a partial homomorphism from $\mathbf{Fm}$ to $\mathbf{B}$, we can define a set of homomorphisms $\bar{f} : \mathbf{Fm} \to \mathbf{W}^{+}$ that extend the composition of f and $\{\_\}^{\lhd}$. For each such $\bar{f}$, Lemma 5.1 holds. Validity of a sequent in a (cut-free) Gentzen frame is defined with the precondition that all operations are defined; Theorem 5.2 remains valid.

Corollary 5.4. (*Adequacy*) *If a sequent is valid in* RLUG*, then it is valid in all cut-free Gentzen ru-frames.*

Completeness and cut elimination. Combining the soundness of the various Gentzen systems given in Lemmas 3.2, 3.3, 3.4, 3.6 with respect to RLUG, and their adequacy given as part of Corollary 5.4, we have the completeness of these systems.

Corollary 5.5. (*Completeness, cf.* [10]) *A sequent is provable in* **GL** (**PL** *or* **ML**) *iff it is valid in* RLUG*. The same holds for* FL *and the systems* $\mathbf{GL_a}$*,* **FL***,* $\mathbf{PL_a}$ *and* $\mathbf{PL'_a}$.

A Gentzen frame is said to have the *cut-elimination property* if it satisfies the same sequents as its cut-free version.

Theorem 5.6. (*Cut elimination*) *If a given* (*associative*) *Gentzen ru-frame is sound for* RLUG (*or* FL)*, then the frame has the cut-elimination property.*

Proof. By assumption, every sequent that is valid in the frame is also valid in RLUG. By Corollary 5.4 it is also valid in the cut-free version of the frame. The converse direction is obvious. □

An inference rule is *admissible* if its addition does not lead to more provable sequents. Note that only the admissibility of the rules (of **GL**) is used in the proof.

Now, combining the soundness results with Theorem 5.6 we obtain the following.

Corollary 5.7. *The systems* **GL***,* $\mathbf{GL_a}$*,* **FL***,* **PL***,* $\mathbf{PL_a}$*,* $\mathbf{PL'_a}$ *and* **ML** *enjoy the cut elimination property.*

Proof. For **GL**, note that if $\vdash_{\mathbf{GL}} x \Rightarrow a$, then $\models_{\mathsf{RLUG}} x \Rightarrow a$, so $\mathbf{W}^+_{\mathrm{cf}\mathbf{GL}} \models x \Rightarrow a$, where $\mathbf{W}^+_{\mathrm{cf}\mathbf{GL}}$ is the frame associated with cut-free **GL**. By Theorem 5.2, we have $(\mathbf{W}_{\mathrm{cf}\mathbf{GL}}, \mathbf{Fm}) \models x \Rightarrow a$, so $\vdash_{\mathrm{cf}\mathbf{GL}} x \Rightarrow a$. □

Note that validity in the (cut-free) Gentzen system is equivalent to validity in the class RLUG, or FL in the associative case.

Decidability. The cut elimination property allows for an effective decision procedure for determining if a given sequent is provable or not.

Corollary 5.8. *Each of the systems* **GL***,* $\mathbf{GL_a}$*,* **FL***,* **PL***,* $\mathbf{PL_a}$ $\mathbf{PL'_a}$ *and* **ML** *has a decidable set of provable sequents.*

Proof. Given a sequent we perform a exhaustive proof search in the cut free system by constructing all possible proof figures with the sequent as the end result. This is done by matching the sequent with the denominators of the rules, which can only be done in finitely many ways, and applying the rule (working upwards) for each step. This process terminates for cut-free **GL**, $\mathbf{GL_a}$, **FL**, **PL** and $\mathbf{PL_a}$, because all sequents in the numerators of the rules have fewer connectives than the sequent in the denominator. In $\mathbf{PL'_a}$ the identification of the formulas by the relation $\equiv_m$ adds only a finite number of matching terms within each equivalence class. Likewise, in **ML** the application of the bidirectional rules produces only a finite number of sequents to be investigated at each step. □

Corollary 5.9. *The varieties* RLUG *and* FL *have decidable equational theories.*

Cut elimination with simple structural rules. It is well known that for example $\mathbf{FL_e}$ also has the cut elimination property. In [10], and independently in [21], the same is shown for systems obtained from **GL** by adding certain structural rules, including the basic ones. Here we prove cut elimination for many extensions of the systems we have considered in the last section (see Cor. 5.16).

Let $t_0, t_1, \ldots, t_n$ be elements of the absolutely free algebra in the signature $\{\circ, \varepsilon\}$ over a countable set of variables, with t_0 a linear term, and let $(\mathbf{W}, \mathbf{B})$ be a Gentzen frame. As usual, $t_i^{\mathbf{W}}$ denotes the term function on $\mathbf{W}$ defined by t_i.

A *simple* rule is of the form

$$\frac{t_1 \ N \ q \quad \cdots \quad t_n \ N \ q}{t_0 \ N \ q} \ (\mathrm{r})$$

where q is a variable not occurring in $t_0, t_1, \ldots, t_n$. For example exchange, contraction, integrality and associativity are simple structural rules. We say that $(\mathbf{W}, \mathbf{B})$

satisfies this rule if for all $z \in W'$, and for all sequences $\bar{x}$ of elements of W matching the variables involved in $t_0, t_1, \ldots, t_n$, the conjunction of the conditions $t_i^{\mathbf{W}}(\bar{x})\ N\ z$, for $i \in \{1, \ldots, n\}$, implies $t_0^{\mathbf{W}}(\bar{x})\ N\ z$.

Note that t_i and the term function $t_i^{\mathbf{Fm}}$ on the algebra $(Fm, \cdot, 1)$ are interdefinable, once we fix a bijection between the sets of variables of the two related algebras. Also, the rule (r) and the inequality $\varepsilon = (t_0^{\mathbf{Fm}_\mathcal{L}} \leq t_1^{\mathbf{Fm}} \vee \cdots \vee t_n^{\mathbf{Fm}})$ are interdefinable. We denote by ε(r) the inequality corresponding to the above rule and by $\mathrm{R}(\varepsilon)$ the rule corresponding to the above inequality. Such equations are called *simple*.

In RLUG, every equation ε over $\{\vee, \cdot, 1\}$ is equivalent to a conjunction of inequalities of the form above. To show this we distribute all products over all joins to reach a form $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$, where s_i, t_j are unital groupoid terms. Such an equation is in turn equivalent to the conjunction of the two inequalities $s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$. Finally, the first one is equivalent to the conjunctions of the inequalities $s_j \leq t_1 \vee \cdots \vee t_n$. Likewise, the second inequality is written as a conjunction, as well. We now rewrite each of the conjuncts, say $s \leq t_1 \vee \cdots \vee t_n$, in a form for which s is a linear term. For each variable x that appears $k > 1$ times in s, we replace each occurrence of x in the equation by $x_1 \vee x_2 \vee \cdots \vee x_k$, where $x_1, \ldots, x_k$ are fresh variables. As multiplication distributes over join, the new equation can be written in the form $s'_1 \vee \cdots \vee s'_p \leq t'_1 \vee \cdots \vee t'_q$, where all the terms are products of variables. Let s'_l be one of the $k!$-many linear terms among $s'_1, \ldots, s'_p$. The last equation clearly implies the equation $s'_l \leq t'_1 \vee \cdots \vee t'_q$, but it is actually equivalent to it, as the latter implies $s \leq t_1 \vee \cdots \vee t_n$ by setting all duplicate copies of each variable equal to each other. Given an equation ε, let $\mathrm{R}(\varepsilon)$ denote the set of rules associated with each of these conjuncts (inequalities) obtained from ε in the way described above.

In the way of transforming simple rules to equations over $\{\vee, \cdot, 1\}$ and vice versa we established the following lemma, whose proof-theoretic analogue appears in [21].

Lemma 5.10. *Every equation over $\{\vee, \cdot, 1\}$ is equivalent to a conjunction of simple equations.*

Lemma 5.11. *Every equation ε over $\{\vee, \cdot, 1\}$ is equivalent, relative to* RLUG*, to $\mathrm{R}(\varepsilon)$. More precisely, for every $\mathbf{G} \in$ RLUG, $\mathbf{G}$ satisfies ε iff $\mathbf{W_G}$ satisfies $\mathrm{R}(\varepsilon)$.*

Proof. It suffices to show the lemma for the case where ε is of the form $t_0^{\mathbf{Fm}} \leq t_1^{\mathbf{Fm}} \vee \cdots \vee t_n^{\mathbf{Fm}}$. Clearly $\mathbf{W_G}$ satisfies $\mathrm{R}(\varepsilon)$ iff $\mathbf{G}$ satisfies the implication: if $t_i^{\mathbf{Fm}} \leq z$ for all $i \in \{1, \ldots, n\}$, then $t_0^{\mathbf{Fm}} \leq z$, for all propositional variables z, which by lattice-theoretic considerations is equivalent to ε. □

Theorem 5.12. *Let $(\mathbf{W}, \mathbf{B})$ be a cut free Gentzen frame and let ε be an equation over $\{\vee, \cdot, 1\}$. Then $(\mathbf{W}, \mathbf{B})$ satisfies $\mathrm{R}(\varepsilon)$ iff $\mathbf{W}^+$ satisfies ε.*

Proof. Clearly it suffices to show the lemma for the case where ε is simple, namely of the form $t_0^{\mathbf{Fm}} \leq t_1^{\mathbf{Fm}} \vee \cdots \vee t_n^{\mathbf{Fm}}$, where t_0 is linear.

Assume that $(\mathbf{W}, \mathbf{B})$ satisfies $\mathrm{R}(\varepsilon)$. Let $\bar{X} = (X_j)_{j \in J}$ be a sequence of elements in $\mathbf{W}^+$. We will show that $\varepsilon^{\mathbf{W}^+}(\bar{X})$ holds, i.e., $t_0^{\mathbf{W}^+}(\bar{X}) \subseteq t_1^{\mathbf{W}^+}(\bar{X}) \vee \cdots \vee t_n^{\mathbf{W}^+}(\bar{X})$. Assume that $t_1^{\mathbf{W}^+}(\bar{X}) \vee \cdots \vee t_n^{\mathbf{W}^+}(\bar{X}) \subseteq z^{\lhd}$, for some $z \in W'$. It suffices to show that $t_0^{\mathbf{W}^+}(\bar{X}) \subseteq z^{\lhd}$. We have $t_1^{\mathbf{W}^+}(\bar{X}) \cup \cdots \cup t_n^{\mathbf{W}^+}(\bar{X}) \subseteq t_1^{\mathbf{W}^+}(\bar{X}) \vee \cdots \vee t_n^{\mathbf{W}^+}(\bar{X})$, so for every $i \in \{1, \ldots n\}$, we have $t_i^{\mathbf{W}^+}(\bar{X}) \subseteq z^{\lhd}$. If $x_j \in X_j$, for all $j \in J$, (we abbreviate this by $\bar{x} \in \bar{X}$) and $\bar{x} = (x_j)_{j \in J}$, then

$$\begin{array}{lll} t_i^{\mathbf{W}}(\bar{x}) & = t_i^{\mathbf{W}}((x_j)_{j\in J}) & \\ & \in t_i^{\mathcal{P}(\mathbf{W})}((\{x_j\})_{j\in J}) & \text{(by the definition of } \circ \text{ in } \mathcal{P}(\mathbf{W})) \\ & \subseteq t_i^{\mathcal{P}(\mathbf{W})}(\bar{X}) & \text{(operations are element-wise)} \\ & \subseteq \gamma_N(t_i^{\mathcal{P}(\mathbf{W})}(\bar{X})) & (\gamma_N \text{ is a closure operator)} \\ & = t_i^{\mathbf{W}^+}(\bar{X}) \subseteq z^{\lhd} & \text{(Lemma 2.6(ii))} \end{array}$$

It follows that $t_i^{\mathbf{W}}(\bar{x})\ N\ z$, for all $i \in \{1, \ldots n\}$. Hence $t_0^{\mathbf{W}}(\bar{x})\ N\ z$, by $r(\varepsilon)^{\mathbf{W}}$, and $t_0^{\mathbf{W}}(\bar{x}) \in z^{\lhd}$, for all $\bar{x} \in \bar{X}$. Since t_0 is a linear term, we obtain $t_0^{\mathcal{P}(\mathbf{W})}(\bar{X}) \subseteq z^{\lhd}$. Since $z^{\lhd}$ is a closed set, we have $t_0^{\mathbf{W}^+}(\bar{X}) = \gamma_N(t_0^{\mathcal{P}(\mathbf{W})}(\bar{X})) \subseteq z^{\lhd}$.

Conversely, assume that $\mathbf{W}^+$ satisfies ε. For every sequence $\bar{X} = (X_j)_{j\in J}$ of elements in $\mathbf{W}^+$, we have $t_0^{\mathbf{W}^+}(\bar{X}) \subseteq t_1^{\mathbf{W}^+}(\bar{X}) \vee \cdots \vee t_n^{\mathbf{W}^+}(\bar{X})$. In particular, for $X_j = \gamma_N(\{x_j\})$, where $x_j \in W$, we have

$$t_0^{\mathbf{W}^+}((\gamma_N(\{x_j\}))_{j\in J}) \subseteq t_1^{\mathbf{W}^+}((\gamma_N(\{x_j\}))_{j\in J}) \vee \cdots \vee t_n^{\mathbf{W}^+}((\gamma_N(\{x_j\}))_{j\in J}).$$

By Lemma 2.6(ii)

$$\gamma_N(t_0^{\mathcal{P}(\mathbf{W})}((\{x_j\})_{j\in J})) \subseteq \gamma_N(t_1^{\mathcal{P}(\mathbf{W})}((\{x_j\})_{j\in J}) \cup \cdots \cup t_n^{\mathcal{P}(\mathbf{W})}((\{x_j\})_{j\in J})),$$

hence

$$\gamma_N(\{t_0^{\mathbf{W}}(\bar{x})\}) \subseteq \gamma_N(\{t_1^{\mathbf{W}}(\bar{x}), \ldots, t_n^{\mathbf{W}}(\bar{x})\}).$$

Therefore, for all $z \in W'$, $\gamma_N(\{t_1^{\mathbf{W}}(\bar{x}), \ldots, t_n^{\mathbf{W}}(\bar{x})\}) \subseteq z^{\lhd}$ implies $\gamma_N(\{t_0^{\mathbf{W}}(\bar{x})\}) \subseteq z^{\lhd}$, namely $\{t_1^{\mathbf{W}}(\bar{x}), \ldots, t_n^{\mathbf{W}}(\bar{x})\} \subseteq z^{\lhd}$ implies $t_0^{\mathbf{W}}(\bar{x}) \in z^{\lhd}$. Consequently, ($t_1^{\mathbf{W}}(\bar{x})\ N\ z$ and $\ldots$ $t_n^{\mathbf{W}}(\bar{x})\ N\ z$) implies $t_0^{\mathbf{W}}(\bar{x})\ N\ z$, and $r(\varepsilon)$ holds in $(\mathbf{W}, \mathbf{B})$. ☐

It follows from Lemma 5.11 that if $(\mathbf{W}, \mathbf{B})$ is a Gentzen frame, then $\mathbf{W}^+$ satisfies ε iff $\mathbf{W}^+$ satisfies $\mathrm{R}(\varepsilon)$.

We say that a set R of rules is *preserved by* $(\_)^+$, if for every cut-free Gentzen frame $(\mathbf{W}, \mathbf{B})$, if $(\mathbf{W}, \mathbf{B})$ satisfies R then $\mathbf{W}^+$ satisfies R. The following corollary follows directly from Theorem 5.12.

Corollary 5.13. *All simple rules are preserved by* $(\_)^+$.

For example the rules of exchange, weakening, contraction and associativity are preserved by $(\_)^+$. For a given set R of rules a *Gentzen* R*-frame* is simply a Gentzen frame that satisfies R. We denote by $\mathsf{RLUG}_{\mathrm{R}}$ the subvariety of RLUG axiomatized by $\varepsilon(R)$. By Theorem 5.2 we have the following.

Corollary 5.14. *If a sequent is valid in* $\mathsf{RLUG}_{\mathrm{R}}$*, then it is valid in all cut-free* R*-Gentzen frames.*

Theorem 5.15. *Let* R *be a set of rules that are preserved by* $(\_)^+$*. If a given Gentzen* R*-frame is sound for* $\mathsf{RLUG}_{\mathrm{R}}$*, then the frame has the cut-elimination property.*

Proof. By assumption, every sequent that is valid in the frame is also valid in $\mathsf{RLUG}_{\mathrm{R}}$. By Corollary 5.14 it is also valid in the cut-free version of the frame. The converse direction is obvious. ☐

Recall that if R is a set of rules, and $\mathbf{L}$ a sequent calculus, then $\mathbf{L}_{\mathrm{R}}$ denotes the system obtained from $\mathbf{L}$ by adding the set R.

Corollary 5.16. *The systems* $\mathbf{GL}_{\mathrm{R}}$, $\mathbf{FL}_{\mathrm{R}}$, $\mathbf{PL}_{\mathrm{R}}$, $\mathbf{PL}'_{\mathbf{a}\mathrm{R}}$, $\mathbf{ML}_{\mathrm{R}}$ *enjoy the cut elimination property, for every set* R *of rules that are preserved by* $(\_)^+$*, and in particular for the set* $\mathrm{R} = \mathrm{R}(\varepsilon)$ *with simple rules for an equation* ε *over* $\{\vee, \cdot, 1\}$.

Corollary 5.17. *The basic systems* $\mathbf{GL}_{\mathrm{R}}$*, where* R *is a subset of* $\{a, e, c, i\}$ *have the cut elimination property.*

Finite model property. We say that a Gentzen system has the *finite model property* (FMP), if for every sequent that is not provable there exists a finite countermodel. We will show the FMP for extensions by simple rules of the Gentzen systems that we have considered.

Recall that a *variety* is a class of algebras closed under homomorphic images, subalgebras and direct products. Equivalently, it is an *equational class*, namely the class of all algebras that satisfy a given set of equations. A variety is *generated* by a class of algebras, if it is the smallest variety that contains this class. We say that a variety has the *finite model property* if every non-valid equation is actually falsified in a finite algebra in the variety. Clearly a variety has the *finite model property* if it is generated by (the class of) its finite members.

Let $\mathbf{L}$ be a sequent system that enjoys cut elimination and gives rise to a Gentzen frame. For a sequent s, let $s^{\leftarrow}$ be the least set of sequents such that

- $s \in s^{\leftarrow}$
- if $(\{t_1, t_2\}, t)$ is an instance of a rule of $\mathbf{L}$ (other than cut) and $t \in s^{\leftarrow}$, then $t_1, t_2 \in s^{\leftarrow}$. (Also, if there is no t_2.)

In other words, $s^{\leftarrow}$ is the set of all sequents involved in an exhaustive proof search for s.

We say that a simple rule in a sequent system *does not increase complexity* if the complexity of each sequent in the numerator is at most as big as the complexity of the denominator. For simple structural rules, complexity of a sequent can be defined to be, for example, its length. As there are only finitely many sequents of the same complexity the addition of such a rule contributes only a finite number of sequents to $s^{\leftarrow}$.

If a system has logical rules with the subformula property and the structural rules do not increase complexity (i.e., there is an upper bound on the number of commas), then all possible sequents in a proof search will be among sequents with a bounded number of commas and formulas taken from the finite set of subformulas of the original sequent. So, $s^{\leftarrow}$ will be finite.

Theorem 5.18. *The systems* $\mathbf{GL}$, $\mathbf{FL}$, $\mathbf{PL}$, $\mathbf{PL}'_{\mathbf{a}}$ *and* $\mathbf{ML}$*, as well as their extensions with simple rules that do not increase complexity, have the FMP.*

Proof. We will give the proof for $\mathbf{GL}$, as the arguments for the other systems are analogous. We consider the involutive frame $\mathbf{W_{GL}} = (W, W', N, \circ, E)$ and a sequent s that is not provable in $\mathbf{GL}$. Let N' be the relation defined by

$$x \; N' \; (u, a) \qquad \text{iff} \qquad x \; N \; (u, a) \;\text{ or }\; (u(x) \Rightarrow a) \notin s^{\leftarrow}.$$

To see that N' is nuclear, let $x, y \in W$, $(u, a) \in W'$ and define $v(z) = u(z \circ y)$, for all $z \in W$. We have $x \circ y \; N' \; (u, a)$ iff $x \circ y \; N \; (u, a)$ or $[u(x \circ y) \Rightarrow a] \notin s^{\leftarrow}$ iff $x \; N \; (v, a)$ or $(v(x) \Rightarrow a) \notin s^{\leftarrow}$ iff $x \; N' \; (v, a)$. Also, N' satisfies the conditions $\mathbf{GN}$. Indeed, let $(\{t_1, t_2\}, t_0)$ be a rule of $\mathbf{GN}$ and assume that $t_1, t_2 \in N'$. If $t_1, t_2 \in N$, then $t_0 \in N$ (since N satisfies $\mathbf{GN}$) and $t_0 \in N'$. Otherwise, $t'_1 \notin s^{\leftarrow}$

or $t_2' \notin s^{\leftarrow}$. Here, if $t_i = (x, (u, a))$, by t_i' we denote the sequent $u(x) \Rightarrow a$. By the (contrapositive of the) second condition for $s^{\leftarrow}$, we have $t_0' \notin s^{\leftarrow}$ and thus again $t_0 \in N'$. So, $(\mathbf{W}_s, \mathbf{Fm})$ is a Gentzen frame, where $\mathbf{W}_s = (W, W', N', \circ, \backslash\!\backslash, /\!/, E)$.

Since $s^{\leftarrow}$ is finite, there are only finitely many x, u, a such that $(u(x) \Rightarrow a) \in s^{\leftarrow}$. Therefore, the complement $(N')^c$ of N' is finite, hence also its image $Im((N')^c) = \{z \in W' : x\ (N')^c\ z, \text{ for some } x \in W\}$. If $z \notin Im((N')^c)$, then $W\ N'\ z$ and $z^{\lhd} = W$, where $^{\lhd}$ is with respect to N'. Therefore, $z^{\lhd} \neq W$ only for the finitely many $z \in Im((N')^c)$. Consequently, there are only finitely many basic closed sets and $\mathbf{W}_s^+$ is finite.

Moreover, s fails in $\mathbf{W}_s^+$. Indeed, let s be the sequent $x \Rightarrow a$ and let $b = x^{\mathbf{Fm}}$. Since s is not provable in $\mathbf{GL}$, then $b \Rightarrow a$ is not provable either (the two sequents are interderivable in $\mathbf{GL}$). Moreover, if $b \Rightarrow a$ would be in $s^{\leftarrow}$, then s would be also, in view of the rule $(\cdot\mathrm{L})$. Hence, $b \Rightarrow a$ is not in N', namely $b \not\!N'\ a$ and $b \notin a^{\lhd}$. Since $b \in b^{\lhd}$, we have $b^{\lhd} \not\subseteq a^{\lhd}$. Since $(\mathbf{W}_s, \mathbf{Fm})$ is a Gentzen frame, the map $^{\lhd} : \mathbf{Fm} \to \mathbf{W}_s^+$ is a homomorphism by Cor. 4.3. Consequently, the inequality $b \leq a$ is not valid in $\mathbf{W}_s^+$, so neither is the sequent $x \Rightarrow a$. $\square$

Given a sequent s, we can perform a (necessarily terminating) exhaustive proof search. If on the way we construct a proof of s, then of course s is provable. If all partial proofs fail to be proofs, then we know s is not provable. Along the way we have constructed the set $s^{\leftarrow}$. This can be used, in turn, to construct the finite algebra $\mathbf{W}_s^+$. Therefore, we have described a constructive method that yields either a proof or a counterexample for a given sequent.

The result for $\mathbf{FL}$ was proved in [16] in the setting of phase spaces, for $\mathbf{PL_a'}$ it was shown in [14], and for $\mathbf{GL}$ it was proved in [10] in the setting of Gentzen matrices. It is clear from our proof that the result applies to any sequent system with the cut elimination property, that has rules that do not increase complexity, and gives rise to a Gentzen frame.

We note that in the proof of the above theorem in [16], definition of $s^{\leftarrow}$ includes the extra condition:

- If $u(x) \Rightarrow a \in s^{\leftarrow}$, then $u(\varepsilon) \Rightarrow a \in s^{\leftarrow}$

As we saw in the proof of Theorem 5.18, this condition is not needed, but it simplifies the argument of finiteness in [16]. See [10] for a comparison of the two definitions in the setting of Gentzen matrices.

Note that the addition of the extra condition in the definition of $s^{\leftarrow}$ simulates the effect of having the rule (i) of left-weakening (or integrality) in the generation of sequents in $s^{\leftarrow}$, *without* adding the rule in the system $\mathbf{FL}$. The latter is witnessed by the fact that in the definition of the relation N', the relation N still corresponds to provability in $\mathbf{FL}$ (not in $\mathbf{FL_i}$). Surprisingly, even though the additional condition enlarges $s^{\leftarrow}$, it does so in such a way that $\mathbf{W}^+$ becomes smaller. (Note that there is no direct relation between the size of the Galois relation N and the size of $\mathbf{W}^+$.) Alternatively, or additionally, one could add further conditions to the definition of $s^{\leftarrow}$. If $s^{\leftarrow}$ remains finite (for example by stipulating closure under any simple rule that does not increase complexity) then the counter-model $\mathbf{W}^+$ will be finite, and actually simpler. Moreover, even in situations where $s^{\leftarrow}$ is infinite it may happen that $\mathbf{W}^+$ is finite. However, proving that $\mathbf{W}^+$ is always finite in such cases would require additional techniques.

Finite embeddability property. Let $\mathbf{A}$ be a $r\ell u$-groupoid and $\mathbf{B}$ a partial subalgebra of $\mathbf{A}$. Recall that $(\mathbf{W}_{\mathbf{A},\mathbf{B}}, \mathbf{B})$ is a Gentzen frame. By Corollary 4.4 we obtain the following result that was originally proved in [4].

Corollary 5.19. *The map* $\{\_\}^{\lhd} : \mathbf{B} \to \mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ *is an embedding of the partial subalgebra* $\mathbf{B}$ *of the* $r\ell u$*-groupoid* $\mathbf{A}$ *into the* $r\ell u$*-groupoid* $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$.

Theorem 5.20. *If an equation over* $\{\vee, \cdot, 1\}$ *is valid in the* $r\ell u$*-groupoid* $\mathbf{A}$*, then it is also valid in* $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$*, for every partial subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$.

Proof. By Lemma 5.10 it is enough to consider simple equations ε, namely of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_0 is a linear term. Assume that ε is valid in $\mathbf{A}$, and let $\mathbf{B}$ be a partial subalgebra of $\mathbf{A}$. By Theorem 5.12 and Remark 5.3, to show that ε is valid in $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ is enough to show that the rule

$$\frac{t_1\ N\ (u,c) \quad \cdots \quad t_n\ N\ (u,c)}{t_0\ N\ (u,c)}\ \mathrm{R}(\varepsilon)$$

is valid in the Gentzen frame $(\mathbf{W}, \mathbf{B})$, namely that if $u(t_i) \leq_{\mathbf{A}} c$, for all $i \in \{1, \ldots, n\}$, then $u(t_i) \leq_{\mathbf{A}} c$; here we abused notation slightly by using, for example, c initially as a metavariable and then as an element of B. The latter implication follows directly from the fact that $\mathbf{A}$ satisfies ε. $\square$

A class of algebras $\mathcal{K}$ is said to have the *finite embeddability property* (FEP) if for every algebra $\mathbf{A}$ in $\mathcal{K}$ and every *finite* partial subalgebra $\mathbf{B}$ of $\mathbf{A}$, there exists an algebra $\mathbf{D}$ in $\mathcal{K}$ such that $\mathbf{B}$ embeds into $\mathbf{D}$.

Theorem 5.21. *Every variety of integral* $r\ell u$*-groupoids axiomatized by equations over the signture* $\{\vee, \cdot, 1\}$ *has the FEP.*

Proof. We follow the ideas in [4] to establish the finiteness of $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$. Let k be the cardinality of the set $B = \{b_1, \ldots, b_k\}$ and F the free groupoid with unit over k generators $x_1, \ldots, x_k$ (so all non-unit elements of F are just products of generators). For $s, t \in F$, we write $s \leq^F t$ iff t is obtained from s by deleting some (possibly none) of the generators. As a degenerate case we allow $s \leq^F 1$. In [4] it is shown that this relation is a partial order on F such that F has no infinite antichains and no infinite ascending chains (it is dually well-ordered), using Higman's Lemma. Moreover it is shown that under the above order and multiplication F can be expanded to an integral ru-groupoid $\mathbf{F}$.

Let $h : F \to W$ be the (surjective) homomorphism that extends the assignment $x_i \mapsto b_i$. Consider the new frame $\mathbf{W}^{\mathbf{F}}_{\mathbf{A},\mathbf{B}} = (F, W', h{\circ}N, \cdot^{\mathbf{F}}, \backslash\!\backslash_h, /\!/_h)$, where $x\ (h \circ N)\ z$ iff $h(x)\ N\ z$, and $x \backslash\!\backslash_h z = h(x) \backslash\!\backslash z$ and $z /\!/_h y = z /\!/ h(y)$. It is easy to see that $h \circ N$ is nuclear, so $\mathbf{W}^{\mathbf{F}}_{\mathbf{A},\mathbf{B}}$ is a residuated frame.

To prove that $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ is finite, it suffices to prove that it possesses a finite basis of sets $z^{\lhd_N} = Nz$, for $z \in W'$, where the polarity is calculated with respect to N. As h is surjective, it suffices to show that there are finitely many sets of the from $z^{\lhd}$, for $z \in W'$, where in this case the polarity is calculated with respect to $h \circ N$.

For $x \in F$, and $(u, b) \in W'$, we have $x \in (u, b)^{\lhd}$ iff $u(h(x)) \leq b$ iff $h(v(x)) \leq b$, for some $v \in S_F$ such that $h(v) = u$, since h is a surjective homomorphism (here we have extended h to a map from S_F to S_W). Equivalently, $v(x) \in h^{-1}({\downarrow_A} b)$, for some $v \in h^{-1}(u)$. Now, $h^{-1}(u)$ is a downset in F and, because F is dually well-ordered, this downset is equal to ${\downarrow} M_b$, for some *finite* $M_b \subseteq F$. So, the above statement is equivalent to $v(x) \leq m$, or to $x \leq \frac{m}{v}$, for some $v \in h^{-1}(u)$ and some $m \in M_b$. Here $\frac{m}{v}$ is defined inductively by $\frac{m}{1} = m$, $\frac{m}{v \cdot x} = \frac{m/x}{v}$ and $\frac{m}{x \cdot v} = \frac{x \backslash m}{v}$, where the divisions are calculated in $\mathbf{F}$. Consequently, $(u, b)^{\lhd} = {\downarrow}\{\frac{m}{v} : m \in M_b, h(v) = u\}$.

Note that the set $\{\frac{m}{v} : m \in M_b, b \in B, h(v) = u, u \in S_W\}$ is finite, being a subset of the finite set ${\uparrow} \bigcup_{b \in B} M_b$, as $m \leq \frac{m}{v}$ (or $v(m) \leq m$), by integrality. Thus, there are only finitely many choices for $(u, b)^{\lhd}$. $\square$

In [4] the FEP is estalished for the cases of whole variety of $r\ell u$-groupoids, as well as for the associative and the commutative subvarieties. This result was further extended in [22] for case of subvarieties axiomatized by equations of the form $x^n \leq x^m$. The result also specializes to associative and/or commutative $r\ell u(z)$-groupoids. It is well known that if a finitely axiomatized class of algebras has the FEP, then it has a decidable universal theory. Hence it is decidable whether each of these varieties of integral residuated groupoids satisfies a given universal formula.

The results we have proved so far in the paper work also for poorer signatures that do not include the connectives $\wedge$, $\backslash$, $/$, as the proof of Theorem 4.2 handles each connective separately. The connectives $\vee, \cdot, 1$ need to be present, however, for all results that rely on the linearization process.

6. Involutive frames

Definition and examples. An *involutive (residuated) frame* is a structure of the form $\mathbf{W} = (W, N, \circ, E, {}^{\sim}, {}^{-})$, where

(i) $(W, W, N, \circ, E)$ is an ru-frame,
(ii) $x^{\sim -} = x = x^{-\sim}$, for all $x \in W$ (${}^{\sim}$ and ${}^{-}$ are maps on W)
(iii) $(y^{\sim} \circ x^{\sim})^{-} = (y^{-} \circ x^{-})^{\sim}$ $[= x \oplus y]$, for all $x, y \in W$ (weak involution)
(iv) $x \circ y \; N \; z$ iff $y \; N \; x^{\sim} \oplus z$ iff $x \; N \; z \oplus y^{-}$, for all $x, y, z \in W$ (nuclear)
(v) $(x \circ E)^{\rhd} = x^{\rhd} = (E \circ x)^{\rhd}$, for all $x \in W$ (weak unit)

Recall that $x \circ y = \{z \mid (x, y, z) \in \circ\}$, and we also make use of the convention that $X^{\sim} = \{x^{\sim} \mid x \in X\}$ and $X^{-} = \{x^{-} \mid x \in X\}$. Note that condition (iii) guarantees that N is nuclear and we do not need to stipulate it in (i). In particular, we have $x \backslash\!\backslash y = x^{\sim} \oplus y$ and $y /\!/ x = y \oplus x^{-}$.

Note that, in view of (ii), condition (iii) can be written equivalently in the form $(x \circ y)^{\sim\sim} = x^{\sim\sim} \circ y^{\sim\sim}$ or $(x \circ y)^{--} = x^{--} \circ y^{--}$.

An *involutive groupoid with unit* is an algebra $\mathbf{G} = (G, \circ, 1, \sim, -)$ such that $(G, \circ, 1)$ is a groupoid with unit, and for all $x, y \in G$, we have ${\sim}{-}x = x = {-}{\sim}x$ and ${\sim}(x \circ y) = {\sim}y \circ {\sim}x$. It follows that $-(x \circ y) = -y \circ -x$, $-({\sim}y \circ {\sim}x) = {\sim}(-y \circ -x)$ and ${\sim}1 = -1 = 1$. We often prefer the notation $x^{\sim}$ and x^{-} instead of ${\sim}x$ and $-x$. We will use this notation when considering the free involutive groupoid with unit over the set of formulas, to avoid confusion with the connectives $\sim$ and $-$.

A *weakly involutive groupoid with unit* is an algebra $\mathbf{G} = (G, \circ, 1, \sim, -)$ such that $(G, \circ, 1)$ is a groupoid with unit, and for all $x, y \in G$, we have ${\sim}{-}x = x = {-}{\sim}x$ and $-({\sim}y \circ {\sim}x) = {\sim}(-y \circ -x)$. It follows that ${\sim\sim}(x \circ y) = {\sim\sim}x \circ {\sim\sim}y$, $--(x \circ y) = --x \circ --y$ and ${\sim}1 = -1$. The operation $\oplus$ is defined by $x \oplus y = -({\sim}y \circ {\sim}x) = {\sim}(-y \circ -x)$ and $(G, \oplus, 1, \sim, -)$ is also a weakly involutive groupoid with unit.

Note that structures that satisfy conditions (ii) and (iii) are related to weakly involutive groupoids, which in turn are more general than involutive groupoids.

$$\frac{x \Rightarrow a \quad a \Rightarrow z}{x \Rightarrow z}\ (\mathrm{CUT}) \qquad \frac{}{a \Rightarrow a}\ (\mathrm{Id})$$

$$\frac{x \Rightarrow a \quad b \Rightarrow z}{x \circ (a\backslash b) \Rightarrow z}\ (\backslash\mathrm{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a\backslash b}\ (\backslash\mathrm{R})$$

$$\frac{x \Rightarrow a \quad b \Rightarrow z}{(b/a) \circ x \Rightarrow z}\ (/\mathrm{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\mathrm{R})$$

$$\frac{a \circ b \Rightarrow z}{a \cdot b \Rightarrow z}\ (\cdot\mathrm{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot\mathrm{R})$$

$$\frac{a \Rightarrow z}{a \wedge b \Rightarrow z}\ (\wedge\mathrm{L}\ell) \qquad \frac{b \Rightarrow z}{a \wedge b \Rightarrow z}\ (\wedge\mathrm{L}r) \qquad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge\mathrm{R})$$

$$\frac{a \Rightarrow z \quad b \Rightarrow z}{a \vee b \Rightarrow z}\ (\vee\mathrm{L}) \qquad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee\mathrm{R}\ell) \qquad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee\mathrm{R}r)$$

$$\frac{\varepsilon \Rightarrow z}{1 \Rightarrow z}\ (1\mathrm{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathrm{R})$$

$$\frac{a^{\sim} \Rightarrow z}{{\sim}a \Rightarrow z}\ ({\sim}\mathrm{L}) \qquad \frac{x \Rightarrow a^{\sim}}{x \Rightarrow {\sim}a}\ ({\sim}\mathrm{R}) \qquad \frac{a^{-} \Rightarrow z}{-a \Rightarrow z}\ (-\mathrm{L}) \qquad \frac{x \Rightarrow a^{-}}{x \Rightarrow -a}\ (-\mathrm{R})$$

$$\frac{x \circ y \Rightarrow z}{\overline{y \Rightarrow x^{\sim} \circ z}}\ ({}^{\sim}) \qquad \frac{x \circ y \Rightarrow z}{\overline{x \Rightarrow z \circ y^{-}}}\ ({}^{-})$$

FIGURE 5. The system **InGL**.

The Gentzen system **InGL** is defined in Figure 5 and is an involutive analogue of **GL**. One important difference is that the rules are written without any explicit reference to a context u. However, the bidirectional rules $({}^{\sim})$ and $({}^{-})$ essentially allow any context to be moved back and forth between the two sides of a sequent.

Metavariables a, b, c range over formulas and x, y over elements of the free involutive groupoid with unit over the set of formulas. We also consider the Gentzen system **InFL**, defined by taking the free involutive *monoid*, instead. Note that in **InFL**, for $x = a_1, \ldots, a_n$ (written using comma in place of $\circ$), we have $\varepsilon^{\sim} = \varepsilon^{-} = \varepsilon$, $x^{\sim} = a_n^{\sim}, \ldots, a_1^{\sim}$ and $x^{-} = a_n^{-}, \ldots, a_1^{-}$. The next result is proved by checking that each **InGL**-rule (resp. **InFL**-rule) is valid in involutive $r\ell u$-groupoids (resp. InFL-algebras).

Lemma 6.1. (*Soundness*) *Every sequent that is provable in* **InGL** *is valid in all involutive $r\ell u$-groupoids. The same holds for* **InFL** *and InFL-algebras.*

Recall that in involutive $r\ell u$-groupoids, the negation and the division operations are interdefinable. Likewise, in the system **InGL** we can omit the negation or the division rules and define the omitted connectives in terms of the remaining ones.

Then the omitted rules become derivable in the system. Futhermore, in the same sense, we can conservatively add the rules:

$$\frac{a \Rightarrow z \quad b \Rightarrow w}{a + b \Rightarrow z \circ w} \text{ (+L)} \qquad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a + b} \text{ (+R)} \qquad \frac{}{0 \Rightarrow \varepsilon} \text{ (0L)} \qquad \frac{x \Rightarrow \varepsilon}{x \Rightarrow 0} \text{ (0R)}$$

and even rules for $\fallingdotseq$ and $\risingdotseq$, by taking the duals of the division rules.

The involutive frames $\mathbf{W_{InGL}}$ and $\mathbf{W_{InFL}}$ are defined in the obvious way, where $x\ N\ z$ iff $\vdash x \Rightarrow z$. In both cases $(W, \circ, 1, {}^{\sim}, {}^{-}\}$ is a involutive groupoid with unit and $x \circ y = x \oplus y$.

It is easy to see that if $\mathbf{G}$ is an involutive $r\ell u$-groupoid, then $\mathbf{W_G} = (G, \leq, \cdot, \sim, -, 1)$ is an involutive frame. Note that $(G, \cdot, 1, \sim, -)$ is not a involutive groupoid with unit, since in general $x \cdot y \neq x + y$.

Galois algebra. If $Y \subseteq W$, we define $Y^{\sim} = \{y^{\sim} : y \in Y\}$ and $Y^{-} = \{y^{-} : y \in Y\}$. Also, in involutive frames, we define $-Y = Y^{-\lhd}$ and $\sim Y = Y^{\sim\lhd}$ for $Y \subseteq W$. Note that $-Y$ and $\sim Y$ are Galois closed sets.

Lemma 6.2. *Let $\mathbf{W}$ be an involutive frame, and let $x, y \in W$ and $X, Y, Z \subseteq W$. Then,*

(i) *$x\ N\ y^{-}$ iff $y\ N\ x^{\sim}$.*
(ii) *$\sim E = -E$.*
(iii) *$-Y = Y^{\rhd -} = Y^{-\lhd}$ and $\sim Y = Y^{\rhd\sim} = Y^{\sim\lhd}$.*
(iv) *$X \circ Y \subseteq Z$ iff $Y \subseteq \sim(-Z \circ X)$ iff $X \subseteq -(Y \circ \sim Z)$, if Z is Galois closed.*

Proof. (i) $x\ N\ y^{-}$ iff $x \circ E\ N\ y^{-}$ iff $E\ N\ x^{\sim} \oplus y^{-}$ iff $E \circ y\ N\ x^{\sim}$ iff $y\ N\ x^{\sim}$.

(ii) $x \in \sim E = E^{\sim\lhd}$ iff $x\ N\ E^{\sim}$ iff $x \circ E\ N\ E^{\sim}$ iff $x\ N\ E^{\sim} \oplus E^{-}$ iff $E \circ x\ N\ E^{-}$ iff $x\ N\ E^{-}$ iff $x \in E^{-\lhd} = -E$.

(iii) For all $x \in W$, we have $x \in Y^{-\lhd}$iff $x\ N\ Y^{-}$ iff $Y\ N\ x^{\sim}$ iff $x^{\sim} \in Y^{\rhd}$ iff $x \in Y^{\rhd -}$, by (i).

(iv) We have $X \circ Y \subseteq Z = Z^{\rhd\lhd}$ iff $X \circ Y\ N\ Z^{\rhd}$ iff $Y\ N\ X^{\sim} \oplus Z^{\rhd}$ iff $Y \subseteq (X^{\sim} \oplus Z^{\rhd})^{\lhd}$. Also, $(X^{\sim} \oplus Z^{\rhd})^{\lhd} = (X^{\sim} \oplus Z^{\rhd})^{-\sim\lhd} = (Z^{\rhd -} \circ X^{\sim -})^{\sim\lhd} = \sim(-Z \circ X)$, by (iii). Likewise, we prove the second equivalence. ☐

For an involutive frame $\mathbf{W}$, we know that $(W, W, N, \circ, E)^{+}$ is a $r\ell u$-groupoid (we will also write 1 for E). We will show that $\sim 1 = -1$ and denote the common value by 0. We write $\mathbf{W}^{+}$ for the extension of that $r\ell u$-groupoid with the element 0.

Corollary 6.3. *For any involutive frame $\mathbf{W}$, $\mathbf{W}^{+}$ is an involutive $r\ell u$-groupoid.*

Proof. Note that the operations $\sim$ and $-$ are defined independently of the operations in a $r\ell u$-groupoid. We will show that they coincide with the usual negation operations of an $r\ell u$-groupoid, namely that $\sim Z = Z \backslash 0$ and $-Z = 0/Z$, for all $Z \in \mathbf{W}^{+}$.

We have $\sim -Z = Z^{\rhd - \sim \lhd} = Z^{\rhd\lhd} = Z$, by Lemma 6.2(iii). Likewise, $-\sim Z = Z$. Also, $\sim 1 = -1$, by Lemma 6.2(i). Finally, by Lemma 6.2(iv), $Z \backslash 0 = \sim(-0 \circ Z) = (E \circ Z)^{\rhd\sim} = Z^{\rhd\sim} = \sim Z$, and likewise $0/Z = \sim Z$. Consequently, $\mathbf{W}^{+}$ is an involutive $r\ell u$-groupoid. ☐

Involutive Gentzen frames. An *involutive Gentzen frame* is a pair $(\mathbf{W}, \mathbf{B})$ where

(i) $\mathbf{W} = (W, N, \circ, \{\varepsilon\}, {}^{\sim}, {}^{-})$ is an involutive frame, where $\circ$ is an operation
(ii) $\mathbf{B}$ is a partial algebra of the type of InGL,
(iii) B is a subset of W that generates $(W, \circ, \varepsilon, {}^{\sim}, {}^{-})$ and

(iv) N satisfies **GN** and the following rules, for all $a, b \in B$, $x, y \in W$ and $z \in W'$.

$$\frac{x\ N\ a}{x \circ ({\sim}a)\ N\ \varepsilon}\ ({\sim}\mathrm{L}) \qquad \frac{a \circ x\ N\ \varepsilon}{x\ N\ {\sim}a}\ ({\sim}\mathrm{R})$$

$$\frac{x\ N\ a}{(-a) \circ x\ N\ z}\ (-\mathrm{L}) \qquad \frac{x \circ a\ N\ \varepsilon}{x\ N\ -a}\ (-\mathrm{R})$$

For example, $(\mathbf{W_{InGL}}, \mathbf{Fm})$ and $(\mathbf{W_{InFL}}, \mathbf{Fm})$ are involutive Gentzen frames.

Theorem 6.4. *Let $(\mathbf{W}, \mathbf{B})$ be a cut free involutive Gentzen frame. Then the conditions in Theorem 4.2 hold. Moreover, for all $a \in B$, $X \in \mathbf{W}^+$, if ${\sim}a$ and $-a$ are defined and $a \in X \subseteq \{a\}^{\lhd}$, then*

(i) ${\sim}a \in {\sim}X \subseteq \{{\sim}a\}^{\lhd}$ *and* $-a \in -X \subseteq \{-a\}^{\lhd}$.
(ii) *In particular,* ${\sim}a \in {\sim}\{a\}^{\lhd} \subseteq \{{\sim}a\}^{\lhd}$ *and* $-a \in -\{a\}^{\lhd} \subseteq \{-a\}^{\lhd}$.
(iii) *If, additionally, N satisfies* (CUT) *then* ${\sim}\{a\}^{\lhd} = \{{\sim}a\}^{\lhd}$ *and* $-\{a\}^{\lhd} = \{-a\}^{\lhd}$.

Proof. If $\mathbf{B}_r$ is the 0-free reduct of $\mathbf{B}$ and $\mathbf{W}_r = (W, W, N, \circ, \varepsilon)$, then $(\mathbf{W}_r, \mathbf{B}_r)$ satisfies the conditions of a Gentzen frame except for the fact that B_r generates $(W, \circ, \varepsilon)$. Nevertheless, this condition is not used in the proof of Theorem 4.2. Since $\mathbf{W}^+$ is simply the expansion of $\mathbf{W}_r^+$ by 0, the conclusion of Theorem 4.2 holds.

We want to show that ${\sim}X \subseteq \{{\sim}a\}^{\lhd}$. If $x \in {\sim}X = X^{\sim\lhd}$, then $x\ N\ X^{\sim}$. Since $a \in X$, we have $x\ N\ a^{\sim}$. By (${\sim}$R) we obtain $x\ N\ {\sim}a$, or $x \in \{{\sim}a\}^{\lhd}$.

To show that ${\sim}a \in {\sim}X = X^{\sim\lhd}$ we need to prove that ${\sim}a\ N\ X^{\sim}$. We have $X \subseteq \{a\}^{\lhd}$, so $X\ N\ a$ and $a^{\sim}\ N\ X^{\sim}$, by ($^{\sim}$) and ($^{-}$). Finally, by (${\sim}$L) we get ${\sim}a\ N\ X^{\sim}$. $\square$

Corollary 6.5. *If $(\mathbf{W}, \mathbf{B})$ is an involutive Gentzen frame, the map $\{\_\}^{\lhd} : \mathbf{B} \to \mathbf{W}^+$ is a homomorphism of the partial algebra $\mathbf{B}$ into the involutive $r\ell u$-groupoid $\mathbf{W}^+$.*

Embedding into the DM-completion and representation.

Corollary 6.6. *Let $\mathbf{G}$ be an involutive $r\ell u$-groupoid. The map $\{\_\}^{\lhd} : \mathbf{G} \to \mathbf{W}_{\mathbf{G}}^+$ is an embedding.*

Therefore, every involutive FL-algebra can be embedded into a complete one.

Corollary 6.7. *Every InFL-algebra is a subalgebra of the nucleus image of the power set of a weakly involutive monoid.*

Cut elimination. For this section a *(classical) sequent* is an element of $Fm^i \times Fm^i$, where Fm^i denotes the free groupoid with unit over the set of negated formulas. Here negated formulas are elements of the form $a^{\sim n}$ or a^{-n}, for $a \in Fm$, defined inductively by $a^{\sim 0} = a$ and $a^{\sim(n+1)} = (a^{\sim n})^{\sim}$ (and likewise for a^{-n}). We use the notation $x \Rightarrow y$ for sequents.

Let $(\mathbf{W}, \mathbf{B})$ be an involutive Gentzen frame. Note that every map $f : Fm \to B$ extends to a groupoid (with unit) homomorphism $f^{\circ} : (Fm^i, \circ, \varepsilon) \to (W, \circ, \varepsilon)$ by $f^{\circ}(x^{\sim}) = (f^{\circ}(x))^{\sim}$, $f^{\circ}(x^{-}) = (f^{\circ}(x))^{-}$ and $f^{\circ}(x \circ^{Fm^i} y) = f^{\circ}(x) \circ^W f^{\circ}(y)$. Likewise, every assignment $f : Fm \to B$ extends to a groupoid (with unit) homomorphism $f^{\oplus} : (Fm^i, \circ, \varepsilon) \to (W, \oplus, \varepsilon^{\sim})$ by $f^{\oplus}(x^{\sim}) = (f^{\oplus}(x))^{\sim}$, $f^{\oplus}(x^{-}) = (f^{\oplus}(x))^{-}$ and $f^{\oplus}(x \circ^{Fm^i} y) = f^{\oplus}(x) \oplus^W f^{\oplus}(y)$.

Also, every homomorphism $f : \mathbf{Fm} \to \mathbf{G}$ into an $\mathcal{L}$-algebra $\mathbf{G}$ extends to a groupoid (with unit) homomorphism $f : \mathbf{Fm}^i \to \mathbf{G}$. A sequent $x \Rightarrow y$ is said to be *valid* in $(\mathbf{W}, \mathbf{B})$, if for every assignment $f : Fm \to B$, we have $f^{\circ}(x)\ N\ f^{\oplus}(y)$. A sequent $x \Rightarrow y$ is said to be *valid* in an involutive $r\ell u$-groupoid $\mathbf{G}$, if it is valid in the involutive Gentzen frame $(\mathbf{W_G}, \mathbf{G})$, namely if for all homomorphisms $f : \mathbf{Fm} \to \mathbf{G}$ we have $f^{\circ}(x) \leq f^{\oplus}(y)$.

We would like to mention that if p, q are propositional variables, then the sequent $p \circ q \Rightarrow p \circ q$ is not provable in **InGL**. This is in agreement with the fact that the equation $p \cdot q \leq p + q$ is not valid in InGL. This indicates why $\circ$ needs to be interpreted as $\cdot$ in the left hand side and as $+$ in the right hand side of a sequent.

Also, note that $(p \circ q)^{\sim} \Rightarrow {\sim}(p + q)$ is provable in **InGL**, but the equation ${\sim}(p \cdot q) \leq {\sim}(p{+}q)$ is not valid in InGL. On the other hand the equation ${\sim}(p{+}q) \leq {\sim}(p{+}q)$ is clearly valid. Therefore, the above comment about the interpretation of $\circ$ on the two sides of a sequent holds only for occurrences of $\circ$ not under negations. This is the reason why we defined (classical) sequents is such a way that all occurrences of $\circ$ are outermost and all negations are applied to formulas.

Note that we have different interpretations of the comma (also denoted by $\circ$) on the two sides of a sequent: comma is interpreted by $\circ$ on the left and by $\oplus$ on the right. This means that for interpretations in an algebra, comma is interpreted as $\cdot$ and $+$, respectively. This agrees with the usual practice in semantics for sequent calculus systems. For classical logic we obtain interpretations by using $\wedge$ and $\vee$, respectively.

Also note that the set W in **InGL** is taken to be a the free involutive groupoid with unit. On the other hand, in the definition of an involutive frame the set W is only weakly involutive, whenever the operation $\circ$ is a function. This should not create the impression that we can assume the stronger involutive condition in the definition of a frame.

To clarify things, we mention that we could have simply taken the free weakly involutive groupoid with unit $\mathbf{W} = (W, \circ, {}^{\sim}, {}^{-}, \varepsilon)$ on a countable set X of variables. By the weakly involutive law we can define an operation $\oplus$, which can be easily shown to be associative and have as unit the element $\delta = \varepsilon^{\sim} = \varepsilon^{-}$. We can also define iterated negations of elements of X, by taking repeated applications of the operations ${}^{\sim}, {}^{-}$ on elements of X. The resulting set is denoted $X^{\neg}$. It is easy to see that the underlying set of W can be concretely realized as the underlying set V of the free bi-groupoid with unit $\mathbf{V} = (V, \circ, \varepsilon, \oplus, \delta)$ on the set $X^{\neg}$. In other words V supports the free weakly involutive monoid by defining involution functions that satisfy $(x \circ y)^{\sim} = y^{\sim} \oplus x^{\sim}$.

Let $\mathbf{V}_{\circ}$ denote the $\circ$-submonoid of $\mathbf{V}$ generated by $X^{\neg}$ and $\mathbf{V}_{\oplus}$ the $\oplus$-submonoid of $\mathbf{V}$ generated by $X^{\neg}$. Clearly $\mathbf{V}_{\circ}$ and $\mathbf{V}_{\oplus}$ are isomorphic. Actually, they both support the free involutive monoid generated by X, the first with involutive functions satisfying $(x \circ y)^{\sim} = y^{\sim} \circ x^{\sim}$ and the second with $(x \oplus y)^{\sim} = y^{\sim} \oplus x^{\sim}$. We could have taken sequents as elements of $V_{\circ} \times V_{\oplus}$, but since the two monoids are isomorphic and support the free involutive monoid generated by X, we chose to identify them.

Lemma 6.8. *A classical sequent $x \Rightarrow y$ is valid in $(\mathbf{W_{InGL}}, \mathbf{Fm})$ iff $f^{\circ}(x) \Rightarrow f^{\oplus}(y)$ is provable in* **InGL**, *where f is the identity map.*

Let $\mathbf{W}$ be an involutive frame and $X \subseteq W$. We define $({\sim}n)X$ and $X^{\sim n}$ inductively by

$$({\sim}0)X = X^{\sim 0} = X,\ ({\sim}(n+1))X = {\sim}({\sim}n)X \text{ and } X^{\sim(n+1)} = (X^{\sim n})^{\sim}.$$

Likewise, we define $(-n)X$ and X^{-n}.

Lemma 6.9. *Let $\mathbf{W}$ be a residuated frame.*

(i) *The operation $+$ on $\mathbf{W}^+$ is order preserving on both coordinates.*
(ii) *If $X, Y \in \mathbf{W}^+$, then $X^{\lhd} + Y^{\lhd} \subseteq (X \oplus Y)^{\lhd}$.*
(iii) *If $X \in W^+$, then $({\sim}n)X = X^{\sim n}$ and $(-n)X = X^{-n}$, for every even natural number n.*

Proof. (i) For subsets X_1, X_2, Y_1, Y_2 of W with $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ we have $X_2^{\rhd} \subseteq X_1^{\rhd}$ and $Y_2^{\rhd} \subseteq Y_1^{\rhd}$. Therefore, $X_2^{\rhd} Y_2^{\rhd} \subseteq X_1^{\rhd} Y_1^{\rhd}$ and $(X_1^{\rhd} Y_1^{\rhd})^{\lhd} \subseteq (X_2^{\rhd} Y_2^{\rhd})^{\lhd}$, namely $X_1 + Y_1 \subseteq X_2 + Y_2$. We used that $X + Y = -[({\sim}Y)({\sim}X)] = (Y^{\rhd\sim} X^{\rhd\sim})^{-\lhd} = (X^{\rhd\sim-} Y^{\rhd\sim-})^{\lhd} = (X^{\rhd} Y^{\rhd})^{\lhd}$.

(ii) Recall that ${}^{\lhd\rhd}$ is a closure operator, ${}^{\lhd}$ is order reversing and that $X \circ Y \subseteq X \cdot Y$, for all $X, Y \subseteq W$. Also, we define $X \oplus Y$ element-wise. We have $X^{\lhd} + Y^{\lhd} = -[({\sim}Y^{\lhd}) \cdot ({\sim}X^{\lhd})] = (Y^{\lhd\rhd\sim} \cdot X^{\lhd\rhd\sim})^{-\lhd} \subseteq (Y^{\sim} \cdot X^{\sim})^{-\lhd} \subseteq (Y^{\sim} \circ X^{\sim})^{-\lhd} = (X \oplus Y)^{\lhd}$, so $X^{\lhd} + Y^{\lhd} \subseteq (X \oplus Y)^{\lhd}$.

(iii) For $w \in W$, we have $w \in {\sim}X = X^{\rhd\sim}$ iff $w^- \in X^{\rhd}$ iff $X\ N\ w^-$. We also have ${\sim}X = X^{\rhd\sim}\ N\ w$ iff $w^-\ N\ X^{\rhd}$ iff $w^- \in X^{\rhd\lhd} = X$.

Consequently, $w \in {\sim}{\sim}X$ iff ${\sim}X\ N\ w^-$ iff $w^{--} \in X$ iff $w \in X^{\sim\sim}$. So, ${\sim}{\sim}X = X^{\sim\sim}$ and for every even number n, $({\sim}n)X = X^{\sim n}$. □

Theorem 6.10. *If $(\mathbf{W}, \mathbf{B})$ is a cut free involutive Gentzen frame, then every sequent that is valid in $\mathbf{W}^+$ is also valid in $(\mathbf{W}, \mathbf{B})$.*

Proof. Assume that $s = (x, y)$ is valid in $\mathbf{W}^+$ and let $f : \mathbf{Fm} \to \mathbf{B}$ be a homomorphism. We will show that $f^{\circ}(x)\ N\ f^{\oplus}(y)$. Since s is valid in $\mathbf{W}^+$, for every homomorphism $\bar{f} : \mathbf{Fm} \to \mathbf{W}^+$ with $\bar{f} \in H(f)$, we have $\bar{f}^{\circ}(x) \subseteq \bar{f}^{\oplus}(y)$.

For brevity we adopt the notation $\neg n = {\sim}n$, for non-negative integers and $\neg n = -|n|$ for negative integers. By definition, there exist formulas $a_1, \ldots, a_n, b_1, \ldots, b_m \in Fm$ and groupoid (with unit) terms t_x, t_y such that $x = t_x^{\mathbf{Fm}^i}(a_1^{\neg k_1}, \ldots, a_n^{\neg k_n})$ and $y = t_y^{\mathbf{Fm}^i}(b_1^{\neg l_1}, \ldots, b_m^{\neg l_m})$. Then $\bar{f}^{\circ}(x) = t_x^{(\mathbf{W}^+,\cdot)}((\neg k_1)\bar{f}(a_1), \ldots, (\neg k_n)\bar{f}(a_n))$ and $\bar{f}^{\oplus}(y) = t_y^{\mathbf{W}^+,+)}((\neg l_1)\bar{f}(b_1), \ldots, (\neg l_n)\bar{f}(b_m))$.

By Lemma 5.1, $\bar{f}(c) \subseteq f(c)^{\lhd}$ and $f(c) \in \bar{f}(c)$, for all formulas $c \in Fm$. From $\bar{f}(c) \subseteq f(c)^{\lhd}$ we also obtain $f(c) \in \bar{f}(c)^{\rhd}$, so $f(c)^{\sim} \in \bar{f}(c)^{\rhd\sim} = {\sim}\bar{f}(c)$ and $f(c)^- \in -\bar{f}(c)$. Thus, using Lemma 6.9(iii), we can show that for all k we have

$$f(c)^{\neg k} \in (\neg k)\bar{f}(c). \qquad (*)$$

So,

$$\begin{aligned}
f^{\circ}(x) &= f^{\circ}(t_x^{\mathbf{Fm}^i}(a_1^{\neg k_1}, \ldots, a_n^{\neg k_n})) \\
&= t_x^{(W,\circ)}(f(a_1)^{\neg k_1}, \ldots, f(a_n)^{\neg k_n}) && (f^{\circ} \text{ extends } f) \\
&\in t_x^{(\mathcal{P}(W),\circ)}((\neg k_1)\bar{f}(a_1), \ldots, (\neg k_n)\bar{f}(a_n)) && (*), (\circ \text{ in } \mathcal{P}(W) \text{ is element-wise}) \\
&\subseteq t_x^{(\mathbf{W}^+,\cdot)}((\neg k_1)\bar{f}(a_1), \ldots, (\neg k_n)\bar{f}(a_n)) && (\gamma_N \text{ is a closure operator}) \\
&= \bar{f}^{\circ}(x)
\end{aligned}$$

From $f(c) \in \bar{f}(c)$ we have $f(c)^{\neg} \in \bar{f}(c)^{\neg}$, hence $\bar{f}(c)^{\neg\lhd} \subseteq f(c)^{\neg\lhd}$, namely $\neg\bar{f}(c)\ N\ f(c)^{\neg}$. By the negation rules for N, we have $(\neg\bar{f}(c))^{\neg k}\ N\ f(c)^{\neg(\neg k)}$ for every even integer k, so $(\neg\bar{f}(c))^{\neg k} \subseteq f(c)^{\neg(k+1)\lhd}$. Using Lemma 6.9(iii), we have $(\neg(k+1))\bar{f}(c) \subseteq f(c)^{\neg(k+1)\lhd}$, for every even integer k.

On the other hand, from $\bar{f}(c) \subseteq f(c)^{\lhd}$ we have $\bar{f}(c)\ N\ f(c)$. By the negation rules for N, we have $\bar{f}(c)^{\neg k}\ N\ f(c)^{\neg k}$, for every even integer k, so $\bar{f}(c)^{\neg k} \subseteq f(c)^{(\neg k)\lhd}$. In view of Lemma 6.9(iii), we obtain $(\neg k)\bar{f}(c) \subseteq f(c)^{(\neg k)\lhd}$, for every even integer k.

Consequently, we have

$$(**) \qquad (\neg k)\bar{f}(c) \subseteq f(c)^{(\neg k)\lhd}, \text{ for every integer } k.$$

$$\begin{aligned}
\bar{f}^{\oplus}(y) &= t_y^{(\mathbf{W}^+,+)}((\neg l_1)\bar{f}(b_1), \dots, (\neg l_n)\bar{f}(b_m)) \\
&\subseteq t_y^{(\mathbf{W}^+,+)}(f(b_1)^{(\neg l_1)\lhd}, \dots, f(b_m)^{(\neg l_m)\lhd}) && (**), \text{(Lemma 6.9(i))} \\
&= [t_y^{(W,\oplus)}(f(b_1)^{(\neg l_1)}, \dots, f(b_m)^{(\neg l_m)})]^{\lhd} && \text{(Lemma 6.9(ii))} \\
&= [f^{\oplus}(y)]^{\lhd}
\end{aligned}$$

Consequently, $f^{\circ}(x) \in \bar{f}^{\circ}(x) \subseteq \bar{f}^{\oplus}(y) \subseteq \{f^{\oplus}(y)\}^{\lhd}$. Thus, $f^{\circ}(x) \in \{f^{\oplus}(y)\}^{\lhd}$, i.e., $f^{\circ}(x)\ N\ f^{\oplus}(y)$. □

Corollary 6.11. *(Adequacy) If a sequent is valid in* InGL*, then it is valid in all cut-free involutive Gentzen frames.*

Combining the soundness of **InGL** (**InFL**) with respect to involutive $r\ell u$-groupoids (InFL-algebras) given by Lemma 6.1, and their adequacy given as part of Corollary 6.11, we have the completeness of these systems.

Corollary 6.12. *(Completeness) A sequent is provable in* **InGL** *iff it is valid in* RLUG*. The same holds for* **InFL** *and* InFL*.*

Theorem 6.13. *(Cut elimination) If* InGL *(or* InFL*) is sound for a given (associative) involutive Gentzen frame, then the frame has the cut-elimination property.*

Proof. By assumption, every sequent that is valid in the frame is also valid in InGL. By Corollary 6.11 it is also valid in the cut-free version of the frame. The converse direction is obvious. □

Now, combining the soundness results with Theorem 6.13 we obtain the following.

Corollary 6.14. *The systems* **InGL** *and* **InFL** *enjoy the cut elimination property.*

Decidability. Even though an exhaustive proof search for a given sequent in **InGL** or **InFL** is never finite, we can still restrict our attention to a finite part.

Theorem 6.15. *The equational theory of InFL-algebras (and InGL-algebras) is decidable.*

Proof. At every step of a proof search there are only finitely many sequents, obtained by applying $(^{\sim})$ and $(^{-})$, such that not all formulas have external negations on them. If all formulas have external negations then no logical rule can be applied (upwards). Therefore we need to explore, by using $(^{\sim})$ and $(^{-})$, only finitely many sequents between applications of logical rules.

Indeed, for the associative case, if $a_1 \circ a_2 \circ \cdots \circ a_m \Rightarrow b_1 \circ b_2 \circ \cdots \circ b_n$ is a sequent, then the possible sequents that can be obtained using the rules for external negations are given by moving the formulas according to the following diagram:

$$\begin{array}{ccc} a_1 & & b_1 \\ \circ & & \circ \\ a_2 & & b_2 \\ \circ & \Rightarrow & \circ \\ \vdots & & \vdots \\ \circ & & \circ \\ a_m & & b_n \end{array}$$

For example the above sequent is equivalent to the following sequents:

$$\begin{array}{rcl} a_2 \circ \cdots \circ a_m & \Rightarrow & a_1^{\sim} \circ b_1 \circ \cdots \circ b_n \\ a_2 \circ \cdots \circ a_m \circ b_n^{\sim} & \Rightarrow & a_1^{\sim} \circ b_1 \circ \cdots \circ b_{n-1} \\ & \vdots & \\ b_n^{\sim} \circ \cdots \circ b_2^{\sim} \circ b_1^{\sim} & \Rightarrow & a_m^{\sim} \circ \cdots \circ a_2^{\sim} \circ a_1^{\sim} \\ & \vdots & \\ a_1^{\sim\sim} \circ a_2^{\sim\sim} \circ \cdots \circ a_m^{\sim\sim} & \Rightarrow & b_1^{\sim\sim} \circ b_2^{\sim\sim} \circ \cdots \circ b_n^{\sim\sim} \\ & & \\ a_2 \circ \cdots \circ a_m & \Rightarrow & a_1^{\sim} \circ b_1 \circ \cdots \circ b_n \\ a_2 \circ \cdots \circ a_{m-1} & \Rightarrow & a_1^{\sim} \circ b_1 \circ \cdots \circ b_n \circ a_m^{-} \\ & \vdots & \\ a_2 & \Rightarrow & a_1^{\sim} \circ b_1 \circ \cdots \circ b_n \circ a_m^{-} \circ \cdots \circ a_3^{-} \end{array}$$

In the non-associative case the possible moves are even more restricted, therefore we can obtain only a finite set of possible sequents with at least one non-negated formula. ☐

A cut-free system for cyclic involutive FL-algebras, from which decidability is derived, is given in [24]. Also, a cut-free system for involutive FL-algebras is presented in [1]. A complicated argument is given to establish cut-elimination for this system, but no decision procedure is derived.

The (unital) involutive frames we have defined satisfy the property: $x \mathrel{N} y^{-}$ iff $y \mathrel{N} x^{\sim}$ (Lemma 6.2(i)). Also the rule (followed by its derivation)

$$\frac{x \Rightarrow y^{-}}{y \Rightarrow x^{\sim}}\ (\mathrm{G}) \qquad\qquad \cfrac{\cfrac{x \circ \varepsilon \Rightarrow y^{-}}{\varepsilon \Rightarrow x^{\sim} \circ y^{-}}\ ({}^{\sim})}{\varepsilon \circ y \Rightarrow x^{\sim}}\ ({}^{-})$$

holds in **InGL**. Note that these derivations require the existence of E and ε, respectively. It is possible to define involutive frames without E and modify **InGL** to not include 1 and to not allow sequents with an empty side. In this case, if we simply stipulate the above frame condition and the displayed rule, our results extend in a natural way.

Cut elimination with simple structural rules. We will extend the notion of a simple structural rule to the involutive case.

Let $t_0, t_1, \ldots, t_n$ be elements of the free groupoid with unit (using the signature if $\{\circ, \varepsilon\}$) over a countable set of possibly negated variables, i.e., formal expressions of the form $q^{\neg n}$, for even integers n, with the usual conventions for $\neg$ adopted in the proof of Theorem 6.10. Note that the construction is similar to Fm^i. We further assume that t_0 is a linear term.

A *simple* rule is of the form

$$\frac{t_1 \ N \ q \quad \cdots \quad t_n \ N \ q}{t_0 \ N \ q} \ (\mathrm{r})$$

where q is a variable not occurring in $t_0, t_1, \ldots, t_n$. Satisfaction of a rule in a frame, as well as the correspondence between simple rules and equations, are defined as before. So, a *simple* equation over $\{\vee, \cdot, 1, {\sim}, -\}$ is of the form $t_0^{\mathbf{Fm}} \leq t_1^{\mathbf{Fm}} \vee \cdots \vee t_n^{\mathbf{Fm}}$, where t_0 is linear and the negations are applied directly and an even number of times to the variables.

Theorem 6.16. *Let* $(\mathbf{W}, \mathbf{B})$ *be a cut free involutive Gentzen frame and let* ε *be a simple. Then* $(\mathbf{W}, \mathbf{B})$ *satisfies* $\mathrm{R}(\varepsilon)$ *iff* $\mathbf{W}^+$ *satisfies* ε.

Proof. Clearly it suffices to show the lemma for the case where ε is simple, namely of the form $t_0^{\mathbf{Fm}} \leq t_1^{\mathbf{Fm}} \vee \cdots \vee t_n^{\mathbf{Fm}}$, where t_0 is linear.

As in Theorem 6.10, $\varepsilon^{\mathbf{W}^+}(\bar{X})$ holds iff $t_0^{\mathbf{W}^+}(\bar{X}) \subseteq t_1^{\mathbf{W}^+}(\bar{X}) \vee \cdots \vee t_n^{\mathbf{W}^+}(\bar{X})$, for all sequences $\bar{X} = (X_j)_{j \in J}$ of elements in $\mathbf{W}^+$. By Corollary 2.4, this is equivalent to the stipulation that for all $z \in W'$, $t_1^{\mathbf{W}^+}(\bar{X}) \vee \cdots \vee t_n^{\mathbf{W}^+}(\bar{X}) \subseteq z^{\lhd}$ implies $t_0^{\mathbf{W}^+}(\bar{X}) \subseteq z^{\lhd}$. As $t_1^{\mathbf{W}^+}(\bar{X}) \vee \cdots \vee t_n^{\mathbf{W}^+}(\bar{X}) = \gamma_N(t_1^{\mathbf{W}^+}(\bar{X}) \cup \cdots \cup t_n^{\mathbf{W}^+}(\bar{X}))$, $t_i^{\mathbf{W}^+}(\bar{X}) = \gamma_N(t_i^{\mathcal{P}(\mathbf{W})}(\bar{X}))$ and in view of Lemma 6.9(iii), this is equivalent to asking that

$$(*) \qquad \forall z \in W', \text{ if } t_i^{\mathcal{P}(\mathbf{W})}(\bar{X}) \subseteq z^{\lhd} \text{ for every } i \in \{1, \ldots n\}, \text{ then } t_0^{\mathcal{P}(\mathbf{W})}(\bar{X}) \subseteq z^{\lhd}.$$

Assume first that $(\mathbf{W}, \mathbf{B})$ satisfies $\mathrm{R}(\varepsilon)$ and let $x_j \in X_j$, for all $j \in J$ (we write $\bar{x} \in \bar{X}$ and $\bar{x} = (x_j)_{j \in J}$). Note that

$$t_i^{\mathbf{W}}(\bar{x}) = t_i^{\mathbf{W}}((x_j)_{j \in J}) \in t_i^{\mathcal{P}(\mathbf{W})}((\{x_j\})_{j \in J}) \subseteq t_i^{\mathcal{P}(\mathbf{W})}(\bar{X}),$$

by the definition of the operations in $\mathcal{P}(\mathbf{W})$. Note further that for all $z \in W'$, if $t_i^{\mathcal{P}(\mathbf{W})}(\bar{X}) \subseteq z^{\lhd}$ for every $i \in \{1, \ldots n\}$, then $t_i^{\mathbf{W}}(\bar{x}) \ N \ z$, for all $i \in \{1, \ldots n\}$, $t_0^{\mathbf{W}}(\bar{x}) \ N \ z$, by $\mathrm{R}(\varepsilon)$, and $t_0^{\mathbf{W}}(\bar{x}) \in z^{\lhd}$, for all $\bar{x} \in \bar{X}$. Since t_0 is a linear term, we obtain $t_0^{\mathcal{P}(\mathbf{W})}(\bar{X}) \subseteq z^{\lhd}$, namely (*) holds and $\mathbf{W}^+$ satisfies ε.

Conversely, assume that (*) holds for $X_j = \gamma_N(\{x_j\})$, where $x_j \in W$, namely for all $z \in W'$, $\{t_1^{\mathbf{W}}(\bar{x}), \ldots, t_n^{\mathbf{W}}(\bar{x})\} \subseteq z^{\lhd}$ implies $t_0^{\mathbf{W}}(\bar{x}) \in z^{\lhd}$. Consequently, $(t_1^{\mathbf{W}}(\bar{x}) \ N \ z$ and $\ldots$ $t_n^{\mathbf{W}}(\bar{x}) \ N \ z)$ implies $t_0^{\mathbf{W}}(\bar{x}) \ N \ z$, and $\mathrm{R}(\varepsilon)$ holds in $(\mathbf{W}, \mathbf{B})$. $\square$

Corollary 6.17. *All simple rules are preserved by* $(\_)^+$.

Corollary 6.18. *All extensions of* **InGL** *by simple rules enjoy the cut elimination property.*

7. Finite model property for **InGL** and extensions

Clearly every finite involutive $r\ell u$-groupoid satisfies the equation $(\sim(k+n))x = (\sim k)x$ for some natural numbers k and n. By taking k many right negations $(-)$ of both sides we have $(\sim n)x = x$. (Note that this equation is in turn equivalent to the equation $x = (-n)x$.) If n is an odd number, then $f(x) = (\sim n)x$ is an order antimorphism, so the involutive $r\ell u$-groupoid is trivial. Consequently, every finite non-trivial involutive $r\ell u$-groupoid satisfies an equation of the form $(\sim n)x = x$, and n can be taken to be both even and minimal non-zero. Clearly $(\sim 2)x = x$ is equivalent to cyclicity $\sim x = -x$.

Consider the following bi-directional rule.

$$\frac{x^{\sim n} \Rightarrow z}{x \Rightarrow z} \ (\sim n)$$

We will call two sequences of formulas *n-equivalent* if one is obtained from the other by adding or removing exactly n-many negations (of the same kind) to/from some of the formulas in the sequence. Clearly this equivalence (in the presence of the external negations rules) simulates the effect of the rule $(\sim n)$ to the one side of the sequents involved. We extend the equivalence also to sequents in the obvious way. For every equivalence class there are representatives such that all formulas in them contain no more than n negations. We will call them *minimal representatives.* The following result follows from Theorem 6.16.

Corollary 7.1. *Let* $\mathbf{W}$ *be an involutive frame for* **InGL** *and* n *an even natural number. The frame* $\mathbf{W}$ *satisfies the rule* $(\sim n)$ *iff* $\mathbf{W}^+$ *satisfies the equation* $(\sim n)x = x$.

Corollary 7.2. *The system* $\mathbf{InGL} + (\sim n)$ *has the cut elimination property, for every even natural number* n. *Furthermore, it is decidable.*

Proof. Soundness of the calculus is routine. The proof proceeds as for the cut elimination of **InGL**, by using Corollary 7.1. Decidability follows from the observation that, due to the rule $(\sim n)$, there are finitely many sequents with up to n negations that need to be considered at each step of the proof search. $\square$

Corollary 7.3. *The system* $\mathbf{InGL} + (\sim n)$ *has the FMP, for every even natural number* n.

Proof. We consider the involutive frame $\mathbf{W}$, where $W = W'$ is the free monoid over Fm and $x\ N\ z$ iff $\vdash_{\mathbf{InGL}+(\sim n)} x \Rightarrow z$. Consider a sequent s that is not provable and let $s^{\leftarrow}$ be the set of all the sequents involved in a thorough proof search for s. Note that $s^{\leftarrow}$ is infinite, since the $(^{\sim})$ and $(^{-})$ rules can be applied an arbitrary number of times. We also consider the relation $N' = N \cup (s^{\leftarrow})^c$. It is easy to see that N' is a nuclear relation and $\mathbf{W}' = (W, W', N', \circ, \varepsilon)$ is an involutive frame for $\mathbf{InGL} + (\sim n)$.

Clearly $s^{\leftarrow}$ is a union of n-equivalence classes. Moreover, there are only finitely many such classes. Note that if $y, z \in W$ are n-equivalent then $y^{\lhd} = z^{\lhd}$ because of the rule $(\sim n)$. Consequently, the basic closed sets are finitely many and $\mathbf{W}'^+$ is finite. Moreover, s fails in $\mathbf{W}'^+$. $\square$

Recall that for $x \in W$ and $m \in \mathbb{Z}^+$, $x^{\neg m}$ denotes the element $x^{\sim m}$ if $m > 0$, the element x^{-m} if $m < 0$, and x if $m = 0$.

Theorem 7.4. *Every sequent provable in* **InGL** $+ (\sim n)$ *has a proof in* **InGL** *augmented by intial sequents of the form* $p^{\neg kn} \Rightarrow p^{\neg mn}$, $\varepsilon \Rightarrow 1^{\neg mn}$ *and* $0^{\neg kn} \Rightarrow \varepsilon$ *where* p *ranges over the propositional variables of the sequent and* $m, n \in \mathbb{Z}$.

Proof. Assume that a sequent s is provable in **InGL** $+ (\sim n)$. By cut-elimination (Corollary 7.2) it is provable in the cut free systems. All of the systems we will mention in this proof will be considered in their cut-free versions. Note that in the presence of $({}^{\sim})$ and $({}^{-})$, the rules

$$\dfrac{u(x) \Rightarrow z}{u(x^{\sim n}) \Rightarrow z}\ (\sim n\mathrm{L}) \qquad \dfrac{x \Rightarrow u(z)}{x \Rightarrow u(z^{\sim n})}\ (\sim n\mathrm{R})$$

are derivable from $(\sim n)$, and vice versa. Here u is such that $u(x)$ is an element of W in which no negations are applied to x. Note that these rules could have been called $(-n\mathrm{L})$ and $(-n\mathrm{R})$, as the versions with $-$ instead of $\sim$ are simple the upward direction of the rule, since $x^{-\sim} = x$. So, s is provable in **InGL** $+ (\sim n\mathrm{L}) + (\sim n\mathrm{R})$.

We will first prove, inductively, that the rules $(\sim n\mathrm{L})$ and $(\sim n\mathrm{R})$ can be moved to the top of the proof in **InGL** $+ (\sim n\mathrm{L}) + (\sim n\mathrm{R})$, namely that there is a proof of s in which all applications of the rules $(\sim n\mathrm{L})$ and $(\sim n\mathrm{R})$ precede all application of rules in **InGL**. We proceed by focusing on the rule applied immediately before $(\sim n\mathrm{L})$ or $(\sim n\mathrm{R})$. Below, we give a proof and its rewritten version. We will be using often the following instances of $({}^{\sim})$ and $({}^{-})$

$$\dfrac{x \Rightarrow z^{-n}}{x^{\sim n} \Rightarrow z}\ ({}^{\sim}n) \qquad \dfrac{x^{-n} \Rightarrow z}{x \Rightarrow z^{\sim n}}\ ({}^{-}n)$$

which hold for even n.

We first deal with the case where $(\sim n\mathrm{L})$, applied upward, is preceded by a left rule. For $(/\mathrm{L})$, we have

$$\dfrac{\dfrac{x \Rightarrow a \quad b \Rightarrow z}{(b/a) \circ x \Rightarrow z}\ (/\mathrm{L})}{(b/a)^{\sim n} \circ x \Rightarrow z}\ (\sim n\mathrm{L}) \quad \to \quad \dfrac{\dfrac{\dfrac{x \Rightarrow a}{x^{\sim n} \Rightarrow a}\ (\sim n\mathrm{L}) \quad \dfrac{b \Rightarrow z}{b \Rightarrow z^{-n}}\ (\sim n\mathrm{R})}{b/a \circ x^{-n} \Rightarrow z^{-n}}\ (/\mathrm{L})}{(b/a)^{\sim n} \circ x \Rightarrow z}\ ({}^{\sim}n)$$

Note that this case illustrates the necessity of replacing $(\sim n)$ by $(\sim n\mathrm{L})$, and $(\sim n\mathrm{R})$, in our arguments. If ${}^{\sim n}$ is applied to x, the situation is even simpler. The rewriting for $(\backslash\mathrm{L})$ is similar. The situation for all other left rules is somehow simpler, and all of them are handled in exactly the same way. For example, For $(\vee\mathrm{L})$, we have

$$\dfrac{\dfrac{a \Rightarrow z \quad b \Rightarrow z}{a \vee b \Rightarrow z}\ (\vee\mathrm{L})}{(a \vee b)^{\sim n} \Rightarrow z}\ (\sim n\mathrm{L}) \quad \to \quad \dfrac{\dfrac{\dfrac{a \Rightarrow z}{a \Rightarrow z^{-n}}\ (\sim n\mathrm{R}) \quad \dfrac{b \Rightarrow z}{b \Rightarrow z^{-n}}\ (\sim n\mathrm{R})}{a \vee b \Rightarrow z^{-n}}\ (\vee\mathrm{L})}{(a \vee b)^{\sim n} \Rightarrow z}\ ({}^{\sim}n)$$

For the case where $(\sim n\mathrm{R})$ is preceded by a right rule, of slight interest are $(/\mathrm{L})$ and $(\cdot\mathrm{R})$.

$$\dfrac{\dfrac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\mathrm{R})}{x \Rightarrow (b/a)^{\sim n}}\ (\sim n\mathrm{R}) \quad \to \quad \dfrac{\dfrac{\dfrac{x \circ a \Rightarrow b}{x^{-n} \circ a \Rightarrow b}\ (\sim n\mathrm{L})}{x^{-n} \Rightarrow b/a}\ (/\mathrm{R})}{x \Rightarrow (b/a)^{\sim n}}\ ({}^{-}n)$$

$$\dfrac{\dfrac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\,(\cdot\mathrm{R})}{x \circ y \Rightarrow (a \cdot b)^{\sim n}}\,(\sim n\mathrm{R}) \qquad \to \qquad \dfrac{\dfrac{\dfrac{x \Rightarrow a}{x^{-n} \Rightarrow a}\,(\sim n\mathrm{L}) \quad \dfrac{y \Rightarrow b}{y^{-n} \Rightarrow b}\,(\sim n\mathrm{L})}{x^{-n} \circ y^{-n} \Rightarrow a \cdot b}\,(\cdot\mathrm{R})}{x \circ y \Rightarrow (a \cdot b)^{\sim n}}\,({}^{-}n)$$

We used $(x \circ y)^{-n} = x^{-n} \circ y^{-n}$, which is true for even n. The remaining right rules are handled in the same way.

The cases where $(\sim n\mathrm{R})$ is preceded by a left rule and $(\sim n\mathrm{L})$ is preceded by a right rule are much simpler. We show only one example.

$$\dfrac{\dfrac{x \Rightarrow a}{x \Rightarrow a \vee b}\,(\vee\mathrm{L})}{x^{\sim n} \Rightarrow a \vee b}\,(\sim n\mathrm{R}) \qquad \to \qquad \dfrac{\dfrac{x \Rightarrow a}{x^{\sim n} \Rightarrow a}\,(\sim n\mathrm{L})}{x^{\sim n} \Rightarrow a \vee b}\,(\vee\mathrm{R})$$

So far we have considered the bidirectional rules $(\sim n\mathrm{L})$ and $(\sim n\mathrm{R})$ in the downward direction. In the inverse direction they take the same form, but with ${}^{\sim n}$ replaced by ${}^{-n}$; this shows that the rules could have been called $(-n\mathrm{L})$ and $(-n\mathrm{R})$. The proof rewriting for these cases is completely analogous to the cases handled above.

Finally, we show that $(\sim n\mathrm{L})$ and $(\sim n\mathrm{R})$ commute with the rules $({}^{\sim})$ and $({}^{-})$. We show two illustrative cases.

$$\dfrac{\dfrac{x \circ u(y) \Rightarrow z}{u(y) \Rightarrow x^{\sim} \circ z}\,({}^{\sim})}{u(y^{\sim n}) \Rightarrow x^{\sim} \circ z}\,(\sim n\mathrm{L}) \qquad \to \qquad \dfrac{\dfrac{x \circ u(y) \Rightarrow z}{x \circ u(y^{\sim n}) \Rightarrow z}\,(\sim n\mathrm{L})}{u(y^{\sim n}) \Rightarrow x^{\sim} \circ z}\,({}^{\sim})$$

$$\dfrac{\dfrac{x \circ u(y) \Rightarrow z}{x \Rightarrow z \circ u^{-}(y^{-})}\,({}^{-})}{x \Rightarrow z \circ u^{-}((y^{-})^{\sim n})}\,(\sim n\mathrm{L}) \qquad \to \qquad \dfrac{\dfrac{x \circ u(y) \Rightarrow z}{x \circ u(y^{\sim n}) \Rightarrow z}\,(\sim n\mathrm{L})}{x \Rightarrow z \circ u^{-}((y^{-})^{\sim n})}\,({}^{\sim})$$

We used $(y^{\sim n})^{-} = (y^{-})^{\sim n}$, and $u(y)^{-} = u^{-}(y^{-})$, where u^{-} is obtained from u by reversing the order and applying ${}^{-}$ to every factor.

We have shown that there is a proof of the sequent s in which $(\sim n\mathrm{L})$ and $(\sim n\mathrm{R})$ are applied before rules of **InGL**. As the only initial sequents are of the form $p \Rightarrow p$, $\varepsilon \Rightarrow 1$ and $0 \Rightarrow \varepsilon$, the seqents obtained by applications of only $(\sim n\mathrm{L})$ and $(\sim n\mathrm{R})$ are of the form $p^{\neg kn} \Rightarrow p^{\neg mn}$, $\varepsilon \Rightarrow 1^{\neg mn}$ and $0^{\neg kn} \Rightarrow \varepsilon$, where p ranges over the propositional variables of the sequent and $m, n \in \mathbb{Z}$. Consequently, we have obtained a proof of s in **InGL** from these initial sequents. $\square$

Corollary 7.5. *Given a sequent that is not provable in* **InGL***, there is an upper bound on the number n such that the sequent is provable in* **InGL** $+ (\sim n)$.

Proof. Assume that the sequent is provable in **InGL** $+ (\sim n)$. Then there is a proof of the sequent in **GL** from the sequents of the form $p^{\neg kn} \Rightarrow p^{\neg mn}$, $\varepsilon \Rightarrow 1^{\neg mn}$ and $0^{\neg kn} \Rightarrow \varepsilon$, where p is a variable and $m, n \in \mathbb{Z}$. Reading the proof in **InGL** upward, note that between applications of logical rules there is a bounded number of external negations that can be added via the rules $({}^{\sim})$ and $({}^{-})$; for the other sequents obtained by applications of these rules no logical rule would be applicable (again upward). Consequently, there is only a bounded number of external negations that can appear in the initial sequents of the form $p^{\neg kn} \Rightarrow p^{\neg mn}$, $\varepsilon \Rightarrow 1^{\neg mn}$ and $0^{\neg kn} \Rightarrow \varepsilon$ used in the proof of the sequent. $\square$

Corollary 7.6. *For every sequent s there exists an even natural number n_s such that s is provable in* $\mathbf{InGL} + ({\sim}n_s)$ *iff s is provable in* $\mathbf{InGL}$.

Corollary 7.7. *The system* $\mathbf{InGL}$ *has the FMP.*

Proof. If s is not provable $\mathbf{InGL}$, then s is not provable in $\mathbf{InGL}+({\sim}n_s)$. Since the latter has the FMP, there exists a finite involutive $r\ell u$-groupoid (that also satisfies $x^{\sim n_s} = x$) where s fails. $\square$

Lemma 7.8. *The rules* $({\sim}n\mathrm{L})$ *and* $({\sim}n\mathrm{R})$ *commute with all simple structural rules.*

Proof. Let

$$\frac{t_1 \Rightarrow z \quad \cdots \quad t_k \Rightarrow z}{t_0 \Rightarrow z}\ (r)$$

be a (multiple conclusion) simple structural rule. Also, let x appear in t_0. We will write $t_0(x)$ for t_0, treating t_0 as a unary linear polynomial.

$$\dfrac{\dfrac{t_1(x) \Rightarrow z \quad \cdots \quad t_k(x) \Rightarrow z}{t_0(x) \Rightarrow z}\ (r)}{t_0(x^{\sim n}) \Rightarrow z}\ ({\sim}n\mathrm{L}) \qquad \to \qquad \dfrac{\dfrac{t_1(x) \Rightarrow z}{t_1(x^{\sim n}) \Rightarrow z} \quad \cdots \quad \dfrac{t_k(x) \Rightarrow z}{t_k(x^{\sim n}) \Rightarrow z}\ ({\sim}n\mathrm{L})}{t_0(x^{\sim n}) \Rightarrow z}\ (r)$$

where the uppermost steps in the second proofs indicate repeated applications of $({\sim}n\mathrm{L})$ to each occurence of x in $t_i(x)$. $\square$

Corollary 7.9. *The systems* $\mathbf{InGL}_R$ *have the FMP, for every set R of simple rules for which the complexity does not increase. In particular,* $\mathbf{InGL}_a$ *and* $\mathbf{InGL}_{ae}$ *have the FMP.*

8. Further results

We conclude the paper with two sections devoted to exploring certain links with work on one-sided sequent systems and with representation theorems for perfect algebras.

One-sided involutive frames. A *one-sided involutive frame* is a structure of the form $\mathbf{W} = (W, \circ, E, D, {}^{\sim}, {}^{-})$, where W is a set, $\circ$ is a ternary relation on W, ${}^{\sim}, {}^{-}$ are unary maps on W, and E, D are subsets of W, such that

(i) $x^{\sim -} = x = x^{-\sim}$
(ii) $(x^{\sim} \circ y^{\sim})^{-} = (x^{-} \circ y^{-})^{\sim}$ (weak involution)
(iii) $x \circ y^{\sim} \in D$ iff $y^{-} \circ x \in D$ (nuclear)
(iv) $w \circ [(x \circ y) \circ z] \subseteq D$ iff $w \circ [x \circ (y \circ z)] \subseteq D$ (weak associativity)
(v) $w \circ (E \circ x) \subseteq D$ iff $w \circ x \subseteq D$ (weak unit)
(vi) $(x \circ y) \circ z \subseteq D$ iff $x \circ (y \circ z) \subseteq D$ (associativity under D)

Theorem 8.1. *Weakly associative involutive frames and one-sided involutive frames are interdefinable.*

Proof. Assume that $(W, \circ, E, D, {}^\sim, {}^-)$ is a one-sided involutive frame and define $x\ N\ y$ iff $x \circ y^\sim \subseteq D$. We then have $x \circ y\ N\ z$ iff $(x \circ y) \circ z^\sim \subseteq D$ iff $x \circ (y \circ z^\sim) \subseteq D$ iff $x\ N\ (y \circ z^\sim)^- = z \oplus y^-$, where we used associativity under D. Also, we have $x \circ y\ N\ z$ iff $z^- \circ (x \circ y) \subseteq D$ iff $(z^- \circ x) \circ y \subseteq D$ iff $y\ N\ (z^- \circ x)^\sim = x^\sim \oplus z$. Consequently, N is nuclear.

For all $w, x, y, z \in W$, we have $w \in [(x \circ y) \circ z]^\rhd$ iff $(x \circ y) \circ z\ N\ w$ iff $w^\sim \circ [(x \circ y) \circ z] \subseteq D$ iff $w^\sim \circ [x \circ (y \circ z)] \subseteq D$ iff $w \in [x \circ (y \circ z)]^\rhd$.

For all $w, x \in W$, we have $w \in x^\rhd$ iff $x\ N\ w$ iff $w^\sim \circ x \subseteq D$ iff $w^\sim \circ (E \circ x) \subseteq D$ iff $w \in (E \circ x)^\rhd$.

Therefore, $(W, N, \circ, E, {}^\sim, {}^-)$ is an involutive frame.

Conversely, assume that $(W, N, \circ, E, {}^\sim, {}^-)$ is an involutive frame and define the set $D = \{w \in W : w\ N\ E^\sim\} = {\sim}E$.

We have $x \circ y^\sim \subseteq D$ iff $x \circ y^\sim\ N\ E^\sim$ iff $x\ N\ E^\sim \oplus y$ iff $x\ N\ y$ iff $x\ N\ y \oplus E^\sim$ iff $y^- \circ x\ N\ E^\sim$ iff $y^- \circ x \subseteq D$.

Note that $w \circ [(x \circ y) \circ z] \subseteq D$ iff $w \in [(x \circ y) \circ z]^\rhd$ iff $w \in [(x \circ y) \circ z]^\rhd$ iff $w \circ [x \circ (y \circ z)] \subseteq D$. Also, $w \circ (E \circ x) \subseteq D$ iff $w \in (E \circ x)^\lhd = x^\lhd$ iff $w \circ x \subseteq D$. $\square$

Such one-sided Gentzen frames include applications to sequent systems as the ones in [25] and [24].

Perfect involutive frames. Assume that $\mathbf{L}$ is a perfect involutive $r\ell u$-groupoid and ${}^\sim$ is a map on the set $J = J^\infty(\mathbf{L})$ of completely join irreducibles of $\mathbf{L}$ such that $x^{\sim\sim} = {\sim\sim}x$, for all $x \in J$; note that the map $x \mapsto {\sim\sim}x$ is a lattice isomorphism, which restricts to an isomorphism on the poset $(J, \leq)$. We define the structure $\mathbf{W}_\mathbf{L}^\sim = (W, N, \circ, {}^\sim, {}^-, E)$, where $W = J$, $x \circ y = {\downarrow_J}(xy)$, ${}^-$ is the inverse of ${}^\sim$, $x\ N\ y$ iff $x \leq -(y^\sim)$ and $E = {\downarrow_J}1$. Note that $x^{\sim\sim} = {\sim\sim}x$ is equivalent to $x^{--} = --x$ and to $-(y^\sim) = {\sim}(y^-)$, hence we also have $x\ N\ y$ iff $x \leq {\sim}(y^-)$.

Theorem 8.2. *Let $\mathbf{L}$ be a perfect involutive $r\ell u$-groupoid and let ${}^\sim$ be a map on J such that $x^{\sim\sim} = {\sim\sim}x$, for all $x \in J$. Then $\mathbf{W}_\mathbf{L}^\sim$ is an involutive frame and $\mathbf{L}$ is embedded in $(\mathbf{W}_\mathbf{L}^\sim)^+$ via the map $x \mapsto \{x\}^\lhd = {\downarrow_J}x$. If $\mathbf{L}$ is complete then this embedding is an isomorphism.*

Proof. For every $x, y, z \in W$, we have the following series of equivalent statements

$$\begin{gathered}
z \in x^{\sim\sim} \circ y^{\sim\sim} = {\downarrow_J}(x^{\sim\sim} \cdot y^{\sim\sim}) \\
z \leq x^{\sim\sim} \cdot y^{\sim\sim} = {\sim\sim}x \cdot {\sim\sim}y = {\sim\sim}(xy) \\
--z \leq xy \\
z^{--} \leq xy \\
z^{--} \in {\downarrow_J}(xy) \\
z \in {\downarrow_J}(xy)^{\sim\sim} = (x \circ y)^{\sim\sim}
\end{gathered}$$

Therefore, we have $x^{\sim\sim} \circ y^{\sim\sim} = (x \circ y)^{\sim\sim}$. Moreover, for $x, y, z \in W$,

$$\begin{array}{cc}
x \circ y\ N\ z & x \circ y\ N\ z \\
{\downarrow_J}(xy) \leq {\sim}(z^-) & {\downarrow_J}(xy) \leq -(z^\sim) \\
xy \leq {\sim}(z^-) & xy \leq -(z^\sim) \\
z^- \leq -(xy) = -y/x & z^\sim \leq {\sim}(xy) = y\backslash{\sim}x \\
z^- x \leq -y & yz^\sim \leq {\sim}x \\
z^- \circ x = {\downarrow_J}(z^- x) \leq -y & y \circ z^\sim = {\downarrow_J}(yz^\sim) \leq -x \\
y \leq {\sim}(z^- \circ x) & x \leq -(y \circ z^\sim) \\
y\ N\ (z^- \circ x)^\sim = x^\sim \oplus z & x\ N\ (y \circ z^\sim)^- = z \oplus y^-
\end{array}$$

Let $x, y, z \in W$. Note that $xy = \bigvee {\downarrow_J}(xy)$, so for every $w \in J$, $w \leq ({\downarrow_J}(xy))z$ iff $w \leq (xy)z$, as multiplication distributes over join. Therefore, $({\downarrow_J}(xy))z = x({\downarrow_J}(yz))$ and $(x \circ y) \circ z = {\downarrow_J}[({\downarrow_J}(xy))z] = {\downarrow_J}[x({\downarrow_J}(yz))] = x \circ (y \circ z)$. Thus, we have (weak) associativity. For similar reasons, we have $(x \circ E) = {\downarrow_J}(x{\downarrow_J}1) = {\downarrow_J}x$, which gives (weak unit).

We claim that the map $f : L \to (\mathbf{W}_{\mathbf{L}}^{\sim})^+$ defined by $f(x) = {\downarrow_J}x$ is an embedding. Note that ${\downarrow_J}x$ is indeed a closed element. Also, if $X \subseteq J$, $\gamma_N(X) = {\downarrow_J} \bigvee X$. It is shown in [7] that, in general, f is a lattice embedding, and if $\mathbf{L}$ is complete then it is a lattice isomorphism. To show that it is an involutive $r\ell u$-groupoid isomorphism it suffices to show that it is a groupoid isomorphism, which in turn follows from proving that multiplication agrees on the completely join irreducible elements, namely that $f(xy) = f(x) \circ_{\gamma_N} f(y)$, for $x, y \in J$. Indeed we have $xy = \bigvee {\downarrow_J}x \cdot \bigvee {\downarrow_J}y = \bigvee({\downarrow_J}x \cdot {\downarrow_J}y)$, by the distributivity of multiplication over join. Therefore, $f(xy) = {\downarrow_J}(xy) = {\downarrow_J} \bigvee({\downarrow_J}x \cdot {\downarrow_J}y) = \gamma_N(f(x) \circ f(y))$. $\square$

We also note that $D = {\downarrow_J}0$, since for $w \in W$, we have $w \in D = {\sim}^{(\mathbf{W}_{\mathbf{L}}^{\sim})^+}E$ iff $w\ N\ E^{\sim} = {\sim}(E^{\sim -}) = {\sim}E = {\sim}({\downarrow_J}1) = {\uparrow_M}{\sim}1 = {\uparrow}0$ iff $w \in {\downarrow_J}({\uparrow_M}0) = {\downarrow_J}0$.

Let id_J be the identity map on J, and let $\mathbf{W}_{\mathbf{L}}^{id_J} = (W, N, id_J, id_J, E)$.

Corollary 8.3. *For any perfect cyclic InFL-algebra $\mathbf{L}$, the structure $\mathbf{W}_{\mathbf{L}}^{id_J}$ is an involutive frame and L is embedded in $(\mathbf{W}_{\mathbf{L}}^{id_J})^+$. If L is complete then this embedding is an isomorphism.*

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