CUT ELIMINATION AND STRONG SEPARATION FOR SUBSTRUCTURAL LOGICS: AN ALGEBRAIC APPROACH.

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Abstract. We develop a general algebraic and proof-theoretic study of substructural logics that may lack associativity, along with other structural rules. Our study extends existing work on (associative) substructural logics over the full Lambek Calculus **FL** (see e.g. [36, 19, 18]). We present a Gentzen-style sequent system **GL** that lacks the structural rules of contraction, weakening, exchange and associativity, and can be considered a non-associative formulation of **FL**. Moreover, we introduce an equivalent Hilbert-style system **HL** and show that the logic associated with **GL** and **HL** is algebraizable, with the variety of residuated lattice-ordered groupoids with unit serving as its equivalent algebraic semantics.

Overcoming technical complications arising from the lack of associativity, we introduce a generalized version of a logical matrix and apply the method of quasicompletions to obtain an algebra and a quasiembedding from the matrix to the algebra. By applying the general result to specific cases, we obtain important logical and algebraic properties, including the cut elimination of **GL** and various extensions, the strong separation of **HL**, and the finite generation of the variety of residuated lattice-ordered groupoids with unit.

1. Introduction

Substructural logics are generally understood as extensions of logics obtained by removing some *structural rules* from intuitionistic logic in its sequent formulation **LJ**, and thus they are extensions of full Lambek calculus **FL**—the calculus defining the basic substructural logic without the rules of exchange, weakening and contraction. In algebraic terms, they are logics determined by subvarieties of the variety of FL-algebras, i.e., residuated lattices with a constant 0. More precisely, in terms of abstract algebraic logic: *the variety of FL-algebras is an equivalent algebraic semantics for the deducibility relation determined by* **FL**. Substructural logics over **FL** and residuated lattices have been extensively studied in recent years both from algebraic and proof-theoretic view points. For general information, see [18].

One main purpose of the present paper is to extend the current study to substructural logics that may lack associativity, and in particular to explore to what extent algebraic methods—already developed for substructural logics over **FL**—are applicable. Obvious modifications include, moving from monoids to groupoids with unit, for the algebraic structures, and considering a non-associative version of comma in sequents, for the syntactic objects. However, we will show that although

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many results, requiring more involved proofs, generalize to the non-associative setting, some facts fail in the general case.

The second, equally important, aim of the paper is to provide a setting, consistent with the theory of abstract algebraic logic, that unifies many constructions in the literature. In particular, we show that *logical matrices*, appropriately generalized, serve as a unifying object for the comprehensive study of (non-associative) substructural logics and admit quasicompletions that yield important logical and algebraic properties for the corresponding logics.

Throughout the paper, we assume some familiarity with substructural logics over **FL** and residuated lattices (see e.g. [18]), as well as with basic notions from universal algebra (see e.g. [9] for more information). To avoid disrupting the flow of the paper, the proofs of some of technical results are given in the appendices. To facilitate navigation through the paper, we give a table of contents before the bibliography.

1.1. **Main results.** In Section 2.1, we introduce the Gentzen-style system **GL**. The rules of the system are specified in terms of metarules (rule schemes); see Figure 1 in Section 2.1. This presentation has the advantage that the same set of metarules, by appropriate interpretations, can specify, for example, the system **FL** of full Lambek calculus, $\mathbf{FL_e}$ (**FL** with the rule of exchange), or even Gentzen's original system **LJ** for intutionistic logic. Sequents, the main syntactic object of **GL**, involve non-associative sequences of formulas, while in the cases of **FL**, $\mathbf{FL_e}$ or **LJ**, they involve sequences, multisets or sets, respectively. By considering these different data types for sequents the same set of metarules serves as a definition for all of the above systems.

Alternatively, these systems can be defined by adding *structural* rules (see Figure 2) to **GL**. If we add associativity we obtain (a system equivalent to) **FL**; if we add all basic structural rules of associativity, exchange, weakening and contraction we obtain (a system equivalent to) **LJ**.

It is easily seen that it is decidable whether a sequent is provable in cut-free **GL** (**GL** without the cut rule). The *cut elimination property* states that the cut rule does not contribute at all to the provable (without assumptions) sequents of the system. The proof of this property (Theorem 4.8) implies the decidability of **GL**. The *basic* structural rules are among the ones (*simple* structural rules) that can be added to **GL** without affecting the cut elimination property (see Section 4.3). Therefore, the property holds for all the systems mentioned above (see Corollary 4.14), with the understanding that the rule of contraction is formulated for sequences of formulas. In Section 4.4, we prove that **GL**, as well as its extensions with simple structural rules, has the finite model property.

We introduce the Hilbert-style systems **HL** (Figure 6) and **sHL** (Figure 5), and prove that both are equivalent (Theorems 2.1 and 2.3), in the sense of [22], to **GL**; the equivalence holds also for extensions of the systems with simple structural rules (Theorem 2.3). The *strong separation property* for **HL** (Theorem 4.19) states that every proof in **HL** can be rewritten in a way that it only uses the connectives already in the assumptions and conclusion of the proof plus maybe the basic connective $\backslash$ of left implication. As a consequence, the system is a strong conservative extension of each of its fragments. [The adjective 'strong' here refers to the existence of assumptions in the derivation.] We prove that **HL**, as well as its expansions that correspond to simple structural rules, enjoy the strong separation property

(Theorem 4.19). The system **HL** is not finitely axiomatized (Theorem 2.5) while it enjoys the strong separation property. On the other hand its equivalent version **sHL** is finitely axiomatized, but enjoys a restricted version of the strong separation property for the case where the set of basic connectives is $\{\backslash, \wedge\}$. [More generally, **sHL** has the strong separation property (Theorem 2.2) under the understanding that the connective $\wedge$ needs to be included when we include the connective $\vee$.] The associative version $\mathbf{HL_a}$ of **HL**(see Section 2.2.4) can be simplified to a system (**HL** plus associativity) equivalent to **FL** that has the strong separation property with respect to the set of basic connectives $\{\backslash, /\}$ (see Corollary 4.19 and Lemma 4.20). Given the separation property for **HL**, the general algebraization theory yields axiomatizations for the classes of subreducts of the algebraic semantics.

Having developed the necessary algebraic background in the beginning of Section 3, we proceed to show that the algebraic semantics (Theorem 4.23) of **GL** (and **HL**) are residuated lattice-ordered groupoids with unit (see Section 3.1) and they form a variety RLUG. We prove that RLUG has a decidable equational theory and is actually generated by its finite members (Theorem 4.24). We also give a list of subvarieties, corrsponding to simple structural rules, that have the same properties (Corollary 4.23 and Theorem 4.24).

Most of Section 3 is devoted to introducing *generalized logical matrices*, the main and unifying object to which the quasicompletion will be performed, and to developing, in the non-associative setting, the necessary background theory for these matrices. The type of logical matrix that we consider generalizes the notion of a matrix from abstract algebraic logic—a pair of an algebra $\mathbf{A}$ and a subset F of A—to allow for $\mathbf{A}$ to be a partial algebra and for F to be a set of *sequents* over A.

In Section 4, the *quasicompletion* method is applied to an arbitrary generalized logical matrix $\boldsymbol{\mathcal{A}}$ to yield a residuated lattice-ordered groupoid with unit $\mathbf{R}(\boldsymbol{\mathcal{A}})$ (Theorem 4.1) and is followed by the construction of a *quasiembedding* into $\mathbf{R}(\boldsymbol{\mathcal{A}})$ (Lemma 4.4). This is the main technical part of the paper and is applied to obtain all the main results by instantiating the generalized logical matrix according to the particular application. In particular, the cut elimination theorem for the Gentzen system **GL**, the strong separation theorem for the corresponding Hilbert system **HL**, the finite model property and the finite embeddability property for various systems (see Section 4.6) are all obtained by means of the quasicompletion theorem.

We mention that the notion of a *nucleus* is the main tool in the quasicompletion construction. A nucleus on a residuated lattice is a closure operator on the underlying lattice that is compatible with multiplication and the division operations. The concept has its origins in topological frames and Heyting algebras (e.g., see [40]), but has been also extended in the context of quantales [39]. Moreover, it has been used in many different and diverse applications (see [21], [22], [20]). En route to our goal (see Appendix B), we present natural systems, which we call (*residuated*) *action systems* and which produce a residuated lattice-ordered groupoid with unit when a nucleus is applied to them.

1.2. **Background of the main idea.** To place the paper in context, we review briefly some of the relevant literature. In particular, we show how our work subsumes and generalizes diverse and seemingly unrelated results.

Okada and Terui [31]—relying on ideas of Maehara [29] and Okada [30], who describes a method for proving cut elimination for various logics using phase semantics for linear logic intoduced by Girard [23] (and expanded by Abrusci [1])—prove the finite model property (FMP) for certain fragments of intuitionistic linear logic.

Blok and van Alten, in a series of papers [4, 5, 6], further extend the method of Okada and Terui to prove stronger results like the finite embeddability property (FEP) for various varieties and quasivarieties of residuated structures. In particular, they describe a construction for embedding a partial subalgebra $\mathbf{B}$ of an algebra $\mathbf{A}$ into an algebra $\mathbf{D}(\mathbf{A}, \mathbf{B})$, which remains in the variety in certain cases; also, if $\mathbf{B}$ is finite, then $\mathbf{D}$ is also finite, in particular situations, hence the construction then yields the FEP. By modifying the construction of $\mathbf{D}$, Kowalski and Ono [28] obtain the FEP for certain fuzzy logics. Also, Buszkowski [10, 11, 12] obtains the FMP for BCI logics and action logic.

In connection to residuated lattices (models of $\mathbf{FL}$), Bernadineli, Jipsen and Ono [2], introduce quasi-residuated lattices (essentially models of cut-free $\mathbf{FL}$) and give an algebraic proof of the cut elimination theorem for various Gentzen systems related to $\mathbf{FL}$. More precisely, given a sequent that is not provable in cut-free $\mathbf{FL}$, and hence fails in a quasi-residuated lattice, it is shown that the sequent also fails in a residuated lattice, obtained from the quasi-residuated lattice via a *quasi-completion* construction (that resembles the constructions of Blok and van Alten, and of Okada and Terui); thus the sequent is not provable even using the cut rule.

Raftery and van Alten [43] present a Hilbert-style system that has the strong separation property and is equivalent to $\mathbf{FL_e}$; in other words it applies to the associative, commutative case and its algebraic semantics is the variety of commutative residuated lattices. In order to prove the strong separation property the authors assume that a formula is not provable from a given set of assumptions in the appropriate fragment and they show that it is not provable in the whole system. To achieve this, they construct a commutative residuated lattice associated with the set of assumptions into which the formula fails. The construction is again based on the quasi-completion idea. The result in [43] is preceded by work of Ono and Komori [37], who obtain a (weak) separation theorem (which refers only to proofs without assumptions) for the associative, integral case (equivalent to $\mathbf{FL_w}$), for a system that may involve only one of the division (implication) connectives. The (weak) separation property is obtained from the equivalence to the corresponding Genzen system and the fact that the latter has the *subformula property.* Also, K. Došen [14] discusses the non-associative case with one division operation, and proves cut elimination using proof-theoretic argunemts, but the proposed system fails even the (weak) separation property.

As mentioned before, the constructions in the above papers make use of the quasi-completion/quasi-embedding idea to construct a residuated lattice and quasi-embed a certain structure to it. Nevertheless, the constructions apply to different objects/ingredients: to a set of sequents in [31], to a partial subalgebra of a residuated lattice in [4, 5, 6], to a quasi-residuated lattice in [2] and to a set of formulas in [43]. We show that a logical matrix serves as a single unifying object to which the construction applies in a way that it instantiates to the examples above. It should be stressed that we develop this general construction in the absence of all the basic structural rules of contraction, weakening, exchange and associativity. At the same time these rules, as well as any other simple structural rule, can be added in a modular way, hence the construction becomes applicable to a wide range of situations.

2. Syntactic consequence relations

In this section, we define four consequence relations, all presented syntactically; one by a Gentzen-style system, two by Hilbert-style systems and one by an algebraic system. They all turn out to be equivalent in the sense of [22].

Recall that a *consequence relation* $\vdash$ on a set S is a subset of $\mathcal{P}(S) \times S$ such that for all $X \cup Y \cup \{x, y, z\} \subseteq S$, (we write $X \vdash x$ for $(X, x) \in \vdash$)

(1) if $x \in X$, then $X \vdash x$, and
(2) if $X \vdash y$, for all $y \in Y$, and $Y \vdash z$, then $X \vdash z$.

2.1. **The non-associative Gentzen system GL.** Sequent calculi were introduced by Gentzen [24], who proved the decidability of intuitionistic logic. This is done via a proof search algorithm in the cut free system (after having shown cut elimination).

A sequent, for the purposes of intuitionistic logic, is made up of formulas, commas and the separator $\Rightarrow$. More precisely, a sequent is a (non-associative) possibly empty sequence of formulas (separated by commas), concatenated with the separator symbol $\Rightarrow$ and concatenated with another formula. For example,

$$(p, p \to (q \wedge r)), q \;\Rightarrow\; p \vee r$$

is a sequent, where p, q, r are propositional variables; note the double role of the parentheses in the formula and the sequent level. In the original formulation the left-hand side of a sequent (what comes before $\Rightarrow$) was just a set of formulas, but it can be taken to be a multiset, or a sequence or a groupoid word (non-associative sequence) of formulas. This freedom in the choice of the syntactic type of a sequent is due to the fact that intuitionistic logic has all of the structural rules; the latter are responsible for the the left-hand side of a sequent behaving like a set, even in the case when it is formulated under a different syntactic type.

In order to consider substructural systems one needs to identify the structural rules and separate them both from the syntax of a sequent and from the logical rules. Depending on the degree of substructurality that one wants to achieve there is some flexibility in the choice. We will consider the system without any of the four structural rules of (contraction, weakening, exchange and associativity), so the left-hand sides of the sequents will be groupoid words (non-associative sequents). Our approach works also if we consider systems with some structural rules, by modifying the data type of the sequents.

Another complication introduced by considering sequent calculi is related to the rule schemes. In Hilbert style systems, one can usually consider a finite number of axiom and rule schemes expressed over an alphabet of metavariables, for which formulas can be substituted. Alternatively, the axiom and rule schemes can be expressed over the propositional variables, and substitution can be encoded in the definition of a proof. In the Gentzen systems we will consider, the second approach cannot be applied and even the first one needs modifications. The rule schemes considered require more types of metavariables (one for formulas, one for non-associative sequences of formulas, and one for non-associative sequences of formulas with an extra place-holder). For example, we will differentiate between rules and metarules (or rule schemes) in the deductive systems. Therefore, we will have an alphabet P for propositional variables and an alphabet F of metavariables (for formulas), as well as other alphabets for sequences of formulas etc.

We start by specifying the appropriate syntax for the general substructural case.

2.1.1. *Groupoid words and sequents.* Consider a set Q and distinct symbols ε and $\_$ not in Q. We define the set Q^γ of *groupoid words* over the set Q, relative to ε, as the smallest set such that

- $Q \cup \{\varepsilon\} \subseteq Q^\gamma$ and
- if $x, y \in Q^\gamma - \{\varepsilon\}$, then $(x, y) \in Q^\gamma$.

Alternatively, we consider the free groupoid $\langle FG(Q), \circ\rangle$ over Q and we expand it by a new element ε subject to the conditions $x \circ \varepsilon = \varepsilon \circ x = x$, for all $x \in FG(Q)$, in order to obtain the free groupoid with unit $\langle FG(Q) \cup \{\varepsilon\}, \circ, \varepsilon\rangle$. We identify $FG(Q) \cup \{\varepsilon\}$ with Q^γ, and set $\mathbf{Q}^\gamma = \langle Q^\gamma, \circ, \varepsilon\rangle$ becomes the free groupoid with unit. For example, if $Q = \{a, b, c\}$, then

$$((a, c), ((a, b), a)) = (a \circ c) \circ ((a \circ b) \circ a) \in Q^\gamma,$$

but $((a, c), (a, b), a) \notin Q^\gamma$, since it is a triple. Note that comma and $\circ$ are almost interchangeable; we simply omit the external parentheses when using $\circ$ and note that elements like (a, ε) do not exist. Therefore, Q^γ is the set of possibly empty (oriented) binary trees with leaves from Q, or the set of possibly-empty non-associative sequents of elements from Q. The element ε is called the *empty* groupoid word.

The set Q^α of *augmented groupoid words* over Q, relative to $\_$, is defined to be the set of all groupoid words over $Q \cup \{\_\}$ with exactly one occurrence of the element $\_$. More precisely, Q^α is defined recursively by the clauses

- $\_ \in Q^\alpha$ and
- if $u \in Q^\alpha$, $x \in Q^\gamma$, then $u \circ x, x \circ u \in Q^\alpha$.

For example, $((a, c), ((\_, b), a)) \in Q^\alpha$, but $((a, c), ((\_, b), \_)) \notin Q^\alpha$.

For $u \in (Q^\gamma - \{\varepsilon\}) \cup Q^\alpha$ and $x \in Q^\gamma - \{\varepsilon\}$, we define $x \circ u = (x, u)$, $u \circ x = (u, x)$ and $u \circ \varepsilon = \varepsilon \circ u = u$; we use the same symbol $\circ$, since it extends the operation in $\mathbf{Q}^\gamma$. For example, if $x = (a, b)$ and $u = (a, (\_, a))$, then $x \circ u = ((a, b), (a, (\_, a)))$ and $u \circ x = ((a, (\_, a)), (a, b))$. Also, $x \circ x = ((a, b), (a, b))$.

If $u \in Q^\alpha$ and $v \in Q^\gamma \cup Q^\alpha$, we denote by $u[v]$ the element of $Q^\gamma \cup Q^\alpha$ obtained from u by substituting v for $\_$. For example, if $x = (a, b)$ and $u = (a, (\_, a))$, then

$$u[x] = (a, ((a, b), a)) \text{ and } u[u] = (a, ((a, (\_, a)), a)).$$

Obviously, $u = u[\_]$ for all $u \in Q^\alpha$. Note that for $v = \varepsilon$, $u[\varepsilon]$ is evaluated after all commas in u have been replaced by $\circ$. So, if $u = (a, (\_, a)) = a \circ (\_ \circ a)$, then $u[\varepsilon] = a \circ (\varepsilon \circ a) = (a, a)$. We set $|u| = u[\varepsilon]$. Essentially, the absolute value of an element in Q^α is the same element (now in Q^γ) but without $\_$. To make the operation more explicit we allow ourselves to denote the element $u[x]$ also by $u \star x$ and $x \star u$.

An (*intuitionistic* or *single conclusion*) *sequent over Q* or a *Q-sequent* is an element of $Q^\gamma \times Q$. We write the sequent (x, a) as $x \Rightarrow a$. For example, $((a, c), ((a, b), a)) \Rightarrow c$ is a sequent. We usually drop the external parentheses of a groupoid word in a sequent, so the last sequent will be usually written as $(a, c), ((a, b), a) \Rightarrow c$.

An *inference rule (instance)* is a pair $r = (S, s)$, where $S \cup s$ is a set of sequents. We usually denote r in fractional notation $\frac{S}{s}(r)$, and put the name of the rule in parentheses next to the fraction. If $S = \{s_1, s_2, \ldots, s_n\}$, then we write

$$\frac{s_1 \ s_2 \ \cdots \ s_n}{s}(r).$$

If S is empty, then r is called *axiomatic* or an *axiom*; in fractional notation we leave the numerator empty.

2.1.2. *Propositional formulas.* By a *propositional* (or *algebraic*) *language* we understand a pair $\mathcal{L} = (L, \alpha)$, where L is a set of *connectives* and $\alpha : L \to \omega$ is the *arity* function. When α is understood, we often identify $\mathcal{L}$ and L. Given a propositional language $\mathcal{L}$ and a countable set P of *propositional variables*, the set $Fm_{\mathcal{L}}(\mathsf{P})$, or simply $Fm_{\mathcal{L}}$, of (propositional) formulas over $\mathcal{L}$ (and over P) is defined in the usual way and will play the role of the set Q above in the sequent calculus discussed below; the set $Fm_{\mathcal{L}}$ is also called the set of all *terms* in the context of algebra. We will be interested in formulas over sublanguages of $\mathcal{L} = \{\wedge, \vee, \cdot, \backslash, /, 1, 0\}$; 1 and 0 are constants and all other connectives are binary. In writing formulas, we abbreviate $a \cdot b$ to ab, and assume that the priority order of the connectives is as follows: multiplication ($\cdot$) is performed first, followed by the division (or implication) connectives ($\backslash$ and $/$) and by the lattice connectives ($\wedge$ and $\vee$). Thus, $pq \wedge pr/q$ is short for $(p \cdot q) \wedge ((p \cdot r)/q)$, if $p, q, r \in \mathsf{P}$.

In the following, we will refer to an $Fm_{\mathcal{L}}$-sequent, simply as an $\mathcal{L}$-sequent.

2.1.3. *Metasequents and metarules.* In the presentation of our sequent calculus, we need to specify the axioms and the rules of inference. As mentioned before, the system will have infinitely many rules of inference organized in sets (called metarules) of rules. Alternatively, a metarule is a syntactic object, of a different level than that of a rule, that describes all the rules in the set by specifying their common form. As an example, we mention that ($\backslash$L)

$$\frac{x \Rightarrow a \quad u[b] \Rightarrow c}{u[x \circ (a\backslash b)] \Rightarrow c}\ (\backslash \mathrm{L})$$

in Figure 1 is a metarule for the system **GL** that includes all the rules of the same 'form' as ($\backslash$L), where $a, b, c \in Fm_{\mathcal{L}}$, $x \in (Fm_{\mathcal{L}})^{\gamma}$ and $u \in (Fm_{\mathcal{L}})^{\alpha}$.

To formally define metarules, a necessary complication as we need to syntactically manipulate metarules, we need to define metasequents and metagroupoid words. The latter are made up from three different sorts of metavariables A (of sort S_{A}), X (of sort S_{X}) and U (of sort S_{U}), where $S_{\mathsf{A}} \subseteq S_{\mathsf{X}}$, the constant ε (of sort S_{X}) and the operators $\circ : S_{\mathsf{X}} \times S_{\mathsf{X}} \to S_{\mathsf{X}}$ and $\star : S_{\mathsf{U}} \times S_{\mathsf{X}} \to S_{\mathsf{X}}$ (we denote $u \star x$ simply by $u[x]$); we assume that the sets A, X and U are pairwise disjoint. In our systems, we will take the elements of A to have some internal structure; in particular, A will be the set $Fm_{\mathcal{L}}(\mathsf{F})$ of $\mathcal{L}$-formulas over a set F (different and disjoint from the set P of propositional variables). *Metagroupoid words* are defined as the terms of sort S_{X} of the above multi-sorted language. For example, $u[v[\varepsilon] \circ x] \circ u[a\backslash b]$ is a metagroupoid word, if $u, v \in \mathsf{U}$, $x \in \mathsf{X}$ and $a, b \in \mathsf{F}$, but u is not (because it is a term of sort S_{U}) and $u[v]$ is not even defined. *Metasequents* are simply sequences of the form $g \Rightarrow a$, where g is a metagroupoid word and $a \in A$. The fact that we used the same symbols ($\circ$, $\star$ and ε) for the different operators in defining metasequents and sequents should create no confusion.

A *metarule* is a pair $r = (S, s)$, where $S \cup s$ is a set of metasequents. The same fractional notation conventions used for rules, apply also to metarules. A rule is said to be an *instance* of a metarule, if all metavariables from F, X and U are instantiated to elements of $Fm_{\mathcal{L}}$, $(Fm_{\mathcal{L}})^{\gamma}$ and $(Fm_{\mathcal{L}})^{\alpha}$, respectively, and the metasequent operators $\circ$, $\star$ and ε are replaced by the corresponding sequent operators. For example, if p, q, r are propositional variables, then

$$\frac{p \wedge q, q \Rightarrow p \quad q, p \vee r \Rightarrow q \vee r}{q, ((p \wedge q, q), (p \backslash (p \vee r))) \Rightarrow q \vee r}$$

is an instance of ($\backslash$L) for $a = p$, $b = p \vee r$, $c = q \vee r$, $x = (p \wedge q, q)$ and $u = (q, \_)$.

It should be clear that to express ($\backslash$L) formally, we need to allow metavariables a, b in F (to be evaluated as formulas in $Fm_{\mathcal{L}}$), while $a \backslash b$ is a formal object in $\mathsf{A} = Fm_{\mathcal{L}}(\mathsf{F})$ (also eventually to be evaluated in $Fm_{\mathcal{L}}$).

2.1.4. *The Gentzen system* **GL**. The sequent calculus **GL** over the language $\mathcal{L} = \{\wedge, \vee, \cdot, \backslash, /, 1, 0\}$ is specified by the metarules of Figure 1. Instances of the metarules are obtained by replacing the metavariables a, b, c by formulas over $\mathcal{L}$, the metavariables x, y by groupoid words in $(Fm_{\mathcal{L}})^{\gamma}$ and u by an augmented groupoid word in $(Fm_{\mathcal{L}})^{\alpha}$; recall that $|u| = u[\varepsilon]$. In what follows we will use **GL** to refer to both the set of metarules specifying it and to the actual set of rules (instances of the metarules).

With the exception of the first two rules of the system **GL**, every rule introduces a connective to the left or right-hand side of a sequent; depending on the side on which the connective is introduced, we distinguish between *left* and *right* rules. Note that the left rules of **GL** can be simplified in the presence of cut, but we loose the cut elimination property. For example, $u[a]$ in ($\vee$L) can be replaced by groupoid words, where a is a (left or right) outermost formula; to prove the equivalence we use ($\backslash$R) and (/R).

If R is a set of metarules, not to be confused with the notation used for right rules, then $\mathbf{GL}_{\mathrm{R}}$ denotes the expansion of **GL** by the metarules from R. The system $\mathbf{GL}^{\mathrm{f}}_{\mathrm{R}}$, called cut-free $\mathbf{GL}_{\mathrm{R}}$, is obtained from $\mathbf{GL}_{\mathrm{R}}$ by removing the metarule (CUT).

2.1.5. *Proofs.* We define *proofs (from assumptions)* in $\mathbf{GL}_{\mathrm{R}}$, their *conclusions* and their *(set of) assumptions* by mutual induction.

- A sequent is a proof, whose conclusion and assumption is itself.
- A rule $\frac{s_1 \; s_2 \; \cdots \; s_n}{s}(r)$ in $\mathbf{GL}_{\mathrm{R}}$ is a proof, whose conclusion is s and whose assumptions are $s_1, s_2, \ldots s_n$ (more precisely, whose set of assumptions is $\{s_1, s_2, \ldots s_n\}$).
- Let $\Pi_1, \Pi_2, \ldots, \Pi_n$ be proofs in $\mathbf{GL}_{\mathrm{R}}$ with conclusions $s_1, \ldots s_n$, respectively, and sets of assumptions $S_1, S_2, \ldots, S_n$, respectively. If $\frac{s_1 \; s_2 \; \cdots \; s_n}{s}(r)$ is a rule in $\mathbf{GL}_{\mathrm{R}}$, then $\frac{\Pi_1 \; \Pi_2 \; \cdots \; \Pi_n}{s}(r)$ is a proof whose conclusion is s and whose set of assumptions is $S_1 \cup \cdots \cup S_n$.

Metaproofs are defined in a similar way, using the obvious notion for schematic substitution for expressions like $u[x]$. The following notions have analogues for metaproofs and metasequents, as well.

We say that a sequent s is *provable* or *derivable* in $\mathbf{GL}_{\mathrm{R}}$ from a set S of sequents, in symbols $S \vdash_{\mathbf{GL}_{\mathrm{R}}} s$, if there is a proof whose conclusion is s and whose set of assumptions is *contained* in S. It is easy to see that $\vdash_{\mathbf{GL}_{\mathrm{R}}}$ is a consequence relation on the set of sequents; we will call it the *deducibility* or *provability* relation of the Gentzen system.

If s is provable in $\mathbf{GL}_{\mathrm{R}}$ from an empty set of assumptions, then we simply say that s is *provable* in $\mathbf{GL}_{\mathrm{R}}$. Proofs from assumptions that have an empty set of assumptions are simply called *proofs*.

$$\frac{x \Rightarrow a \quad u[a] \Rightarrow c}{u[x] \Rightarrow c}\ (\mathrm{CUT}) \qquad \frac{}{a \Rightarrow a}\ (\mathrm{Id})$$

$$\frac{x \Rightarrow a \quad u[b] \Rightarrow c}{u[x \circ (a\backslash b)] \Rightarrow c}\ (\backslash\mathrm{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a\backslash b}\ (\backslash\mathrm{R})$$

$$\frac{x \Rightarrow a \quad u[b] \Rightarrow c}{u[(b/a) \circ x] \Rightarrow c}\ (/\mathrm{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\mathrm{R})$$

$$\frac{u[a \circ b] \Rightarrow c}{u[a \cdot b] \Rightarrow c}\ (\cdot\mathrm{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot\mathrm{R})$$

$$\frac{u[a] \Rightarrow c}{u[a \wedge b] \Rightarrow c}\ (\wedge\mathrm{L}\ell) \qquad \frac{u[b] \Rightarrow c}{u[a \wedge b] \Rightarrow c}\ (\wedge\mathrm{L}r) \qquad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge\mathrm{R})$$

$$\frac{u[a] \Rightarrow c \quad u[b] \Rightarrow c}{u[a \vee b] \Rightarrow c}\ (\vee\mathrm{L}) \qquad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee\mathrm{R}\ell) \qquad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee\mathrm{R}r)$$

$$\frac{|u| \Rightarrow a}{u[1] \Rightarrow a}\ (1\mathrm{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\mathrm{R})$$

FIGURE 1. The system **GL**.

Depending on whether a, b, c are formulas (in $\mathbf{Fm}_{\mathcal{L}}$) or metavariables for formulas (in F), the following is an example of a proof or a metaproof in **GL**.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \Rightarrow a}\ (\mathrm{Id}) \quad \dfrac{}{b \Rightarrow b}\ (\mathrm{Id})}{a, b \Rightarrow ab}\ (\cdot\mathrm{R})}{a, b \Rightarrow ab \vee ac}\ (\vee\mathrm{R}\ell) \quad \dfrac{\dfrac{\dfrac{}{a \Rightarrow a}\ (\mathrm{Id}) \quad \dfrac{}{c \Rightarrow c}\ (\mathrm{Id})}{a, c \Rightarrow ac}\ (\cdot\mathrm{R})}{a, c \Rightarrow ab \vee ac}\ (\vee\mathrm{R}r)}{\dfrac{a, b \vee c \Rightarrow ab \vee ac}{a(b \vee c) \Rightarrow ab \vee ac}\ (\cdot\mathrm{L})}\ (\vee\mathrm{L})$$

2.1.6. *Structural rules.* The Gentzen system **FL** is defined in a way similar to **GL**. The essential difference is that the left-hand side of an *associative sequent* is not a groupoid word, but a sequence (a monoid word) of formulas. *Augmented associative sequences* are associative versions of augmented groupoid words, as well, and the operation $\circ$ in the definition of metasequents is taken to be associative for associative metasequents; see [36] for more on **FL**. With the understanding that they are defined over different syntactic objects (sequents), the metarules of the systems **GL** and **FL** are the same; the difference lies in the instances of the metarules. Obviously,

GL is more expressive than **FL** and it can be shown that **FL** is equivalent to a restricted version of **GL**.

$$\frac{u[(x \circ y) \circ z] \Rightarrow a}{u[x \circ (y \circ z)] \Rightarrow a}\ (\mathrm{a}) \qquad \frac{u[y \circ x] \Rightarrow a}{u[x \circ y] \Rightarrow a}\ (\mathrm{e}) \qquad \frac{u[x \circ x] \Rightarrow a}{u[x] \Rightarrow a}\ (\mathrm{c}) \qquad \frac{|u| \Rightarrow a}{u[x] \Rightarrow a}\ (\mathrm{i})$$

$$\frac{}{0 \Rightarrow a}\ (\mathrm{o}) \qquad (\mathrm{w}) = (\mathrm{i}) + (\mathrm{o})$$

FIGURE 2. The basic metarules

Let $\mathbf{GL}_{\mathrm{a}}$ denote the expansion of **GL** by the rule (a) of Figure 2; the double line in (a) means that the metarule can be applied in both directions. Given a sequent, an associative sequent can be obtained by ignoring the parentheses. It can be shown that a sequent is provable in $\mathbf{GL}_{\mathrm{a}}$ iff the corresponding associative sequent is provable in **FL**. Actually, $\mathbf{GL}_{\mathrm{a}}$ is equivalent to **FL** in the sense of [22].

We refer to the rules of Figure 2 as (global) *associativity*, *exchange*, *contraction*, *integrality* or *right weakening*, and *left weakening*; we also refer to the combination of (i) and (o) as *weakening* and we denote it by (w). We call these metarules *basic*. Note that our basic metarules are different than the ones usually considered. For example exchange is usually written with the metagroupoid words $x, y \in \mathsf{X}$ replaced by $b, c \in \mathsf{F}$, respectively. This means that in its application only formulas can be commuted while commutation of groupoid words is not assumed; we use boldface ($\mathbf{e}$) for this restricted version of the 'global' metarule (e). These rules can also be applied to **FL**, yielding the systems $\mathbf{FL}_{\mathrm{e}}$ and $\mathbf{FL}_{\mathbf{e}}$. It can be shown that these two systems have exactly the same deducibility relation; the same holds for $\mathbf{FL}_{\mathrm{e}}^{\mathrm{f}}$ and $\mathbf{FL}_{\mathbf{e}}^{\mathrm{f}}$. Nevertheless, even though $\mathbf{FL}_{\mathrm{c}}$ and $\mathbf{FL}_{\mathbf{c}}$ have the same deducibility relation, the systems $\mathbf{FL}_{\mathrm{c}}^{\mathrm{f}}$ and $\mathbf{FL}_{\mathbf{c}}^{\mathrm{f}}$ do not. Therefore, it matters whether the metarules refer to groupoid words or formula metavariables. As for the case of $\mathbf{GL}_{\mathrm{a}}$ and **FL**, the systems $\mathbf{GL}_{\mathrm{R} \cup \{\mathrm{a}\}}$ and $\mathbf{FL}_{\mathrm{R}}$ are equivalent, for every set R of metarules. In particular, $\mathbf{GL}_{\mathrm{aecw}}$ is equivalent to Gentzen's original system **LJ** for intuitionistic logic.

Observe that the basic metarules do not involve any connectives; metarules with this property are called *structural*. Basic metarules are special cases of what we will call *simple* structural metarules. Recall the formal definition of a metarule from Section 2.1.3, as well as the special meaning of the sets $\mathsf{F}, \mathsf{A}, \mathsf{X}, \mathsf{U}$. A metagroupoid word (a term of sort S_{X}) t that involves only $\circ$ (and not $\star$) and only metavariables from X (not from A) will be called *simple*. In other words, simple metagroupoid words are groupoid words over the set X of metavariables. For example, $(x \circ y) \circ x$ is such a term, for $x, y \in \mathsf{X}$. Fix metavariables $u \in \mathsf{U}$ and $a \in \mathsf{F}$. If $t_0, t_1, \ldots, t_n$ are simple metagroupoid words and t_0 is *linear* (every metavariable occurs once), the metarule

$$\frac{u[t_1] \Rightarrow a \quad \cdots \quad u[t_n] \Rightarrow a}{u[t_0] \Rightarrow a}\ (\mathrm{r})$$

is called *simple*.

2.1.7. *Decidability and cut elimination.* As mentioned above, a is a theorem of intuitionistic logic iff $\vdash_{\mathbf{GL}_{\mathrm{aecw}}} \varepsilon \Rightarrow a$. Therefore, deciding theoremhood in intuitionistic logic reduces to deciding provability in $\mathbf{GL}_{\mathrm{aecw}}$. Note that with the exception of (a), (e), (c) and (CUT), all the rules reduce the complexity of a sequent as we search upwards for a proof. Rules (a) and (e) rearrange the formulas in the sequent and can be responsible for an infinite loop in the proof search, but with their careful application this effect can be controlled without changing provability. The same can be done, with much more care, for the rule (c) that otherwise increases the complexity as we search upwards; see [34] for details. The rule (CUT) causes considerably more complications as it introduces a new formula. Nevertheless, the system obtained from $\mathbf{GL}_{\mathrm{aecw}}$ by removing (CUT) has the same provable sequents as the original one (this holds only for provability without assumptions) and this is the content of the *cut-elimination property* originally established by Gentzen. Cut elimination has been established by proof-theoretic methods for all the systems $\mathbf{GL}_{\mathrm{R}}$, where R is a set of basic rules, see [34], [14]; it is important that we select the global versions of the simple structural rules, as for example $\mathbf{FL}_{\mathrm{c}}$ enjoys cut elimination, but $\mathbf{FL}_{\mathbf{c}}$ does not. We will present a semantical (algebraic) proof of this fact in Section 4.2.

2.1.8. *The external consequence relation.* If $B \cup \{c\}$ is a set of formulas, and R is a set of metarules, we write $B \vdash^{\mathbf{GL}_{\mathrm{R}}} c$ if $\{\varepsilon \Rightarrow b \mid b \in B\} \vdash_{\mathbf{GL}_{\mathrm{R}}} \varepsilon \Rightarrow c$. Note the difference in the position of $\mathbf{GL}_{\mathrm{R}}$ (superscript or subscript) in the two relations. It is not hard to see that $\vdash^{\mathbf{GL}_{\mathrm{R}}}$ is a consequence relation on $Fm_{\mathcal{L}}$, called the *external consequence relation* of $\vdash_{\mathbf{GL}_{\mathrm{R}}}$. We will show that the consequence relations $\vdash^{\mathbf{GL}_{\mathrm{R}}}$ and $\vdash_{\mathbf{GL}_{\mathrm{R}}}$ are actually equivalent in the sense of [22] (see Section 2.2.5 and Appendix A) thus the former can actually be defined in terms of the latter. Moreover, in the next section we will introduce a Hilbert system and prove that the consequence relation associated with it is equal to $\vdash^{\mathbf{GL}}$.

2.1.9. *Solvability.* Given a *deductive system* $\mathbf{D}$ (for example $\mathbf{GL}$) and a sublanguage $\mathcal{K}$ (for example, $\{\wedge, \vee\}$) of the language $\mathcal{L}$ used in $\mathbf{D}$, we can consider subsystems of $\mathbf{D}$ associated with $\mathcal{K}$. A natural choice for such a subsystem is the set of all the rules of inference of $\mathbf{S}$ that involve connectives only from $\mathcal{K}$ plus possibly a fixed set (for example $\{\backslash, /\}$) of basic connectives. Traditionally, implication is such a basic connective for Hilbert-style systems, since otherwise we would not allow modus ponens. As long as the set of basic connectives contains $\cdot$ and at least one of $\backslash$ or $/$, then this notion of subsystem behaves well for $\mathbf{GL}$. For example, the external consequence relation of such a subsystem is equivalent to the consequence relation of the subsystem. Although, such a definition works well for $\mathbf{FL}$, for a smaller set of basic connectives (just $\{\backslash\}$ or $\{/\}$), it needs some fine tuning for $\mathbf{GL}$, so as to yield the desired results (equivalence with the external relation and the associated Hilbert system) for such a small set of basic connectives.

To motivate the definition of a subsystem of $\mathbf{GL}$, we mention the following. In order to prove the equivalence between the deducibility relation of a subsystem of $\mathbf{GL}$ and its external consequence relation, or the deducibility relation of the corresponding subsystem of the Hilbert system to be introduced, it is necessary to be able to translate (transform) a sequent into a formula. In the presence of $\cdot$ and at least one of $\backslash$ or $/$, we can translate a sequent $x \Rightarrow a$ into the formula $\phi(x)\backslash a$, or $a/\phi(x)$, where $\phi(x)$ is the formula obtained from the groupoid word x by replacing all occurrences of $\circ$ by $\cdot$; this works essentially because the sequents $x \Rightarrow a$, $\varepsilon \Rightarrow \phi(x)\backslash a$ and $\varepsilon \Rightarrow a/\phi(x)$ are mutually derivable in (the $\{\cdot, \backslash, /\}$ subsystem of) $\mathbf{GL}$. If we lack multiplication, the translation is still possible in the case of **FL**; we simply translate the sequent $a_1, a_2, \ldots a_n \Rightarrow a$ to the formula $a_n\backslash \ldots (a_2\backslash(a_1\backslash a))$; note that the order is reversed. Again this works because the sequents $a_1, a_2, \ldots a_n \Rightarrow a$ and $\varepsilon \Rightarrow a_n\backslash \ldots (a_2\backslash(a_1\backslash a))$ are mutually derivable in (the $\{\backslash\}$ subsystem of) **FL**. Unfortunately, because of the lack of associativity, the same is not possible for **GL**. For example, there is no sequent of the form $\varepsilon \Rightarrow f$ that is mutually derivable with the sequent $(a, b), (c, d) \Rightarrow e$ in the multiplication-free subsystem of **GL**. It is, therefore, necessary to identify the actual subsystem of **GL** whose deducibility relation is equivalent to its external consequence relation.

We define the set of *solvable* groupoid words inductively:

(1) Every element in $Q \cup \{\varepsilon\}$ is a solvable groupoid word.
(2) If x is a solvable groupoid word and $a \in Q$, then $x \circ a$ and $a \circ x$ are solvable groupoid words.

For example the groupoid word $(a, (((a, b), c), d))$ is solvable, but $(((a, b), c), (a, b))$ is not. Thus, solvable groupoid words over formulas are exactly the ones that can be translated into a formula, namely they are exactly the right hand-sides of sequents that can be *solved* [by means of the rules (\R) and (/R)] for ε on the left hand side without using multiplication. Note that $a, (((a, b), c), d) \Rightarrow e$ is solvable into (i.e., mutually derivable in the multiplication-free subsystem of **GL** with) $\varepsilon \Rightarrow ((((a\backslash e)/d)/c)/b)/a$. The 'solution' is not unique;

$$\varepsilon \Rightarrow a\backslash((((a\backslash e)/d)/c)/b) \text{ and } \varepsilon \Rightarrow (a\backslash(((a\backslash e)/d)/c))/b$$

are solutions, as well, obtained by a different order of application of the rules (\R) and (/R). Nevertheless, $\varepsilon \Rightarrow (a\backslash(c\backslash((a\backslash e)/d)))/b$ is not a solution, as the only freedom is given after the step $a, b \Rightarrow ((a\backslash e)/d)/c$. Note that the term tree (the tree associated with a term) corresponding to a solvable groupoid word has a distinct shape; there is a main branch such that only leaves stem out of it.

We define the set of *solvable* augmented groupoid words over a set Q inductively:

(1) The constant $\_$ is a solvable augmented groupoid word.
(2) If u is a solvable augmented groupoid word and $a \in Q$, then $u \circ a$ and $a \circ u$ are solvable augmented groupoid words.

For example, the augmented groupoid words $(a, (((\_, a), c), d))$ and $(a, (((b, \_), c), d))$ are solvable, but $(a, (((a, b), \_), d))$ and $(((a, \_), c), (a, b))$ are not. Thus, solvable augmented groupoid words over formulas are exactly the right hand-sides of (augmented) sequents that can be *solved* for $\_$ on the left hand side without using multiplication. Here the solution is unique; for example the unique solution to the augmented sequent $(a, (((b, \_), c), d)) \Rightarrow e$ is the augmented sequent $\_ \Rightarrow b\backslash(((a\backslash e)/d)/c)$. Here we used the term *augmented sequent* for a sequent that allows $\_$ on the left-hand side.

Left solvable (augmented) groupoid words are defined in a similar way, if in (2) we allow only $a \circ x$ ($a \circ u$) to be left solvable. A groupoid word is left solvable iff it is completely associated to the right. For example the groupoid word $(a, (a, (a, a)))$ is left solvable, but $((a, (b, a)), a)$ is not. The augmented groupoid word $(a, (a, (b, \_)))$ is left solvable, but $(a, (a, (\_, a)))$ is not. Note that left solvable (augmented) groupoid words are exactly the ones that are solvable by using only the left division operation \. For example, $(a, (a, (b, a))) \Rightarrow c$ is left-solvable into $\varepsilon \Rightarrow a\backslash(b\backslash(a\backslash(a\backslash c)))$ and $(a, (a, (b, \_))) \Rightarrow c$ is left-solvable into $\_ \Rightarrow b\backslash(a\backslash(a\backslash c))$.

Obviously, every left solvable groupoid word is solvable. Likewise, we define *right solvable* (augmented) groupoid words.

According to the connectives needed for solving a groupoid word, the latter is called *fit* with respect to the corresponding connectives. More precisely, let $\mathcal{K}$ be a sublanguage of $\mathcal{L}$ that contains at least one of the connectives $\backslash$ and $/$. An (augmented) groupoid word x is called *fit* for $\mathcal{K}$ or an *(augmented) $\mathcal{K}$-groupoid word*, if it involves only connectives contained in $\mathcal{K}$ and the following conditions are satisfied:

(1) If $\mathcal{K}$ does not contain $\cdot$, then x is solvable.
(2) If $\mathcal{K}$ contains neither $\cdot$ nor $/$, then x is left solvable.
(3) If $\mathcal{K}$ contains neither $\cdot$ nor $\backslash$, then x is right solvable.

For example, $((((p \wedge q\backslash p, p), q \wedge p), p), q)$ is fit for $\{\backslash, \wedge, /\}$, but not for $\{\backslash, \wedge\}$. Also, $(((p \wedge q\backslash p, p), q \wedge p), (p, q))$ is fit for $\{\backslash, \wedge, \cdot\}$, but not for $\{\backslash, \wedge, /\}$.

We denote by $Q^{\gamma\mathcal{K}}$ and $Q^{\alpha\mathcal{K}}$ the sets of groupoid and augmented groupoid words over Q fit for $\mathcal{K}$. A sequent $x \Rightarrow a$ is called *fit* for $\mathcal{K}$ or a *$\mathcal{K}$-sequent*, if x is a $\mathcal{K}$-groupoid word and a is a $\mathcal{K}$-formula.

As explained above a sequent calculus can be specified by a set of metarules together with a way to obtain their instances; to define the subsystems of **GL**, we restrict the instances of the metarules of **GL**. If $\mathcal{K}$ is a sublanguage of $\mathcal{L}$ that contains at least one of the connectives $\backslash$ and $/$, then the *$\mathcal{K}$-subsystem* $\mathcal{K}$**GL** of **GL** is specified by the metarules of **GL** that do not involve connectives outside of $\mathcal{K}$; the allowed instances of those metarules are ones in which all the resulting sequents are fit for $\mathcal{K}$. For example, the instance

$$\frac{(c, d), (a, f) \ \Rightarrow\ e}{(c, d), (a \wedge b, f) \ \Rightarrow\ e}$$

of the rule $(\wedge \mathrm{L}\ell)$ is not included in $\{\wedge, \backslash, /\}$**GL**, because the sequents involved are not solvable and multiplication is not included in the language.

The consequence relations $\vdash^{\mathcal{K}\mathbf{GL}}$ and $\vdash_{\mathcal{K}\mathbf{GL}}$, for different choices of $\mathcal{K}$, are defined in the obvious way. Recall that if R is a set of metarules, then $\mathbf{GL}_{\mathrm{R}}$ denotes the system obtained from **GL** by adding the set R. If $\mathcal{K}$ is a sublanguage of $\mathcal{L}$ that contains $\backslash$, the system $\mathcal{K}\mathbf{GL}_{\mathrm{R}}$, is obtained by adding to the rules of $\mathcal{K}$**GL** all rules that are instances of the metarules in R so that all the resulting sequents are fit for $\mathcal{K}$.

In the case of **FL** the $\mathcal{K}$-subsystem $\mathcal{K}$**FL** does not put any restrictions on the instances of the metarules, since in all instances the resulting sequents are fit for a sublanguage $\mathcal{K}$ that contains at least one of the connectives $\backslash$ and $/$.

2.2. **Hilbert systems.** In this section we will define a Hilbert-style system **HL** with deducibility relation equivalent to the relation $\vdash_{\mathbf{GL}}$. The system contains (infinitely) many rules (schemes) of inference, but it enjoys the *strong separation property* (with respect to $\{\backslash\}$), which states that for every proof only the rules that involve the connectives in the assumptions and the conclusion (and possibly $\backslash$) are needed in the derivation. In Section 4.5, we present extensions of **HL** (to the associative, commutative and other cases) which also enjoy the strong separation property; see also Lemma 4.20. We first present simplified versions $\mathbf{HL}'_{\mathbf{ae}}$ (Figure 3) and $\mathbf{HL}'_{\mathbf{a}}$ (Figure 4) of **HL** that correspond to **FL** and $\mathbf{FL}_{\mathbf{e}}$, but do not have the strong separation property.

(id)	$\alpha \to \alpha$	(identity)
(pf)	$(\alpha \to \beta) \to [(\delta \to \alpha) \to (\delta \to \beta)]$	(prefixing)
(per)	$[\alpha \to (\beta \to \delta)] \to [\beta \to (\alpha \to \delta)]$	(permutation)
$(\cdot\wedge)$	$[(\alpha \wedge 1)(\beta \wedge 1)] \to (\alpha \wedge \beta)$	(fusion conjunction)
$(\wedge \to)$	$(\alpha \wedge \beta) \to \alpha$	(conjunction implication)
$(\wedge \to)$	$(\alpha \wedge \beta) \to \beta$	(conjunction implication)
$(\to \wedge)$	$[(\alpha \to \beta) \wedge (\alpha \to \delta)] \to [\alpha \to (\beta \wedge \delta)]$	(implication conjunction)
$(\to \vee)$	$\alpha \to (\alpha \vee \beta)$	(implication disjunction)
$(\to \vee)$	$\beta \to (\alpha \vee \beta)$	(implication disjunction)
$(\vee \to)$	$[(\alpha \to \delta) \wedge (\beta \to \delta)] \to [(\alpha \vee \beta) \to \delta]$	(disjunction implication)
$(\to \cdot)$	$\beta \to (\alpha \to \alpha\beta)$	(implication fusion)
$(\cdot \to)$	$[\beta \to (\alpha \to \delta)] \to (\alpha\beta \to \delta)$	(fusion implication)
(1)	1	(unit)
$(1\to)$	$1 \to (\alpha \to \alpha)$	(unit implication)

$$\frac{\alpha \quad \alpha \to \beta}{\beta}\ (\mathrm{mp}) \qquad\qquad \frac{\alpha}{\alpha \wedge 1}\ (\mathrm{adj}_u)$$

(modus ponens) (adjunction unit)

FIGURE 3. The system $\mathbf{HL'_{ae}}$.

2.2.1. *The Hilbert system* **sHL**. The Hilbert-style system **sHL** is an equivalent variant of **HL** with finitely many rules. It enjoys the strong separation property for signatures that contain $\wedge$ whenever they contain $\vee$ (Corollary 2.4), but does not have the property for other signatures. We introduce the systems $\mathbf{HL'_{ae}}$, $\mathbf{HL'_{a}}$ and **sHL** before **HL**, as the latter is more complicated.

The system **sHL** is specified by the metarules of Figure 5. To define (Hilbert-style) metarules formally, as before let F be the set (disjoint from the set P of propositional variables) of formula metavariables and let A be the set of all $\mathcal{L}$-formulas over F. A Hilbert-style metarule is a pair (S, s), where $S \cup \{s\}$ is a subset of A. An instance of a metarule is obtained by replacing elements of F by formulas in $Fm_{\mathcal{L}}(P)$.

If (r) is a simple structural metarule involving the simple metagroupoid words $t_0, t_1, \ldots, t_n$ (see Section 2.1.6) then we define the axiom $t_0^{\mathbf{Fm}_{\mathcal{L}}} \backslash (t_1^{\mathbf{Fm}_{\mathcal{L}}} \vee \cdots \vee t_n^{\mathbf{Fm}_{\mathcal{L}}})$; here $t^{\mathbf{Fm}_{\mathcal{L}}}$ denotes the formula resulting from t by replacing $\circ$ by $\cdot$. If R is a set of simple structural metarules, then $\mathbf{sHL}_{\mathrm{R}}$ denotes the expansion of **sHL** by the axioms corresponding to R.

Given a sequent $x \Rightarrow b$, we define the formula $\phi(x \Rightarrow b) = \phi(x)\backslash b$, where $\phi(x)$ is the formula obtained by replacing $\circ$ by $\cdot$ in x. If S is a set of sequents we define $\phi[S] = \{\phi(s) \mid s \in S\}$. If $a \in Fm_{\mathcal{L}}$, we define the sequent $s(a) = (\varepsilon \Rightarrow a)$ and if B is a set of formulas, we define $s[B] = \{s(b) \mid b \in B\}$.

Theorem 2.1. *Let $S \cup \{s\}$ be a set of sequents, let $B \cup \{c\}$ be a set of formulas and let R be a set of simple structural rules. Then*

(1) $S \vdash_{\mathbf{GL}_{\mathrm{R}}} s$ *iff* $\phi[S] \vdash_{\mathbf{sHL}_{\mathrm{R}}} \phi(s)$.
(2) $B \vdash_{\mathbf{sHL}_{\mathrm{R}}} c$ *iff* $s[B] \vdash_{\mathbf{GL}_{\mathrm{R}}} s(c)$.

(id_ℓ)	$\alpha\backslash\alpha$	(identity)
(pf_ℓ)	$(\alpha\backslash\beta)\backslash[(\delta\backslash\alpha)\backslash(\delta\backslash\beta)]$	(prefixing)
$(\mathrm{as}_{\ell\ell})$	$\alpha\backslash[(\beta/\alpha)\backslash\beta]$	(assertion)
(a)	$[(\beta\backslash\delta)/\alpha]\backslash[\beta\backslash(\delta/\alpha)]$	(associativity)
$(\cdot\backslash/)$	$[(\beta(\beta\backslash\alpha))/\beta]\backslash(\alpha/\beta)$	(fusion divisions)
$(\cdot\wedge)$	$[(\alpha\wedge 1)(\beta\wedge 1)]\backslash(\alpha\wedge\beta)$	(fusion conjunction)
$(\wedge\backslash)$	$(\alpha\wedge\beta)\backslash\alpha$	(conjunction division)
$(\wedge\backslash)$	$(\alpha\wedge\beta)\backslash\beta$	(conjunction division)
$(\backslash\wedge)$	$[(\alpha\backslash\beta)\wedge(\alpha\backslash\delta)]\backslash[\alpha\backslash(\beta\wedge\delta)]$	(division conjunction)
$(\backslash\vee)$	$\alpha\backslash(\alpha\vee\beta)$	(division disjunction)
$(\backslash\vee)$	$\beta\backslash(\alpha\vee\beta)$	(division disjunction)
$(\vee\backslash)$	$[(\alpha\backslash\delta)\wedge(\beta\backslash\delta)]\backslash[(\alpha\vee\beta)\backslash\delta]$	(disjunction division)
$(\backslash\cdot)$	$\beta\backslash(\alpha\backslash\alpha\beta)$	(division fusion)
$(\cdot\backslash)$	$[\beta\backslash(\alpha\backslash\delta)]\backslash(\alpha\beta\backslash\delta)$	(fusion division)
(1)	1	(unit)
$(1\backslash)$	$1\backslash(\alpha\backslash\alpha)$	(unit division)
$(\backslash 1)$	$\alpha\backslash(1\backslash\alpha)$	(division unit)

$$\frac{\alpha \quad \alpha\backslash\beta}{\beta}\ (\mathrm{mp}_\ell) \qquad \frac{\alpha}{\alpha\wedge 1}\ (\mathrm{adj}_u) \qquad \frac{\alpha}{\beta\backslash\alpha\beta}\ (\mathrm{pn}_\ell) \quad \frac{\alpha}{\beta\alpha/\beta}\ (\mathrm{pn}_r)$$

(modus ponens) (adjunction unit) (product normality)

FIGURE 4. The system $\mathbf{HL}'_{\mathbf{a}}$.

$$\frac{}{a\backslash a}\ (\mathrm{I}_\ell) \quad \frac{a \quad a\backslash b}{b}\ (\mathrm{MP}_\ell) \quad \frac{a\backslash b}{(c\backslash a)\backslash(c\backslash b)}\ (\mathrm{Rd}\backslash) \quad \frac{a\backslash b}{(b\backslash c)\backslash(a\backslash c)}\ (\mathrm{Rn}\backslash) \quad \frac{a}{(a\backslash b)\backslash b}\ (\mathrm{N}_\ell)$$

$$\frac{}{a\backslash[(b/a)\backslash b]}\ (\mathrm{As}_{\ell\ell}) \qquad \frac{a\backslash(b\backslash c)}{b\backslash(c/a)}\ (\mathrm{RA}r_\ell) \qquad \frac{b\backslash a}{a/b}\ (\mathrm{RC}_r)$$

$$\frac{}{(a\wedge b)\backslash a}\ (\mathrm{ME}\ell) \quad \frac{}{(a\wedge b)\backslash b}\ (\mathrm{ME}r) \quad \frac{a \quad b}{a\wedge b}\ (\mathrm{RM}) \quad \frac{}{[(a\backslash b)\wedge(a\backslash c)]\backslash[a\backslash(b\wedge c)]}\ (\mathrm{M}\backslash)$$

$$\frac{}{a\backslash(a\vee b)}\ (\mathrm{JI}\ell) \qquad \frac{}{b\backslash(a\vee b)}\ (\mathrm{JI}r)$$

$$\frac{}{[(a\backslash c)\wedge(b\backslash c)]\backslash[(a\vee b)\backslash c]}\ (\mathrm{J}\backslash) \qquad \frac{}{[(c/a)\wedge(c/b)]\backslash[c/(a\vee b)]}\ (\mathrm{J}/)$$

$$\frac{}{b\backslash(a\backslash ab)}\ (\mathrm{PI}) \quad \frac{b\backslash(a\backslash c)}{ab\backslash c}\ (\mathrm{RPI}) \quad \frac{}{1}\ (1) \quad \frac{}{1\backslash(a\backslash a)}\ (\mathrm{I}1\ell) \quad \frac{}{a\backslash(1\backslash a)}\ (\mathrm{I}1r)$$

FIGURE 5. The system **sHL**

(3) $s(\phi(s)) \dashv\vdash_{\mathbf{GL}_{\mathrm{R}}} s$.
(4) $\phi(s(c)) \dashv\vdash_{\mathbf{sHL}_{\mathrm{R}}} c$.

Theorem 2.2. *The strong separation property holds for the system* **sHL***, provided that if the language contains* $\vee$*, it also contains* $\wedge$*.*

The proofs of Theorems 2.1 and 2.2 are similar to the proofs of Theorem 2.3 (see Appendix A) and Corollary 4.19, and are left to the reader. A result related to Theorem 2.1 on the (weak) separation property was shown in [37] for a Hilbert system equivalent to $\mathbf{FL_w}$.

We mention that the rules (MP_ℓ) and (N_ℓ) are in the current forms because of the presence of 1. The same applies to (RC_r). ($\mathrm{As}_{\ell\ell}$) is a non-commutative version of the assertion axiom. Non-commutativity dictates the existence of the rules (N_ℓ) and (RC_r). ($\mathrm{RA}r_\ell$) is needed because of the absense of associativity. (Rd$\backslash$) needs to be stated in a non-axiom form because the corresponding axiom of prefixing implies associativity.

2.2.2. *Definable connectives.* Since we want the strong separation property to hold (see Section 4.5) for the Hilbert-style system **HL** we need enough rules for each connective. A main difficulty is presented when a set of connectives under consideration contains $\vee$, but not $\wedge$. In order for the strong separation property to work for this case we need an infinite set of rules organized in two metarules (RJ$\backslash$) and (RJ/) (see Figure 6). To express these metarules, we need to introduce a definable connective $\rightsquigarrow_{\mathcal{K}}$, for each set of connectives $\mathcal{K}$. We will introduce the necessary notation for the definition of **HL** in this section.

Recall from the discussion on the subsystems of **GL** that we have a choice on representing the sequent $x \Rightarrow a$ by either one of the formulas $\phi(x)\backslash a$ and $a/\phi(x)$. In case that we have exactly one of the division connectives in our sublanguage $\mathcal{K}$ together with multiplication, then there is no choice, but if we have both connectives, then we need to be consistent which of the two formulas to consider. Moreover, if x is a solvable groupoid word there are multiple 'solutions' involving the division operations in addition to the two formulas mentioned above. Therefore, we fix a representation $\phi_{\mathcal{K}}(x \Rightarrow b)$ for the sequent $x \Rightarrow b$, relative to the different sublanguages $\mathcal{K}$, and this will be exactly what we will define $x \rightsquigarrow_{\mathcal{K}} b$ as follows.

Let Q be the set of all $\mathcal{L}$ formulas over an alphabet that can be either the set P of propositional variables, or the set F of formula metavariables; so $Q = Fm_{\mathcal{L}}(\mathsf{P})$ or $Q = Fm_{\mathcal{L}}(\mathsf{F})$ (we will need both cases for discussing rules and metarules). First we define the depth $d(x)$ of a groupoid word $x \in Q^\gamma$ by induction:

- $d(\varepsilon) = -1$, $d(a) = 0$, for $a \in Q$, and
- $d(x \circ y) = 1 + max\{d(x), d(y)\}$, for $x, y \in Q^\gamma$.

Now, given a sublanguage $\mathcal{K}$ of $\mathcal{L}$ that contains $\backslash$, and a (meta)sequent $x \Rightarrow b$ ($x \in Q^\gamma$ and $b \in Q$) fit for $\mathcal{K}$, we define $x \rightsquigarrow_{\mathcal{K}} b$ as follows. Here we assume that if $x \Rightarrow b$ is a metasequent, then x is simple.

If $\mathcal{K}$ contains multiplication, then $x \rightsquigarrow_{\mathcal{K}} b = \phi(x)\backslash b$, where $\phi(x)$ is the formula obtained from the groupoid word x by replacing all occurrences of $\circ$ by $\cdot$. For example, $((a,(b,c)),((d,e),f)) \rightsquigarrow_{\{\backslash,\cdot,\wedge\}} g = ((a(bc))((de)f))\backslash g$.

If $\mathcal{K}$ does not contain multiplication (and hence x is solvable), then $x \rightsquigarrow_{\mathcal{K}} b$ is defined by induction on x:

- $\varepsilon \rightsquigarrow_{\mathcal{K}} b = b$;
- for $a \in Q$, $a \rightsquigarrow_{\mathcal{K}} b = a\backslash b$;
- for $x, y \in Q^\gamma$, $(x \circ y) \rightsquigarrow_{\mathcal{K}} b = y \rightsquigarrow_{\mathcal{K}} (\phi(x)\backslash b)$ when $d(x) \leq d(y)$, and $(x \circ y) \rightsquigarrow_{\mathcal{K}} b = x \rightsquigarrow_{\mathcal{K}} (b/\phi(y))$ otherwise.

Note that in the last case at least one of x, y is in Q. By this definition we give preference to $\backslash$ relative to $/$. For example, $(a, ((d, e), f)) \rightsquigarrow_{\{\backslash,/,\wedge\}} g = e\backslash(d\backslash((a\backslash g)/f))$, not $(d\backslash((a\backslash g)/f))/e$.

Note that $x \rightsquigarrow_{\mathcal{K}} b$ is always a 'solution' of the sequent $x \Rightarrow b$. Also, the outermost element of Q in $x \rightsquigarrow_{\mathcal{K}} b$ is the rightmost of all occurrences of subformulas of x of maximum depth. Moreover, if $\mathcal{K}$ contains neither multiplication nor $/$ (and hence x is left solvable), then $x \rightsquigarrow_{\mathcal{K}} b$ contains neither multiplication nor $/$. In general, $x \rightsquigarrow_{\mathcal{K}} b$ is always a $\mathcal{K}$-formula.

2.2.3. *Hilbert-style metarules.* In order to introduce a new type of metarules, including (RJ$\backslash$) and (RJ$/$), we need to modify the definition of metarules for a Hilbert system. As before, let F be the set (disjoint from the set P of propositional variables) of formula metavariables and let A be the set of all $\mathcal{L}$-formulas over F. Also, let A' be the set A together with all formal expressions of the form $x \rightsquigarrow_{\mathcal{M}} b$, where x and $\rightsquigarrow_{\mathcal{M}}$ are new symbols and $b \in \mathsf{A}$. A Hilbert-style metarule is a pair (S, s), where $S \cup \{s\}$ is a subset of A'. An instance of a metarule is obtained by replacing elements of F by formulas in $Fm_{\mathcal{L}}(\mathsf{P})$, and all expressions of the form $z \rightsquigarrow_{\mathcal{M}} b$ by the formulas obtained by replacing $\mathcal{M}$ by a sublanguage of $\mathcal{L}$ that contains $\backslash$, and z by a solvable element of $(Fm_{\mathcal{L}}(\mathsf{P}))^{\gamma}$ that is fit for $\mathcal{M}$.

2.2.4. *The Hilbert system* **HL**. The Hilbert system **HL** consists of the following metarules, where a, b, c denote formulas; for the rules (RJ$\backslash$) and (RJ$/$), $\mathcal{M}$ ranges over all sublanguages of $\mathcal{L}$ that contain $\backslash$, and z ranges over all solvable groupoid words over formulas fit for $\mathcal{M}$.

$$\frac{}{a\backslash a}\ (\mathrm{I}_\ell) \qquad \frac{a \quad a\backslash b}{b}\ (\mathrm{MP}_\ell) \qquad \frac{a\backslash b}{(c\backslash a)\backslash(c\backslash b)}\ (\mathrm{Rd}\backslash) \qquad \frac{a\backslash b}{(b\backslash c)\backslash(a\backslash c)}\ (\mathrm{Rn}\backslash) \qquad \frac{a}{(a\backslash b)\backslash b}\ (\mathrm{N}_\ell)$$

$$\frac{}{a\backslash[(b/a)\backslash b]}\ (\mathrm{As}_{\ell\ell}) \qquad \frac{a\backslash(b\backslash c)}{b\backslash(c/a)}\ (\mathrm{RA}r_\ell) \qquad \frac{b\backslash a}{a/b}\ (\mathrm{RC}_r)$$

$$\frac{}{(a\wedge b)\backslash a}\ (\mathrm{ME}\ell) \qquad \frac{}{(a\wedge b)\backslash b}\ (\mathrm{ME}r) \qquad \frac{a \quad b}{a\wedge b}\ (\mathrm{RM}) \qquad \frac{}{[(a\backslash b)\wedge(a\backslash c)]\backslash[a\backslash(b\wedge c)]}\ (\mathrm{M}\backslash)$$

$$\frac{}{a\backslash(a\vee b)}\ (\mathrm{JI}\ell) \qquad \frac{}{b\backslash(a\vee b)}\ (\mathrm{JI}r)$$

$$\frac{z \rightsquigarrow_{\mathcal{M}} (a\backslash c) \quad z \rightsquigarrow_{\mathcal{M}} (b\backslash c)}{z \rightsquigarrow_{\mathcal{M}} [(a\vee b)\backslash c]}\ (\mathrm{RJ}\backslash) \qquad \frac{z \rightsquigarrow_{\mathcal{M}} (c/a) \quad z \rightsquigarrow_{\mathcal{M}} (c/b)}{z \rightsquigarrow_{\mathcal{M}} [c/(a\vee b)]}\ (\mathrm{RJ}/)$$

$$\frac{}{b\backslash(a\backslash ab)}\ (\mathrm{PI}) \qquad \frac{b\backslash(a\backslash c)}{ab\backslash c}\ (\mathrm{RPI}) \qquad \frac{}{1}\ (1) \qquad \frac{}{1\backslash(a\backslash a)}\ (\mathrm{I}1\ell) \qquad \frac{}{a\backslash(1\backslash a)}\ (\mathrm{I}1r)$$

FIGURE 6. The system **HL**

The de Morgan style axioms (J$\backslash$) and (J$/$) of **sHL** are replaced in **HL** by the rules (RJ$\backslash$) and (RJ$/$), which are important to the proof of the strong separation property (Theorem 2.3).

Also, note that for every sublanguage $\mathcal{K}$ of $\mathcal{L}$ that contains the connective $\backslash$ and for every formula a, $a \dashv\vdash_{\mathcal{K}-\mathbf{HL}} (a\backslash a)\backslash a$; (N_ℓ) justifies one direction, and (I_ℓ) and (MP_ℓ) justify the other.

It is possible to replace some of the rules by the following

$$\frac{c}{ab\backslash a(cb)}\ (\mathrm{N}_1) \qquad \frac{c}{a\backslash[(ab)c/b]}\ (\mathrm{N}_2) \qquad \frac{c}{[a\backslash(ab)c]/b}\ (\mathrm{N}_3)$$

However, this simplification destroys the strong separation property, as multiplication is needed for these rules.

Given a sublanguage $\mathcal{K}$ of $\mathcal{L}$ that contains the connective $\backslash$, the the $\mathcal{K}$-*subsystem* $\mathcal{K}$**HL** of **HL** is defined to be the Hilbert system containing only the rules of **HL** that involve connectives over $\mathcal{K}$.

The notion of a *(meta)proof with assumptions* in a Hilbert system is similar to that in a sequent calculus. The only difference is that instead of (meta)sequents, we have (meta)formulas. If a formula c is provable in $\mathcal{K}$**HL** from assumptions B, then we write $B \vdash_{\mathcal{K}\mathbf{HL}} v$.

A simple structural metarule (r) is called *fit* for $\mathcal{K}$, if t_i is fit for $\mathcal{K}$ for every i. If (r) is fit for $\mathcal{K}$, then we define the Hilbert rule (for a fixed $b \in \mathsf{F}$)

$$\frac{t_1 \rightsquigarrow_{\mathcal{K}} b \quad \dots \quad t_n \rightsquigarrow_{\mathcal{K}} b}{t_0 \rightsquigarrow_{\mathcal{K}} b}\ \mathrm{h(r)}$$

If R is a set of simple structural metarules, then $\mathcal{K}\mathbf{HL}_{\mathrm{R}}$ denotes the extension of **HL** by the rules h(r).

2.2.5. *Equivalence.* Given a sublanguage $\mathcal{K}$ of $\mathcal{L}$ that contains the connective $\backslash$ and a sequent $x \Rightarrow b$ fit for $\mathcal{K}$, we define the formula $\phi_{\mathcal{K}}(x \Rightarrow b) = x \rightsquigarrow_{\mathcal{K}} b$. If S is a set of sequents we set $\phi_{\mathcal{K}}[S] = \{\phi_{\mathcal{K}}(s) \mid s \in S\}$.

Recall that if $a \in Fm_{\mathcal{L}}$, we define the sequent $s(a) = (\varepsilon \Rightarrow a)$ and if B is a set of formulas, we define $s[B] = \{s(b) \mid b \in B\}$.

Theorem 2.3. *Let $S \cup \{s\}$ be a set of sequents, $\mathcal{K}$ a sublanguage of $\mathcal{L}$ that contains $\backslash$, $B \cup \{c\}$ a set of $\mathcal{K}$-formulas and* R *a set of simple structural metarules fit for $\mathcal{K}$. Then*

1. $S \vdash_{\mathcal{K}\mathbf{GL}_{\mathrm{R}}} s$ *iff* $\phi_{\mathcal{K}}[S] \vdash_{\mathcal{K}\mathbf{HL}_{\mathrm{R}}} \phi_{\mathcal{K}}(s)$.
2. $B \vdash_{\mathcal{K}\mathbf{HL}_{\mathrm{R}}} c$ *iff* $s[B] \vdash_{\mathcal{K}\mathbf{GL}_{\mathrm{R}}} s(c)$.
3. $s(\phi_{\mathcal{K}}(s)) \dashv\vdash_{\mathcal{K}\mathbf{GL}_{\mathrm{R}}} s$.
4. $\phi_{\mathcal{K}}(s(c)) \dashv\vdash_{\mathcal{K}\mathbf{HL}_{\mathrm{R}}} c$.

In the terminology of [22], the theorem states that the two consequence relations are equivalent under the above transformations.

As the proof of Theorem 2.3 is long and would interrupt the flow of the paper we include it, together with the necessary lemmas, in Appendix A (see Corollary A.5).

Corollary 2.4. *The results of Theorem 2.3 hold also for* **sHL** *in place of* **HL**, *for signatures $\mathcal{K}$ that contain $\wedge$ whenever they contain $\vee$.*

Proof. It suffices to show that, for signatures that contain $\wedge$ whenever they contain $\vee$, the rules (RJ$\backslash$) and (RJ/) can be replaced by the axioms (J$\backslash$) and (J/).

It is clear that in the presence of $\wedge$ in the signature the rules imply the axioms, by instantiating $z = (a\backslash c) \wedge (b\backslash c)$. For the converse, starting from the axioms and using repeatedly (Rd$\backslash$) and its companion version (Rd/), which is shown to be derivable (Lemma A.2 in Appendix A), we can obtain

$$\{z \rightsquigarrow_{\mathcal{K}} [(a\backslash c) \wedge (b\backslash c)]\}\backslash\{z \rightsquigarrow_{\mathcal{K}} [(a \vee b)\backslash c]\}.$$

Note that
$$\{[z \rightsquigarrow_{\mathcal{K}} (a\backslash c)] \wedge [z \rightsquigarrow_{\mathcal{K}} (b\backslash c)]\}\backslash\{z \rightsquigarrow_{\mathcal{K}} [(a\backslash c) \wedge (b\backslash c)]\}$$
is provable by using (RM$\rightsquigarrow_{\mathcal{K}}$), (ME$\ell$), (ME$r$) and (MP$_\ell$), so by (T$_\ell$) we get
$$\{[z \rightsquigarrow_{\mathcal{K}} (a\backslash c)] \wedge [z \rightsquigarrow_{\mathcal{K}} (b\backslash c)]\}\backslash\{z \rightsquigarrow_{\mathcal{K}} [(a \vee b)\backslash c]\}.$$
Rules (RM$\rightsquigarrow_{\mathcal{K}}$) and (T$_\ell$) are derived in Lemma A.2 in Appendix A. By a combination of (MEℓ), (MEr) and (MP$_\ell$) we obtain (RJ$\backslash$). □

Theorem 2.5. *There is no Hilbert-style system with finitely many rule schemes that is equivalent to* **HL** *and has the strong separation property.*

Proof. By way of contradiction assume that there is a Hilbert-style system **H** with finitely many rule schemes that is equivalent to **HL** and has the strong separation property. Then the same holds for the extension $\mathbf{H}_i$ of **H** by the axiom $a\backslash(b\backslash a)$. Put differently, the consequence relation $\vdash_{\mathbf{H}_i}$ is finitely axiomatizable. In particular, the $\{\backslash, \vee\}$-fragment of $\vdash_{\mathbf{H}_i}$ is finitely axiomatizable. However, Corollary 3.6 of [42] shows that this fragment is not finitely axiomatizable. □

It is obvious that in **HL** the role of $\backslash$ is different than that of $/$. Nevertheless, if we interchange the roles of the two division operation, by interchanging all occurrences of $a\backslash b$ with b/a, then we obtain rules that are derivable in **HL**; these rules are called *opposite*. Recall that a rule is called *derivable* if the deducibility relation of the system expanded by the rule is the same as the original one. If we include these opposite rules (and axioms) we obtain an equivalent Hilbert system that is symmetric with respect to the two division operations. All the statements, like Theorem 2.3, that we have made for **HL** and $\backslash$ hold for the new system with respect to either of the division operations.

2.3. **Algebraic presentations of sequent systems.** Sequent systems that do not contain $\circ$ and do not allow an empty left hand side (in other words the left-hand side is always a single formula) are called *algebraic*. Usually, we write $\leq$ for $\Rightarrow$ and we refer to sequents as *inequalities*. These systems have the advantage that groupoid words can be avoided and they deal only with formulas, so the syntax is much easier to handle.

In the following we introduce the algebraic systems **PL** (Figure 7) and **ML** (Figure 8) considered in [27] and [26], respectively. Both of them are equivalent to **GL** and enjoy the cut elimination property. The cut elimination property was established semantically for **PL** in [27] and using proof theoretic methods for **ML** in [26]. For more on these systems, see [17]. Computation in **PL** closely parallels that of **GL**. On the other hand, **ML** has two bidirectional rules and is reminiscent of display calculi. The system **ML** is very convenient for algebraic calculations.

If s is a sequent, we denote by $s^\bullet$ the sequent (inequality) resulting from s by replacing $\circ$ by $\cdot$ and ε by 1. Also, we denote by s° the sequent resulting from s by replacing all external occurrences of $\cdot$ in the left-hand side of S by $\circ$; here an occurrence of $\cdot$ in a formula is called external if all connectives in the formula tree above the particular occurrence of $\cdot$ are also $\cdot$. For example, we replace the inequality $(p \cdot q) \cdot [(p \cdot q) \vee r)] \leq p \cdot q$ by the sequent $(p \circ q) \circ [(p \cdot q) \vee r)] \Rightarrow p \cdot q$.

Theorem 2.6. *The systems* **GL** *and* **PL** *are equivalent. In particular, for every set of sequents $S \cup \{s\}$,*

- $S \vdash_{\mathbf{GL}} s$ *iff* $S^\bullet \vdash_{\mathbf{PL}} s^\bullet$.

$$\frac{a \le b \quad u[b] \le c}{u[a] \le c}\ (\mathrm{cut}) \qquad \frac{}{a \le a}\ (\mathrm{id}) \qquad \frac{a \le b \quad c \le d}{ac \le bd}\ (\cdot\mathrm{r})$$

$$\frac{a \le b \quad u[c] \le d}{u[a(b\backslash c)] \le d}\ (\backslash\mathrm{l}) \qquad \frac{ab \le c}{b \le a\backslash c}\ (\backslash\mathrm{r})$$

$$\frac{a \le b \quad u[c] \le d}{u[(c/b)a] \le d}\ (/\mathrm{l}) \qquad \frac{ab \le c}{a \le c/b}\ (/\mathrm{r})$$

$$\frac{u[a] \le c}{u[a \wedge b] \le c}\ (\wedge\mathrm{l}\ell) \qquad \frac{u[b] \le c}{u[a \wedge b] \le c}\ (\wedge\mathrm{l}r) \qquad \frac{a \le b \quad a \le c}{a \le b \wedge c}\ (\wedge\mathrm{r})$$

$$\frac{u[a] \le c \quad u[b] \le c}{u[a \vee b] \le c}\ (\vee\mathrm{l}) \qquad \frac{a \le b}{a \le b \vee c}\ (\vee\mathrm{r}\ell) \qquad \frac{a \le c}{a \le b \vee c}\ (\vee\mathrm{r}r)$$

$$\frac{|u| \le a}{u[1] \le a}\ (1\mathrm{l}) \qquad \frac{a \le b}{a \le 1b}\ (1\mathrm{r}\ell) \qquad \frac{a \le b}{a \le b1}\ (1\mathrm{r}r)$$

FIGURE 7. The system **PL**.

- *$s \dashv\vdash_{\mathbf{PL}} s^{\circ\bullet}$ (actually, $s = s^{\circ\bullet}$).*

The same holds for the systems involving fragments of the language that contain multiplication and 1*, where the rule instance are restricted appropriately.*

Proof. If we are given a proof of s in **GL** from assumptions S, we replace every sequent t by the inequality $t^\bullet$ and contract all applications of ($\cdot$L). Also, the axiom (1R) by an instance of (id). The resulting proof figure is obviously a proof in **PL**.

Conversely, given a a proof of s in **PL** from assumptions S, we first replace every inequality t by t° in the proof. The resulting proof figure might not be a proof in **GL**. For example, if an application of the rule (\r) in the original proof has assumption $(ab)c \le d$ and conclusion $c \le (ab)\backslash d$, then the translation will yield a rule step with assumption $(a \circ b) \circ c \Rightarrow d$ and conclusion $c \Rightarrow (ab)\backslash d$; this is not an instance of the rule (\R), but it is the combination of ($\cdot$L), which yields $(a \cdot b) \circ c \Rightarrow d$, and of (\R). Therefore, in the proof figure, we insert applications of ($\cdot$L) before applications of the rules (\R) and (/R), so that x (in these rules) becomes a formula. Likewise, for (1rℓ) and (1rℓ), we use (1R) and ($\cdot$R). Also, for the axioms in the original proof we provide proofs in **GL** from axioms of the form (Id) applied to formulas. It is not difficult to verify that the resulting proof figure is a proof of s° in **GL** from S°.

Finally, by using (cut) it is easy to see that $s \dashv\vdash_{\mathbf{GL}} s^{\bullet\circ}$. □

Moreover, the following relation holds between the cut-free systems: $\vdash_{\mathbf{GL}^{\mathrm{f}}} s$ iff $\vdash_{\mathbf{PL}^{\mathrm{f}}} s^\bullet$. The idea is, by moving from bottom upward, in every occurrence of (\r) and (/r) to replace ab with $a \circ b$ and propagate this change all the way up in the proof. Moreover, we replace every occurrence of (\l) by an application of (\L) to

$$\frac{a \le b \quad b \le c}{a \le c}\ (\mathrm{tr}) \qquad \frac{}{a \le a}\ (\mathrm{id}) \qquad \frac{a \le b \quad c \le d}{ac \le bd}\ (\cdot)$$

$$\frac{a \le b \quad c \le d}{b\backslash c \le a\backslash d}\ (\backslash\mathrm{o}) \qquad \frac{ab \le c}{b \le a\backslash c}\ (\backslash\mathrm{res})$$

$$\frac{a \le b \quad c \le d}{c/b \le d/a}\ (/\mathrm{o}) \qquad \frac{ab \le c}{a \le c/b}\ (/\mathrm{res})$$

$$\frac{a \le c}{a \wedge b \le c}\ (\wedge\mathrm{lt}\ell) \qquad \frac{b \le c}{a \wedge b \le c}\ (\wedge\mathrm{lt}r) \qquad \frac{a \le b \quad a \le c}{a \le b \wedge c}\ (\wedge\mathrm{rt})$$

$$\frac{a \le c \quad b \le c}{a \vee b \le c}\ (\vee\mathrm{lt}) \qquad \frac{a \le b}{a \le b \vee c}\ (\vee\mathrm{rt}\ell) \qquad \frac{a \le c}{a \le b \vee c}\ (\vee\mathrm{rt}r)$$

$$\frac{a \le c \quad b \le 1}{ab \le c}\ (1\mathrm{r}\ell) \qquad \frac{a \le 1 \quad b \le c}{ab \le c}\ (1\mathrm{r}r)$$

$$\frac{a \le b \quad 1 \le c}{a \le bc}\ (1\mathrm{r}\ell) \qquad \frac{1 \le b \quad a \le c}{a \le bc}\ (1\mathrm{r}r)$$

FIGURE 8. The system **ML**.

get $u[a \circ (b\backslash c)] \Rightarrow d$ and an application of $(\cdot\mathrm{L})$ to get $u[a \cdot (b\backslash c)] \Rightarrow d$; likewise, we modify the occurrences of (/l). Similarly, every application of $(\cdot\mathrm{r})$ is replaced by an application of $(\cdot\mathrm{R})$, followed by an application of $(\cdot\mathrm{l})$. Finally, we replace every occurrence of (1l) by an application of (1L) to get $u^{\circ}[1] \Rightarrow d$ and an application of $(\cdot\mathrm{L})$ to get $u[1] \Rightarrow d$; here u° is the same as u, except that the $\cdot$ next to $\_$ is replaced by $\circ$.

3. SEMANTICAL CONSEQUENCE RELATIONS

3.1. **Residuated lattice-ordered groupoids with unit.** A *residuated lattice-ordered groupoid with unit* or *$r\ell u$-groupoid*, is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ such that

- $\langle L, \wedge, \vee\rangle$ is a lattice,
- $\langle L, \cdot, 1\rangle$ is a groupoid with unit, and
- $a \cdot b \le c \;\Leftrightarrow\; a \le c/b \;\Leftrightarrow\; b \le a\backslash c$, for all $a, b, c \in L$.

We will often assume that the language contains an additional constant 0, of which nothing is assumed. Here $\le$ is the order relation associated with the lattice $\langle L, \wedge, \vee, \rangle$; so, $a \le b$ stands for $a = a \wedge b$. Note that $x/y = max\{z \mid zy \le x\}$ and $y\backslash x = max\{z \mid yz \le x\}$. The class RLUG of all $r\ell u$-groupoids is an *equational class*; i.e., the class of models of a set of equations. In particular, the identities

$$\begin{array}{lll} x \approx x \wedge ((xy \vee z)/y), & x(y \vee z) \approx xy \vee xz, & (x/y)y \vee x \approx x, \\ y \approx y \wedge (x\backslash(yx \vee z)), & (y \vee z)x \approx yx \vee zx, & y(y\backslash x) \vee x \approx x. \end{array}$$

together with the lattice and the unit identities form an axiomatization for it. Consequently, RLUG is a *variety*; i.e., a class of algebras closed under taking subalgebras, homomorphic images and direct products of the algebras in the class. For basic results and terminology in universal algebra, see [9].

Lemma 3.1. *If x, y, y_i, where $i \in I$, are elements of a $r\ell u$-groupoid and $\bigvee y_i, \bigwedge y_i$ exist, then*

(1) $x(\bigvee y_i) = \bigvee(xy_i)$ *and* $(\bigvee y_i)x = \bigvee(y_i x)$
(2) $(\bigwedge y_i)/x = \bigwedge(y_i/x)$ *and* $x\backslash(\bigwedge y_i) = \bigwedge(x\backslash y_i)$
(3) $x/(\bigvee y_i) = \bigwedge(x/y_i)$ *and* $(\bigvee y_i)\backslash x = \bigwedge(y_i\backslash x)$
(4) $(x/y)y \leq x$ *and* $y(y\backslash x) \leq x$
(5) $x/1 = x = 1\backslash x$
(6) $1 \leq x/x$ *and* $1 \leq x\backslash x$.

A *residuated lattice*, or *residuated lattice-ordered monoid*, is an associative $r\ell u$-groupoid. A residuated lattice is called *commutative*, if its underlying monoid is commutative. We denote by RL and CRL the varieties of residuated lattices and commutative residuated lattices, respectively. A residuated lattice is commutative iff $x\backslash y = y/x$ for all elements x, y; we denote the common value by $x \to y$.

Lemma 3.2. *If x, y, z are elements of a residuated lattice, then*

(1) $x(y/z) \leq xy/z$ *and* $(z\backslash y)x \leq z\backslash yx$
(2) $(x/y)/z = x/zy$ *and* $z\backslash(y\backslash x) = yz\backslash x$
(3) $x\backslash(y/z) = x\backslash(y/z)$

For more on residuated lattices and $r\ell u$-groupoids, see [7], [27] and [18].

3.2. **Logical matrices.** Logical matrices are pairs of an algebra and a set and can been used to define logics in the setting of algebraic logic. Here we generalize the standard matrices in two directions. We will generalize the notion of a logical matrix to allow for pairs of a *partial* algebras and a sets. Also, together with the algebra, we will consider a set that is not a subset of the underlying set of the (partial) algebra, but a set of more complex objects.

3.2.1. *Multidimensional matrices.* We start by reviewing the standard notion of a logical matrix. Recall that if $\mathcal{L}$ is a propositional (or algebraic) language, as considered in Section 2.1.2, then an *$\mathcal{L}$-algebra* is a structure $\mathbf{A} = \langle A, (f^{\mathbf{A}})_{f\in L}\rangle$, where A is a set and for every $f \in L$ of arity $\alpha(f)$, $f^{\mathbf{A}}$ is an operation on A of arity $\alpha(f)$; we also write $\mathcal{L}^{\mathbf{A}}$ or $L^{\mathbf{A}}$ for $(f^{\mathbf{A}})_{f\in L}$, and $\mathbf{A} = \langle A, L^{\mathbf{A}}\rangle$. Sometimes, we omit the superscript $\mathbf{A}$ from $f^{\mathbf{A}}$ and write $\mathbf{A} = \langle A, L\rangle$. If $L = \{f_1, \ldots, f_n\}$, we usually write $\mathbf{A} = \langle A, f_1, \ldots, f_n\rangle$. Also, recall that if $\mathbf{A}$ and $\mathbf{B}$ are $\mathcal{L}$-algebras, then a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$, in symbols $h : \mathbf{A} \to \mathbf{B}$, is a map $h : A \to B$, such that for every $f \in L$ and $a \in A^{\alpha(f)}$, $h(f^{\mathbf{A}}(a)) = f^{\mathbf{B}}(h(a))$, where $f(a) = (f(a_i))_{1\leq i\leq\alpha(f)}$ and $h(a) = (h(a_i))_{1\leq i\leq\alpha(f)}$, for $a = (a_i)_{1\leq i\leq\alpha(f)}$.

If P is the set of propositional variables, usually taken to be infinitely countable, then $\mathbf{Fm}_{\mathcal{L}}(\mathsf{P}) = \langle Fm_{\mathcal{L}}(\mathsf{P}), L\rangle$ is an $\mathcal{L}$-algebra, called the *absolutely free $\mathcal{L}$-algebra over* P or the *$\mathcal{L}$-formula algebra over* P; we often write simply $\mathbf{Fm}_{\mathcal{L}}$. An *assignment* (from $\mathbf{Fm}_{\mathcal{L}}(\mathsf{P})$) to an $\mathcal{L}$-algebra $\mathbf{A}$ is an arbitrary map $f : P \to A$. Such a map extends uniquely to a *homomorphism* $f : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$.

A *(1-dimensional) $\mathcal{L}$-matrix* is a pair $\boldsymbol{\mathcal{A}} = (\mathbf{A}, S)$, where $\mathbf{A}$ is an $\mathcal{L}$-algebra and $S \subseteq A$. The elements of S are called *designated* or *true* elements of A. For every subset $B \cup \{c\}$ of $Fm_{\mathcal{L}}$, we write $B \models_{\langle \mathbf{A}, S \rangle} c$ (or $(B, c) \in \models_{\langle \mathbf{A}, S \rangle}$) if, for every homomorphism $f : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, $h[B] \subseteq S$ implies $h(c) \in S$, where $h[B] = \{h(b) \mid b \in B\}$. If $\mathcal{M}$ is a class of $\mathcal{L}$-matrices, then $\models_{\mathcal{M}}$ is defined to be the intersection of all relations $\models_{\boldsymbol{\mathcal{A}}}$, over all $\boldsymbol{\mathcal{A}} \in \mathcal{M}$. It is easy to see that $\models_{\mathcal{M}}$ is a consequence relation on $Fm_{\mathcal{L}}$.

The $\mathcal{L}$-matrix $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, S \rangle$, is called a *matrix model* of a consequence relation $\vdash$ on $Fm_{\mathcal{L}}$, if $\vdash \subseteq \models_{\boldsymbol{\mathcal{A}}}$; in this case S is called a *deductive filter for* $\vdash$ (or a $\vdash$-*filter*) of $\mathbf{A}$. A class $\mathcal{M}$ of matrices is called a *matrix semantics* for a consequence relation $\vdash$, if $\vdash = \models_{\mathcal{M}}$. For example, if $\mathbf{B}$ is a Boolean algebra and $\vdash_{\mathbf{CPL}}$ is the deducibility relation of Classical Propositional Logic, then $\langle \mathbf{B}, \{1_{\mathbf{B}}\} \rangle$ is a matrix model of $\vdash_{\mathbf{CPL}}$. It is well known that $\vdash_{\mathbf{CPL}} = \models_{\langle \mathbf{2}, \{1\} \rangle}$, where $\mathbf{2}$ is the two-element Boolean algebra. So, $\{\langle \mathbf{2}, \{1\} \rangle\}$ and $\{\langle \mathbf{B}, \{1_{\mathbf{B}}\} \rangle \mid \mathbf{b} \in \mathcal{BA}\}$, where $\mathcal{BA}$ is the class of all Boolean algebras, are matrix semantics for $\vdash_{\mathbf{CPL}}$. See [16] for more on matrices.

Generalizations of 1-dimensional matrices include n-dimensional ones. An *n-dimensional $\mathcal{L}$-matrix* is a pair $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, S \rangle$, where $\mathbf{A}$ is an $\mathcal{L}$-algebra and $S \subseteq A^n$. For every subset $B \cup \{c\}$ of $(Fm_{\mathcal{L}})^n$, we define $B \models_{\boldsymbol{\mathcal{A}}} c$ iff, for every homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, $h[B] \subseteq S$ implies $h(c) \in S$; here $h(c)$ is defined coordinatewise. It is clear that the 1-dimensional $\mathcal{L}$-matrix $\langle \mathbf{A}^n, S \rangle$ has exactly the same information content with $\boldsymbol{\mathcal{A}}$. If $\mathcal{M}$ is a class of n-dimensional $\mathcal{L}$-matrices, the relation $\models_{\mathcal{M}}$ is defined in the obvious way. Clearly, $\models_{\mathcal{M}}$ is a consequence relation on $(Fm_{\mathcal{L}})^n$, or an *n-dimensional consequence relation on* $\mathbf{Fm}_{\mathcal{L}}$.

If $\mathbf{A}$ is an $\mathcal{L}$-algebra, then the 2-dimensional $\mathcal{L}$-matrix $\langle \mathbf{A}, =_{\mathbf{A}} \rangle$, where $=_{\mathbf{A}}$ denotes the equality relation on A, plays a special role and we simply write $\models_{\mathbf{A}}$ for $\models_{\langle \mathbf{A}, =_{\mathbf{A}} \rangle}$; we refer to elements of $(Fm_{\mathcal{L}})^2$ as *$\mathcal{L}$-equations* and to the elements of $=_{\mathbf{A}}$ as *true equalities*. In detail, if $\mathbf{A}$ is an $\mathcal{L}$-algebra and $E \cup \{\varepsilon_0\}$ is a set of $\mathcal{L}$-equations, then we write $E \models_{\mathbf{A}} \varepsilon_0$ iff for every homomorphism $f : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, if $f(\varepsilon)$ is true for all $\varepsilon \in E$, then $f(\varepsilon_0)$ is true, as well. Similarly, if $\mathcal{K}$ is a class of $\mathcal{L}$-algebras, we write $\models_{\mathcal{K}}$ for the relation defined relative to the corresponding class of matrices.

Another example of 2-dimensional $\mathcal{L}$-matrices are ordered algebras $\langle \mathbf{A}, \leq_{\mathbf{A}} \rangle$. The elements of $\leq_{\mathbf{A}}$ are called *true* inequalities.

3.2.2. *Sequent matrices.* We, now, want to capture the notion of a *true* sequent over an algebra. The way to do this is to define as a sequent matrix a pair of an algebra $\mathbf{A}$ and a set of sequents over A, namely a subset of $A^{\gamma} \times A$, designated as true sequents. We mention that this notion of a matrix does not fit into the definition of an n-dimensional matrix, because we have an unbounded number of different dimensions and because n-dimensional matrices presuppose the presence of associativity.

Although this definition completely captures the intented meaning of the terms, we will need it to be more general for technical reasons. For example, we will want to concentrate on only some of all possible sequents, when we discuss a $\mathcal{K}$-subsystem of $\mathbf{GL}$; in this case we will allow only sequents fit for $\mathcal{K}$ to be considered. In a different direction, to prove the strong separation property for $\mathbf{HL}$, which will be discussed in Section 4.5, we will need to considerer the set of subformulas of a set of formulas and view it as a partial subalgebra of $Fm_{\mathcal{L}}$. The notion of partial subalgebra also appears naturally, when we consider the application of our results to the finite embeddability property, which will be discussed in Section 4.6.2. Therefore, our definition will need to allow for partial algebras.

Recall that a *partial $\mathcal{L}$-algebra* is a structure $\mathbf{A} = \langle A, (f^{\mathbf{A}})_{f \in L}\rangle$, where A is a set and for every $f \in L$ of arity $\alpha(f)$, $f^{\mathbf{A}}$ is a partial operation on A of arity $\alpha(f)$. A *partial map* from A to B is a relation $f \subseteq A \times B$, that is functional, i.e. if $(x, y), (x, z) \in f$, then $y = z$. As usual we write $f(x) = y$ for $(x, y) \in f$; when there exists a $y \in B$ such that $(x, y) \in f$, we say that $f(x)$ is *defined* and write $f(x) \in B$ or $x \in f^{-1}[B]$; if $f(x)$ is not defined, we say that it is *undefined*. Also, we write $f : A \rightharpoonup B$ for a partial map from A to B. A *partial operation* on A is partial map from from a power of A to A.

Let $\mathcal{K}$ be a sublanguage of $\mathcal{L}$. A *(partial) assignment* from $\mathbf{Fm}_{\mathcal{K}}$ to a partial $\mathcal{K}$-algebra $\mathbf{A}$ is a map $f : Y \to A$, where Y is a subset of the set P of propositional variables. We extend such a map as much as possible to a partial map $f : Fm_{\mathcal{K}} \rightharpoonup A$, also called a *(partial) assignment*. In detail, f is extended by the following clause:

- if $t_i \in Fm_{\mathcal{K}}(Y)$, where $1 \leq i \leq n$, $t \in Fm_{\mathcal{K}}(P)$ has arity n and all of $f(t_1), \dots, f(t_n), t^{\mathbf{A}}(f(t_1), \dots, f(t_n))$ are elements of A, then we define $f(t(t_1, \dots, t_n)) = t^{\mathbf{A}}(f(t_1), \dots, f(t_n))$. Else, $f(t(t_1, \dots, t_n))$ is undefined.

Moreover, f extends to a partial map from the set of groupoid and augmented groupoid words fit for $\mathcal{K}$, by the clauses:

- $f(\varepsilon) = \varepsilon$ and $f(\_) = \_$;
- if $x_1, x_2 \in Fm_{\mathcal{K}}^{\gamma_{\mathcal{K}}} \cup Fm_{\mathcal{K}}^{\alpha_{\mathcal{K}}}$ and all of $f(x_1), f(x_2), f(x_1) \circ f(x_2)$ are in $A^{\gamma_{\mathcal{K}}} \cup A^{\alpha_{\mathcal{K}}}$, then $f(x_1 \circ x_2) = f(x_1) \circ f(x_2)$. Otherwise, $f(x_1 \circ x_2)$ is undefined.

Finally, f naturally extends to a partial map from the set $Fm_{\mathcal{K}}^{\gamma_{\mathcal{K}}} \times Fm_{\mathcal{K}}$ of $\mathcal{K}$-sequents to the set $A^{\gamma_{\mathcal{K}}} \times A$ of A-sequents (fit for $\mathcal{K}$), by $f(x, a) = (f(x), f(a))$ whenever $a \in f^{-1}[A]$ and $x \in f^{-1}[A^{\gamma_{\mathcal{K}}}]$.

For every sublanguage $\mathcal{K}$ of $\mathcal{L}$, a *sequent $\mathcal{K}$-matrix* is a pair $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \preceq\rangle$, where $\mathbf{A}$ is a partial $\mathcal{K}$-algebra and $\preceq$ is a set of A-sequents fit for $\mathcal{K}$. We often write $x \preceq a$ for $(x, a) \in \preceq$ and say that the A-sequent $x \Rightarrow a$ is *true*. The set $\preceq$ is called *the set of true (or designated) A-sequents of $\boldsymbol{\mathcal{A}}$*.

If $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \preceq\rangle$ is a sequent $\mathcal{K}$-matrix, for every set of $\mathcal{K}$-sequents $S \cup \{s\}$, we define $S \models_{\langle \mathbf{A}, \preceq\rangle} s$ iff, for every partial assignment $f : \mathbf{Fm}_{\mathcal{L}} \rightharpoonup \mathbf{A}$ such that $S \cup \{s\} \subseteq f^{-1}[A^{\gamma} \times A]$, $f[S] \subseteq \preceq$ implies $f(s) \in \preceq$; namely, if all A-sequents in $f[S]$ are true, then the A-sequent $f[s]$ is true.

We say that an $\mathcal{L}$-sequent s *holds* or that it is *valid* in $\boldsymbol{\mathcal{A}}$, if $\models_{\boldsymbol{\mathcal{A}}} s$. A sequent $\mathcal{K}$-matrix $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \preceq\rangle$ is a *matrix model* of a $\mathcal{K}$-sequent consequence relation $\vdash$, if $\vdash \subseteq \models_{\boldsymbol{\mathcal{A}}}$. We define a *$\mathcal{K}$-sequent consequence relation* to be a consequence relation on the set of $\mathcal{K}$-sequents. A sequent $\mathcal{K}$-matrix can be a matrix model of a sequent consequence relation $\vdash$ in a trivial way; for example if all operations in the underlying algebra are nowhere-defined. Note that, unless all operations in $\mathbf{A}$ are full, $\models_{\boldsymbol{\mathcal{A}}}$ may fail to be a sequent consequence relation. The relation $\models_{\mathcal{M}}$ associated with a class $\mathcal{M}$ of sequent matrices is defined in the usual way. The class $\mathcal{M}$ is called a *matrix semantics* for a sequent consequence relation $\vdash$, if $\vdash = \models_{\mathcal{M}}$.

3.2.3. *Algebraic semantics.* A class of $\mathcal{L}$-algebras $\mathcal{K}$ is called an *algebraic semantics* (in the sense of Blok and Pigozzi [3]) for an (1-dimensional) $\mathcal{L}$-consequence relation $\vdash$, if there are $\mathcal{L}$-equations $\varepsilon_i(p)$, where $1 \leq i \leq n$, on one variable such that for all $B \cup \{c\} \subseteq Fm_{\mathcal{L}}$ and for all $1 \leq j \leq n$,

$$B \vdash c \text{ iff } \{\varepsilon_i(b) \mid 1 \leq i \leq n, b \in B\} \models_{\mathcal{K}} \varepsilon_j(c).$$

If $\mathcal{K}$ is a class of $\mathcal{L}$-algebras and E is set of $\mathcal{L}$-equations on one variable, define $\mathcal{M}(\mathcal{K}, E)$ as the class of all 1-dimensional $\mathcal{L}$-matrices of the form $(\mathbf{A}, E_{\mathbf{A}})$, where $\mathbf{A} \in \mathcal{K}$ and $E_{\mathbf{A}} = \{a \in A \mid \varepsilon(a) \text{ is true in } \mathbf{A}\}$. It is easy to observe that $\mathcal{K}$ is an algebraic semantics for $\vdash$ iff there exists a finite set E of equations on one variable such that $\mathcal{M}(\mathcal{K}, E)$ is a matrix semantics for $\vdash$. Therefore, algebraic semantics can be thought of as a special case of matrix semantics.

The class $\mathcal{K}$ is called an *equivalent algebraic semantics* for $\vdash$, if it is an algebraic semantics for $\vdash$ and there are binary terms $\Delta_j(p, q)$, where $1 \leq j \leq m$, such that

$$u \approx v =\mid\models_{\mathcal{K}} \{\varepsilon_i(\Delta_j(u, v)) \mid 1 \leq i \leq n, 1 \leq j \leq m\}.$$

The equations ε_i are called the defining equations and Δ_j are called the equivalence formulas.

The class $\mathcal{BA}$ is an equivalent algebraic semantics for $\vdash_{\mathbf{CPL}}$, for $\varepsilon(p) = (p \approx 1)$ and $\Delta(p, q) = (p \leftrightarrow q)$.

The notion of (equivalent) algebraic semantics is also defined for sequent consequence relations in a natural way. The definition given in [38] for associative-sequent consequence relations does not apply to our case, as we work in the non-associative setting. The definition we give here as well as the previous two are a special case of the definition of (finitary) equivalence between two consequence relations given in [22], which also instantiates to the equivalence of the syntactic consequence relations of the previous section.

To specialize the definition in [22] to our case we need to enter into technicalities. We fix an enumeration $p_1, p_2, \ldots$ of the set P of propositional variables. For every Q-sequent s (Q is any set) we define its *type* $tp(s)$ to be the sequent obtained from s by replacing (all occurrences of) the elements of Q in s from left to right by the variables from P in order. In detail, we start by defining the *n-type* $tp_n(x)$ for every $x \in Q^\gamma$ inductively:

- $tp_n(\varepsilon) = \varepsilon$; $tp_n(a) = p_n$, for every $a \in Q$;
- if p_m is the first variable not in $tp_n(x)$ with $m \geq n$, then $tp_n(x \circ y) = (tp_n(x), tp_{n+m}(y))$.

We define the *type* of a sequent $x \Rightarrow a$ to be the sequent $tp(x \Rightarrow a) = (tp_1(x) \Rightarrow p_m)$, where p_m is the first variable not in $tp_1(x)$. For example, $tp((a, b), (b, a) \Rightarrow b) = ((p_1, p_2), (p_3, p_4) \Rightarrow p_5)$. We denote by Tp the set of all types. It is clear that given a sequent s of type t, there exists a map σ_s from Tp to $Q^\gamma \times Q$ such that $\sigma_s(t) = s$; note that σ_s is not unique. If Q is the set $\mathbf{Fm}_{\mathcal{L}}$, then σ_s is essentially an $\mathcal{L}$-substitution.

An $\mathcal{L}$-equation of type $t \in Tp$ is an equation on the variables appearing in t. Obviously, an equation may have many types: $p_1 \wedge p_2 \approx p_1 \vee (p_3 \vee p_4)$ has types $((p_1, p_2), p_3 \Rightarrow p_4)$, $(p_1, (p_2, p_3) \Rightarrow p_4)$, $(p_2, (p_4, p_3) \Rightarrow p_1)$ etc. If ε is an equation (here we use the same symbol as for the the empty groupoid word, as ε is also traditional for equations) and s a sequent of the same type, then we define the equation $\varepsilon(s) = \sigma_s(\varepsilon)$; i.e., the equation obtained by replacing the variables in ε by the corresponding formulas in s. For example, if s is $(a, b) \Rightarrow b$ and ε is $(x \cdot y) \vee y = y$, then $\sigma_s(\varepsilon) = (a \cdot b) \vee b = b$. A *typed equation* is a pair (ε, t), where ε is an equation of type t. If E is a set of typed equations and S a set of sequents, we define $E[S] = \{\varepsilon(s) \mid (\varepsilon, tp(s)) \in E, s \in S\}$; note that even if ε and s are of the same type, $\varepsilon(s)$ will not be in $E[S]$ if $(\varepsilon, tp(s)) \notin E$. For example, the equation $(x \cdot y) \vee y = y$, call it ε, has both type $(p_1, p_2) \Rightarrow p_2$ and type $(p_1, p_2) \Rightarrow p_1$, call these types t_1 and t_2, but maybe only the typed equation (ε, t_1) is in the set E. Considering typed equation is a way of knowing which variables of ε to replace by which variables of s in the expression $\varepsilon(s)$.

A class of $\mathcal{L}$-algebras $\mathcal{K}$ is called an *algebraic semantics* for a sequent consequence relation $\vdash$, if there exists a finite set E of typed $\mathcal{L}$-equations, such that for all sets $S \cup \{s\}$ of $\mathcal{L}$-sequents

$$S \vdash s \text{ iff } E[S] \models_{\mathcal{K}} E(s).$$

The class $\mathcal{K}$ is called an *equivalent algebraic semantics* for $\vdash$, if it is an algebraic semantics for $\vdash$ and there is a finite set $S(p, q)$ of $\mathcal{L}$-sequents in two variables, such that

$$u \approx v =\mid\models_{\mathcal{K}} E[S(u, v)].$$

The class $\mathcal{HA}$ of Heyting algebras is an equivalent algebraic semantics for Gentzen's original system **LJ**, for $E(x \Rightarrow a) = (x^{\mathbf{Fm}_{\mathcal{L}}} \leq a)$ and $S(p, q) = \{(p \Rightarrow q), (q \Rightarrow p)\}$.

3.3. **Matrix models of GL.** The class $\mathcal{KGL}$ ($\mathcal{KGL}^{\mathrm{f}}$) is defined to contain all sequent $\mathcal{K}$-matrices $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \preceq \rangle$, that are models of $\mathcal{K}\mathbf{GL}$ ($\mathcal{K}\mathbf{GL}^{\mathrm{f}}$, respectively). If R is a set of structural metarules, we denote the matrix models of $\mathcal{K}\mathbf{GL}_{\mathrm{R}}$ by $\mathcal{KGL}_{\mathrm{R}}$ and of $\mathcal{K}\mathbf{GL}^{\mathrm{f}}_{\mathrm{R}}$ by $\mathcal{KGL}^{\mathrm{f}}_{\mathrm{R}}$. We will provide a condition for checking if a matrix is in $\mathcal{KGL}_{\mathrm{R}}$ or $\mathcal{KGL}^{\mathrm{f}}_{\mathrm{R}}$.

Given a sequent $\mathcal{K}$-matrix $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \preceq \rangle$ and a non-structural metarule (r) of $\mathcal{K}\mathbf{GL}$ that involves metavariables among $\{a, b, c, x, u\}$ and concerns the connective $\bullet \in \mathcal{K}$, we define the interpretation $(r)^{\boldsymbol{\mathcal{A}}}$ of (r) in $\boldsymbol{\mathcal{A}}$ to be the following statement:

> $\forall a, b, c \in A, x \in A^{\gamma_{\mathcal{K}}}, u \in A^{\alpha_{\mathcal{K}}}$, if $a \bullet_{\mathbf{A}} b$ is defined, then the conjunction of the assumptions of the metarule (with the metavariables evaluated and $\Rightarrow$ replaced by $\preceq$) implies the conclusion of the metarule.

For example, $(\backslash\mathrm{L})^{\boldsymbol{\mathcal{A}}}$ is

> $\forall a, b, c \in A, x \in A^{\gamma_{\mathcal{K}}}, u \in A^{\alpha_{\mathcal{K}}}$, if $a \backslash_{\mathbf{A}} b$ is defined, then
> $x \preceq a$ and $u[b] \preceq c$ implies $u[x \circ (a \backslash b)] \preceq c$.

If (r) is a structural metarule, then $(\mathrm{r})^{\boldsymbol{\mathcal{A}}}$ is defined by a clause like the above but without any mention to a connective.

Lemma 3.3. *For a sublanguage $\mathcal{K}$ that contains $\backslash$ and for a set* R *of structural metarules, a $\mathcal{K}$-sequent matrix $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \preceq \rangle$ is in $\mathcal{KGL}_{\mathrm{R}}$ ($\mathcal{KGL}^{\mathrm{f}}_{\mathrm{R}}$) iff the interpretation $(\mathrm{r})^{\boldsymbol{\mathcal{A}}}$ of every metarule* (r) *of $\mathcal{K}\mathbf{GL}_{\mathrm{R}}$ ($\mathcal{K}\mathbf{GL}^{\mathrm{f}}_{\mathrm{R}}$, respectively) holds.*

Proof. We first mention that if the metavariables $\{a', b', c', x', u'\}$ of (r) are evaluated appropriately in the sets A, $A^{\gamma_{\mathcal{K}}}$ and $A^{\alpha_{\mathcal{K}}}$, and $a \bullet_{\mathbf{A}} b$ is defined, then the assignments of the expressions in (r) are in the appropriate sets, for the different possibilities for $\mathcal{K}$. For example, for ($\vee$L), if u is in $A^{\alpha_{\mathcal{K}}}$ and $\mathcal{K} = \{\vee, \backslash, /\}$, then u is a solvable augmented groupoid word over A, since $\boldsymbol{\mathcal{A}}$ is a sequent $\mathcal{K}$-matrix. Also, if $a, b \in A$, then $a, b, a \vee b$ are in A. It is easy to see that then $u[a]$, $u[b]$ and $u[a \vee b]$ are all solvable groupoid words over A and are, therefore, in $A^{\gamma_{\mathcal{K}}}$. As another example, we consider the case of ($\backslash$L) for $\mathcal{K} = \{\vee, \backslash\}$. If $a, b, a\backslash b \in A$, $x \in A^{\gamma_{\mathcal{K}}}$ and $u \in A^{\alpha_{\mathcal{K}}}$, then x is a left-solvable groupoid word over A and u is a left-solvable augmented groupoid word over A, since $\boldsymbol{\mathcal{A}}$ is a sequent $\mathcal{K}$-matrix. It is easy to observe that then $u[x \circ (a\backslash b)]$ is a left-solvable groupoid word over A and is therefore in $A^{\gamma\mathcal{K}}$.

Assume that $\boldsymbol{\mathcal{A}} \in \mathcal{K}\mathcal{G}\mathcal{L}$ and (r) is a metarule of $\mathcal{K}\mathbf{GL}$; we will show that $(r)^{\boldsymbol{\mathcal{A}}}$ holds. We proceed in the proof by considering the representative case where (r) is $(\backslash\mathrm{L})$. To show that $(\backslash\mathrm{L})^{\boldsymbol{\mathcal{A}}}$ holds, let $a, b, c \in A, x \in A^{\gamma\mathcal{K}}, u \in A^{\alpha\mathcal{K}}$ and assume that $a\backslash_{\mathbf{A}} b$ is defined, $x \preceq a$ and $u[b] \preceq c$. We want to show that $u[x \circ (a\backslash b)] \preceq c$. Assume that x is a groupoid word on the elements $d_1, \dots, d_n$ of A, $x = x(d_1, \dots, d_n)$ in short, and that $u = u(e_1, \dots, e_m)$. Pick propositional variables $a', b', c', d_1', \dots, d_n', e_1', \dots, e_m'$ in P and define a partial assignment f into A that maps the propositional variables to the corresponding elements in A. There exists a groupoid word $x' = x'(d_1', \dots, d_n')$ in $\mathbf{Fm}_{\mathcal{K}}^{\gamma\mathcal{K}}$ and an augmented groupoid word $u' = u'(e_1', \dots, e_m')$ in $\mathbf{Fm}_{\mathcal{K}}^{\alpha\mathcal{K}}$ such that $f(x') = x$ and $f(u') = u$ under the associated partial assignment. Now, we have $(f(x'), f(a')) \in \preceq$ and $(f(u'[b']), f(c')) \in \preceq$, so $(f(u'[x' \circ (a'\backslash b')]), f(c')) \in \preceq$, since $\boldsymbol{\mathcal{A}} \in \mathcal{K}\mathcal{G}\mathcal{L}$. Hence $u[x \circ (a\backslash b)] \preceq c$ holds.

Conversely, assume that the interpretation $(r)^{\boldsymbol{\mathcal{A}}}$ of every metarule (r) of $\mathcal{K}\mathbf{GL}$ holds for a sequent $\mathcal{K}$-matrix $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \preceq \rangle$; we will prove that $\boldsymbol{\mathcal{A}}$ is in $\mathcal{K}\mathcal{G}\mathcal{L}$. Consider a metarule (r) of $\mathcal{K}\mathbf{GL}$ that involves (possibly) the connective $\bullet \in \mathcal{K}$ and an instance $(r') = (S, s)$ of it; we will show that $S \models_{\langle \mathbf{A}, \preceq \rangle} s$. To this end, consider a partial assignment $f : \mathbf{Fm}_{\mathcal{K}} \rightharpoonup \mathbf{A}$ such that $S \cup \{s\} \subseteq f^{-1}[A^{\gamma\mathcal{K}} \times A]$ and $f[S] \subseteq \preceq$; we need to verify that $f(s) \in \preceq$. For the sake of concreteness, let (r) be the metarule $(\vee\mathrm{L})$ and $(r') = (\{u'[a'] \Rightarrow c', u'[b'] \Rightarrow c'\}, u'[a' \vee b'] \Rightarrow c')$, where $a', b', c' \in Fm_{\mathcal{K}}$ and $u' \in Fm_{\mathcal{K}}^{\alpha\mathcal{K}}$. Also, let $a = f(a'), b = f(b'), c = f(c')$ be the elements of A and $u = f(u')$ be the element of $A^{\alpha\mathcal{K}}$, in the image of f. From $f[S] \in \preceq$, we obtain $u[a] \preceq c$ and $u[b] \preceq c$. Also, from $s \in f^{-1}[A^{\gamma\mathcal{K}} \times A]$ we have that the join $a \vee_{\mathbf{A}} b$ is defined in A. Therefore, $(r)^{\boldsymbol{\mathcal{A}}}$ yields $u[a \vee b] \preceq c$; i.e., $f(s) \in \preceq$.

The argument for structural metarules is similar. □

Lemma 3.4. *$\mathcal{K}\mathcal{G}\mathcal{L}_{\mathrm{R}}$ ($\mathcal{K}\mathcal{G}\mathcal{L}_{\mathrm{R}}^{\mathrm{f}}$) is a matrix semantics for $\mathcal{K}\mathbf{GL}_{\mathrm{R}}$ ($\mathcal{K}\mathbf{GL}_{\mathrm{R}}^{\mathrm{f}}$, respectively); i.e. $\models_{\mathcal{K}\mathcal{G}\mathcal{L}_{\mathrm{R}}} = \vdash_{\mathcal{K}\mathbf{GL}_{\mathrm{R}}}$ and $\models_{\mathcal{K}\mathcal{G}\mathcal{L}_{\mathrm{R}}^{\mathrm{f}}} = \vdash_{\mathcal{K}\mathbf{GL}_{\mathrm{R}}^{\mathrm{f}}}$.*

Proof. Let $S \cup s$ be a set of $\mathcal{K}$-sequents. By definition, $S \vdash_{\mathcal{K}\mathbf{GL}_{\mathrm{R}}} s$ implies $S \models_{\mathcal{K}\mathcal{G}\mathcal{L}_{\mathrm{R}}} s$, and $S \vdash_{\mathcal{K}\mathbf{GL}_{\mathrm{R}}^{\mathrm{f}}} s$ implies $S \models_{\mathcal{K}\mathcal{G}\mathcal{L}_{\mathrm{R}}^{\mathrm{f}}} s$. To show the converse implications (we will do only the first one) assume that $S \nvdash_{\mathcal{K}\mathbf{GL}_{\mathrm{R}}} s$. We define the sequent $\mathcal{K}$-matrix $\boldsymbol{\mathcal{A}} = \langle \mathbf{Fm}_{\mathcal{K}}, \preceq \rangle$, where $s \in \preceq$ iff $S \vdash_{\mathcal{K}\mathbf{GL}_{\mathrm{R}}} s$. The set $\preceq$ is called the *sequent-theory* generated by S.

Obviously, $S \not\models_{\boldsymbol{\mathcal{A}}} s$ (for the identity partial assignment) and it can be easily checked that $\boldsymbol{\mathcal{A}} \in \mathcal{K}\mathcal{G}\mathcal{L}_{\mathrm{R}}$, by using Lemma 3.3. As an example, we check that $(\backslash\mathrm{L})^{\boldsymbol{\mathcal{A}}}$ holds. Let $a, b, c \in Fm_{\mathcal{K}}$, $x \in Fm_{\mathcal{K}}^{\gamma\mathcal{K}}$ and $u \in Fm_{\mathcal{K}}^{\alpha\mathcal{K}}$. Since $\mathcal{K}$ contains $\backslash$, the formula $a\backslash b$ is in $Fm_{\mathcal{K}}^{\gamma\mathcal{K}}$. If $x \preceq a$ and $u[b] \preceq c$, then $S \vdash_{\mathcal{K}\mathbf{GL}_{\mathrm{R}}} x \Rightarrow a$ and $S \vdash_{\mathcal{K}\mathbf{GL}_{\mathrm{R}}} u[b] \Rightarrow c$; note that $x \Rightarrow a$ is both an $\mathcal{K}$-sequent and an $\mathbf{Fm}_{\mathcal{K}}$-sequent. If Π_1 is a proof of $x \Rightarrow a$ from S and Π_2 is a proof of $u[b] \Rightarrow c$ from S, then

$$\frac{\Pi_1 \qquad \Pi_2}{u[x \circ (a\backslash b)] \Rightarrow c} \ (\backslash\mathrm{L})$$

is a proof of $u[x \circ (a\backslash b)]$ from S. Consequently, we have $S \vdash_{\mathcal{K}\mathbf{GL}_{\mathrm{R}}} u[x \circ (a\backslash b)] \Rightarrow c$, so $u[x \circ (a\backslash b)] \preceq c$. As a second example consider $(\vee\mathrm{L})^{\boldsymbol{\mathcal{A}}}$ for the case where $\mathcal{K}$ does not contain $\vee$. If $a, b \in Fm_{\mathcal{K}}$, then the formula $a \vee b$ is not in $Fm_{\mathcal{K}}$, hence $(\vee\mathrm{L})^{\boldsymbol{\mathcal{A}}}$ is vacuously true. □

If $\mathbf{A}$ is an $r\ell u$-groupoid, we define the sequent $\mathcal{K}$-matrix $\mathbf{A}^{\mathcal{K}} = \langle \mathbf{A}, \preceq_{\mathbf{A}^{\mathcal{K}}} \rangle$, where $x \preceq_{\mathbf{A}^{\mathcal{K}}} a$ iff $x^{\mathbf{A}} \leq_{\mathbf{A}} a$; here $x^{\mathbf{A}}$ denotes the element of $\mathbf{A}$ obtained from x by replacing $\circ$ by $\cdot^{\mathbf{A}}$. We say that an A-sequent s is *true* in $\mathbf{A}$, if it is true in $\mathbf{A}^{\mathcal{L}}$. Also, a $\mathcal{K}$-sequent s is *valid* (or *holds*) in $\mathbf{A}$, if it is valid in $\mathbf{A}^{\mathcal{K}}$. If s is a $\mathcal{K}$-sequent, we define the equation $\varepsilon(x \Rightarrow a) = (x^{\mathbf{Fm}_{\mathcal{L}}} \leq a)$. The next lemma implies that the sequent s is valid in $\mathbf{A}^{\mathcal{K}}$ iff the equation $\varepsilon(s)$ is valid in $\mathbf{A}$.

Lemma 3.5. *If $S \cup \{s\}$ is a set of $\mathcal{K}$-sequents and $\mathbf{A}$ is an $r\ell u$-groupoid, then $S \models_{\mathbf{A}^{\mathcal{K}}} s$ iff $\varepsilon[S] \models_{\mathbf{A}} \varepsilon(s)$.*

Proof. The proof follows from the definitions. In detail, $S \models_{\mathbf{A}^{\mathcal{K}}} s$ iff for all partial assignments $f : \mathbf{Fm}_{\mathcal{K}} \rightharpoonup \mathbf{A}$, $f[S] \in \preceq_{\mathbf{A}^{\mathcal{K}}}$ implies $f(s) \in \preceq_{\mathbf{A}^{\mathcal{K}}}$. If $s = (x \Rightarrow a)$, then $f(s) = (f(x) \Rightarrow f(a))$; assume that $x = x(a_1, \ldots, a_n)$. Now, $f(s) \in \preceq_{\mathbf{A}^{\mathcal{K}}}$ iff $f(x) \preceq_{\mathbf{A}^{\mathcal{K}}} f(a)$ iff $x(f(a_1), \ldots, f(a_n)) \preceq_{\mathbf{A}^{\mathcal{K}}} f(a)$ iff $x^{\mathbf{A}}(f(a_1), \ldots, f(a_n)) \leq_{\mathbf{A}} f(a)$ iff $(x^{\mathbf{Fm}_{\mathcal{L}}})^{\mathbf{A}}(f(a_1), \ldots, f(a_n)) \leq_{\mathbf{A}} f(a)$ iff $\varepsilon^{\mathbf{A}}(f(a_1), \ldots, f(a_n), f(a))$ holds iff $\varepsilon^{\mathbf{A}}(a_1(f(\bar{p})), \ldots, a_n(f(\bar{p})), a(f(\bar{p})))$, where $\bar{p}$ is the list of all propositional variables from P that appear in $S \cup \{s\}$. We will abbreviate the last expression by $\varepsilon^{\mathbf{A}}(s(f(\bar{p})))$. Therefore, we obtain the equivalent statement:

> for all maps $f : Y \to A$, where Y is a subset of the set P of propositional variables containing $\bar{p}$, $\varepsilon^{\mathbf{A}}[S(f(\bar{p}))]$ holds implies $\varepsilon^{\mathbf{A}}(s(f(\bar{p})))$ holds.

It is clear that in this statement the range of the set Y can be replaced by just $\bar{p}$ or by just P. For the latter choice, in the resulting statement the map $f : \mathsf{P} \to A$ extends uniquely to a homomorphism $f : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, since $\mathbf{A}$ is a total $\mathcal{L}$-algebra, and the expression $\varepsilon^{\mathbf{A}}(s(f(\bar{p})))$ can be replaced by $f(\varepsilon(s(\bar{p})))$ or simply $f(\varepsilon(s))$. This is precisely the definition of $\varepsilon[S] \models_{\mathbf{A}} \varepsilon(s)$. $\square$

Theorem 3.6. *If $\mathbf{A}$ is an $r\ell u$-groupoid, then $\mathbf{A}^{\mathcal{K}}$ is a matrix model of $\mathcal{K}\mathbf{GL}$.*

Proof. It is routine to check that $\varepsilon[S] \models_{\mathbf{A}} \varepsilon(s)$, for all the rules (S, s) in $\mathcal{K}\mathbf{GL}$. $\square$

Without further discussion we mention that we can define (partial) two-dimensional $\mathcal{L}$-matrix models $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \leq \rangle$ of $\mathbf{PL}$.

4. Quasicompletion and applications

After developing the main theorem of the paper, we will apply it to various cases. The logical property of cut elimination, the finite model property and the strong separation property will follow as particular applications of the main result.

4.1. **Quasicompletion.**

We will first develop the main tools for the quasicompletion method, which we will apply in the following sections.

4.1.1. *The $r\ell u$-groupoid of a sequent matrix.* Given a sequent matrix, we will construct a residuated lattice associated with it that will play a key role the proofs in this section.

Let $\mathcal{K}$ be a sublanguage of $\mathcal{L}$ that contains $\backslash$ and let $\langle \mathbf{A}, \preceq \rangle$ be a sequent $\mathcal{K}$-matrix. We define the algebra

$$\mathbf{R}(\boldsymbol{\mathcal{A}}) = \langle \mathcal{P}(A^{\gamma\mathcal{K}})_g, \cap, \vee_g, \cdot_g, \backslash, /, \varepsilon_g \rangle,$$

where for $X \subseteq A^{\gamma\mathcal{K}}$,

$$g(X) = \{y \in A^{\gamma\mathcal{K}} | \text{for all } u \in A^{\alpha\mathcal{K}}, a \in A, \text{ if } u[x] \preceq a, \text{ for all } x \in X, \text{ then } u[y] \preceq a\},$$

$\mathcal{P}(A^{\gamma\mathcal{K}})_g = g[\mathcal{P}(A^{\gamma\mathcal{K}})]$ is the image of the powerset $\mathcal{P}(A^{\gamma\mathcal{K}})$ under the map g, $\varepsilon_g = g(\{\varepsilon\})$ and for $X, Y \subseteq A^{\gamma\mathcal{K}}$, (we set $X \circ Y = \{x \circ y \mid x \in X, y \in Y\}$; $x \circ y$ is not always defined)

$$X \vee_g Y = g(X \cup Y), \qquad X \cdot_g Y = g(X \circ Y),$$
$$X \backslash Y = \{z \in A^{\gamma\mathcal{K}} \mid X \circ \{z\} \subseteq Y\}, \qquad Y/X = \{z \in A^{\gamma\mathcal{K}} \mid \{z\} \circ X \subseteq Y\}.$$

We note that $g(X)$ is the set of all groupoid words that fit in the same contexts that all elements of X fit; here by *x fits in the context* (u, a) we mean $u[x] \preceq a$. It is easy to see that g is a closure operator on $\mathcal{P}(A^{\gamma\mathcal{K}})$.

Theorem 4.1. *If $\langle \mathbf{A}, \preceq\rangle$ is a sequent $\mathcal{K}$-matrix, then $\mathbf{R}(\boldsymbol{\mathcal{A}})$ is a residuated ℓ-groupoid with unit.*

Theorem 4.1 can be proved directly, but its proof relies on many fundamental notions and constructions, including that of a nucleus. Since a discussion on these topics would disrupt the flow of the paper, we include the background and the proof of the theorem in Appendix B; see Corollary B.7.

For every $a \in A$ and $u \in A^{\alpha\mathcal{K}}$, we define

$$[u, a] = \{x \in A^{\gamma\mathcal{K}} \mid u[x] \preceq a\}$$

and $\downarrow a = [\_, a] = \{x \in A^{\gamma\mathcal{K}} \mid x \preceq a\}$. Using this notation, for $X \subseteq A^{\gamma\mathcal{K}}$, we can express $g(X)$ as follows:

$$g(X) = \bigcap\{[u, a] \mid a \in A, u \in A^{\alpha\mathcal{K}} \text{ and } X \subseteq [u, a]\}.$$

In particular, we have $[u, a] \in R(\boldsymbol{\mathcal{A}})$, so the assignment $q(a) = \downarrow a$ defines a map $q : A \to R(\boldsymbol{\mathcal{A}})$. The following lemma follows directly form the above expression for g.

Lemma 4.2. *If $k \in \mathbf{R}(\boldsymbol{\mathcal{A}})$ and $x \in A^{\gamma\mathcal{K}}$, then $x \in k$ iff $x \in [u, a]$, for all $u \in A^{\alpha\mathcal{K}}$ and $a \in A$ such that $k \subseteq [u, a]$.*

The following lemma follows from Corollary B.6 in Appendix B (see also Lemma B.2).

Lemma 4.3. *Let $\boldsymbol{\mathcal{A}}$ be a sequent $\mathcal{K}$-matrix. Then the map g is a $\{\vee, \cdot, 1\}$ homomorphism from $\mathcal{P}(\mathbf{A}^\gamma) = \langle \mathcal{P}(A^\gamma), \cap, \cup, \circ, \backslash, /, \{\varepsilon\}\rangle$ to $\mathbf{R}(\boldsymbol{\mathcal{A}})$.*

4.1.2. *Quasiembedding.* We are now ready to present the main technical result of the paper.

Lemma 4.4. *Assume that $\mathcal{K}$ is a subset of $\mathcal{L}$ that contains the connective $\backslash$, $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \preceq\rangle$ is a sequent matrix in $\mathcal{KGL}^{\mathrm{f}}$, $a, b \in A$ and $k, l \in R(\boldsymbol{\mathcal{A}})$. Also, assume that $\bullet$ is one of the connectives in $\mathcal{K}$, $a \bullet_{\mathbf{A}} b$ is defined, $a \in k \subseteq \downarrow a$ and $b \in l \subseteq \downarrow b$. Then*

(1) *$1_{\mathbf{A}} \in \varepsilon_g \subseteq \downarrow 1_{\mathbf{A}}$ ($1_{\mathbf{A}}$ is defined, for $\bullet = 1$) and*
(2) *$a \bullet_{\mathbf{A}} b \in k \bullet_{\mathbf{R}(\boldsymbol{\mathcal{A}})} l \subseteq \downarrow (a \bullet_{\mathbf{A}} b)$.*
(3) *In particular, $a \bullet_{\mathbf{A}} b \in \downarrow a \bullet_{\mathbf{R}(\boldsymbol{\mathcal{A}})} \downarrow b \subseteq \downarrow (a \bullet_{\mathbf{A}} b)$.*
(4) *If, additionally, $\boldsymbol{\mathcal{A}}$ is in $\mathcal{KGL}$, then $k = \downarrow a$ and $\downarrow a \bullet_{\mathbf{R}(\boldsymbol{\mathcal{A}})} \downarrow b = \downarrow (a \bullet_{\mathbf{A}} b)$.*

Proof. (1) By $(1\mathrm{R})^{\boldsymbol{\mathcal{A}}}$, we have $\varepsilon_{A^{\gamma\mathcal{K}}} \in \downarrow 1_{\mathbf{A}}$, so $\varepsilon_g = g(\varepsilon_{A^{\gamma\mathcal{K}}}) \subseteq \downarrow 1_{\mathbf{A}}$. On the other hand, if $\varepsilon_g = g(\varepsilon_{A^{\gamma\mathcal{K}}}) \subseteq [u, c]$, then $\varepsilon \in [u, c]$ and $|u| = u[\varepsilon] \preceq c$; so $u[1_{\mathbf{A}}] \preceq c$, by $(1\mathrm{L})^{\boldsymbol{\mathcal{A}}}$, hence $1_{\mathbf{A}} \in [u, c]$. Thus, $1_{\mathbf{A}} \in \varepsilon_g$, by Lemma 4.3.

(2) We will give the proof for the connectives $\vee$, $\cdot$ and $\backslash$. The proof for the remaining two connectives follows the same ideas.

Let $\bullet = \vee$. If $x \in k$, then $x \in \downarrow a$; so $x \preceq a$ and $x \preceq a \vee b$, by $(\vee\mathrm{R}\ell)^{\boldsymbol{\mathcal{A}}}$ (see Lemma 3.3); hence $x \in \downarrow (a \vee b)$. Consequently, $k \subseteq \downarrow (a \vee b)$. Similarly, we obtain $l \subseteq \downarrow (a \vee b)$ using $(\vee\mathrm{R}r)^{\boldsymbol{\mathcal{A}}}$; so $k \cup l \subseteq \downarrow (a \vee b)$, hence $k \vee l = g(k \cup l) \subseteq \downarrow (a \vee b)$.

On the other hand, let $k \vee l \subseteq [u, c]$, for some $u \in A^{\alpha\mathcal{K}}$ and $b \in A$. Then, $a \in k \subseteq k \vee l \subseteq [u, c]$, so $u[a] \preceq c$. Similarly, $u[b] \preceq c$, so $u[a \vee b] \preceq c$, by $(\vee\mathrm{L})^{\boldsymbol{\mathcal{A}}}$, hence $a \vee b \in [u, b]$. Thus, $a \vee b \in k \vee l$, by Lemma 4.2.

Let $\bullet = \cdot$. If $x \in k$ and $y \in l$, then $x \in \downarrow a$ and $y \in \downarrow b$; i.e. $x \preceq a$ and $y \preceq b$. So $x \circ y \preceq a \cdot b$, by $(\cdot\mathrm{R})^{\boldsymbol{\mathcal{A}}}$; hence $x \circ y \in \downarrow (a \cdot b)$. Consequently, $k \circ l \subseteq \downarrow (a \cdot b)$ and $k \cdot_{\mathbf{R}(\boldsymbol{\mathcal{A}})} l = g(k \circ l) \subseteq \downarrow (a \cdot b)$.

On the other hand, let $k \cdot_{\mathbf{R}(\boldsymbol{\mathcal{A}})} l \subseteq [u, c]$, for some $u \in A^{\alpha\mathcal{K}}$ and $c \in A$. Since $a \circ b \in k \circ l \subseteq g(k \circ l) = k \cdot_{\mathbf{R}(\boldsymbol{\mathcal{A}})} l$, we have $a \circ b \in [u, c]$, so $u[a \circ b] \preceq c$. Consequently, $u[a \cdot b] \preceq c$, by $(\cdot\mathrm{L})^{\boldsymbol{\mathcal{A}}}$, hence $a \cdot b \in [u, b]$. Thus, $a \cdot b \in k \cdot l$.

Let $\bullet = \backslash$. If $x \in k \backslash_{\mathbf{R}(\boldsymbol{\mathcal{A}})} l$ then $k \circ \{x\} \subseteq l$. Since $a \in k$ and $l \subseteq \downarrow b$, we have $a \circ x \in \downarrow b$; i.e., $a \circ x \preceq b$. By $(\backslash\mathrm{R})^{\boldsymbol{\mathcal{A}}}$ we obtain $x \preceq a \backslash b$; hence $x \in \downarrow (a \backslash b)$.

On the other hand, if $l \subseteq [u, c]$, then $b \in [u, c]$, so $u[b] \preceq c$. For all $x \in \downarrow a$, $x \preceq a$, so $u[x \circ (a \backslash b)] \preceq c$, by $(\backslash\mathrm{L})^{\boldsymbol{\mathcal{A}}}$; i.e., $x \circ (a \backslash b) \in [u, c]$, for all $x \in \downarrow a$. Consequently, $\downarrow a \circ \{a \backslash b\} \subseteq [u, c]$, for all $[u, c]$ that contain l, so $\downarrow a \circ \{a \backslash b\} \subseteq l$. Since $k \subseteq \downarrow a$, we have $k \circ \{a \backslash b\} \in l$, so $a \backslash b \in k \backslash_{\mathbf{R}(\boldsymbol{\mathcal{A}})} l$.

Statement (3) is a direct consequence of (2) for $k = \downarrow a$ and $l = \downarrow b$.

(4) We will show that $\downarrow a \subseteq k$. If $x \in \downarrow a$, then $x \preceq a$. To show that $x \in k$, let $k \subseteq [u, b]$, for some $u \in A^{\alpha\mathcal{K}}$ and $b \in A$. Since $a \in k$, by assumption, we get $a \in [u, b]$, that is $u[a] \preceq b$. By (CUT) we obtain $u[x] \preceq b$, namely $x \in [u, b]$. Consequently, $x \in k$, by Lemma 4.2.

In the last paragraph, we have shown that if $c \in m \subseteq \downarrow c$, for some $c \in A$ and $m \in R(\boldsymbol{\mathcal{A}})$, then $m = \downarrow c$. For $c = a \bullet b$ and $m = \downarrow a \bullet \downarrow b$, we obtain $\downarrow a \bullet \downarrow b = \downarrow (a \bullet b)$ from (2). $\square$

It follows from Lemma 4.4(2) that if $\boldsymbol{\mathcal{A}} \in \mathcal{KGL}$, then the map $q : A \to R(\boldsymbol{\mathcal{A}})$ is an homomorphism from the partial $\mathcal{L}$-algebra $\mathbf{A}$ into the $r\ell u$-groupoid $\mathbf{R}(\boldsymbol{\mathcal{A}})$. In certain cases, q is actually an embedding. If $\boldsymbol{\mathcal{A}} \in \mathcal{KGL}^{\mathrm{f}}$, then q comes close to being an homomorphism, but it is not in general. Therefore, we call it a *quasi-homomorphism*.

For every partial assignment $f : \mathbf{Fm}_{\mathcal{K}} \rightharpoonup \mathbf{A}$, we let $H(f)$ be the set of all $\mathcal{L}$-homomorphisms $\bar{f} : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{R}(\boldsymbol{\mathcal{A}})$ that extend the assignment $\bar{f}(p) = \downarrow f(p)$, for all variables p of $\mathbf{Fm}_{\mathcal{L}}$ in $f^{-1}[A]$.

Lemma 4.5. *If $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \preceq \rangle$ is a sequent matrix in $\mathcal{KGL}^{\mathrm{f}}$, then for every partial assignment $f : \mathbf{Fm}_{\mathcal{K}} \rightharpoonup \mathbf{A}$, we have $f(a) \in \bar{f}(a) \subseteq \downarrow f(a)$, for every $a \in f^{-1}[A]$ and every $\bar{f} \in H(f)$. If $\boldsymbol{\mathcal{A}}$ is in $\mathcal{KGL}$, then $\bar{f}(a) = \downarrow f(a)$.*

Proof. Let $f : \mathbf{Fm}_{\mathcal{L}} \rightharpoonup \mathbf{A}$ be partial assignment and $\bar{f} \in H(f)$. By definition of $H(f)$ and $(\mathrm{id})^{\boldsymbol{\mathcal{A}}}$, the statement holds for the propositional variables in $f^{-1}[A]$. For $a = 1$, by Lemma 4.4(1), we have $f(1) = 1_{\mathbf{A}} \in \varepsilon_g = 1_{\mathbf{R}(\boldsymbol{\mathcal{A}})} \subseteq \downarrow 1_{\mathbf{A}} = \downarrow f(1)$, if $1_{\mathbf{A}}$ is defined. We proceed by induction; this is possible because $f^{-1}[A]$ is closed under subformulas. Let $a, b \in f^{-1}[A]$. Also, assume that $f(a) \in \bar{f}(a) \subseteq \downarrow f(a)$, $f(b) \in \bar{f}(b) \subseteq \downarrow f(b)$, $\bullet \in \mathcal{K}$ and $a \bullet b \in f^{-1}[A]$. By Lemma 4.4(2), we have $f(a) \bullet_A f(b) \in \bar{f}(a) \bullet_{\mathbf{R}(\boldsymbol{\mathcal{A}})} \bar{f}(b) \subseteq \downarrow (f(a) \bullet_{\mathbf{A}} f(b))$. Since f and $\bar{f}$ are homomorphisms, we have $f(a \bullet b) \in \bar{f}(a \bullet b) \subseteq \downarrow f(a \bullet b)$. Finally, if $\boldsymbol{\mathcal{A}}$ is in $\mathcal{KGL}$, then $\bar{f}(a) = \downarrow f(a)$, by Lemma 4.4(4). $\square$

We will use Lemma 4.5 to transform a failure of a property in sequent matrices to a failure of the property in $r\ell u$-groupoids.

4.2. **Cut elimination.** Here we prove the cut elimination property for **GL** and its subsystems. Recall the definition of validity of a sequent in a $r\ell u$-groupoid preceding Lemma 3.5.

Theorem 4.6. *Assume that $\mathcal{K}$ is a subset of $\mathcal{L}$ that contains the connective $\backslash$, s is a sequent fit for $\mathcal{K}$ and $\boldsymbol{\mathcal{A}} \in \mathcal{KGL}^{\mathrm{f}}$. If s is valid in $\mathbf{R}(\boldsymbol{\mathcal{A}})$ then it is valid in $\boldsymbol{\mathcal{A}}$.*

Proof. Assume that s is $(x \Rightarrow a)$ and that it holds in $\mathbf{R}(\boldsymbol{\mathcal{A}})$. Also, let $f : \mathbf{Fm}_{\mathcal{K}} \rightharpoonup \mathbf{A}$ be a partial assignment such that $f(s) \in A^{\gamma\mathcal{K}} \times A$. We will show that $f(x) \preceq_{\boldsymbol{\mathcal{A}}} f(a)$. Since s is valid in $\mathbf{R}(\boldsymbol{\mathcal{A}})$, for every homomorphism $\bar{f} : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{R}(\boldsymbol{\mathcal{A}})$ (which is total assignment) with $\bar{f} \in H(f)$, we have $(\bar{f}(x))^{\mathbf{R}(\boldsymbol{\mathcal{A}})} \subseteq \bar{f}(a)$. If $x = x^{\mathbf{Fm}_{\mathcal{L}}^{\gamma}}(b_1, \ldots, b_n)$, then $\bar{f}(x) = \bar{f}(x^{\mathbf{Fm}_{\mathcal{L}}^{\gamma}}(b_1, \ldots, b_n)) = x^{\mathbf{R}(\boldsymbol{\mathcal{A}})^{\gamma}}(\bar{f}(b_1), \ldots, \bar{f}(b_n))$ and $(\bar{f}(x))^{\mathbf{R}(\boldsymbol{\mathcal{A}})} = x^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{f}(b_1), \ldots, \bar{f}(b_n))$, since $\bar{f}$ is an assignment. We assumed that $f(s) \in A^{\gamma\mathcal{K}} \times A$, so $a, b_1, \ldots, b_n \in f^{-1}[A]$ and, by Lemma 4.5, $\bar{f}(a) \subseteq \downarrow f(a)$ and $f(b) \in \bar{f}(b)$, for all subformulas b of x. So,

$$\begin{array}{lll}
f(x) & = f(x^{\mathbf{Fm}_{\mathcal{L}}^{\gamma}}(b_1, \ldots, b_n)) & \\
& = x^{\mathbf{A}^{\gamma\mathcal{K}}}(f(b_1), \ldots, f(b_n)) & (f \text{ is a partial assignment}) \\
& \in x^{\mathcal{P}(\mathbf{A}^{\gamma\mathcal{K}})}(\bar{f}(b_1), \ldots, \bar{f}(b_n)) & (\circ \text{ in } \mathcal{P}(\mathbf{A}^{\gamma\mathcal{K}}) \text{ is element-wise}) \\
& \subseteq x^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{f}(b_1), \ldots, \bar{f}(b_n)) & (g \text{ is a closure operator}) \\
& = (\bar{f}(x))^{\mathbf{R}(\boldsymbol{\mathcal{A}})} &
\end{array}$$

Consequently, $f(x) \in (\bar{f}(x))^{\mathbf{R}(\boldsymbol{\mathcal{A}})} \subseteq \bar{f}(a) \subseteq \downarrow f(a)$. Thus, $f(x) \in \downarrow f(a)$ and $f(x) \preceq_{\boldsymbol{\mathcal{A}}} f(a)$. □

Theorem 4.7. *For every subset $\mathcal{K}$ of $\mathcal{L}$ that contains the connective $\backslash$ and for every sequent s fit for $\mathcal{K}$, s is valid in $\mathcal{KGL}^{\mathrm{f}}$ iff s (equivalently, $\varepsilon(s)$) is valid in* RLUG *iff s is valid in $\mathcal{KGL}$.*

Proof. If s is valid in $\mathcal{KGL}$, then it is valid in $\mathcal{KGL}^{\mathrm{f}}$, since $\mathcal{KGL}^{\mathrm{f}} \subseteq \mathcal{KGL}$, by Lemma 3.3. Conversely, if s is valid in $\mathcal{KGL}$, then it is valid in $\mathbf{R}(\boldsymbol{\mathcal{A}})$, for all $\boldsymbol{\mathcal{A}} \in \mathcal{KGL}^{\mathrm{f}}$. By Theorem 4.6, s is valid in all sequent matrices $\boldsymbol{\mathcal{A}}$ in $\mathcal{KGL}^{\mathrm{f}}$, so it is valid in $\mathcal{KGL}^{\mathrm{f}}$. Finally, the validity in RLUG of s is equivalent to the validity of $\varepsilon(s)$, by Lemma 3.5. □

The following result was proved in [14], using syntactical methods, in the special case where $\mathcal{K}$ is the full language $\mathcal{L}$.

Corollary 4.8. *The Gentzen system $\mathcal{K}$**GL** enjoys the cut elimination property, for all sublanguages $\mathcal{K}$ of $\mathcal{L}$ that include the connective $\backslash$.*

Proof. The corollary is a direct consequence of Theorem 4.7 and Lemma 3.4. □

Corollary 4.8 states that every sequent provable in **GL** without assumptions can be proved without the use of (CUT). The corresponding statement about sequents provable from assumptions is, however, not true. For example, (CUT) itself is not a derivable rule in $\mathbf{GL}^{\mathrm{f}}$. This can be shown either syntactically, by performing a proof search, or semantically by exhibiting a matrix in $\mathcal{KGL}$ but not in $\mathcal{KGL}^{\mathrm{f}}$.

A sequent calculus is said to have the *subformula property* if in any proof without assumptions all the formulas in the numerator of a rule are subformulas of formulas appearing in the denominator.

Corollary 4.9. *The system* **GL** *enjoys the subformula property.*

4.3. **Adding structural rules.** It is well known that for example **FL** and $\mathbf{FL_e}$ also enjoy the cut elimination property (see [32, 33]). As discussed in Section 2.1.6, these systems are equivalent to $\mathbf{GL}_{\mathrm{a}}$ and $\mathbf{GL}_{\mathrm{ae}}$, respectively. Using our previous results, we show in this section that, among others, the basic systems $\mathbf{GL}_{\mathrm{R}}$, where $\mathrm{R} \subseteq \{\mathrm{a}, \mathrm{e}, \mathrm{c}, \mathrm{i}, \mathrm{o}\}$, have the cut elimination property. (The result for (o) follows by an easy modification to the right-hand sides of the sequents.

Recall that a simple metarule is of the form

$$\frac{u[t_1] \Rightarrow a \quad \cdots \quad u[t_n] \Rightarrow a}{u[t_0] \Rightarrow a} \ (\mathrm{r})$$

for fixed elements $u \in \mathsf{U}$ and $a \in \mathsf{F}$, where $t_0, t_1, \ldots, t_n$ are simple metagroupoid words and t_0 is linear. For example the rules of exchange, weakening, contraction and associativity are simple. If t is a simple metagroupoid word, we write $t^{\mathbf{Fm}_{\mathcal{L}}}$ for the formula obtained from t by replacing $\circ$ with $\cdot$ and the metavariables from X with propositional variables from P; we assume that there is a fixed bijection between X and P. Clearly, t and $t^{\mathbf{Fm}_{\mathcal{L}}}$ are interdefinable. Clearly, the metarule (r) and the inequality $\varepsilon = (t_0^{\mathbf{Fm}_{\mathcal{L}}} \leq t_1^{\mathbf{Fm}_{\mathcal{L}}} \vee \cdots \vee t_n^{\mathbf{Fm}_{\mathcal{L}}})$ are interdefinable, as well. We denote by $\varepsilon(\mathrm{r})$ the inequality corresponding to the above rule and by $\mathrm{R}(\varepsilon)$ the rule corresponding to the above inequality.

Recall that if R is a set of metarules, then $\mathbf{GL}_{\mathrm{R}}$ denotes the system obtained from **GL** by adding the set R. If $\mathcal{K}$ is a sublanguage of $\mathcal{L}$ that contains $\backslash$, the metarule (r) is called *fit* for $\mathcal{K}$ if the metagroupoid words t_i are fit for $\mathcal{K}$, for all i. The system $\mathcal{K}\mathbf{GL}_{\mathrm{R}}$, is obtained by adding to the rules of $\mathcal{K}\mathbf{GL}$ all rules that are instances of the metarules in R and u, a evaluate so that all the resulting sequents are fit for $\mathcal{K}$. We denote the matrix models of $\mathcal{K}\mathbf{GL}_{\mathrm{R}}$ by $\mathcal{K}\mathcal{G}\mathcal{L}_{\mathrm{R}}$.

In RLUG, every equation ε over $\{\vee, \cdot, 1\}$ is equivalent to a conjunction of inequalities of the form above. To do this we distribute all products over all joins to reach a form $s_1 \vee \cdots \vee s_m \approx t_1 \vee \cdots \vee t_n$, where s_i, t_j are groupoid words with unit terms. Such an equation is in turn equivalent to the conjunction of the two inequalities $s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$. Finally, the fist one is equivalent to the conjunctions of the inequalities $s_j \leq t_1 \vee \cdots \vee t_n$; likewise, the second inequality is written as a conjunction, as well. If ε is an equation, by $\mathrm{R}(\varepsilon)$ we understand the set of rules associated with each of the conjuncts (inequalities) associated with ε.

Lemma 4.10. *Every equation ε over $\{\vee, \cdot, 1\}$ is equivalent, relative to* RLUG*, to* $\mathrm{R}(\varepsilon)$*. More precisely, for every* $\mathbf{A} \in \mathsf{RLUG}$*,* $\mathbf{A}$ *satisfies* ε *iff* $\mathbf{A}^{\mathcal{L}}$ *satisfies* $\mathrm{R}(\varepsilon)$*.*

Proof. It suffices to show the lemma for the case where ε is of the form $t_0^{\mathbf{Fm}_{\mathcal{L}}} \leq t_1^{\mathbf{Fm}_{\mathcal{L}}} \vee \cdots \vee t_n^{\mathbf{Fm}_{\mathcal{L}}}$. Clearly, $\mathbf{A}^{\mathcal{L}}$ satisfies $\mathrm{R}(\varepsilon)$ iff $\mathbf{A}$ satisfies the following implication: if $u[t_i^{\mathbf{Fm}_{\mathcal{L}}}] \leq a$ for all $i \in \{1, \ldots, n\}$, then $u[t_0^{\mathbf{Fm}_{\mathcal{L}}}] \leq a$, for all propositional variables a and all augmented groupoid words u over the set of propositional variables where $\circ$ is replaced by $\cdot$. Now, $u[t_i^{\mathbf{Fm}_{\mathcal{L}}}] \leq a$ is equivalent to $t_i \leq u \rightsquigarrow_{\mathcal{L}} a$, for all i, so the implication becomes: if $t_i^{\mathbf{Fm}_{\mathcal{L}}} \leq b$ for all $i \in \{1, \ldots, n\}$, then $t_0^{\mathbf{Fm}_{\mathcal{L}}} \leq b$, for all propositional variables b. By lattice-theoretic considerations this is equivalent to ε. $\square$

Theorem 4.11. *Let $\boldsymbol{\mathcal{A}}$ be a matrix in $\mathcal{KGL}^{\mathrm{f}}$ and let ε be an equation over $\{\vee, \cdot, 1\}$ such that all rules in $\mathrm{R}(\varepsilon)$ are fit for $\mathcal{K}$. Then, $\boldsymbol{\mathcal{A}}$ satisfies $\mathrm{R}(\varepsilon)$ iff $\mathbf{R}(\boldsymbol{\mathcal{A}})$ satisfies ε.*

Proof. Clearly, it suffices to show the lemma for the case where ε is of the form $t_0^{\mathbf{Fm}_\mathcal{L}} \leq t_1^{\mathbf{Fm}_\mathcal{L}} \vee \cdots \vee t_n^{\mathbf{Fm}_\mathcal{L}}$.

Assume that $\boldsymbol{\mathcal{A}}$ satisfies $\mathrm{R}(\varepsilon)$. Let $\bar{k} = (k_j)_{j\in J}$ be a sequence of elements in $\mathbf{R}(\boldsymbol{\mathcal{A}})$. We will show that $\varepsilon^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k})$ holds; i.e., $t_0^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k}) \subseteq t_1^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k}) \vee \cdots \vee t_n^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k})$. Assume that $t_1^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k}) \vee \cdots \vee t_n^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k}) \subseteq [u,a]$, for some $a \in A$, $u \in A^{\alpha\mathcal{K}}$; we will show that $t_0^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k}) \subseteq [u,a]$. We have $t_1^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k}) \cup \cdots \cup t_n^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k}) \subseteq t_1^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k}) \vee \cdots \vee t_n^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k})$, so for every $i \in \{1, \ldots n\}$, we have $t_i^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k}) \subseteq [u,a]$. If $x_j \in k_j$, for all $j \in J$, (we abbreviate this by $\bar{x} \in \bar{k}$) and $\bar{x} = (x_j)_{j\in J}$, then

$$\begin{array}{lll} t_i^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x}) & = t_i^{\mathbf{A}^{\gamma\mathcal{K}}}((x_j)_{j\in J}) & \\ & \in t_i^{\mathcal{P}(\mathbf{A}^{\gamma\mathcal{K}})}((\{x_j\})_{j\in J}) & \text{(by the definition of } \circ \text{ in } \mathcal{P}(\mathbf{A}^{\gamma\mathcal{K}})) \\ & \subseteq t_i^{\mathcal{P}(\mathbf{A}^{\gamma\mathcal{K}})}(\bar{k}) & \text{(operations are elementwise)} \\ & \subseteq g(t_i^{\mathcal{P}(\mathbf{A}^{\gamma\mathcal{K}})}(\bar{k})) & (g \text{ is a closure operator)} \\ & = t_i^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k}) \subseteq [u,a] & \text{(Lemma 4.3)} \end{array}$$

So, $u[t_i^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x})] \preceq a$, for all $i \in \{1, \ldots n\}$; hence $u[t_0^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x})] \preceq a$, by $r(\varepsilon)^{\boldsymbol{\mathcal{A}}}$, and $t_0^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x}) \in [u,a]$, for all $\bar{x} \in \bar{k}$. Since t_0 is a linear term and since the variables of t_0 are among the ones in $\{t_1, \ldots t_n\}$, we obtain $t_0^{\mathcal{P}(\mathbf{A}^{\gamma\mathcal{K}})}(\bar{k}) \subseteq [u,a]$; hence, by Lemma 4.3, $t_0^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k}) = g(t_0^{\mathcal{P}(\mathbf{A}^{\gamma\mathcal{K}})}(\bar{k})) \subseteq [u,a]$.

Conversely, assume that $\mathbf{R}(\boldsymbol{\mathcal{A}})$ satisfies ε. For every sequence $\bar{k} = (k_j)_{j\in J}$ of elements in $\mathbf{R}(\boldsymbol{\mathcal{A}})$, we have $t_0^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k}) \subseteq t_1^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k}) \vee \cdots \vee t_n^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{k})$. In particular, for $k_j = g(\{x_j\})$, where $x_j \in A^{\gamma\mathcal{K}}$, we have

$$t_0^{\mathbf{R}(\boldsymbol{\mathcal{A}})}((g(\{x_j\}))_{j\in J}) \subseteq t_1^{\mathbf{R}(\boldsymbol{\mathcal{A}})}((g(\{x_j\}))_{j\in J}) \vee \cdots \vee t_n^{\mathbf{R}(\boldsymbol{\mathcal{A}})}((g(\{x_j\}))_{j\in J}).$$

By Lemma 4.3

$$g(t_0^{\mathcal{P}(\mathbf{A}^{\gamma\mathcal{K}})}((\{x_j\})_{j\in J})) \subseteq g(t_1^{\mathcal{P}(\mathbf{A}^{\gamma\mathcal{K}})}((\{x_j\})_{j\in J}) \cup \cdots \cup t_n^{\mathcal{P}(\mathbf{A}^{\gamma\mathcal{K}})}((\{x_j\})_{j\in J})),$$

hence

$$g(\{t_0^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x})\}) \subseteq g(\{t_1^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x}), \ldots, t_n^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x})\}).$$

Therefore, for all $[u,a]$, where $a \in A$ and $u \in A^{\alpha}$, $g(\{t_1^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x}), \ldots, t_n^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x})\}) \subseteq [u,a]$ implies $g(\{t_0^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x})\}) \subseteq [u,a]$; i.e., $\{t_1^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x}), \ldots, t_n^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x})\} \subseteq [u,a]$ implies $t_0^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x}) \in [u,a]$. Consequently, $(u[t_1^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x})] \preceq a$ and $\ldots$ $u[t_n^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x})] \preceq a)$ implies $u[t_0^{\mathbf{A}^{\gamma\mathcal{K}}}(\bar{x})] \preceq a$; i.e., $r(\varepsilon)^{\boldsymbol{\mathcal{A}}}$ holds. □

It follows from Lemma 4.10 and Theorem 4.11 that if $\boldsymbol{\mathcal{A}}$ is a sequent $\mathcal{K}$ matrix, then $\mathbf{R}(\boldsymbol{\mathcal{A}})$ satisfies ε iff $\mathbf{R}(\boldsymbol{\mathcal{A}})^{\mathcal{K}}$ satisfies $\mathrm{R}(\varepsilon)$. Recall that we have agreed to say that in this case $\mathbf{R}(\boldsymbol{\mathcal{A}})$ satisfies $\mathrm{R}(\varepsilon)$.

We say that a set R of metarules is *preserved by* $\mathbf{R}$ with respect to $\mathcal{K}$, if for every sequent matrix $\boldsymbol{\mathcal{A}}$ in $\mathcal{KGL}^{\mathrm{f}}$, if $\boldsymbol{\mathcal{A}}$ satisfies R then $\mathbf{R}(\boldsymbol{\mathcal{A}})$ satisfies R; we naturally extend this definition for sets of metarules. The following corollary follows directly from Theorem 4.11.

Corollary 4.12. *All simple metarules are preserved by* $\mathbf{R}$. *In particular, the metarules of exchange, weakening, contraction and associativity are preserved by* $\mathbf{R}$.

The following theorem, for the case where $\mathcal{K} = \mathcal{L}$ and R contains at least associativity (a), was obtained independently in [41]. (Delays in the submission of the current paper are responsible for the time discrepancy.) Extensions of this result to other sequent calculi appear in [17]. Also, extensions to classes of structural rules that extend simple rules, as well as to hypersequent calculi appear in [13].

Theorem 4.13. *If* R *is a set of metarules that are preserved by* $\mathbf{R}$ *with respect to* $\mathcal{K}$*, then* $\mathcal{K}\mathbf{GL}_{\mathrm{R}}$ *enjoys the cut elimination property. In particular, for every equation* ε *over* $\{\vee, \cdot, 1\}$ *such that all rules in* $\mathrm{R}(\varepsilon)$ *are simple,* $\mathcal{K}\mathbf{GL}_{\mathrm{R}(\varepsilon)}$ *enjoys the cut elimination property.*

Proof. If $\vdash_{\mathcal{K}\mathbf{GL}_{\mathrm{R}}} s$, for a sequent s, then $\models_{\mathcal{K}\mathcal{GL}_{\mathrm{R}}} s$. Let $\boldsymbol{\mathcal{A}}$ be a matrix in $\mathcal{K}\mathcal{GL}^{\mathrm{f}}_{\mathrm{R}}$. Then $\boldsymbol{\mathcal{A}} \in \mathcal{K}\mathcal{GL}^{\mathrm{f}}$ and $\boldsymbol{\mathcal{A}}$ satisfies R. So, $\mathbf{R}(\boldsymbol{\mathcal{A}})^{\mathcal{K}}$ satisfies R, since R is preserved by $\mathbf{R}$ with respect to $\mathcal{K}$, and $\mathbf{R}(\boldsymbol{\mathcal{A}})^{\mathcal{K}} \in \mathcal{K}\mathcal{GL}$ by Theorem 4.1 and Theorem 3.6. Therefore, $\mathbf{R}(\boldsymbol{\mathcal{A}})^{\mathcal{K}} \in \mathcal{K}\mathcal{GL}_{\mathrm{R}}$, so $\models_{\mathbf{R}(\boldsymbol{\mathcal{A}})^{\mathcal{K}}} s$. By Theorem 4.6, $\models_{\boldsymbol{\mathcal{A}}} s$. Hence $\models_{\mathcal{K}\mathcal{GL}^{\mathrm{f}}_{\mathrm{R}}} s$ and $\vdash_{\mathcal{K}\mathbf{GL}^{\mathrm{f}}_{\mathrm{R}}} s$, by Lemma 3.4. □

Corollary 4.14. *The basic systems* $\mathbf{GL}_{\mathrm{R}}$*, where* R *is a subset of* $\mathrm{R} \subseteq \{\boldsymbol{a}, \boldsymbol{e}, \boldsymbol{c}, \boldsymbol{i}, \boldsymbol{o}\}$ *have the cut elimination property.*

We should clarify that $\mathbf{FL}_c$, unlike $\mathbf{FL_c}$ (boldface), does not enjoy the cut elimination property. Note that contraction for formulas (c) is not a simple metarule, so our results do not apply. In general, if a rule is formulated for formulas as opposed to groupoid words then the corresponding equation mentioned in Theorem 4.11 is properly stronger than the rule.

4.4. **The finite model property.** Let $\mathcal{K}$ be a sublanguage of $\mathcal{L}$ that contains the connective $\backslash$ and let R be a set of structural rules.

If s is a $\mathcal{K}$-sequent, we define $s^{\leftarrow}$ to be the smallest set of $\mathcal{K}$-sequents such that

- $s^{\leftarrow}$ contains s
- if $t \in s^{\leftarrow}$ and $(\{t_1, \ldots, t_n\}, t)$ is an instance of a metarule in $\mathcal{K}\mathbf{GL}^{\mathrm{f}}_{\mathrm{R}}$, for $n \in \{0, 1, 2\}$, then $t_1, \ldots, t_n \in s^{\leftarrow}$.

Let $\mathcal{K}$ be a sublanguage of $\mathcal{L}$ and let s be a $\mathcal{K}$-sequent. Consider the partial subalgebra $\mathbf{A}_{\mathcal{K}}(s) = \mathbf{A}_{\mathcal{K}}$ of the algebra $\mathbf{Fm}_{\mathcal{K}}$ of all subformulas of s. Consider the sequent $\mathcal{K}$-matrix $\boldsymbol{\mathcal{A}}_{\mathcal{K}}(s) = \boldsymbol{\mathcal{A}}_{\mathcal{K}} = \langle \mathbf{A}_{\mathcal{K}}, \preceq \rangle$, where $x \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}} a$ iff $\vdash_{\mathcal{K}\mathbf{GL}^{\mathrm{f}}_{\mathrm{R}}} x \Rightarrow a$. Also, consider the sequent $\mathcal{K}$-matrix $\boldsymbol{\mathcal{A}}'_{\mathcal{K}}(s) = \boldsymbol{\mathcal{A}}'_{\mathcal{K}} = \langle \mathbf{A}_{\mathcal{K}}, \preceq_{\boldsymbol{\mathcal{A}}'_{\mathcal{K}}} \rangle$, where $\preceq_{\boldsymbol{\mathcal{A}}'_{\mathcal{K}}} = \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}} \cup (s^{\leftarrow})^c$.

Lemma 4.15. *Let* $\mathcal{K}$ *be a sublanguage of* $\mathcal{L}$ *and let* s *be a* $\mathcal{K}$*-sequent. Then,* $\boldsymbol{\mathcal{A}}_{\mathcal{K}}(s)$ *and* $\boldsymbol{\mathcal{A}}'_{\mathcal{K}}(s)$ *are matrix models of* $\mathcal{K}\mathbf{GL}^{\mathrm{f}}_{\mathrm{R}}$.

Proof. For $\boldsymbol{\mathcal{A}}_{\mathcal{K}}$, it suffices to check the interpretations $(\mathrm{r})^{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$ of every metarule (r) of $\mathcal{K}\mathbf{GL}^{\mathrm{f}}_{\mathrm{R}}$ in $\boldsymbol{\mathcal{A}}_{\mathcal{K}}(s)$. Recall that $(\mathrm{r})^{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$ is of the form (see Section 3.3):

$\forall a, b, c \in A_{\mathcal{K}}, x \in A_{\mathcal{K}}^{\gamma\mathcal{K}}, u \in A_{\mathcal{K}}^{\alpha\mathcal{K}}$, (if $a \bullet_{\mathbf{A}_{\mathcal{K}}} b$ is defined,) then $t_1 \in \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$ and $t_2 \in \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$ implies $t \in \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$.

Assume that $a \bullet_{\mathbf{A}_{\mathcal{K}}} b$ is defined and both $t_1 \in \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$ and $t_2 \in \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$; we need to show that $t \in \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$. We have $\vdash_{\mathcal{K}\mathbf{GL}^{\mathrm{f}}_{\mathrm{R}}} t_1$ and $\vdash_{\mathcal{K}\mathbf{GL}^{\mathrm{f}}_{\mathrm{R}}} t_2$, so $\vdash_{\mathcal{K}\mathbf{GL}^{\mathrm{f}}_{\mathrm{R}}} t$, because $(\{t_1, t_2\}, t)$ is an instance of (r). Consequently, $t \in \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$, since $a \bullet_{\mathbf{A}_{\mathcal{K}}} b$ is defined.

Likewise, for $\boldsymbol{\mathcal{A}}'_{\mathcal{K}}$, we assume that $a \bullet_{\mathbf{A}_{\mathcal{K}}} b$ is defined, $t_1 \in \preceq_{\boldsymbol{\mathcal{A}}'_{\mathcal{K}}}$ and $t_2 \in \preceq_{\boldsymbol{\mathcal{A}}'_{\mathcal{K}}}$. Recall that $\preceq_{\boldsymbol{\mathcal{A}}'_{\mathcal{K}}} = \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}} \cup (s^{\leftarrow})^c$. If $t_1 \in \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$ and $t_2 \in \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$, then $t \in \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}} \subseteq \preceq_{\boldsymbol{\mathcal{A}}'_{\mathcal{K}}}$, by the argument above. Otherwise, without loss of generality, $t_1 \in (s^{\leftarrow})^c$. Since $(\{t_1, t_2\}, t)$ is an instance of (r), $t \in s^{\leftarrow}$ would imply $t_1 \in s^{\leftarrow}$, a contradiction. So, $t \in (s^{\leftarrow})^c \subseteq \preceq_{\boldsymbol{\mathcal{A}}'_{\mathcal{K}}}$. □

Lemma 4.16. *Let X, Y be sets, Z a subset of Y such that $Z^c = Y - Z$ is finite and F a set of 1-1 maps from X to Y such that $F^{-1}(y) = \{f^{-1}(y) \mid f \in F\}$ is finite for all $y \in Y$. Then, $F^{-1}[Z] = \{f^{-1}[Z] \mid f \in F\}$ is finite.*

Proof. We first show that $F^{-1}[W] = \{f^{-1}[W] \mid f \in F\}$ is finite, for every finite subset W of Y. Since $f^{-1}[W] \subseteq \bigcup_{g\in F} g^{-1}[W]$, it suffices to show that $\bigcup_{g\in F} g^{-1}[W]$ is finite. We have $\bigcup_{g\in F} g^{-1}[W] = \bigcup_{g\in F} \bigcup_{w\in W} g^{-1}(w) = \bigcup_{w\in W} \bigcup_{g\in F} g^{-1}(w) = \bigcup_{w\in W} F^{-1}(w)$. Since both W and $F^{-1}(w)$ are finite, $\bigcup_{g\in F} g^{-1}[W]$ is finite. Thus, $F^{-1}[W]$ is finite, if W is finite.

For all $f \in F$, and $x \in X$, we have $x \in f^{-1}[Z]$ iff $f(x) \notin Z^c$ iff $x \notin f^{-1}[Z^c]$ iff $x \in (f^{-1}[Z^c])^c$. Consequently, for all $f \in F$, $f^{-1}[Z] = (f^{-1}[Z^c])^c$; so $F^{-1}[Z] = \{(f^{-1}[Z^c])^c \mid f \in F\}$. Thus, $F^{-1}[Z]$ is bijective, under the bijection $U \mapsto U^c$, with the set $F^{-1}[Z^c] = \{f^{-1}[Z^c] \mid f \in F\}$, which is finite since Z^c is finite. □

A set R of simple structural metarules is called *reducing* if for every sequent s, $s^{\leftarrow}$ is finite. Note that the empty set of metarules is reducing.

Theorem 4.17. *The system $\mathcal{K}\mathbf{GL}^{\mathrm{f}}_{\mathrm{R}}$ has the finite model property for all subsets $\mathcal{K}$ of $\mathcal{L}$ and for all reducing sets R of metarules.*

Proof. Consider the matrix $\boldsymbol{\mathcal{A}}'_{\mathcal{K}}(s)$. Let s be a sequent such that $\not\vdash_{\mathcal{K}\mathbf{GL}^{\mathrm{f}}} s$. Then, $s \notin \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$, so $s \notin \preceq_{\boldsymbol{\mathcal{A}}'_{\mathcal{K}}}$, since $s \in s^{\leftarrow}$. So, $\not\models_{\boldsymbol{\mathcal{A}}'_{\mathcal{K}}(s)} s$ and $\not\models_{\mathbf{R}(\boldsymbol{\mathcal{A}}'_{\mathcal{K}}(s))} s$, by Theorem 4.6. We will show that $\mathbf{R}(\boldsymbol{\mathcal{A}}'_{\mathcal{K}}(s))$ is finite. It follows from Lemma 4.16 for $X = \mathbf{A}_{\mathcal{K}}^{\gamma\mathcal{K}}$, $Y = \mathbf{A}_{\mathcal{K}}^{\gamma\mathcal{K}}(s) \times \mathbf{A}_{\mathcal{K}}(s)$, $Z = \preceq_{\boldsymbol{\mathcal{A}}'_{\mathcal{K}}}$ and $F = \{f_{(u,a)} \mid (u, a) \in \mathbf{A}_{\mathcal{K}}^{\alpha\mathcal{K}} \times \mathbf{A}_{\mathcal{K}}(s)\}$, where $f_{(u,a)} = ((u, a) \star x) = (u[x] \Rightarrow a)$, that $F^{-1}[\preceq_{\boldsymbol{\mathcal{A}}'_{\mathcal{K}}}] = \{[u, a]_{\boldsymbol{\mathcal{A}}'_{\mathcal{K}}(s)} \mid u \in \mathbf{A}_{\mathcal{K}}^{\alpha\mathcal{K}}, a \in \mathbf{A}_{\mathcal{K}}\}$ is finite; the fact that $Y - Z$ is finite follows from the fact that $s^{\leftarrow}$ is finite. Every set in $\mathbf{R}(\boldsymbol{\mathcal{A}}'_{\mathcal{K}}(s))$ is an intersection of elements of $F^{-1}[\preceq_{\boldsymbol{\mathcal{A}}'_{\mathcal{K}}}]$, so $\mathbf{R}(\boldsymbol{\mathcal{A}}'_{\mathcal{K}}(s))$ is finite. □

Given a sequent s not provable in $\mathbf{GL}^{\mathrm{f}}$, using the method described in the proof of Theorem 4.17 we can construct a finite $r\ell u$-groupoid in which s fails. We will present a very simple example of this.

It is easy to see that the sequent $p \Rightarrow p \cdot p$ is not provable in $\mathbf{GL}^{\mathrm{f}}$, if p is a propositional variable. Actually, the only rule that can be applied in a proof search is $(\cdot\mathrm{R})$ and we obtain the only (up to permutation of the assumptions) incomplete proof:

$$\frac{p \Rightarrow p \quad \varepsilon \Rightarrow p}{p \Rightarrow p \cdot p}\ (\cdot\mathrm{R})$$

So, $(p \Rightarrow p \cdot p)^{\leftarrow} = \{(\varepsilon \Rightarrow p), (p \Rightarrow p), (p \Rightarrow p \cdot p)\}$. In order to construct $\mathbf{R}(\boldsymbol{\mathcal{A}}')$, we need to consider all subsets of A^{γ} of the form $[u, a]$, for $u \in A^{\alpha}$ and $a \in A$, and their intersections. Recall that $A = \{p, p \cdot p\}$ is the set of all subformulas of $p \Rightarrow p \cdot p$ and A^{γ} is the free groupoid over A. Also recall that $[u, a] = \{x \in A^{\gamma\mathcal{K}} \mid u[x] \preceq_{\boldsymbol{\mathcal{A}}'} a\}$ and a sequent is in $\preceq_{\boldsymbol{\mathcal{A}}'}$ iff it is provable in $\mathbf{GL}^{\mathrm{f}}$ or it is not in $(p \Rightarrow p \cdot p)^{\leftarrow}$. So the only way that $[u, a]$ is not all of A^{γ} is that for some x, $u[x] \Rightarrow a$ is in $(p \Rightarrow p \cdot p)^{\leftarrow}$ and $u[x] \Rightarrow a$ is not provable. Therefore, $[u, a] = A^{\gamma}$ except possibly for $[\_, p]$, $[\_ \circ p, p]$, $[p \circ \_, p]$, $[\_, p \cdot p]$, $[\_ \circ p, p \cdot p]$ and $[p \circ \_, p \cdot p]$. Note that $x \Rightarrow p$ is not in $(p \Rightarrow p \cdot p)^{\leftarrow}$, unless x is ε or p; also, $p \Rightarrow p$ is provable and $\varepsilon \Rightarrow p$ is not. Therefore, $x \preceq_{\boldsymbol{\mathcal{A}}'} p$ iff

$x \neq \varepsilon$, and $[\_, p] = \{\varepsilon\}^c$. Similarly, we can see that $[\_ \circ p, p] = [p \circ \_, p] = A^\gamma$, $[\_, p \cdot p] = \{p\}^c$ and $[\_ \circ p, p \cdot p] = [p \circ \_, p \cdot p] = \{\varepsilon\}^c$. Consequently,

$$R(\boldsymbol{\mathcal{A}}') = \{A^\gamma, \downarrow p = \{\varepsilon\}^c, \downarrow (p \cdot p) = \{p\}^c, \{\varepsilon, p\}^c\},$$

since it contains all intersections of the sets $[u, a]$. The order relation is set inclusion. Also, we have $\{\varepsilon\}^c \circ \{\varepsilon\}^c = (A \cup \{\varepsilon\})^c$; so $(\downarrow p) \cdot (\downarrow p) = \{\varepsilon\}^c \cdot \{\varepsilon\}^c = g(\{\varepsilon\}^c \circ \{\varepsilon\}^c) = g((A \cup \{\varepsilon\})^c) = \{\varepsilon, p\}^c$, since $(A \cup \{\varepsilon\})^c \subseteq \{\varepsilon, p\}^c$ and g is a closure operator. Therefore, $\downarrow p \not\subseteq (\downarrow p)^2$ and $p \Rightarrow p \cdot p$ is not valid in $\mathbf{R}(\boldsymbol{\mathcal{A}}')$.

Obviously, the construction in the proof of Theorem 4.17 did not produce the smallest counterexample to the equation $p \leq p^2$, since a three-element chain would also work. We present an alternative proof of Theorem 4.17 that produces smaller counterexamples. The proof is along the same lines as the proof in [31].

First for every $\mathcal{K}$-sequent s, we define $s^{\Leftarrow}$ as the smallest set of $\mathcal{K}$-sequents that satisfies the conditions in the definition of $s^{\leftarrow}$ plus the condition

- If $(u[x] \Rightarrow a) \in s^{\Leftarrow}$, then $(|u| \Rightarrow a) \in s^{\Leftarrow}$.

It is easy to see that if $s^{\leftarrow}$ is finite, then $s^{\Leftarrow}$ is also finite. Also, we define the sequent $\mathcal{K}$-matrix $\boldsymbol{\mathcal{B}}_{\mathcal{K}}(s) = \boldsymbol{\mathcal{B}}_{\mathcal{K}} = \langle \mathbf{A}_{\mathcal{K}}, \preceq_{\boldsymbol{\mathcal{B}}_{\mathcal{K}}} \rangle$, where $\preceq_{\boldsymbol{\mathcal{B}}_{\mathcal{K}}} = \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}} \cup (s^{\Leftarrow})^c$. It can be easily shown, as in Lemma 4.15, that $\boldsymbol{\mathcal{B}}_{\mathcal{K}}$ is a matrix model of $\mathcal{K}\mathbf{GL}^{\mathrm{f}}$. We can now prove Theorem 4.17 using the matrix $\boldsymbol{\mathcal{B}}_{\mathcal{K}}$.

Proof. Let s be a sequent such that $\not\vdash_{\mathcal{K}\mathbf{GL}^{\mathrm{f}}} s$. Then, $s \notin \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$, so $s \notin \preceq_{\boldsymbol{\mathcal{B}}_{\mathcal{K}}}$, since $s \in s^{\Leftarrow}$. So, $\not\models_{\boldsymbol{\mathcal{B}}_{\mathcal{K}}(s)} s$ and $\not\models_{\mathbf{R}(\boldsymbol{\mathcal{B}}_{\mathcal{K}}(s))} s$, by Theorem 4.6. We will show that $\mathbf{R}(\boldsymbol{\mathcal{B}}_{\mathcal{K}}(s))$ is finite. If $(|u| \Rightarrow a) \notin s^{\Leftarrow}$, then $(u[x] \Rightarrow a) \notin s^{\Leftarrow}$, for all $x \in \mathbf{A}_{\mathcal{K}}^{\gamma_{\mathcal{K}}}$, hence $(u[x] \Rightarrow a) \in \preceq_{\boldsymbol{\mathcal{B}}_{\mathcal{K}}}$, for all $x \in \mathbf{A}_{\mathcal{K}}^{\gamma_{\mathcal{K}}}$ and $[u, a]_{\boldsymbol{\mathcal{B}}_{\mathcal{K}}(s)} = \mathbf{A}_{\mathcal{K}}^{\gamma_{\mathcal{K}}}$. Since there are only finitely many sequents in $s^{\Leftarrow}$, the set $D = \{[u, a]_{\boldsymbol{\mathcal{B}}_{\mathcal{K}}(s)} \mid u \in \mathbf{A}_{\mathcal{K}}^{\alpha_{\mathcal{K}}}, a \in \mathbf{A}_{\mathcal{K}}\}$ is finite. Every set in $\mathbf{R}(\boldsymbol{\mathcal{B}}_{\mathcal{K}}(s))$ is an intersection of elements of D, so $\mathbf{R}(\boldsymbol{\mathcal{B}}_{\mathcal{K}}(s))$ is finite. $\square$

We revisit the same example of $p \Rightarrow p \cdot p$ and describe the $r\ell u$-groupoid $\mathbf{R}(\boldsymbol{\mathcal{B}})$. According to the last proof we need only consider sets $[u, a]$ such that $(|u| \Rightarrow a) \in (p \Rightarrow p \cdot p)^{\Leftarrow} = \{(\varepsilon \Rightarrow p), (p \Rightarrow p), (p \Rightarrow p \cdot p), (\varepsilon \Rightarrow p \cdot p)\}$, since all other such sets are equal to A^γ; note that $(p \Rightarrow p \cdot p)^{\Leftarrow}$ is bigger than $(p \Rightarrow p \cdot p)^{\leftarrow}$. Also, note that $x \preceq_{\boldsymbol{\mathcal{B}}} p \cdot p$ iff x is ε or p, since $(\varepsilon \Rightarrow p \cdot p) \in (p \Rightarrow p \cdot p)^{\Leftarrow}$, even though $(\varepsilon \Rightarrow p \cdot p) \notin (p \Rightarrow p \cdot p)^{\leftarrow}$. So, $[\_, p \cdot p]$ is equal to $\{\varepsilon, p\}^c$ and not to $\{p\}$, as in the previous construction. It can be easily verified that $[\_, p] = [\_ \circ p, p \cdot p] = [p \circ \_, p \cdot p] = \{\varepsilon\}^c$ and $[\_ \circ p, p] = [p \circ \_, p] = A^\gamma$. Consequently,

$$R(\boldsymbol{\mathcal{B}}) = \{A^\gamma, \downarrow p = \{\varepsilon\}^c, \downarrow (p \cdot p) = \{\varepsilon, p\}^c\}.$$

Also, $(\downarrow p) \circ (\downarrow p) = \{\varepsilon\}^c \circ \{\varepsilon\}^c = (A \cup \{\varepsilon\})^c$ and $(\downarrow p) \cdot (\downarrow p) = g((A \cup \{\varepsilon\})^c) = \{\varepsilon, p\}^c$; hence $\downarrow p \not\subseteq (\downarrow p)^2$. Observe that $\mathbf{R}(\boldsymbol{\mathcal{B}})$ is a smaller counterexample than $\mathbf{R}(\boldsymbol{\mathcal{A}}')$; actually it is the smallest counterexample to $p \Rightarrow p \cdot p$. Nevertheless, for the sequent $p \Rightarrow 1$ the construction does not produce a counterexample of minimum cardinality. We mention, without details, that for $p \Rightarrow 1$, $\mathbf{R}(\boldsymbol{\mathcal{B}}) = \{A^{\gamma_{\mathcal{K}}}, [p \circ \_, 1] = \{\varepsilon\}^c, \downarrow p = \{p\}^c, \{\varepsilon, p\}^c\}$, but the smallest counterexample has 3 elements.

4.5. **Strong separation.** Let $\mathcal{K}$ be a sublanguage of $\mathcal{L}$ that contains the connective $\backslash$. The *strong separation property* for **HL** states that $B \vdash_{\mathbf{HL}} c$ iff $B \vdash_{\mathcal{K}\mathbf{HL}} c$, for all sets of formulas $B \cup \{c\}$ over $\mathcal{K}$. Also, the *separation property* for **HL** states that $\vdash_{\mathbf{HL}} c$ iff $\vdash_{\mathcal{K}\mathbf{HL}} c$, for all formulas c over $\mathcal{K}$.

The separation property for **HL** follows from the cut elimination property of $\mathcal{K}\mathbf{GL}$ and the equivalence of the systems $\mathcal{K}\mathbf{HL}$ and $\mathcal{K}\mathbf{GL}$. In detail, if $\vdash_{\mathbf{HL}} c$, then $\vdash_{\mathbf{GL}} \varepsilon \Rightarrow c$, by Theorem 2.3, and $\vdash_{\mathbf{GL}^{\mathrm{f}}} \varepsilon \Rightarrow c$, by Corollary 4.8. Since $\varepsilon \Rightarrow c$ is provable without (CUT), then we can obtain a proof of it by a proof search. It is not hard to see that resulting proof will involve only the rules in $\mathcal{K}\mathbf{GL}$. Since $\vdash_{\mathcal{K}\mathbf{GL}} \varepsilon \Rightarrow c$, we get $\vdash_{\mathcal{K}\mathbf{HL}} c$, by Theorem 2.3. The converse direction is obvious.

A proof of the strong separation property cannot be obtained by a similar argument, since the systems $\mathcal{K}\mathbf{GL}$ and $\mathcal{K}\mathbf{GL}^{\mathrm{f}}$ are not equivalent. Nevertheless, **HL** has the strong separation property, as we prove below.

Let $\mathcal{K}$ be a sublanguage of $\mathcal{L}$ that contains the connective $\backslash$ and let $B \cup \{c\}$ be a set of formulas over $\mathcal{K}$; also, let R be a set of simple structural metarules fit for $\mathcal{K}$. We denote by $\mathbf{A}_{\mathcal{K}} = \mathbf{A}_{\mathcal{K}}(B, c)$ the partial subalgebra of $\mathbf{Fm}_{\mathcal{K}}$ of all subformulas of $B \cup \{c\}$. Consider the sequent $\mathcal{K}$-matrix $\boldsymbol{\mathcal{A}}_{\mathcal{K}}(B, c) = \boldsymbol{\mathcal{A}}_{\mathcal{K}} = \langle \mathbf{A}_{\mathcal{K}}, \preceq \rangle$, where $x \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}} a$ iff $B \vdash_{\mathcal{K}\mathbf{HL}_{\mathrm{R}}} \phi_{\mathcal{K}}(x \Rightarrow a)$.

Corollary 4.18. *Let $\mathcal{K}$ be a sublanguage of $\mathcal{L}$ that contains the connective $\backslash$, let $B \cup \{c\}$ be a set of $\mathcal{K}$-formulas and let* R *be a set of simple structural metarules fit for $\mathcal{K}$. The sequent matrix $\boldsymbol{\mathcal{A}}_{\mathcal{K}}(B, c)$ is in $\mathcal{K}\mathcal{GL}_{\mathrm{R}}$.*

Proof. Let (r) be a metarule of $\mathcal{K}\mathbf{GL}_{\mathrm{R}}$ that may involve the connective $\bullet \in \mathcal{K}$. Recall that $(\mathrm{r})^{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$ is of the form (see Section 3.3):

$$\forall a, b, c \in A_{\mathcal{K}}, x \in A_{\mathcal{K}}^{\gamma_{\mathcal{K}}}, u \in A_{\mathcal{K}}^{\alpha_{\mathcal{K}}}, \text{ if } a \bullet_{\mathbf{A}_{\mathcal{K}}} b \text{ is defined, then } s_1 \in \preceq_{\mathbf{A}_{\mathcal{K}}} \text{ and } \ldots \text{ and } s_n \in \preceq_{\mathbf{A}_{\mathcal{K}}} \text{ implies } s \in \preceq_{\mathbf{A}_{\mathcal{K}}}.$$

By Lemma 3.3, we need to show that $(r)^{\boldsymbol{\mathcal{A}}_{\mathcal{K}}}$ holds, so assume that $a \bullet_{\mathbf{A}_{\mathcal{K}}} b$ is defined and $s_i \in \preceq_{\mathbf{A}_{\mathcal{K}}}$, for all i. By definition, we get $B \vdash_{\mathcal{K}\mathbf{HL}_{\mathrm{R}}} \phi_{\mathcal{K}}(s_i)$, for all i. Since (r) is a metarule of $\mathcal{K}\mathbf{GL}_{\mathrm{R}}$, its instance $(r') = (\{s_1, \ldots, s_n\}, s)$ holds in $\mathcal{K}\mathbf{GL}_{\mathrm{R}}$; i.e. $\{s_1, \ldots, s_n\} \vdash_{\mathcal{K}\mathbf{GL}_{\mathrm{R}}} s$. By Theorem 2.3 we get $\{\phi_{\mathcal{K}}(s_1), \ldots, \phi_{\mathcal{K}}(s_n)\} \vdash_{\mathcal{K}\mathbf{HL}_{\mathrm{R}}} \phi_{\mathcal{K}}(s)$; let Π be a proof in $\mathcal{K}\mathbf{HL}_{\mathrm{R}}$ of this deduction. Let Π_i be a proof of $\phi_{\mathcal{K}}(s_i)$ from B in $\mathcal{K}\mathbf{HL}_{\mathrm{R}}$ for all i. Then

$$\frac{\Pi_1 \quad \Pi_2 \quad \cdots \quad \Pi_n}{\Pi}$$

is a proof of $\phi_{\mathcal{K}}(s)$ in $\mathcal{K}\mathbf{HL}_{\mathrm{R}}$ from B. Hence $s \in \preceq_{\mathbf{A}_{\mathcal{K}}}$. $\square$

Corollary 4.19. *If $B \cup \{c\}$ is a set of formulas over a sublanguage $\mathcal{K}$ of $\mathcal{L}$ that contains $\backslash$ and and let* R *is a set of simple structural metarules fit for $\mathcal{K}$, then $B \vdash_{\mathbf{HL}_{\mathrm{R}}} c$ iff $\{1 \leq b \,|\, b \in B\} \models_{\mathsf{RLUG}_{\mathrm{R}}} 1 \leq c$ iff $B \vdash_{\mathcal{K}\mathbf{HL}_{\mathrm{R}}} c$. In particular, the Hilbert system* **HL** *enjoys the strong separation property.*

Proof. If $B \vdash_{\mathbf{HL}_{\mathrm{R}}} c$, then $s[B] \vdash_{\mathbf{GL}_{\mathrm{R}}} s(c)$ by Theorem 2.3. If $\boldsymbol{\mathcal{A}}_{\mathcal{K}} = \boldsymbol{\mathcal{A}}_{\mathcal{K}}(B, b)$ then $\boldsymbol{\mathcal{A}}_{\mathcal{K}}(B, c) \in \mathcal{K}\mathcal{GL}_{\mathrm{R}}$ by Corollary 4.18, so $\mathbf{R}(\boldsymbol{\mathcal{A}}_{\mathcal{K}}) \in \mathsf{RLUG}_{R}$, by Theorem 4.1 and Theorem 4.11. So $\mathbf{R}(\boldsymbol{\mathcal{A}}_{\mathcal{K}})^{\mathcal{L}} \in \mathcal{GL}_{\mathrm{R}}$ by Lemma 4.10, and $s[B] \vdash_{\mathbf{R}(\boldsymbol{\mathcal{A}}_{\mathcal{K}})^{\mathcal{L}}} s(c)$. Consequently, $\{1 \leq b \mid b \in B\} \models_{\mathbf{R}(\boldsymbol{\mathcal{A}}_{\mathcal{K}})} 1 \leq c$, in view of Lemma 3.5 and the fact that $\varepsilon(s(c)) = (1 \leq c)$. Consider the identity partial map $f : \mathbf{Fm}_{\mathcal{K}} \rightharpoonup \mathbf{A}_{\mathcal{K}}$ on the subformulas of $B \cup \{c\}$ and let $\bar{f} : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{R}(\boldsymbol{\mathcal{A}}_{\mathcal{K}})$ be a homomorphism in $H(f)$ (recall the definition of $H(f)$ preceding Lemma 4.5). So,

$$\text{if } \bar{f}(1) \subseteq_{\mathbf{R}(\boldsymbol{\mathcal{A}}_{\mathcal{K}})} \bar{f}(b), \text{ for all } b \in B, \text{ then } \bar{f}(1) \subseteq_{\mathbf{R}(\boldsymbol{\mathcal{A}}_{\mathcal{K}})} \bar{f}(c).$$

Since $\bar{f}$ is a $\mathcal{L}$-homomorphism we have $\bar{f}(1) = \varepsilon_g = g_{\preceq}(\{\varepsilon\})$. Moreover, since $\boldsymbol{\mathcal{A}}_{\mathcal{K}} \in \mathcal{K}\mathcal{GL}_{\mathrm{R}}$, for every subformula d of $B \cup \{c\}$, $\bar{f}(d) = \downarrow f(d) = \downarrow d$, by Lemma 4.5.

Consequently, $\bar{f}(1) \subseteq_{\mathbf{R}(\boldsymbol{\mathcal{A}}_{\mathcal{K}})} \bar{f}(d)$ iff $g_{\preceq}(\{\varepsilon\}) \subseteq_{\mathbf{R}(\boldsymbol{\mathcal{A}}_{\mathcal{K}})} \downarrow d$ iff $\varepsilon \in \downarrow d$ iff $\varepsilon \preceq_{\boldsymbol{\mathcal{A}}_{\mathcal{K}}} d$. This is in turn equivalent to $B \vdash_{\mathcal{K}\mathbf{HL}_{\mathrm{R}}} d$ by definition, so we have that $B \vdash_{\mathcal{K}\mathbf{HL}_{\mathrm{R}}} b$, for all $b \in B$ implies $B \vdash_{\mathcal{K}\mathbf{HL}_{\mathrm{R}}} c$. Consequently, we obtain $B \vdash_{\mathcal{K}\mathbf{HL}_{\mathrm{R}}} c$. $\square$

Note that $u \approx v =\mid\models_{\mathsf{RLUG}} 1 \leq u\backslash v \wedge v\backslash u$, for all pairs of terms u, v. Consequently, RLUG is the equivalent algebraic semantics of $\vdash_{\mathbf{HL}}$ for $\varepsilon(p) = (p \wedge 1 \approx 1)$ and $\Delta(p, q) = p\backslash q \wedge q\backslash p$.

Note that the terms $x \circ (y \circ z)$ and $(x \circ y) \circ z$ are fit for any language that contains $\backslash$ and at least one of $/$ and multiplication. It follows from Corollary 4.19 that for such a languages $\mathcal{K}$, the system $\mathcal{K}\mathbf{HL_a}$ enjoys the strong separation property. Recall that the (bidirectional) Hilbert-style rule that corresponds to the simple Gentzen-style rule of associativity is

$$\frac{x \circ (y \circ z) \rightsquigarrow_{\mathcal{K}} d}{(x \circ y) \circ z \rightsquigarrow_{\mathcal{K}} d} \ \mathrm{h}(\mathbf{a})$$

The next result simplifies this rule to an axiom.

Lemma 4.20. *Let $\mathcal{K}$ be a set of connectives that contains $\backslash$.*

(1) *If $\mathcal{K}$ contains multiplication, then* $\mathrm{h}(\mathbf{a})$ *is equivalent to the combination of the axioms* $[(ab)c]\backslash[a(bc)]$ *and* $[a(bc)]\backslash[(ab)c]$.

(2) *If $\mathcal{K}$ contains $/$, then* $\mathrm{h}(\mathbf{a})$ *is equivalent to the combination of the axioms* $[(a\backslash d)/c]\backslash[a\backslash(d/c)]$ *and* $[a\backslash(d/c)]\backslash[(a\backslash d)/c]$.

Proof. (1) Since $\mathcal{K}$ contains multiplication, $(x \circ y) \circ z \rightsquigarrow_{\mathcal{K}} d = [\phi_{\mathcal{K}}(x)\phi_{\mathcal{K}}(y)]\phi_{\mathcal{K}}(z)\backslash d$ and $x \circ (y \circ z) \rightsquigarrow_{\mathcal{K}} d = \phi_{\mathcal{K}}(x)[\phi_{\mathcal{K}}(y)\phi_{\mathcal{K}}(z)]\backslash d$. Therefore, $\mathrm{h}(\mathbf{a})$ is equivalent to

$$\frac{[(ab)c]\backslash d}{[a(bc)]\backslash d} \ (\cdot\mathrm{h}(\mathbf{a}))$$

This implies the two axioms, by instantiating d to $a(bc)$ and $(ab)c$. The converse is also true by (T_ℓ), a rule that is shown to be derivable in Lemma A.2.

(2) We first consider the case where $\mathcal{K}$ contains $/$, but not multiplication; clearly both $x \circ (y \circ z)$ and $(x \circ y) \circ z$ need to be solvable. The only case where the rule does not trivialize is when x and z are formulas; for example, if x consists of more than one formula then at least one of y and z need to be empty.

If y is also a single formula, then $\mathrm{h}(\mathbf{a})$ is equivalent to the instance

$$\frac{c\backslash[b\backslash(a\backslash d)]}{b\backslash[a\backslash(d/c)]}$$

Instantiating this for $b = (a\backslash d)/c$, yields the target formula $[(a\backslash d)/c]\backslash[a\backslash(d/c)]$ in the denominator and $c\backslash\{[(a\backslash d)/c]\backslash(a\backslash d)\}$ in the numerator, which is just an instance of $(\mathrm{As}_{\ell\ell})$. Likewise, $\mathrm{h}(\mathbf{a})$ implies $[a\backslash(d/c)]\backslash[(a\backslash d)/c]$.

Conversely, starting from $c\backslash[b\backslash(a\backslash d)]$ we first obtain $b\backslash[(a\backslash d)/c]$, by $(\mathrm{RA}r_\ell)$. Using the first axiom and $(\mathrm{Rd}\backslash)$ we obtain $\{b\backslash[(a\backslash d)/c]\}\backslash\{b\backslash[a\backslash(d/c)]\}$. Hence by (MP_ℓ), we have $b\backslash[a\backslash(d/c)]$, which completes the derivation of the downward direction of $\mathrm{h}(\mathbf{a})$. Likewise, we obtain the upward direction.

If y consists of at least two formulas, then $\mathrm{h}(\mathbf{a})$ is equivalent to the instance

$$\frac{y \rightsquigarrow_{\mathcal{K}} a\backslash(d/c)}{y \rightsquigarrow_{\mathcal{K}} (a\backslash d)/c}$$

which, by similar arguments, is equivalent to the combination of the two axioms.

We will show that if $\mathcal{K}$ contains both / and multiplication, then the two sets of axioms are equivalent, and invoke (1). We saw that the two axioms in (1) are equivalent to the bidirectional rule ($\cdot$h($\mathbf{a}$)). Likewise, by instantiating b appropriately, it is easy to see that the two axioms in (2) are equivalent to the bidirectional rule

$$\genfrac{}{}{}{}{b\backslash[(a\backslash d)/c]}{\overline{b\backslash[a\backslash(d/c)]}}\ (/\mathrm{h}(\mathbf{a}))$$

We claim that the rules ($\cdot$h($\mathbf{a}$)) and (/h($\mathbf{a}$)) are equivalent. As an example, we demonstrate one of the four directions.

$$\dfrac{\dfrac{\dfrac{}{c\backslash[b\backslash(bc)]}\ (\mathrm{PI}) \qquad \dfrac{\dfrac{}{(bc)\backslash[a\backslash(a(bc))]}\ (\mathrm{PI}) \qquad [a(bc)]\backslash d}{(bc)\backslash(a\backslash d)}\ (\mathrm{T}_\ell \rightsquigarrow_{\mathcal{K}})}{c\backslash[b\backslash(a\backslash d)]}\ (\mathrm{T}_\ell \rightsquigarrow_{\mathcal{K}})}{\overline{b\backslash[(a\backslash d)/c]}}\ (\mathrm{RA}r_\ell)$$

where the rule ($\mathrm{T}_\ell \rightsquigarrow_{\mathcal{K}}$) is shown to be derivable in Lemma A.2. □

Lemma 4.20(2) provides a Hilbert-style system that is equivalent to **FL** and has the strong separation property with respect to the set $\{\backslash, /\}$ of basic connectives.

4.6. **Further results.** We conclude the section by presenting two more applications of the quasiembedding result.

4.6.1. *Algebraic semantics.* The next theorem is a strengthening of Theorem 4.6 for matrices in $\boldsymbol{\mathcal{A}} \in \mathcal{K}\mathcal{G}\mathcal{L}$.

Theorem 4.21. *Assume that $\mathcal{K}$ is a subset of $\mathcal{L}$ that contains the connective $\backslash$, $S \cup \{s_0\}$ is a set of sequents fit for $\mathcal{K}$ and $\boldsymbol{\mathcal{A}} \in \mathcal{K}\mathcal{G}\mathcal{L}$. If $S \models_{\mathbf{R}(\boldsymbol{\mathcal{A}})} s_0$ then $S \models_{\boldsymbol{\mathcal{A}}} s_0$.*

Proof. Let $f : \mathbf{Fm}_{\mathcal{K}} \rightharpoonup \mathbf{A}$ be a partial assignment such that $f[S \cup \{s_0\}] \subseteq A^{\gamma_{\mathcal{K}}} \times A$. We will show that $f[S] \subseteq \preceq_{\boldsymbol{\mathcal{A}}}$ implies $f(s_0) \in \preceq_{\boldsymbol{\mathcal{A}}}$. Since $S \models_{\mathbf{R}(\boldsymbol{\mathcal{A}})} s_0$, for every homomorphism $\bar{f} : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{R}(\boldsymbol{\mathcal{A}})$ with $\bar{f} \in H(f)$ (see definition before Lemma 4.5), we have that $\bar{f}[S] \subseteq \preceq_{\mathbf{R}(\boldsymbol{\mathcal{A}})}$ implies $\bar{f}(s_0) \in \preceq_{\mathbf{R}(\boldsymbol{\mathcal{A}})}$.

It suffices to show that $\bar{f}(s) \in \preceq_{\mathbf{R}(\boldsymbol{\mathcal{A}})}$ iff $f(s) \in \preceq_{\boldsymbol{\mathcal{A}}}$, for all $s \in S \cup \{s_0\}$. Let $s = (x \Rightarrow a_0)$ and $x = x^{\mathbf{Fm}_{\mathcal{K}}^{\gamma_{\mathcal{K}}}}(a_1, \ldots, a_n)$. We have $f(s) \in \preceq_{\boldsymbol{\mathcal{A}}}$ iff $f(x) \preceq_{\boldsymbol{\mathcal{A}}} f(a_0)$ iff $f(x) \in\downarrow f(a_0)$. On the other hand, $\bar{f}(s) \in \preceq_{\mathbf{R}(\boldsymbol{\mathcal{A}})}$ iff $(\bar{f}(x))^{\mathbf{R}(\boldsymbol{\mathcal{A}})} \subseteq \bar{f}(a_0)$ iff $\downarrow f(x) \subseteq\downarrow f(a_0)$ iff $f(x) \in\downarrow f(a_0)$.

We used above that $(\bar{f}(x))^{\mathbf{R}(\boldsymbol{\mathcal{A}})} =\downarrow f(x)$. Indeed, we have $\bar{f}(x) = \bar{f}(x^{\mathbf{Fm}_{\mathcal{K}}^{\gamma_{\mathcal{K}}}}(a_1, \ldots, a_n))$ $= x^{\mathbf{R}(\boldsymbol{\mathcal{A}})^{\gamma_{\mathcal{K}}}}(\bar{f}(a_1), \ldots, \bar{f}(a_n))$ and $(\bar{f}(x))^{\mathbf{R}(\boldsymbol{\mathcal{A}})} = x^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\bar{f}(a_1), \ldots, \bar{f}(a_n))$, since $\bar{f}$ is an assignment, and $\bar{f}(a_i) =\downarrow a_i$, for all i, by Lemma 4.5. Moreover, by Lemma 4.4(4) and the fact that f is a assignment, we have $x^{\mathbf{R}(\boldsymbol{\mathcal{A}})}(\downarrow f(a_1), \ldots, \downarrow f(a_n)) =\downarrow x^{A^{\gamma_{\mathcal{K}}}}(f(a_1), \ldots, f(a_n)) =\downarrow f(x^{\mathbf{Fm}_{\mathcal{K}}^{\gamma_{\mathcal{K}}}}(a_1, \ldots, a_n)) =\downarrow f(x)$. □

Lemma 4.22. *The variety RLUG_R of all rℓu-groupoids that satisfy a set of simple rules R is an algebraic semantics for $\mathbf{GL}_R$; i.e., for all sets of sequents $S \cup \{s\}$, we have $S \vdash_{\mathbf{GL}_R} s$ iff $\varepsilon[S] \models_{\mathsf{RLUG}_R} \varepsilon(s)$.*

Proof. In view of Theorem 3.4, it suffices to show that $S \models_{\mathcal{K}\mathcal{G}\mathcal{L}_R} s$ iff $\varepsilon[S] \models_{\mathsf{RLUG}_R} \varepsilon(s)$.

If $\varepsilon[S] \models_{\mathsf{RLUG}_R} \varepsilon(s)$, then $\varepsilon[S] \models_{\mathbf{R}(\boldsymbol{\mathcal{A}})} \varepsilon(s)$ for all $\boldsymbol{\mathcal{A}} \in \mathcal{K}\mathcal{G}\mathcal{L}_R$, by Corollary 4.12. In view of Lemma 3.5 and Theorem 4.1, $S \models_{\mathbf{R}(\boldsymbol{\mathcal{A}})^{\mathcal{K}}} s$, for all $\boldsymbol{\mathcal{A}} \in \mathcal{K}\mathcal{G}\mathcal{L}_R$. By Theorem 4.21, $S \models_{\boldsymbol{\mathcal{A}}} s$ for all $\boldsymbol{\mathcal{A}} \in \mathcal{K}\mathcal{G}\mathcal{L}_R$, so $S \models_{\mathcal{K}\mathcal{G}\mathcal{L}_R} s$.

Conversely, if $S \models_{\mathcal{K}\mathcal{G}\mathcal{L}_R} s$, then $S \models_{\mathbf{A}^{\mathcal{K}}} s$, for all $\mathbf{A} \in \mathsf{RLUG}_R$. By Lemma 3.5, we have $\varepsilon[S] \models_{\mathbf{A}} \varepsilon(s)$, for all $\mathbf{A} \in \mathsf{RLUG}_R$; hence $\varepsilon[S] \models_{\mathsf{RLUG}_R} \varepsilon(s)$. □

For an equation $u \approx v$, we define $s(u \approx v) = \{u \Rightarrow v, v \Rightarrow u\}$. Note that $\varepsilon[s(u \approx v] = \{u \leq v, v \leq u\}$ It is obvious that $u \approx v = \mid\models_{\mathsf{RLUG}} \varepsilon(s(u \approx v))$. If we combine this fact with Lemma 4.22 and the equivalence of **GL** and **HL** given in Theorem 2.3, we obtain the following theorem.

Theorem 4.23. *The variety* RLUG_R *of all* $r\ell u$*-groupoids that satisfy a set of simple rules* R *is an equivalent algebraic semantics for both* $\mathbf{GL}_R$ *and* $\mathbf{HL}_R$*. The same holds for the* $\mathcal{K}$ *reduct of* RLUG_R *and* $\mathcal{K}\mathbf{GL}_R$ *and* $\mathcal{K}\mathbf{HL}_R$*, where* $\mathcal{K}$ *contains* $\backslash$ *for the statement about* $\mathcal{K}\mathbf{HL}_R$*.*

Corollary 4.24. *The variety* RLUG *is generated by its finite members; hence its equational theory is decidable. The same holds for* RLUG_R*, where* R *is a set of reducing simple rules.*

Proof. Generation by finite members follows from Theorems 4.17 and 4.23. Decidability follows by the generation by finite members. Alternatively, an equation ε is valid in RLUG_R iff the sequent $s(\varepsilon)$ is provable in $\mathbf{GL}_R$, by Theorem 4.23, iff $s(\varepsilon)$ is provable in $\mathbf{GL}^{\mathrm{f}}_R$, by Theorem 4.8. Now, by performing an exhaustive proof search for $s(\varepsilon)$, we can decide if it is provable in $\mathbf{GL}^{\mathrm{f}}$. □

4.6.2. *Remarks on the FEP.* Let $\mathbf{A}$ be in $\mathcal{K}\mathsf{RLUG}_R$, for a simple set of rules R and for a sublanguage $\mathcal{K}$ that contains multiplication, and $\mathbf{B}$ a partial subalgebra of $\mathbf{A}$. We define the $\mathcal{K}$-matrix $\boldsymbol{\mathcal{B}}_{\mathbf{A}} = \langle \mathbf{B}^{\mathcal{K}}, \preceq \rangle$, where $x \preceq b$ iff $x^{\mathbf{A}} \leq_{\mathbf{A}} b$.

Lemma 4.25. *If* $\mathbf{A}$ *is in* $\mathcal{K}\mathsf{RLUG}_R$*, for a simple set of rules* R *and for a sublanguage* $\mathcal{K}$ *that contains multiplication, and* $\mathbf{B}$ *a partial subalgebra of* $\mathbf{A}$*, then* $\boldsymbol{\mathcal{B}}_{\mathbf{A}} \in \mathcal{G}\mathcal{L}$ *and the map* $q : \mathbf{B} \to \mathbf{R}(\boldsymbol{\mathcal{B}}_{\mathbf{A}})$*, defined by* $q(b) = \downarrow b$*, is an embedding.*

Proof. In view of Corollary 4.12 and Lemma 3.3, to show that $\boldsymbol{\mathcal{B}}_{\mathbf{A}} \in \mathcal{K}\mathcal{G}\mathcal{L}_R$, it suffices to check the interpretations $(r)^{\boldsymbol{\mathcal{B}}_{\mathbf{A}}}$, for all metarules (r) of $\mathcal{K}\mathbf{GL}$. As an example, we check $(\backslash\mathrm{L})^{\boldsymbol{\mathcal{A}}}$. Let $a, b, c \in B, x \in B^{\gamma}, u \in B^{\alpha}$, and assume that $a \backslash_{\mathbf{B}} b$ is defined, $x \preceq_{\boldsymbol{\mathcal{B}}_{\mathbf{A}}} a$ and $u[b] \preceq_{\boldsymbol{\mathcal{B}}_{\mathbf{A}}} c$. Then $x^{\mathbf{A}} \leq_{\mathbf{A}} a$ and $(u[b])^{\mathbf{A}} \leq_{\mathbf{A}} c$, so $x \preceq_{\mathbf{A}^{\mathcal{M}}} a$ and $u[b] \preceq_{\mathbf{A}^{\mathcal{M}}} c$. Since $\mathbf{A}^{\mathcal{M}} \in \mathcal{G}\mathcal{L}$, we have $u[x \circ (a \backslash b)] \preceq_{\mathbf{A}^{\mathcal{M}}} c$ or $(u[x \circ (a \backslash b)])^{\mathbf{A}} \leq_{\mathbf{A}} c$. Since $a \backslash_{\mathbf{B}} b$ is defined, we have $u[x \circ (a \backslash b)] \preceq_{\boldsymbol{\mathcal{B}}_{\mathbf{A}}} c$. □

A class of algebras is said to have the *finite embeddability property* if every partial subalgebra of an algebra in the class can be (partially) embedded in a finite algebra in the class.

Corollary 4.26. *If for all* $\mathbf{A}$ *is in* RLUG_R*, for a simple set of rules* R*, and for all* $\mathbf{B}$ *a partial subalgebras of* $\mathbf{A}$*,* $\mathbf{R}(\boldsymbol{\mathcal{B}}_{\mathbf{A}})$ *is finite, then* RLUG_R *has the finite embeddability property.*

The following lemma is shown in [4], under a different terminology.

Theorem 4.27. [4] *If* $\mathbf{A}$ *is an integral residuated lattice and* $\mathbf{B}$ *a partial subalgebra of* $\mathbf{A}$*, then* $\mathbf{R}(\boldsymbol{\mathcal{B}}_{\mathbf{A}})$ *is finite. Thus, the variety* $\mathsf{RLUG}_{\mathsf{ai}}$ *of integral residuated lattices has the finite embeddability property.*

Appendix A. Equivalence between GL and HL

Lemma A.1. *For all $a, b, c \in Fm_{\mathcal{L}}$, $x \in (Fm_{\mathcal{L}})^{\gamma}$ and $u \in (Fm_{\mathcal{L}})^{\alpha}$, we have*

(1) $u[a \circ b] \Rightarrow c \dashv\vdash_{\mathbf{GL}} u[a \cdot b] \Rightarrow c$.
(2) $u[x] \Rightarrow a \dashv\vdash_{\mathbf{GL}} u[\phi(x)] \Rightarrow a$.
(3) $u[x] \Rightarrow a \dashv\vdash_{\mathbf{GL}} x \Rightarrow u \rightsquigarrow a$.

Proof. (1) The left-to-right deduction is just $(\cdot\mathrm{L})$. For the converse, we have

$$\dfrac{\dfrac{a \Rightarrow a \quad b \Rightarrow b}{a \circ b \Rightarrow a \cdot b}\,(\cdot\mathrm{R}) \qquad u[a \cdot b] \Rightarrow c}{u[a \circ b] \Rightarrow c}\,(\mathrm{CUT})$$

(2) For $x \in Fm_{\mathcal{L}}$, the statement is obvious. We proceed by induction. Assume that the statement is true for $x, y \in (Fm_{\mathcal{L}})^{\gamma}$ and for all $a \in Fm_{\mathcal{L}}$, $u \in (Fm_{\mathcal{L}})^{\alpha}$; we will show it is true for $x \circ y$. We have, for $u_x[y] = u^y[x] = u[x \circ y]$,

$$\dfrac{\dfrac{\dfrac{u[x \circ y] \Rightarrow a}{u_x[y] \Rightarrow a}\,(=)}{u_x[\phi(y)] \Rightarrow a}\,(\mathrm{ind})}{u[x \circ \phi(y)] \Rightarrow a}\,(=) \quad \text{and} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{u[x \circ \phi(y)] \Rightarrow a}{u^{\phi(y)}[x] \Rightarrow a}\,(=)}{u^{\phi(y)}[\phi(x)] \Rightarrow a}\,(\mathrm{ind})}{u[\phi(x) \circ \phi(y)] \Rightarrow a}\,(=)}{u[\phi(x) \cdot \phi(y)] \Rightarrow a}\,(1)}{u[\phi(x \circ y)] \Rightarrow a}\,(=)$$

All of the above deductions hold upwards as well, so we obtain the converse.

(3) We will use induction on the complexity of u. The statement is obvious for $u = \_$. Assume that the statement holds for u. We have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(y \circ u)[x] \Rightarrow a}{y \circ u[x] \Rightarrow a}\,(=)}{\phi(y) \circ u[x] \Rightarrow a}\,(2)}{u[x] \Rightarrow \phi(y)\backslash a}\,(\backslash\mathrm{R})}{x \Rightarrow u \rightsquigarrow (\phi(y)\backslash a)}\,(\mathrm{ind})}{x \Rightarrow (y \circ u) \rightsquigarrow a}\,(=) \quad \text{and} \quad \dfrac{x \Rightarrow a\backslash b \qquad \dfrac{\dfrac{}{a \Rightarrow a}\,(\mathrm{Id}) \quad \dfrac{}{b \Rightarrow b}\,(\mathrm{Id})}{a \circ (a\backslash b) \Rightarrow b}\,(\backslash\mathrm{L})}{a \circ x \Rightarrow b}\,(\mathrm{CUT})$$

The first sequence of deductions establishes the forward direction. The converse follows from noting that all the deductions except for $(\backslash\mathrm{R})$ hold upwards. The converse of the rule $(\backslash\mathrm{R})$ is given by the second sequence of deductions. Similarly, we obtain $(y \circ u)[x] \Rightarrow a \dashv\vdash_{\mathbf{GL}} x \Rightarrow (y \circ u) \rightsquigarrow a$, a fact that completes the induction. $\square$

A.1. **Derivable rules.** We will show that the following rules and rule schemes are derivable in **HL**. As before, a, b, c denote atomic formulas and $\mathcal{K}$ ranges over all sublanguages of $\mathcal{L}$ that contain $\backslash$ and are such that the rule scheme connectives are contained in $\mathcal{K}$. The variable x ranges over all groupoid words fit for $\mathcal{K}$.

$$\dfrac{a\backslash b \quad b\backslash c}{a\backslash c}\,(\mathrm{T}_\ell) \qquad \dfrac{a \quad b\backslash(a\backslash c)}{b\backslash c}\,(\mathrm{NP}_\ell) \qquad \dfrac{a\backslash b \qquad c\backslash d}{(b\backslash c)\backslash(a\backslash d)}\,(\mathrm{R}\backslash)$$

$$\dfrac{a\backslash b}{(c/b)\backslash(c/a)}\,(\mathrm{Rn}/) \qquad \dfrac{a\backslash b}{(a/c)\backslash(b/c)}\,(\mathrm{Rd}/) \qquad \dfrac{b/a}{a\backslash b}\,(\mathrm{RC}_\ell)$$

$$\frac{}{a\backslash[b/(a\backslash b)]}\ (\mathrm{As}_{\ell r}) \qquad \frac{}{[b/(a\backslash b)]/a}\ (\mathrm{As}_{rr}) \qquad \frac{}{[(b/a)\backslash b]/a}\ (\mathrm{As}_{r\ell})$$

$$\frac{a\backslash(b/c)}{c\backslash(a\backslash b)}\ (\mathrm{RA}\ell_r) \qquad \frac{(b\backslash c)/a}{b\backslash(c/a)}\ (\mathrm{RA}_\ell) \qquad \frac{b\backslash(c/a)}{(b\backslash c)/a}\ (\mathrm{RA}_r)$$

$$\frac{a \quad b/a}{b}\ (\mathrm{MP}_r) \qquad \frac{a}{b/(b/a)}\ (\mathrm{N}_r) \qquad \frac{a \quad (c/a)/b}{c/b}\ (\mathrm{NP}_r)$$

$$\frac{a\backslash b}{(c\wedge a)\backslash(c\wedge b)}\ (\mathrm{R}\ell\wedge) \qquad \frac{a\backslash b}{(a\wedge c)\backslash(b\wedge c)}\ (\mathrm{R}r\wedge) \qquad \frac{c\backslash a \quad c\backslash b}{c\backslash(a\wedge b)}\ (\mathrm{RM}_1)$$

$$\frac{a\backslash b}{(c\vee a)\backslash(c\vee b)}\ (\mathrm{R}\ell\vee) \qquad \frac{a\backslash b}{(a\vee c)\backslash(b\vee c)}\ (\mathrm{R}r\vee)$$

$$\frac{a\backslash b}{ca\backslash cb}\ (\mathrm{R}\ell\cdot) \qquad \frac{a\backslash b}{ac\backslash bc}\ (\mathrm{R}r\cdot) \qquad \frac{a\backslash b \quad c\backslash d}{ac\backslash bd}\ (\mathrm{R}\cdot)$$

$$\frac{x \rightsquigarrow_{\mathcal{K}} a \quad a\backslash b}{x \rightsquigarrow_{\mathcal{K}} b}\ (\mathrm{T}_\ell{\rightsquigarrow_{\mathcal{K}}}) \qquad \frac{x \rightsquigarrow a \quad a\backslash b}{x \rightsquigarrow b}\ (\mathrm{T}_\ell{\rightsquigarrow})$$

$$\frac{x \rightsquigarrow_{\mathcal{K}} a \quad x \rightsquigarrow_{\mathcal{K}} b}{x \rightsquigarrow_{\mathcal{K}} (a\wedge b)}\ (\mathrm{RM}{\rightsquigarrow_{\mathcal{K}}}) \qquad \frac{x \rightsquigarrow_{\mathcal{K}} a \quad b\backslash c}{x \rightsquigarrow_{\mathcal{K}} [c/(a\backslash b)]}\ (\mathrm{R}\backslash{\rightsquigarrow_{\mathcal{K}}}) \qquad \frac{x \rightsquigarrow_{\mathcal{K}} a \quad b\backslash c}{x \rightsquigarrow_{\mathcal{K}} [(b/a)\backslash c]}\ (\mathrm{R}/{\rightsquigarrow_{\mathcal{K}}})$$

$$\frac{}{x \rightsquigarrow \phi(x)}\ (\mathrm{PI}{\rightsquigarrow}) \qquad \frac{x \rightsquigarrow a}{\phi(x)\backslash a}\ (\mathrm{RPI}{\rightsquigarrow}) \qquad \frac{\phi(x)\backslash a}{x \rightsquigarrow a}\ (\mathrm{RPE}{\rightsquigarrow})$$

Lemma A.2.

(1) *The above rules are derivable in* **HL**. *In particular, every rule is derivable from rules of* **HL** *that involve only the connectives of the given rule.*
(2) *The rule (RAr_ℓ) is equivalent to the combination of ($As_{\ell r}$) and (Rn/), in the presence of (I_ℓ), ($As_{\ell\ell}$) and (T_ℓ) [or of (I_ℓ), ($RA\ell_r$), (Rd\) and (MP_ℓ)].*
(3) *The rule ($RA\ell_r$) is equivalent to the combination of ($As_{\ell\ell}$) and (Rn\), in the presence of (I_ℓ), ($As_{\ell r}$) and (T_ℓ) [or of (I_ℓ), (RAr_ℓ), (Rd\) and (MP_ℓ)].*
(4) *The rules (Rd/) and (Rd\) are equivalent, in the presence of ($As_{\ell\ell}$), ($As_{\ell r}$) and (T_ℓ) [or of (I_ℓ), (RAr_ℓ), ($RA\ell_r$), (Rd\) and (MP_ℓ)].*
(5) *Finally, the rules (N_ℓ), (NP_ℓ) and (RC_ℓ) are equivalent, in the presence of ($As_{\ell r}$), ($As_{\ell\ell}$) and (MP_ℓ) [or of (I_ℓ), (RAr_ℓ), ($RA\ell_r$) and (MP_ℓ)].*

Proof. The statements follow from the deductions below.

$$(\mathrm{T}_\ell) \;=\; \dfrac{a\backslash b \quad \dfrac{b\backslash c}{(a\backslash b)\backslash(a\backslash c)}\,(\mathrm{Rd}\backslash)}{a\backslash c}\,(\mathrm{MP}_\ell) \;=\; \dfrac{b\backslash c \quad \dfrac{a\backslash b}{(b\backslash c)\backslash(a\backslash c)}\,(\mathrm{Rn}\backslash)}{a\backslash c}\,(\mathrm{MP}_\ell)$$

$$(\mathrm{Rn}/) \;=\; \dfrac{\dfrac{a\backslash b \quad \dfrac{}{b\backslash[(c/b)\backslash c)]}\,(\mathrm{As}_{\ell\ell})}{a\backslash[(c/b)\backslash c)]}\,(\mathrm{T}_\ell)}{(c/b)\backslash(c/a)}\,(\mathrm{RA}r_\ell) \qquad (\mathrm{RA}r_\ell) \;=\; \dfrac{\dfrac{}{b\backslash[c/(b\backslash c)]}\,(\mathrm{As}_{\ell r}) \quad \dfrac{a\backslash(b\backslash c)}{[c/(b\backslash c)]\backslash(c/a)}\,(\mathrm{Rn}/)}{b\backslash(c/a)}\,(\mathrm{T}_\ell)$$

$$(\mathrm{Rn}\backslash) \;=\; \dfrac{\dfrac{a\backslash b \quad \dfrac{}{b\backslash[c/(b\backslash c)]}\,(\mathrm{As}_{\ell r})}{a\backslash[c/(b\backslash c)]}\,(\mathrm{T}_\ell)}{(b\backslash c)\backslash(a\backslash c)}\,(\mathrm{RA}\ell_r) \qquad (\mathrm{RA}\ell_r) \;=\; \dfrac{\dfrac{}{c\backslash[(b/c)\backslash b]}\,(\mathrm{As}_{\ell\ell}) \quad \dfrac{a\backslash(b/c)}{[(b/c)\backslash b]\backslash(a\backslash b)}\,(\mathrm{Rn}\backslash)}{c\backslash(a\backslash b)}\,(\mathrm{T}_\ell)$$

$$(\mathrm{As}_{\ell\ell}) \;=\; \dfrac{\dfrac{}{(b/a)\backslash(b/a)}\,(\mathrm{I}_\ell)}{a\backslash[(b\backslash a)\backslash b]}\,(\mathrm{RA}\ell_r) \qquad (\mathrm{Rd}/) \;=\; \dfrac{\dfrac{\dfrac{}{c\backslash[(a/c)\backslash a]}\,(\mathrm{As}_{\ell\ell}) \quad \dfrac{a\backslash b}{[(a/c)\backslash a]\backslash[(a/c)\backslash b]}\,(\mathrm{Rd}\backslash)}{c\backslash[(a/c)\backslash b]}\,(\mathrm{T}_\ell)}{(a/c)\backslash(b/c)}\,(\mathrm{RA}r_\ell)$$

$$(\mathrm{As}_{\ell r}) \;=\; \dfrac{\dfrac{}{(a\backslash b)\backslash(a\backslash b)}\,(\mathrm{I}_\ell)}{a\backslash[b/(a\backslash b)]}\,(\mathrm{RA}r_\ell) \qquad (\mathrm{Rd}\backslash) \;=\; \dfrac{\dfrac{\dfrac{}{c\backslash[a/(c\backslash a)]}\,(\mathrm{As}_{\ell r}) \quad \dfrac{a\backslash b}{[a/(c\backslash a)]\backslash[b/(c\backslash a)]}\,(\mathrm{Rd}/)}{c\backslash[b/(c\backslash a)]}\,(\mathrm{T}_\ell)}{(c\backslash a)\backslash(c\backslash b)}\,(\mathrm{RA}\ell_r)$$

$$(\mathrm{R}\backslash) \;=\; \dfrac{\dfrac{a\backslash b}{(b\backslash d)\backslash(a\backslash d)}\,(\mathrm{Rn}\backslash) \quad \dfrac{c\backslash d}{(b\backslash c)\backslash(b\backslash d)}\,(\mathrm{Rd}\backslash)}{(b\backslash c)\backslash(a\backslash d)}\,(\mathrm{T}_\ell)$$

$$(\mathrm{NP}_\ell) \;=\; \dfrac{b\backslash(a\backslash c) \quad \dfrac{\dfrac{a}{(a\backslash c)\backslash c}\,(\mathrm{N}_\ell)}{[b\backslash(a\backslash c)]\backslash(b\backslash c)}\,(\mathrm{Rd}\backslash)}{b\backslash c}\,(\mathrm{MP}_\ell)$$

$$(\mathrm{RC}_\ell) \;=\; \dfrac{b/a \quad \dfrac{}{a\backslash[(b/a)\backslash b]}\,(\mathrm{As}_{\ell\ell})}{a\backslash b}\,(\mathrm{NP}_\ell), \qquad (\mathrm{N}_\ell) \;=\; \dfrac{\dfrac{a \quad \dfrac{}{a\backslash[b/(a\backslash b)]}\,(\mathrm{As}_{\ell r})}{b/(a\backslash b)}\,(\mathrm{MP}_\ell)}{(a\backslash b)\backslash b}\,(\mathrm{RC}_\ell)$$

$$(\mathrm{MP}_r) \;=\; \dfrac{b/a \quad \dfrac{a \quad \dfrac{}{a\backslash[(b/a)\backslash b]}\,(\mathrm{As}_{\ell\ell})}{(b/a)\backslash b}\,(\mathrm{MP}_\ell)}{b}\,(\mathrm{MP}_\ell) \qquad (\mathrm{MP}_\ell) \;=\; \dfrac{\dfrac{a\backslash b \quad \dfrac{a}{1\backslash a}\,(\mathrm{R1a}\ell)}{1\backslash b}\,(\mathrm{T}_\ell)}{b}\,(\mathrm{R1b}\ell)$$

$$(\mathrm{R}\ell\cdot) \;=\; \dfrac{\dfrac{a\backslash b \quad \dfrac{}{b\backslash(c\backslash cb)}\,(\mathrm{PI})}{a\backslash(c\backslash cb)}\,(\mathrm{T}_\ell)}{ca\backslash cb}\,(\mathrm{RPI}) \qquad (\mathrm{R}r\cdot) \;=\; \dfrac{\dfrac{\dfrac{}{c\backslash(b\backslash bc)}\,(\mathrm{PI}) \quad \dfrac{a\backslash b}{(b\backslash bc)\backslash(a\backslash bc)}\,(\mathrm{Rn}\backslash)}{c\backslash(a\backslash bc)}\,(\mathrm{T}_\ell)}{ac\backslash bc}\,(\mathrm{RPI})$$

$$(\mathrm{R}\cdot) \;=\; \dfrac{\dfrac{a\backslash b}{ac\backslash bc}\,(\mathrm{R}r/) \quad \dfrac{c\backslash d}{bc\backslash bd}\,(\mathrm{R}\ell/)}{ac\backslash bd}\,(\mathrm{T}_\ell)$$

$$(\mathrm{R}\ell\wedge) \;=\; \dfrac{\dfrac{}{(c\wedge a)\backslash c}\,(\mathrm{ME}\ell) \quad \dfrac{a\backslash b \quad \dfrac{}{(c\wedge a)\backslash a}\,(\mathrm{ME}r)}{(c\wedge a)\backslash b}\,(\mathrm{T}_\ell)}{(c\wedge a)\backslash(c\wedge b)}\,(\mathrm{RM}_1)$$

$$(\mathrm{R}\ell\vee) \;=\; \dfrac{\dfrac{}{c\backslash(c\vee b)}\,(\mathrm{JI}\ell) \quad \dfrac{a\backslash b \quad \dfrac{}{b\backslash(c\vee b)}\,(\mathrm{JI}r)}{a\backslash(c\vee b)}\,(\mathrm{T}_\ell)}{(c\vee a)\backslash(c\vee b)}\,(\mathrm{RJ}\backslash)$$

The proofs of (R$r\wedge$) and (R$r\vee$) are similar to (R$\ell\wedge$) and (R$\ell\vee$), respectively. Using (RCr), one can show that (As$_{rr}$), (As$_{r\ell}$), (MP$_r$), (N$_r$) and (NP$_r$) follow from the corresponding rules, obtained by interchanging the letters r and ℓ in their names.

Assume that x, y are groupoid words, a, b, c are formulas.

For ($\mathrm{T}_\ell{\rightsquigarrow_{\mathcal{K}}}$), if $\mathcal{K}$ contains $\cdot$, then $x \rightsquigarrow_{\mathcal{K}} a = \phi(x)\backslash a$ and ($\mathrm{T}_\ell{\rightsquigarrow_{\mathcal{K}}}$) is just ($\mathrm{T}_\ell$). Assume now that $\mathcal{K}$ does not contain $\cdot$, in which case x is solvable. If x is equal to a formula or ε, then ($\mathrm{T}_\ell{\rightsquigarrow_{\mathcal{K}}}$) is just ($\mathrm{T}_\ell$) or ($\mathrm{MP}_\ell$). If $x = c \circ y$, y is solvable and ($\mathrm{T}_\ell{\rightsquigarrow_{\mathcal{K}}}$) holds for y, we have

$$\dfrac{\dfrac{(c\circ y) \rightsquigarrow_{\mathcal{K}} a}{y \rightsquigarrow_{\mathcal{K}} (c\backslash a)}\,(=) \qquad \dfrac{a\backslash b}{(c\backslash a)\backslash(c\backslash b)}\,(\mathrm{Rd}\backslash)}{\dfrac{y \rightsquigarrow_{\mathcal{K}} (c\backslash b)}{(c\circ y)\rightsquigarrow_{\mathcal{K}} b}\,(=)}\,(\mathrm{Ind})$$

Note that if $\mathcal{K}$ does not contain $/$, this is the only case. If $x = y \circ c$, y is a solvable groupoid word not equal to a formula or ε, and ($\mathrm{T}_\ell{\rightsquigarrow_{\mathcal{K}}}$) holds for y, then we have

$$\dfrac{\dfrac{(y\circ c) \rightsquigarrow_{\mathcal{K}} a}{y \rightsquigarrow_{\mathcal{K}} (a/c)}\,(=) \qquad \dfrac{a\backslash b}{(a/c)\backslash(b/c)}\,(\mathrm{Rd}/)}{\dfrac{y \rightsquigarrow_{\mathcal{K}} (b/c)}{(y\circ c)\rightsquigarrow_{\mathcal{K}} b}\,(=)}\,(\mathrm{Ind})$$

The proof of ($\mathrm{T}\ell$) is almost identical.

For ($\mathrm{RM}{\rightsquigarrow_{\mathcal{K}}}$), if $\cdot$ is not contained in $\mathcal{K}$, then x is solvable. If $x = c \circ y$, y is solvable and ($\mathrm{RM}{\rightsquigarrow_{\mathcal{K}}}$) holds for y, we have

$$\dfrac{\dfrac{\dfrac{(c\circ y) \rightsquigarrow_{\mathcal{K}} a}{y \rightsquigarrow_{\mathcal{K}} (c\backslash a)}\,(=) \qquad \dfrac{(c\circ y) \rightsquigarrow_{\mathcal{K}} b}{y \rightsquigarrow_{\mathcal{K}} (c\backslash b)}\,(=)}{y \rightsquigarrow_{\mathcal{K}} [(c\backslash a)\wedge(c\backslash b)]}\,(\mathrm{Ind}) \qquad \dfrac{}{[(c\backslash a)\wedge(c\backslash b)]\backslash[c\backslash(a\wedge b)]}\,(\mathrm{M}\backslash)}{\dfrac{y \rightsquigarrow_{\mathcal{K}} (c\backslash(a\wedge b))}{(c\circ y)\rightsquigarrow_{\mathcal{K}} (a\wedge b)}\,(=)}\,(\mathrm{T}_\ell{\rightsquigarrow_{\mathcal{K}}})$$

Note that for $y = \_$ and $y = d$ we obtain

$$\dfrac{c\backslash(a\wedge b) \quad c\backslash(a\wedge b)}{c\backslash(a\wedge b)}\,(\mathrm{RM}_1) \quad \text{and} \quad \dfrac{d\backslash[c\backslash(a\wedge b)] \quad d\backslash[c\backslash(a\wedge b)]}{d\backslash[c\backslash(a\wedge b)]}\,(\mathrm{RM}_2)$$

The proof for the case where $x = y \circ c$, y is a solvable groupoid word not equal to a formula or ε, and ($\mathrm{RM}{\rightsquigarrow_{\mathcal{K}}}$) holds for y is analogous to the previous case, if instead of ($\mathrm{M}\backslash$) we use ($\mathrm{M}/$), which we prove now. We obtain the axiom

$$\dfrac{\dfrac{}{c\backslash[(a/c)\backslash a]}\,(\mathrm{A}_{\ell\ell}) \qquad \dfrac{\dfrac{}{(a/c\wedge b/c)\backslash(a/c)}\,(\mathrm{ME}\ell)}{\{c\backslash[(a/c)\backslash a]\}\backslash[\{c\backslash[(a/c\wedge b/c)\backslash a]\}]}\,(\mathrm{Rn}\backslash), (\mathrm{Rd}\backslash)}{c\backslash[(a/c\wedge b/c)\backslash a]}\,(\mathrm{MP}_\ell)$$

which we call ($\mathrm{Aux}\ell$) and similarly, using ($\mathrm{ME}r$), we obtain the axiom

$$\dfrac{}{c\backslash[(a/c\wedge b/c)\backslash b]}\,(\mathrm{Aux}r)$$

Consequently, we have ($\mathrm{M}/$)

$$\dfrac{\dfrac{}{c\backslash[(a/c\wedge b/c)\backslash a]}\,(\mathrm{Aux}\ell) \qquad \dfrac{}{c\backslash[(a/c\wedge b/c)\backslash b]}\,(\mathrm{Aux}r)}{\dfrac{c\backslash[(a/c\wedge b/c)\backslash(a\wedge b)]}{(a/c\wedge b/c)\backslash[(a\wedge b)/c]}\,(\mathrm{RA}r_\ell)}\,(\mathrm{RM}_2)$$

If x is a formula or ε, then $(\mathrm{RM}{\rightsquigarrow_\mathcal{K}})$ is just (RM_1). The same holds for the case where $\cdot$ is contained in $\mathcal{K}$, since $x \rightsquigarrow_\mathcal{K} a = \phi(x)\backslash a$.

For $(\mathrm{R}\backslash{\rightsquigarrow_\mathcal{K}})$ we have

$$\dfrac{x \rightsquigarrow_\mathcal{K} a \qquad \dfrac{\dfrac{}{a\backslash[b/(a\backslash b)]}\,(\mathrm{As}_{\ell r}) \qquad \dfrac{b\backslash c}{[b/(a\backslash b)]\backslash[c/(a\backslash b)]}\,(\mathrm{Rd}/)}{a\backslash[c/(a\backslash b)]}\,(\mathrm{T}_\ell)}{x \rightsquigarrow_\mathcal{K} [c/(a\backslash b)]}\,(\mathrm{T}_\ell{\rightsquigarrow_\mathcal{K}})$$

For $(\mathrm{R}/{\rightsquigarrow_\mathcal{K}})$ we use $(\mathrm{As}_{\ell\ell})$ and $(\mathrm{Rd}\backslash)$, and work as before.

For $(\mathrm{RPI}{\rightsquigarrow})$, we work by induction on x. If x is a formula or ε, then $x \rightsquigarrow a = \phi(x)\backslash a$. We assume that there are groupoid words y, z such that $x = z \circ y$ or $x = y \circ z$, and y contains the right-most among the occurrences of subformulas or ε of maximal depth. For the two cases, we have respectively

$$\dfrac{\dfrac{\dfrac{\dfrac{(z\circ y) \rightsquigarrow a}{y \rightsquigarrow (\phi(z)\backslash a)}\,(=)}{\phi(y)\backslash(\phi(z)\backslash a)}\,(=)}{\phi(z)\phi(y)\backslash a}\,(\mathrm{RPI})}{\phi(z\circ y)\backslash a}\,(=) \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(y\circ z) \rightsquigarrow a}{y \rightsquigarrow (a/\phi(z))}\,(=)}{\phi(y)\backslash(a/\phi(z))}\,(=)}{\phi(z)\backslash(\phi(y)\backslash a)}\,(\mathrm{RA}\ell_r)}{\phi(y)\phi(z)\backslash a}\,(\mathrm{RPI})}{\phi(y\circ z)\backslash a}\,(=)$$

We apply the same reasoning for $(\mathrm{PI}{\rightsquigarrow})$. If x is a formula or ε, then $(\mathrm{PI}{\rightsquigarrow})$ follows from (I_ℓ). We assume that there are groupoid words y, z such that $x = z \circ y$ or $x = y \circ z$, the rule holds for y and y contains the right-most among the occurrences of subformulas or ε of maximal depth. For the two cases, we have respectively

$$\dfrac{\dfrac{y \rightsquigarrow \phi(y) \qquad \dfrac{}{\phi(y)\backslash[\phi(z)\backslash\phi(z)\phi(y)]}\,(\mathrm{PI})}{y \rightsquigarrow [\phi(z)\backslash\phi(z)\phi(y)]}\,(\mathrm{T}_\ell{\rightsquigarrow})}{z\circ y \rightsquigarrow \phi(z\circ y)}\,(=) \qquad \text{and} \qquad \dfrac{\dfrac{y \rightsquigarrow \phi(y) \qquad \dfrac{\dfrac{}{\phi(z)\backslash[\phi(y)\backslash\phi(y)\phi(z)]}\,(\mathrm{PI})}{\phi(y)\backslash[\phi(z)\phi(y)/\phi(z)]}\,(\mathrm{RA}r_\ell)}{y \rightsquigarrow [\phi(z)\phi(y)/\phi(z)]}\,(\mathrm{T}_\ell{\rightsquigarrow})}{z\circ y \rightsquigarrow \phi(z\circ y)}\,(=)$$

$(\mathrm{RPE}{\rightsquigarrow})$ follows directly from $(\mathrm{PI}{\rightsquigarrow})$ and $(\mathrm{T}_\ell{\rightsquigarrow})$. $\square$

A.2. **Translations between HL and GL.** If (R)$= \langle S, s\rangle$ is a rule of $\mathcal{K}$**GL**, we set (**H**R) to be rule scheme $\langle \phi_\mathcal{K}[S], \phi_\mathcal{K}(s)\rangle$.

Theorem A.3. *For every set $S \cup \{s\}$ of sequents and every sublanguage $\mathcal{K}$ of $\mathcal{L}$ that contains $\backslash$, if $S \vdash_{\mathcal{K}\mathbf{GL}} s$ then $\phi_\mathcal{K}(S) \vdash_{\mathcal{K}\mathbf{HL}} \phi_\mathcal{K}(s)$.*

Proof. Given a rule (R) of $\mathcal{K}$-**GL**s, we will verify that the rule scheme (**H**R) is a derivable rule scheme of $\mathcal{K}$**HL**.

Assume that $\mathcal{K}$ does not contain the connective $\cdot$, and consider $(\mathcal{K}\wedge\mathrm{L}\ell)$. Since $u[a]$ is solvable, there is a solvable groupoid word x and an augmented solvable groupoid word v such that $u[a]$ equals $v[a \circ x]$ or $v[x \circ a]$. If $\mathcal{K}$ does not contain $/$, then only the first case can hold and the terms x and v can be taken to be left solvable. In the first case the rule scheme ($\mathbf{H}\mathcal{K}\wedge\mathrm{L}\ell$) is equal to the following deduction tree.

$$\dfrac{\dfrac{\dfrac{\dfrac{u[a] \rightsquigarrow_{\mathcal{K}} c}{(v[a\circ x]) \rightsquigarrow_{\mathcal{K}} c}\,(=)}{x \rightsquigarrow_{\mathcal{K}} (a\backslash(v \rightsquigarrow_{\mathcal{K}} c))}\,(=) \qquad \dfrac{\dfrac{}{(a\wedge b)\backslash a}\,(\mathrm{ME}\ell)}{[a\backslash(v\rightsquigarrow_{\mathcal{K}} c)]\backslash[(a\wedge b)\backslash(v\rightsquigarrow_{\mathcal{K}} c)]}\,(\mathrm{Rn}\backslash)}{x \rightsquigarrow_{\mathcal{K}} [(a\wedge b)\backslash(v\rightsquigarrow_{\mathcal{K}} c)]}\,(\mathrm{T}_{\ell}{\rightsquigarrow_{\mathcal{K}}})}{u[a\wedge b]\rightsquigarrow_{\mathcal{K}} c}\,(=)$$

The second case uses the rule (Rn/) and the proof is analogous. If $\mathcal{K}$ contains the connective $\cdot$, then $u[a] \rightsquigarrow_{\mathcal{K}} c = \phi(u[a])\backslash c$. Moreover, $\phi(u[a]) = \phi(v[a\circ x])$, for some left solvable groupoid word x and an augmented left solvable groupoid word v. So,

$$\dfrac{\dfrac{\dfrac{u[a] \rightsquigarrow_{\mathcal{K}} c}{\phi(u[a])\backslash c}\,(=)}{\phi(v[a\circ x])\backslash c}\,(=)}{v[a\circ x] \rightsquigarrow c}\,(\mathrm{RPE}{\rightsquigarrow}) \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{\dfrac{v[a\circ x] \rightsquigarrow c}{(v[(a\wedge b)\circ x] \rightsquigarrow c}\,(*)}{\phi(v[(a\wedge b)\circ x])\backslash c}\,(\mathrm{RPI}{\rightsquigarrow})}{\phi(u[a\wedge b])\backslash c}\,(=)}{u[a\wedge b] \rightsquigarrow_{\mathcal{K}} c}\,(=)$$

where (*) follows from the deduction tree for the first case, since $v[a \circ x]$ is left solvable. Likewise, we obtain ($\mathbf{H}\mathcal{K}\wedge\mathrm{L}r$). The rule ($\mathbf{H}\mathcal{K}\wedge\mathrm{R}$) is equal to ($\mathrm{RM}{\rightsquigarrow_{\mathcal{K}}}$).

For ($\mathbf{H}\mathcal{K}\vee\mathrm{R}\ell$), we have

$$\dfrac{x \rightsquigarrow_{\mathcal{K}} a \qquad \dfrac{}{a\backslash(a\vee b)}\,(\mathrm{JI}\ell)}{x \rightsquigarrow_{\mathcal{K}} (a\vee b)}\,(\mathrm{T}_{\ell}{\rightsquigarrow_{\mathcal{K}}})$$

and for ($\mathbf{H}\mathcal{K}\vee\mathrm{R}r$) we use ($\mathrm{JI}r$). For ($\mathbf{H}\mathcal{K}\vee\mathrm{L}$), we use the same reasoning as in ($\mathbf{H}\mathcal{K}\wedge\mathrm{L}\ell$); in the key deduction we use (RJ\) and (RJ/) instead of ($\mathrm{T}_{\ell}{\rightsquigarrow_{\mathcal{K}}}$).

Note that in case that $\mathcal{K}$ does not contain $\cdot$ and $x \neq \varepsilon$, the sequent $u[x\circ(a\backslash b)]$ in ($\mathbf{H}\mathcal{K}\backslash\mathrm{L}$) is solvable for $x\circ(a\backslash b)$, since otherwise it not solvable at all. We consider the two cases, where $x \neq \varepsilon$ and / is contained in $\mathcal{K}$ or not. Note that in the second case x has to be equal to a formula d in order for $x\circ(a\backslash b)$ to be left solvable. For the two cases we have respectively

$$\dfrac{\dfrac{x \rightsquigarrow_{\mathcal{K}} a \qquad \dfrac{u[b] \rightsquigarrow_{\mathcal{K}} c}{b\backslash(u \rightsquigarrow_{\mathcal{K}} c)}\,(=)}{x \rightsquigarrow_{\mathcal{K}} [(u \rightsquigarrow_{\mathcal{K}} c)/(a\backslash b)]}\,(\mathrm{R}\backslash{\rightsquigarrow_{\mathcal{K}}})}{u[x\circ(a\backslash b)] \rightsquigarrow_{\mathcal{K}} c}\,(=) \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{d \rightsquigarrow_{\mathcal{K}} a}{d\backslash a}\,(=) \qquad \dfrac{u[b] \rightsquigarrow_{\mathcal{K}} c}{b\backslash(u \rightsquigarrow_{\mathcal{K}} c)}\,(=)}{(a\backslash b)\backslash[d\backslash(u \rightsquigarrow_{\mathcal{K}} c)]}\,(\mathrm{R}\backslash)}{u[d\circ(a\backslash b)] \rightsquigarrow_{\mathcal{K}} c}\,(=)$$

If $x = \varepsilon$, then we use (N_{ℓ}). Finally, if $\mathcal{K}$ contains $\cdot$, then we reduce the proof to the case where $\mathcal{K}$ contains neither $\cdot$ or /, as we did for the rule ($\mathbf{H}\mathcal{K}\wedge\mathrm{L}\ell$). Likewise we obtain the proof of ($\mathbf{H}\mathcal{K}/\mathrm{L}\ell$), by using ($\mathrm{R}/{\rightsquigarrow_{\mathcal{K}}}$) instead of ($\mathrm{R}\backslash{\rightsquigarrow_{\mathcal{K}}}$). Of course, in this case $\mathcal{K}$ contains /.

For ($\mathbf{H}\mathcal{K}\backslash\mathrm{R}$), we have $a\circ x \rightsquigarrow_{\mathcal{K}} b = x \rightsquigarrow_{\mathcal{K}} a\backslash b$, if $\mathcal{K}$ does not contain $\cdot$, because then x is solvable. If $\mathcal{K}$ contains $\cdot$, then we have

$$\dfrac{\dfrac{\dfrac{\dfrac{(a\circ x) \rightsquigarrow_{\mathcal{K}} b}{\phi(a\circ x)\backslash b}\,(=)}{a\phi(x)\backslash b}\,(=)}{\phi(x)\backslash(a\backslash b)}\,(\mathrm{PRE}{\rightsquigarrow})}{x \rightsquigarrow_{\mathcal{K}} a\backslash b}\,(=)$$

Note that in the case of ($\mathbf{H}\mathcal{K}$/R), $\mathcal{K}$ contains $/$. Assume first that $\mathcal{K}$ does not contain $\cdot$. If x is not equal to a formula or ε, then we have $x \circ a \rightsquigarrow_{\mathcal{K}} b = x \rightsquigarrow_{\mathcal{K}} b/a$, because x has to be solvable. For the case where x is equal to c, where c is either a formula or ε, and for the case where $\mathcal{K}$ contains $\cdot$, we have respectively (here $\hat{c} = c$, if c is a formula, while $\hat{\varepsilon} = 1$)

$$\dfrac{\dfrac{\dfrac{c \circ a \rightsquigarrow_{\mathcal{K}} b}{a\backslash(\hat{c}\backslash b)}\,(=)}{\hat{c}\backslash(b/a)}\,(\mathrm{RA}r_\ell)}{c \rightsquigarrow_{\mathcal{K}} b/a}\,(=) \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{\dfrac{x \circ a \rightsquigarrow_{\mathcal{K}} b}{(\phi(x)\cdot a)\backslash b}\,(=)}{a\backslash(\phi(x)\backslash b)}\,(\mathrm{RPE}{\rightsquigarrow})}{\phi(x)\backslash(b/a)}\,(\mathrm{RA}r_\ell)}{x \rightsquigarrow_{\mathcal{K}} b/a}\,(=)$$

The rule ($\mathbf{H}\mathcal{K}\cdot$R) follows directly from (R$\cdot$) and the fact that $x \rightsquigarrow_{\mathcal{K}} a = \phi(x)\backslash a$, $y \rightsquigarrow_{\mathcal{K}} b = \phi(y)\backslash b$ and $x \circ y \rightsquigarrow_{\mathcal{K}} ab = \phi(x)\phi(y)\backslash ab$. Moreover, the rule ($\mathbf{H}\mathcal{K}\cdot$L) holds trivially since both $u[a \circ b] \rightsquigarrow_{\mathcal{K}} c$ and $u[a \cdot b] \rightsquigarrow_{\mathcal{K}} c$ are equal to $\phi(u[a \circ b])\backslash c$. Also, ($\mathbf{H}\mathcal{K}$1R) follows from (1).

For ($\mathbf{H}\mathcal{K}$1L), if $u = \_$ then $u[1] = 1$ and $|\_| = \varepsilon$, so we have $\dfrac{a}{1\backslash a}$, which follows from (I1r) and (MP$_\ell$). If $\mathcal{K}$ does not contain $\cdot$, then $u[1]$ is solvable, so there exist a solvable groupoid word x and a solvable augmented groupoid word v such that $u[1]$ equals $v[1 \circ x]$ or $v[x \circ 1]$; in both cases $|u| = v[x]$. If $\mathcal{K}$ does not contain $/$, then only the first case can hold and the terms can be taken to be left solvable. We have the following for the two cases

$$\dfrac{\dfrac{\dfrac{\dfrac{|u| \rightsquigarrow_{\mathcal{K}} c}{x \rightsquigarrow_{\mathcal{K}} (v \rightsquigarrow_{\mathcal{K}} c)}\,(=) \qquad \dfrac{}{(v \rightsquigarrow_{\mathcal{K}} c)\backslash[1\backslash(v \rightsquigarrow_{\mathcal{K}} c)]}\,(\mathrm{I}1r)}{x \rightsquigarrow_{\mathcal{K}} (1\backslash(v \rightsquigarrow_{\mathcal{K}} c))}\,(\mathrm{T}_\ell{\rightsquigarrow_{\mathcal{K}}})}{(v[1 \circ x]) \rightsquigarrow_{\mathcal{K}} c}\,(=)}{u[1] \rightsquigarrow_{\mathcal{K}} c}\,(=)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{|u| \rightsquigarrow_{\mathcal{K}} c}{x \rightsquigarrow_{\mathcal{K}} (v \rightsquigarrow_{\mathcal{K}} c)}\,(=) \qquad \dfrac{\dfrac{}{1\backslash[(v \rightsquigarrow_{\mathcal{K}} c)\backslash(v \rightsquigarrow_{\mathcal{K}} c)]}\,(\mathrm{I}1\ell)}{(v \rightsquigarrow_{\mathcal{K}} c)\backslash[(v \rightsquigarrow_{\mathcal{K}} c)/1]}\,(\mathrm{RA}r_\ell)}{x \rightsquigarrow_{\mathcal{K}} ((v \rightsquigarrow_{\mathcal{K}} c)/1)}\,(\mathrm{T}_\ell{\rightsquigarrow_{\mathcal{K}}})}{(v[x \circ 1]) \rightsquigarrow_{\mathcal{K}} c}\,(=)}{u[1] \rightsquigarrow_{\mathcal{K}} c}\,(=)$$

If $\mathcal{K}$ contains $\cdot$, then we reduce the problem to the case where $\mathcal{K}$ does not contain $/$ as in the proof of ($\mathbf{H}\mathcal{K}\wedge$L$\ell$). ($\mathbf{H}\mathcal{K}$Id) is equal to (I$_\ell$). Finally, for ($\mathbf{H}\mathcal{K}$cut), if $\mathcal{K}$ does not contain $\cdot$ and x is not a formula or ε, then $u[x]$ is solvable for x and

$$\dfrac{\dfrac{x \rightsquigarrow_{\mathcal{K}} a \quad a\backslash(u \rightsquigarrow_{\mathcal{K}} c)}{x \rightsquigarrow_{\mathcal{K}} (u \rightsquigarrow_{\mathcal{K}} c)}\,(\mathrm{T}_\ell{\rightsquigarrow_{\mathcal{K}}})}{u[x] \rightsquigarrow_{\mathcal{K}} c}\,(=)$$

If $x = d$, where d is a formula or ε, then $u[a]$ is equal to $v[a \circ y]$ or $v[y \circ a]$, for some solvable groupoid word x and a solvable augmented term v. We have

$$\dfrac{\dfrac{\dfrac{d\backslash a}{(a\backslash v \rightsquigarrow_{\mathcal{K}} c)\backslash(d\backslash v \rightsquigarrow_{\mathcal{K}} c)}\,(\mathrm{Rn}\backslash) \qquad y \rightsquigarrow_{\mathcal{K}} a\backslash(v \rightsquigarrow_{\mathcal{K}} c)}{y \rightsquigarrow_{\mathcal{K}} d\backslash(v \rightsquigarrow_{\mathcal{K}} c)}\,(\mathrm{T}_\ell{\rightsquigarrow_{\mathcal{K}}})}{v[d \circ y] \rightsquigarrow_{\mathcal{K}} c}\,(=)$$

for the first case. Likewise we handle the second case. If $\mathcal{K}$ contains $\cdot$, then we reduce the proof to the case where neither $\cdot$ nor $/$ is in $\mathcal{K}$, as in the proof of ($\mathbf{H}\mathcal{K}\wedge\mathrm{L}\ell$). $\square$

Theorem A.4. *For every set $B \cup \{c\}$ of formulas and every sublanguage $\mathcal{K}$ of $\mathcal{L}$ that contains the connective $\backslash$, if $B \vdash_{\mathcal{K}\mathbf{HL}} c$ then $s[B] \vdash_{\mathcal{K}\mathbf{GL}} s(c)$.*

Proof. First note that by ($\backslash$R) and

$$\dfrac{x \Rightarrow a\backslash b \qquad \dfrac{\dfrac{}{a \Rightarrow a}\,(\mathrm{Id}) \qquad \dfrac{}{b \Rightarrow b}\,(\mathrm{Id})}{a \circ (a\backslash b) \Rightarrow b}\,(\backslash\mathrm{L})}{a \circ x \Rightarrow b}\,(\mathrm{CUT})$$

we obtain the bidirectional rule

$$\dfrac{a \circ x \Rightarrow b}{x \Rightarrow a\backslash b}\,(\backslash\mathrm{R})\updownarrow$$

In particular, for $x = \varepsilon$ we have $\varepsilon \Rightarrow a\backslash b \dashv\vdash_{\mathcal{K}\mathbf{GL}} a \Rightarrow b$; we will be using this fact without explicit reference. For every rule (R) of $\mathbf{HL}$, we will verify that the rule ($\mathbf{G}$R) is derivable in $\mathbf{GL}$. ($\mathbf{G}\mathrm{I}_\ell$) follows from (Id) and ($\mathbf{G}\mathrm{MP}_\ell$) follows from (CUT). For ($\mathbf{G}$Rd$\backslash$) and ($\mathbf{G}$Rn$\backslash$), we have

$$\dfrac{\dfrac{\dfrac{}{c \Rightarrow c}\,(\mathrm{Id}) \quad a \Rightarrow b}{c \circ (c\backslash a) \Rightarrow b}\,(\backslash\mathrm{L})}{c\backslash a \Rightarrow c\backslash b}\,(\backslash\mathrm{R}) \qquad \text{and} \qquad \dfrac{\dfrac{a \Rightarrow b \quad \dfrac{}{c \Rightarrow c}\,(\mathrm{Id})}{a \circ (b\backslash c) \Rightarrow c}\,(\backslash\mathrm{L})}{b\backslash c \Rightarrow a\backslash c}\,(\backslash\mathrm{R})$$

($\mathbf{G}$MEℓ), ($\mathbf{G}$MEr) and ($\mathbf{G}$RM) follow easily from ($\wedge$Lℓ), ($\wedge$Lr) and ($\wedge$R), respectively. ($\mathbf{G}$M$\backslash$) follows from

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \Rightarrow a}\,(\mathrm{Id}) \quad \dfrac{}{b \Rightarrow b}\,(\mathrm{Id})}{a \circ (a\backslash b) \Rightarrow b}\,(\backslash\mathrm{L})}{a \circ [(a\backslash b) \wedge (a\backslash c)] \Rightarrow b}\,(\wedge\mathrm{L}\ell) \quad \dfrac{\dfrac{\dfrac{}{a \Rightarrow a}\,(\mathrm{Id}) \quad \dfrac{}{c \Rightarrow c}\,(\mathrm{Id})}{a \circ (a\backslash c) \Rightarrow b}\,(\backslash\mathrm{L})}{a \circ [(a\backslash b) \wedge (a\backslash c)] \Rightarrow c}\,(\wedge\mathrm{L}r)}{a \circ [(a\backslash b) \wedge (a\backslash c)] \Rightarrow b \wedge c}\,(\wedge\mathrm{R})}{(a\backslash b) \wedge (a\backslash c) \Rightarrow a\backslash(b \wedge c)}\,(\backslash\mathrm{R})$$

($\mathbf{G}$JIℓ), ($\mathbf{G}$JIr) and ($\mathbf{G}$RJ) follow easily from ($\vee$Rℓ), ($\vee$Rr) and ($\vee$L), respectively.

For (PI) and (RPI) we have

$$\dfrac{\dfrac{\dfrac{}{a \Rightarrow a}\,(\mathrm{Id}) \quad \dfrac{}{b \Rightarrow b}\,(\mathrm{Id})}{a \circ b \Rightarrow a \cdot b}\,(\cdot\mathrm{R})}{b \Rightarrow a\backslash a \cdot b}\,(\backslash\mathrm{R}) \qquad \text{and} \qquad \dfrac{\dfrac{b \Rightarrow a\backslash c}{a \circ b \Rightarrow c}\,(\backslash\mathrm{R})\updownarrow}{a \cdot b \Rightarrow c}\,(\cdot\mathrm{L})$$

For ($\mathbf{G}\mathrm{N}_\ell$) and ($\mathbf{G}\mathrm{As}_{\ell\ell}$) we have

$$\dfrac{\dfrac{e \Rightarrow a \quad \dfrac{}{b \Rightarrow b}\,(\mathrm{Id})}{\varepsilon \circ (a\backslash b) \Rightarrow b}\,(\backslash\mathrm{L})}{\varepsilon \Rightarrow (a\backslash b)\backslash b}\,(\backslash\mathrm{R}) \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{}{a \Rightarrow a}\,(\mathrm{Id}) \quad \dfrac{}{b \Rightarrow b}\,(\mathrm{Id})}{(b/a) \circ a \Rightarrow b}\,(\backslash\mathrm{L})}{a \Rightarrow (b/a)\backslash b}\,(\backslash\mathrm{R})$$

For ($\mathbf{G}$PI), ($\mathbf{G}$RAr_ℓ) and ($\mathbf{G}$RPI) we have

$$\dfrac{\dfrac{\dfrac{}{a \Rightarrow a}\,(\mathrm{Id}) \quad \dfrac{}{b \Rightarrow b}\,(\mathrm{Id})}{a \circ b \Rightarrow ab}\,(\cdot\mathrm{R})}{b \Rightarrow a\backslash ab}\,(\backslash\mathrm{R}), \qquad \dfrac{\dfrac{a \Rightarrow b\backslash c}{b \circ a \Rightarrow c}\,(\backslash\mathrm{R})\uparrow}{b \Rightarrow c/a}\,(\backslash\mathrm{R}) \qquad \text{and} \qquad \dfrac{\dfrac{b \Rightarrow a\backslash c}{a \circ b \Rightarrow c}\,(\backslash\mathrm{R})\uparrow}{ab \Rightarrow c}\,(\cdot\mathrm{L})$$

For ($\mathbf{G}$RC$_r$), ($\mathbf{G}$I1ℓ) and ($\mathbf{G}$I1r) we have

$$\dfrac{b \Rightarrow a}{\varepsilon \Rightarrow a/b}\ (\backslash\mathrm{R}), \qquad \dfrac{\dfrac{\overline{a \Rightarrow a}\ (\mathrm{Id})}{\varepsilon \Rightarrow a\backslash a}\ (\backslash\mathrm{L})\!\uparrow}{1 \Rightarrow a\backslash a}\ (1\ell) \qquad \text{and} \qquad \dfrac{\dfrac{\overline{\varepsilon \circ a \Rightarrow a}\ (\mathrm{Id})}{1 \circ a \Rightarrow a}\ (1\mathrm{L})}{a \Rightarrow 1\backslash a}\ (\backslash\mathrm{R})$$

($\mathbf{G}$1) follows from (1R). For (RJ$\backslash$), if $\mathcal{K}$ does not contain $\cdot$, we have

$$\dfrac{\dfrac{b \circ |z| \Rightarrow c \qquad \dfrac{\varepsilon \Rightarrow z \rightsquigarrow_{\mathcal{K}} (a\backslash c)}{a \circ |z| \Rightarrow c}\ (\backslash\mathrm{R})\!\uparrow,\ (/\mathrm{R})\!\uparrow}{(a \vee b) \circ |z| \Rightarrow c}\ (\vee\mathrm{L})}{\varepsilon \Rightarrow z \rightsquigarrow_{\mathcal{K}} (a\backslash c)}\ (\backslash\mathrm{R}),\ (/\mathrm{R})$$

Note that (/R) and (/R)$\uparrow$ are not needed if $\mathcal{K}$ does not contain /. If $\cdot$ is contained in $\mathcal{K}$, the only modification needed in the proof is the replacement of $|z|$ by $\phi(z)$. Likewise, we obtain (RJ/). □

Corollary A.5. *The systems $\mathcal{K}$**HL** and $\mathcal{K}$**GL** are mutually translatable via the maps $\phi_{\mathcal{K}}$ and s.*

Appendix B. Action systems

In this section we define and study the notion of an action, which will be used as a tool in the investigation of matrices appropriate for Gentzen systems.

B.1. **Nuclei.** A *partially-ordered groupoid* (or *po-groupoid*, for brevity) is a structure $\mathbf{K} = \langle K, \leq, \cdot \rangle$ such that $\cdot$ is a binary operation on K, $\leq$ is a partial order on K and multiplication is order preserving ($p \leq q$ implies $pr \leq qr$ and $rp \leq rq$).

A *residuated partially-ordered groupoid* or *residuated po-groupoid* is a structure $\mathbf{K} = \langle K, \leq, \cdot, \backslash, / \rangle$ such that $\langle K, \leq, \cdot \rangle$ is a *po*-groupoid and for all $x, y, z \in K$,

(res) $$xy \leq z \ \Leftrightarrow\ y \leq x\backslash z \ \Leftrightarrow\ x \leq z/y.$$

A *residuated lattice-ordered groupoid* or *residuated ℓ-groupoid* is an algebra $\mathbf{K} = \langle K, \wedge, \vee, \cdot, \backslash, / \rangle$ such that $\langle K, \wedge, \vee \rangle$ is a lattice and $\langle K, \leq, \cdot, \backslash, / \rangle$ is a residuated *po*-groupoid, where $\leq$ is the lattice order.

If $\mathbf{K}$ is one of the above structures, we say that $\mathbf{K}$ has a unit, if there is an element $1 \in K$ such that $1x = x1 = 1$, for all $x \in K$. In this case we add in the type a constant 1 that is interpreted as the unit element. We will refer to a residuated lattice-ordered groupoid with unit as an *rlu-groupoid*. A *po*-groupoid with unit is called *integral*, if the unit is the greatest element; it is called *associative* or *commutative*, if its monoid reduct is. It is called *integral* if $x \leq 1$, for all $x \in K$, and it called *contracting* if $x \leq x^2$, for all $x \in K$.

Lemma B.1. *If $\mathbf{K} = \langle K, \leq, \cdot 1 \rangle$ is a po-groupoid with unit, then the algebra $\mathcal{P}(\mathbf{K}) = \langle \mathcal{P}(K), \cap, \cup, \cdot, \backslash, /, \{1\} \rangle$ is a rlu-groupoid, where for $X, Y \subseteq K$, $X \cdot Y = \{xy \mid x \in X, y \in Y\}$, $X\backslash Y = \{z \mid X \cdot \{z\} \subseteq Y\}$ and $Y/X = \{z \mid \{z\} \cdot X \subseteq Y\}$.*

Recall that a closure operator c on a poset $\mathbf{P} = \langle P, \leq \rangle$ is a map $c : P \to P$ that is extensive ($p \leq c(p)$), monotone (if $p \leq q$, then $c(p) \leq c(q)$) and idempotent ($c(c(p)) = c(p)$, for all $p, q \in P$).

A *nucleus* on a *po*-groupoid $\mathbf{K}$ is a map $g : K \to K$ such that g is a closure operator on $\langle K, \leq \rangle$ and for all $x, y \in K$,

(nuc) $$g(x)g(y) \leq g(xy).$$

A *nucleus* on a residuated *po*-groupoid or ℓ-groupoid is a nucleus on its *po*-groupoid reduct. We denote by $g[K]$ or K_g the image of K under g.

The following lemma essentially generalizes known facts (see [8], [40], [35], [21]). Its proofs can be found in [18].

Lemma B.2.

(1) *If* $\mathbf{K}$ *is a residuated po-groupoid, then* g *is a nucleus on* $\mathbf{K}$ *iff for all* $x, y \in K$, $g(x)/y, y\backslash g(x) \in K_g$ *iff for all* $x, y \in K$, $g(g(x)g(y)) = g(xy)$.
(2) *If* $\mathbf{K} = \langle K, \leq, \cdot \rangle$ *is a po-groupoid and* g *is a nucleus on* $\mathbf{K}$, *then* $\mathbf{K}_g = \langle g[K], \leq, \cdot_g \rangle$, *where* $g[K] = \{g(k) \mid k \in K\}$ *and* $x \cdot_g y = g(xy)$, *is a po-groupoid.*
(3) *If* $\mathbf{K} = \langle K, \leq, \cdot, \backslash, / \rangle$ *is a residuated po-groupoid and* g *is a nucleus on* $\mathbf{K}$, *then* $\mathbf{K}_g = \langle g[K], \leq, \cdot_g, \backslash, / \rangle$ *is a residuated po-groupoid.*
(4) *If* $\mathbf{K} = \langle K, \wedge, \vee, \cdot, \backslash, / \rangle$ *is a residuated ℓ-groupoid and* g *is a nucleus on* $\mathbf{K}$, *then* $\mathbf{K}_g = \langle g[K], \wedge, \vee_g, \cdot_g, \backslash, /, \rangle$, *where* $x \vee_g y = g(x \vee y)$, *is a residuated ℓ-groupoid.*
(5) *In all of the above cases, if* $\mathbf{K}$ *has a unit* 1, *then* $1_g = g(1)$ *is a unit of* $\mathbf{K}_g$.
(6) *In any of the above cases,* g *is a* $\{\cdot, \vee, 1\}$*-homomorphism from* $\mathbf{K}$ *to* $\mathbf{K}_g$ *(if* $\vee$ *and* 1 *exist); also it is order preserving. In particular, if* t *is a* $\{\cdot, \vee, 1\}$*-formula, then* $g(t^{\mathbf{K}}(\bar{x})) = t^{\mathbf{K}_g}(g(\bar{x}))$, *for all appropriate sequences* $\bar{x}$ *of elements in* K.
(7) *If* $\mathbf{K}$ *is associative, commutative, integral or contracting, then so is* $\mathbf{K}_g$.
(8) *If* $\mathbf{K} = \langle K, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$ *is an rlu-groupoid and* $s \in K$, *then the map* $g_s : K \to K$ *defined by* $g_s(x) = (s/x)\backslash s$ *is a closure operator on* $\mathbf{K}$. *If* $\mathbf{K}$ *is associative and commutative, then* g_s *is a nucleus on* $\mathbf{K}$.

Note that for item (6), existing (infinite) meets are preserved and if $\bigvee X$ exists then $\bigvee_g X = g(\bigvee X)$.

B.2. **Action systems.**

B.2.1. *More on groupoid words.* Recall the definition of the set Q^γ of groupoid words and the set Q^α of augmented groupoid words over a set Q. For $u \in Q^\alpha$ and $x \in Q^\gamma$ define $x * u = u[x \circ \_]$ and $u * x = u[\_ \circ x]$. For example, if $x = (a, b)$ and $u = (a, (\_, a))$, then $x * u = (a, ((\_, (a, b)), a))$ and $u * x = (a, (((a, b), \_), a))$. Note that $u * \varepsilon = \varepsilon * u = u$. Recall that we have allowed ourselves to denote the element $u[x]$ also by $u \star x$ and $x \star u$. Note that $u \star x = |u * x|$ and $x \star u = |x * u|$, for all $x \in Q^\gamma$ and $u \in Q^\alpha$. For all $x, y \in Q^\gamma$ and $u \in Q^\alpha$, we have

$$(x * u) \star y = (u * y) \star x = u \star (x \circ y).$$

Indeed, for all $x, y \in Q^\gamma$ and $u \in Q^\alpha$, we have

$$(x * u) \star y = u[x \circ \_] \star y = u[x \circ y]$$

$$(u * y) \star x = u[\_ \circ y] \star x = u[x \circ y]$$

$$u \star (x \circ y) = u[x \circ y]$$

The set $Sub_G(x)$ of *subterms* of a G-term x over Q is defined inductively by $Sub_G(x) = \{x\}$, for $x \in Q \cup \{\varepsilon\}$ and $Sub_G((x, y)) = \{(x, y)\} \cup Sub_G(x) \cup Sub_G(x)$.

B.2.2. *Actions.* A multi-sorted structure $\mathcal{A} = \langle \mathbf{K}_{\mathcal{A}}, L_{\mathcal{A}}, K'_{\mathcal{A}}, *, |\ \ | \rangle$ is called a *partial action system* if

- $\mathbf{K}_{\mathcal{A}} = \langle K_{\mathcal{A}}, \circ, e \rangle$ is a partial groupoid with unit,
- $L_{\mathcal{A}}$ and $K'_{\mathcal{A}}$ are sets,
- $|\ \ | : K'_{\mathcal{A}} \to L_{\mathcal{A}}$ is an onto map,
- $* : K_{\mathcal{A}} \times K'_{\mathcal{A}} \to K'_{\mathcal{A}}$, $* : K'_{\mathcal{A}} \times K_{\mathcal{A}} \to K'_{\mathcal{A}}$ are partial maps which we denote both by the same symbol,
- the partial maps $\star : K_{\mathcal{A}} \times K'_{\mathcal{A}} \to L_{\mathcal{A}}$ and $\star : K'_{\mathcal{A}} \times K_{\mathcal{A}} \to L_{\mathcal{A}}$ are defined by $x \star u = |x * u|$ and $u \star x = |u * x|$, and,
- for all $x \in K_{\mathcal{A}}$, $y \in L_{\mathcal{A}}$ and $u \in K'_{\mathcal{A}}$, we have

$$(x * u) \star y = (u * y) \star x, \tag{act}$$

$$u \star (x \circ y) = (x * u) \star y \quad \text{and} \tag{g-act}$$

$$u * e = e * u = u. \tag{u-act}$$

in the sense: if one of the sides of an equation is defined, then the other side is also defined and they are equal. If $u \star x = x \star u$, for all $x \in K_{\mathcal{A}}$ and $u \in K'_{\mathcal{A}}$, then we denote the common value by $u[x]$. An *action system* is a partial action system, where all partial maps in the definition are full.

If Q is a set and $\varepsilon, \_ \notin Q$, then $\mathcal{Q} = \langle \mathbf{Q}^{\gamma}, Q^{\gamma}, Q^{\alpha}, *, |\ \ | \rangle$ is an action system. We also obtain an action system if instead of groupoid words we consider sequences, multisets or sets of elements of Q. In the last two cases (sequences and multisets) we can actually eliminate $\_$, take $*$ and $\circ$ to be union and ε to be the empty (multi)set.

Also, if $\mathbf{K} = \langle K, \circ, e \rangle$ is a commutative monoid, then $\langle \mathbf{K}, K, K, \circ, |\ \ | \rangle$ is an action system, where $|k| = k$, for all $k \in K$. Note that the assumptions of associativity and commutativity are essential.

In both of the examples given above we have $K_{\mathcal{A}} = L_{\mathcal{A}}$. We allow the two sets to be different in the definition so that partial action systems are closed under the following construction.

If $\mathcal{A} = \langle \mathbf{K}_{\mathcal{A}}, L_{\mathcal{A}}, K'_{\mathcal{A}}, *, |\ \ | \rangle$, is a partial action system and Q any set, consider the structure $\mathcal{A} \times Q = \langle \mathbf{K}_{\mathcal{A}}, L_{\mathcal{A}} \times Q, K'_{\mathcal{A}} \times Q, *, |\ \ | \rangle$, where $k * (k', q) = (k * k', q)$, $(k', q) * k = (k * k', q)$ and $|(k', q)| = (|k'|, q)$.

Lemma B.3. *If $\mathcal{A}$ is a partial action system and Q any set, then $\mathcal{A} \times Q$ is a partial action system, as well.*

A multi-sorted structure $\mathcal{A} = \langle \mathbf{K}_{\mathcal{A}}, \mathbf{L}_{\mathcal{A}}, \mathbf{K}'_{\mathcal{A}}, *, |\ \ |, \backslash_*, /_* \rangle$ is called a *residuated action system* if

- $\mathbf{K}_{\mathcal{A}} = \langle K_{\mathcal{A}}, \leq, \circ, e \rangle$ is a *po*-groupoid with unit,
- $\mathbf{K}'_{\mathcal{A}} = \langle K'_{\mathcal{A}}, \leq' \rangle$ and $\mathbf{L}_{\mathcal{A}} = \langle L_{\mathcal{A}}, \leq'' \rangle$ are posets,
- $\langle \langle K_{\mathcal{A}}, \circ, e \rangle, L_{\mathcal{A}}, K'_{\mathcal{A}}, *, |\ \ | \rangle$, is an action system and
- $\backslash_* : K'_{\mathcal{A}} \times L_{\mathcal{A}} \to K_{\mathcal{A}}$, $/_* : L_{\mathcal{A}} \times K_{\mathcal{A}} \to K'_{\mathcal{A}}$ are maps such that, for all $u \in K'$, $x \in K$ and $y \in L$,

$$u \star x \leq'' y \;\Leftrightarrow\; x \leq u \backslash_* y \;\Leftrightarrow\; u \leq' y /_* x. \tag{r-act}$$

A residuated action system $\mathcal{A}$ is called *lattice-ordered*, if $\mathbf{K}_{\mathcal{A}}$ is lattice-ordered.

If $\langle K, \leq, \circ, \backslash, /, e\rangle$ is a commutative residuated *po*-monoid, then $\langle \mathbf{K}, \mathbf{K'}, \mathbf{K'}, \circ, |\ \ |, \backslash, /\rangle$ is a residuated action system, where $\mathbf{K} = \langle K, \leq, \circ, e\rangle$, $\mathbf{K'} = \langle K, \leq\rangle$ and $|k| = k$, for all $k \in K$.

If $\mathbf{K}$ is lattice ordered, then so is the residuated action system. In this sense, lattice-ordered residuated action systems are generalizations of *commutative* residuated ℓ-monoids.

Assume that $\mathcal{A} = \langle \mathbf{K}_{\mathcal{A}}, L_{\mathcal{A}}, K'_{\mathcal{A}}, *, |\ \ |\rangle$, where $\mathbf{K}_{\mathcal{A}} = \langle K_{\mathcal{A}}, \cdot, e\rangle$, is a partial action system and consider the powersets

$$M_{\mathcal{A}} = \mathcal{P}(K_{\mathcal{A}}),\ N_{\mathcal{A}} = \mathcal{P}(L_{\mathcal{A}}) \text{ and } M'_{\mathcal{A}} = \mathcal{P}(K'_{\mathcal{A}}).$$

For $m_1, m_2, m \in M_{\mathcal{A}}$ $m' \in M'_{\mathcal{A}}$ and $n \in N_{\mathcal{A}}$ define

$$m_1 \circ m_2 = \{k_1 \circ k_2 \mid k_1 \in m_1,\ k_2 \in m_2\},$$

$$m * m' = \{k * k' \mid k \in m,\ k' \in m'\},$$

$$m' * m = \{k' * k \mid k' \in m',\ k \in m\},$$

$$|m'| = \{|k'| \mid k' \in m'\},$$

$$m' \backslash_* n = \{k \in K \mid m' \star \{k\} \subseteq n\}$$

$$n /_* m = \{k' \in K' \mid \{k'\} \star m \subseteq n\}.$$

Consider the structure $\mathcal{P}(\mathcal{A}) = \langle \mathbf{M}_{\mathcal{A}}, \mathbf{N}_{\mathcal{A}}, \mathbf{M}'_{\mathcal{A}}, *, |\ \ |, \backslash_*, /_*\rangle$, where $\mathbf{M}_{\mathcal{A}} = \langle M_{\mathcal{A}}, \subseteq, \circ, \{e\}\rangle$, $\mathbf{N}_{\mathcal{A}} = \langle N_{\mathcal{A}}, \subseteq\rangle$ and $\mathbf{M}'_{\mathcal{A}} = \langle M'_{\mathcal{A}}, \subseteq\rangle$.

Lemma B.4. *If $\mathcal{A}$ is a partial action system, then $\mathcal{P}(\mathcal{A})$ is a residuated action system*

B.2.3. *Nuclei and action systems.*

Lemma B.5. *Let $\mathcal{A} = \langle \mathbf{K}_{\mathcal{A}}, \mathbf{L}_{\mathcal{A}}, \mathbf{K}'_{\mathcal{A}}, *, |\ \ |, \backslash_*, /_*\rangle$ be a residuated action system and s an element of $L_{\mathcal{A}}$. Then the map $g_s : K_{\mathcal{A}} \to K_{\mathcal{A}}$ defined by $g_s(x) = (s/_*x)\backslash_* s$ is a nucleus on $\mathbf{K}_{\mathcal{A}}$.*

Proof. The pair $(x \mapsto s/_*x, x \mapsto x\backslash_* s)$ forms a Galois connection between $\langle K_{\mathcal{A}}, \leq\rangle$ and $\langle K'_{\mathcal{A}}, \leq\rangle$; the two maps are the polarities of the Galois connection. So g_s and g^s, where $g_s(x) = (s/_*x)\backslash_* s$ and $g^s(u) = s/_*(u\backslash_* s)$, are closure operators on $\langle K_{\mathcal{A}}, \leq\rangle$ and $\langle K'_{\mathcal{A}}, \leq\rangle$, respectively. (For more information on Galois connections, see Section 3.1 in [18].)

In detail, we have $s/_*x \leq' s/_*x$, so $(s/_*x) \star x \leq'' s$; hence $x \leq (s/_*x)\backslash_* s = g_s(x)$, for all $x \in K$. Based on the extensivity of g_s we can get that $/_*$ is order reversing in its denominator as follows. If $x \leq y$, then $x \leq y \leq g_s(y) = (s/_*y)\backslash_* s$; so $(s/_*y) \star x \leq'' s$, hence $s/_*y \leq' s/_*x$. Similarly, we can prove that for every $u \in K'$, $u \leq' g^s(u)$, where $g^s(u) = s/_*(u\backslash_* s)$, and that $\backslash_*$ is order reversing in its denominator. Combining these two facts we obtain the monotonicity of g_s. Finally, to show that $g_s(g_s(x)) \leq x$, note that $s/_*x \leq' g^s(s/_*x) = s/_*((s/_*x)\backslash_* s) = s/_* g_s(x)$, so $g_s(g_s(x)) = (s/_* g_s(x))\backslash_* s \leq (s/_*x)\backslash_* s = g_s(x)$. Thus, g_s is a closure operator on $\langle K, \leq\rangle$.

Moreover, for all $x, y \in K$, we have the following implications

$$\begin{array}{lll} & s/_*(x \circ y) \leq' s/_*(x \circ y) & \\ \Rightarrow & (s/_*(x \circ y)) \star (x \circ y) \leq'' s & \text{(r-act)} \\ \Rightarrow & [x * (s/_*(x \circ y))] \star y \leq'' s & \text{(g-act)} \\ \Rightarrow & x * (s/_*(x \circ y)) \leq' s/_*y & \text{(r-act)} \\ \Rightarrow & x * (s/_*(x \circ y)) \leq' s/_*((s/_*y)\backslash_* s) & (u \leq' g^s(u)) \\ \Rightarrow & [x * (s/_*(x \circ y))] \star g_s(y) \leq'' s & \text{(r-act)} \\ \Rightarrow & [(s/_*(x \circ y)) * g_s(y)] \star x \leq'' s & \text{(act)} \\ \Rightarrow & (s/_*(x \circ y)) * g_s(y) \leq' s/_*x & \text{(r-act)} \\ \Rightarrow & (s/_*(x \circ y)) * g_s(y) \leq' s/_*((s/_*x)\backslash_* s) & (u \leq' g^s(u)) \\ \Rightarrow & [(s/_*(x \circ y)) * g_s(y)] \star g_s(x) \leq'' s & \text{(r-act)} \\ \Rightarrow & (s/_*(x \circ y)) \star (g_s(x) \circ g_s(y)) \leq'' s & \text{(g-act), (act)} \\ \Rightarrow & g_s(x) \circ g_s(y) \leq (s/_*(x \circ y))\backslash_* s & \text{(r-act)} \\ \Rightarrow & g_s(x) \circ g_s(y) \leq g_s(x \circ y) & \end{array}$$

Consequently, g_s is a nucleus on $\mathbf{K}$. □

Corollary B.6. *If $\mathcal{A} = \langle \mathbf{K}_\mathcal{A}, \mathbf{K}'_\mathcal{A}, *, |\ \ |, \backslash_*, /_* \rangle$ is a lattice-ordered residuated action system and $s \in L_\mathcal{A}$, then $\mathbf{R}_s(\mathcal{A}) = (\mathbf{K}_\mathcal{A})_{g_s}$—see Lemmas B.2 and B.5—is a residuated ℓ-groupoid with unit. If $\mathbf{K}_\mathcal{A}$ satisfies a given groupoid identity (in particular if it is associative, commutative or idempotent) then so does $\mathbf{R}_s(\mathcal{A})$; more explicitly, for every groupoid word t, $t^{(\mathbf{K}_\mathcal{A})_{g_s}}(\bar{x}) = g_s(t^{\mathbf{K}_\mathcal{A}}(g_s(\bar{x})))$. If e is the unit of $\mathbf{K}_\mathcal{A}$, then $g_s(e)$ is the unit of $\mathbf{R}_s(\mathcal{A})$. If $\mathbf{K}_\mathcal{A}$ is integral, then $\mathbf{R}_s(\mathcal{A})$ is integral and $g_s(e) = e$.*

B.2.4. *The $r\ell u$-groupoid of a G-matrix.* Let $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \preceq \rangle$ be a sequent $\mathcal{K}$-matrix. It is easy to see that $\mathcal{A} = \langle \mathbf{A}^\gamma, A^\gamma, A^\alpha, *, |\ \ | \rangle$ is a partial action system. It follows from Lemmas B.3 and B.4 that $\mathcal{I}(\mathbf{A}) = \mathcal{P}(\mathcal{A} \times A)$ is a lattice-ordered residuated action system. Therefore, by Corollary B.6, we obtain the following result.

Corollary B.7. *Let $\boldsymbol{\mathcal{A}} = \langle \mathbf{A}, \preceq \rangle$ be a sequent $\mathcal{K}$-matrix. Then, $\mathbf{R}(\boldsymbol{\mathcal{A}}) = \mathbf{R}_\preceq(\mathcal{I}(\mathbf{A}))$ is a residuated ℓ-groupoid with unit $g_\preceq(\{\varepsilon\})$*

The algebra $\mathbf{R}(\boldsymbol{\mathcal{A}})$ is called the *residuated ℓ-groupoid of $\boldsymbol{\mathcal{A}}$*.

For every $a \in A$ and $u \in A^\alpha$, set

$$[u, a] = \{(u, a)\}\backslash_* \preceq\ = \{x \in A^\gamma \mid u[x] \preceq a\}$$

and $\downarrow a = [\_, a] = \{x \in A^\gamma \mid x \preceq a\}$. Note that, $[u, a] \in \mathbf{R}(\boldsymbol{\mathcal{A}})$, so the assignment $q(a) = \downarrow a$ defines a map $q : A \to R(\boldsymbol{\mathcal{A}})$.

Lemma B.8. *If $k \in \mathbf{R}(\boldsymbol{\mathcal{A}})$ and $x \in A^\gamma$, then $x \in k$ iff $x \in [u, a]$, for all $u \in A^\alpha$ and $a \in A$ such that $k \subseteq [u, a]$.*

Proof. Since $k \in \mathbf{R}(\boldsymbol{\mathcal{A}})$, we have $k = g_\preceq(k) = (\preceq/_*k)\backslash_*\preceq$. So $x \in k$ iff $x \in (\preceq/_*k)\backslash_*\preceq$ iff $(\preceq/_*k) \star \{x\} \subseteq \preceq$ iff

$$(u, a) \star x \in \preceq, \text{ for all } u \in J_\mathbf{A} \text{ and } a \in Q_\mathbf{A} \text{ such that } (u, a) \in \preceq/_*k.$$

Observe that $(u, a) \star x \in \preceq$ is equivalent to $\{(u, a)\} \star \{x\} \subseteq \preceq$ and $\{x\} \subseteq \{(u, a)\}\backslash_* \preceq\ = [u, a]$. Moreover, $(u, a) \in \preceq/_*k$ is equivalent to $\{(u, a)\} \subseteq \preceq/_*k$ and $k \subseteq \{(u, a)\}\backslash_* \preceq\ = [u, a]$. □

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