COEXISTENCE OF QUANTUM EFFECTS

Stan Gudder
Department of Mathematics
University of Denver
Denver, Colorado 80208
sgudder@math.du.edu

Abstract

The witness set $\mathcal{W}(A,B)$ for two quantum effects A,B is the set of effects that verify the coexistence of A and B. We first study the properties of $\mathcal{W}(A,B)$. We then introduce a measure of sharpness $\mathcal{S}_n(A)$ for an n-dimensional quantum effect A. We show that $\mathcal{S}_2(A)$ has natural properties that a measure of sharpness should possess and conjecture that $\mathcal{S}_n(A)$ also has these properties for $n>2$. Finally, we employ $\mathcal{S}_2(A)$ to obtain a characterization for the coexistence of two qubit effects in terms of operational parameters.

Keywords: coexistence, quantum effects, measure of sharpness.

PACS: 47A53, 47A55.

1 Introduction

Quantum effects represent two-valued quantum measurements that may be unsharp (fuzzy) [2, 3, 4, 6, 7, 9]. Two quantum effects A,B coexist if they can be measured together in the sense that a third measurement can be employed to measure both A and B. An important unsolved problem of quantum measurement theory is to characterize the coexistence of A and B. Of course, it is possible to give trivial characterizations that are close to the definition of coexistence. What is required is a useful operational characterization in terms of physically accessible parameters. That is, parameters that can be obtained from operational procedures in the laboratory. The most direct physical parameters are the eigenvalues and eigenvectors if the effects have pure discrete spectrum. These are the parameters that we shall consider.

Characterizations for the coexistence of two qubit effects have recently appeared [2, 8, 11]. However, these characterizations have not been given in terms of conditions involving operational parameters. We shall translate the conditions in [11] into simpler relations between their eigenvalues and transition probabilities between their eigenvectors. We also discuss a measure of sharpness for finite-dimensional effects. We employ this measure to characterize coexistence of qubit effects and the measure may be useful for characterizing coexistence of higher dimensional effects.

In order to study coexistence more closely we introduce a set $\mathcal{W}(A,B)$ that we call the witness set for A and B. The set $\mathcal{W}(A,B)$ contains the set of effects that witness or verify the coexistence of A and B. One of our goals is to study the properties of $\mathcal{W}(A,B)$. Again, these properties may be useful for characterizing coexistence.

2 The Witness Set

If H is a complex Hilbert space, an **effect** on H is an operator A on H satisfying $0 \le A \le I$ [3, 4, 7, 9]. We denote the set of effects on H by $\mathcal{E}(H)$. The set of projections on H is denoted by $\mathcal{P}(H)$ and elements of $\mathcal{P}(H)$ are called **sharp effects**. Two effects $A, B \in \mathcal{E}(H)$ **coexist** if there are effects $A_1, B_1, C \in \mathcal{E}(H)$ such that $A_1+B_1+C \in \mathcal{E}(H)$ and $A = A_1+C$, $B = B_1+C$. If A and B coexist, we write $A\mathrm{co}B$. We interpret $A\mathrm{co}B$ as meaning that A and B can be measured together [3, 7, 9]. It is easy to check that $A\mathrm{co}B$ if and only if there exists $C \in \mathcal{E}(H)$ such that

$$A + B - I \le C \le A, B \tag{2.1}$$

In general, there may be more than one C satisfying (2.1) and we denote the set of such Cs by $\mathcal{W}(A,B)$. That is,

$$\mathcal{W}(A,B) = \{C \in \mathcal{E}(H) \colon A + B - I \le C \le A, B\}$$

We call $\mathcal{W}(A,B)$ the **witness set** for $A, B \in \mathcal{E}(H)$ and call $C \in \mathcal{W}(A,B)$ a **witness** for A, B. An effect $C \in \mathcal{W}(A,B)$ witnesses or verifies the coexistence of A and B. Of course, $A\mathrm{co}B$ if and only if $\mathcal{W}(A,B) \ne \emptyset$ and clearly $\mathcal{W}(A,B) = \mathcal{W}(B,A)$. We denote the cardinality of $\mathcal{W}(A,B)$ by $|\mathcal{W}(A,B)|$.

Example 1. *This simple example shows that we can have* $|\mathcal{W}(A,B)| = \infty$. *Let*

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \mu & 0 \\ 0 & 1 \end{bmatrix}, \quad 0 \le \lambda, \mu \le 1$$

Then

$$\left\{ A = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} : \alpha \in [\max(\lambda + \mu - 1, 0), \min(\lambda, \mu)] \right\} \subseteq \mathcal{W}(A,B)$$

In general, the set on the left can have infinite cardinality and $\mathcal{W}(A,B)$ *may have more elements than this set.*

Theorem 2.1. $\mathcal{W}(A,B)$ *is convex and closed in the weak operator topology.*

Proof. Suppose $C_1, C_2 \in \mathcal{W}(A,B)$ and $0 \le \lambda \le 1$. Since $C_1, C_2 \le A, B$ we have

$$\lambda C_1 + (1-\lambda)C_2 \le \lambda A + (1-\lambda)A = A$$

and similarly, $\lambda C_1 + (1-\lambda)C_2 \le B$. Since $A + B - I \le C_1, C_2$ we have

$$\lambda(A + B - I) \le \lambda C_1, \quad (1 - \lambda(A + B - I) \le (1-\lambda)C_2$$

Hence,

$$A + B - I = \lambda(A + B - I) + (1-\lambda)(A + B - I) \le \lambda C_1 + (1-\lambda)C_2 \le A, B$$

Therefore, $\lambda C_1 + (1-\lambda)C_2 \in \mathcal{W}(A,B)$ so that $\mathcal{W}(A,B)$ is convex. Suppose C_α is a net in $\mathcal{W}(A,B)$ and $C_\alpha \to C$ weakly. Since $C_\alpha \le A, B$ for every α, we have

$$\langle Cx, x \rangle = \lim \langle C_\alpha x, x \rangle \le \langle Ax, x \rangle$$

Hence, $C \le A$ and similarly $C \le B$. Since $A + B - I \le C_\alpha$ for all α, we have

$$\langle Cx, x \rangle = \lim \langle C_\alpha x, x \rangle \ge \langle (A + B - I)x, x \rangle$$

Hence, $A + B - I \le C \le A, B$ so $C \in \mathcal{W}(A,B)$ Therefore, $\mathcal{W}(A,B)$ is closed in the weak operator topology. □

Theorem 2.2. *If* $A \in \mathcal{P}(H)$, $B \in \mathcal{E}(H)$ *and* $A\mathrm{co}B$, *then* $\mathcal{W}(A,B) = \{AB\}$.

Proof. It is well known that $AB = BA \in \mathcal{E}(H)$ [7, 9]. Clearly, $AB \leq A, B$ and moreover,

$$0 \leq (I - A)(I - B) = I - A - B + AB$$

Hence, $A + B - I \leq AB \leq A, B$ so that $AB \in \mathcal{W}(A, B)$. Now suppose that $C \in \mathcal{W}(A, B)$. Since $C \leq A, B$, we have that $CA = AC = C$. Hence,

$$C = AC = ACA \leq ABA = AB$$

Since $A + B - I \leq C$ we have that $A + AB - A \leq AC = C$. Hence, $AB \leq C$. Therefore, $C = AB$ so that $\mathcal{W}(A, B) = \{AB\}$. $\square$

Example 2. *This example shows that the converse of Theorem 2.2 does not hold. Also, $|\mathcal{W}(A, B)| = 1$ does not imply that A or B is a projection. Let*

$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Then $\mathcal{W}(A, B) = \{0\} = \{AB\}$, yet neither A or B is a projection.

It is an interesting open problem to characterize the pairs $A, B \in \mathcal{E}(H)$ such that $|\mathcal{W}(A, B)| = 1$. We do have the following partial result.

Theorem 2.3. (a) *If $|\mathcal{W}(A, B)| = 1$ and $C \in \mathcal{W}(A, B)$, then C is a maximal lower bound for A and B.* (b) *If $C \in \mathcal{W}(A, B)$ is a maximal lower bound for A and B, then $|\mathcal{W}(A, B)| \neq 1$ in general.*

Proof. (a) Suppose $C \in \mathcal{W}(A, B)$ and $C \leq D$ where $D \leq A, B$. Then

$$A + B - I \leq C \leq D$$

so $D \in \mathcal{W}(A, B)$. Since $|\mathcal{W}(A, B)| = 1$, $C = D$. Hence, C is a maximal lower bound for A, B. (b) To construct a counterexample, suppose that $A + B \leq I$ so $A\text{co}B$ but the greatest lower bound $A \wedge B$ does not exist [5]. Then $0 \in \mathcal{W}(A, B)$ and there exists a $D \in \mathcal{E}(H)$ with $D \neq 0$ and $D \leq A, B$. Hence, $D \in \mathcal{W}(A, B)$. By Zorn's lemma there exists a maximal lower bound C for A and B. Then $C \in \mathcal{W}(A, B)$ but $|\mathcal{W}(A, B)| \neq 1$. $\square$

The following lemma shows that $\mathcal{W}(A, B)$ can be employed to characterize various relationships between A and B. We use the notation $A' = I - A$ and $A \perp B$ whenever $A \leq B'$.

Lemma 2.4. (a) *$A \leq B$ if and only if $A \in \mathcal{W}(A, B)$.* (b) *$A \perp B$ if and only if $0 \in \mathcal{W}(A, B)$.* (c) *$A' \perp B'$ if and only if $A + B - I \in \mathcal{W}(A, B)$.* (d) *$AB = BA$ if and only if $AB \in \mathcal{W}(A, B)$.*

Proof. The proofs of (a) and (b) are straightforward. (c) If $A' \perp B'$ then $A' \leq B$ so that $A + B - I \geq 0$. Hence, $A + B - I \in \mathcal{E}(H)$ and clearly $A+B-I \in \mathcal{W}(A, B)$. Conversely, if $A+B-I \in \mathcal{W}(A, B)$ then $A+B-I \geq 0$ so that $A' \leq B$. Hence, $A' \perp B'$. (d) If $AB \in \mathcal{W}(A, B)$, then

$$AB = (AB)^* = B^*A^* = BA$$

Conversely, if $AB = BA$ then $AB \leq A, B$ and as in the proof of Theorem 2.2, $A + B - I \leq AB$. Hence, $AB \in \mathcal{W}(A, B)$. □

For $\mathcal{A}, \mathcal{B} \subseteq \mathcal{E}(H)$ and $\lambda \in \mathbb{R}$ we define

$$\mathcal{A} + \mathcal{B} = \{A + B \colon A \in \mathcal{A}, B \in \mathcal{B}\}, \quad \lambda\mathcal{A} = \{\lambda A \colon A \in \mathcal{A}\}$$

By convention, $\emptyset + \mathcal{A} = \emptyset$.

Theorem 2.5. *For $0 \leq \lambda \leq 1$ we have that*

$$\lambda\mathcal{W}(A, B) + (1 - \lambda)\mathcal{W}(C, D) \subseteq \mathcal{W}\left(\lambda A + (1 - \lambda)C, \lambda B + (1 - \lambda)D\right)$$

Proof. Suppose that $E \in \mathcal{W}(A, B)$ and $F \in \mathcal{W}(C, D)$. Then

$$\lambda E + (1 - \lambda)F \leq \lambda A + (1 - \lambda)C$$
$$\lambda E + (1 - \lambda)F \leq \lambda B + (1 - \lambda)D$$

Since $A + B - I \leq E$ and $C + D - I \leq F$ we have that

$$\lambda A + \lambda B - \lambda I \leq \lambda E$$

and

$$(1 - \lambda)C + (1 - \lambda)D - (1 - \lambda)I \leq (1 - \lambda)F$$

We conclude that

$$[\lambda A + (1 - \lambda)C] + [\lambda B + (1 - \lambda)D] - I \leq \lambda E + (1 - \lambda)F$$

Therefore,

$$\lambda E + (1 - \lambda)F \in \mathcal{W}\left(\lambda A + (1 - \lambda)C, \lambda B + (1 - \lambda)D\right)$$

and the result follows. □

Corollary 2.6. *For $0 \leq \lambda \leq 1$ we have that $\lambda\mathcal{W}(A,B) \subseteq \mathcal{W}(A,\lambda B)$ and*

$$\lambda\mathcal{W}(A,B) + (1-\lambda)\mathcal{W}(A,C) \subseteq \mathcal{W}\left(A, \lambda B + (1-\lambda)C\right)$$

The inclusions in the other directions for Theorem 2.5 and Corollary 2.6 do not hold in general as the next example shows.

Example 3. *Suppose A and B do not coexist. Then $\mathcal{W}(A,B) = \emptyset$ so that*

$$\frac{1}{2}\,\mathcal{W}(A,B) + \frac{1}{2}\,\mathcal{W}(A,B') = \emptyset$$

However,

$$\mathcal{W}\left(A, \frac{1}{2}\,B + \frac{1}{2}\,B'\right) = \mathcal{W}\left(A, \frac{1}{2}\,I\right) \neq \emptyset$$

We conclude that

$$\mathcal{W}\left(A, \frac{1}{2}\,B + \frac{1}{2}\,B'\right) \not\subseteq \frac{1}{2}(A,B) + \frac{1}{2}\,\mathcal{W}(A,B')$$

In the next result we use the notation $\{A\} = A$.

Theorem 2.7. *$\mathcal{W}(A,B') = A - \mathcal{W}(A,B)$.*

Proof. Since $\mathcal{W}(A,B') = \emptyset$ if and only if $\mathcal{W}(A,B) = \emptyset$, the result holds if one of these witness sets is empty. Hence, we can assume that $\mathcal{W}(A,B)$ and $\mathcal{W}(A,B')$ are nonempty. Suppose that $C \in \mathcal{W}(A,B')$. Then $D = A - C \in \mathcal{E}(H)$ and $C = A - D$. Now $D \leq A$ and since

$$A - B = A + B' - I \leq C = A - D$$

we have that $D \leq B$. Since $C \leq B'$ we have that $B \leq C'$. Hence,

$$A + B - I \leq A + C' - I = A - C = D$$

Therefore, $D \in \mathcal{W}(A,B)$ and

$$C = A - D \in A - \mathcal{W}(A,B)$$

We conclude that $\mathcal{W}(A,B') \subseteq A - \mathcal{W}(A,B)$. Conversely, suppose that $C \in A - \mathcal{W}(A,B)$. Then $C = A - D$ for $D \in \mathcal{W}(A,B)$. Now $C \leq A$ and since

$$A + B - I \leq D = A - C$$

we conclude that $C \le B'$. Finally, since $D \le B$ we have that

$$A + B' - I \le A + D' - I = A - D = C$$

Hence, $C \in \mathcal{W}(A, B')$ so that $A - \mathcal{W}(A, B) \subseteq \mathcal{W}(A, B')$. The result now follows. $\square$

Since $\mathcal{W}(A, I) = A$ we can write Theorem 2.7 in the suggestive notation

$$\mathcal{W}(A, I - B) = \mathcal{W}(A, I) - \mathcal{W}(A, B)$$

It is well known that any bounded self-adjoint operator A has a unique representation $A = A^+ - A^-$ where A^+, A^- are positive operators and $A^+A^- = 0$. The next result gives a sufficient condition for coexistence. We call $C \in \mathcal{W}(A, B)$ a **commuting witness** if $CA = AC$ and $CB = BC$.

Theorem 2.8. (a) $(A + B - I)^+ \le A, B$ *if and only if* $(A + B - I)^+ \in \mathcal{W}(A, B)$. (b) *If* $(A + B - I)^+ \le A, B$ *then* $A\text{co}B$ (c) *If* $\mathcal{W}(A, B)$ *contains a commuting witness, then* $(A + B - I)^+ \in \mathcal{W}(A, B)$.

Proof. (a) Since $A + B - I \le A$, it is clear that $(A + B - I)^+ \in \mathcal{E}(H)$. If $(A + B - I)^+ \in \mathcal{W}(A, B)$ then by definition $(A + B - I)^+ \le A, B$. Conversely, suppose that $(A + B - I)^+ \le A, B$. Since

$$A + B - I \le (A + B - I)^+$$

it follows that $(A + B - I)^+ \in \mathcal{W}(A, B)$. Part (b) follows from (a). (c) We first prove the following result. If E and F are bounded self-adjoint operators satisfying $E \le F$, $EF = FE$ and $F \ge 0$, then $E^+ \le F$. To prove this result, we note that there exists a bounded self-adjoint operator G and Borel functions f, g where $g \ge 0$ that satisfy $E = f(G)$, $F = g(G)$. Moreover, since $E \le F$ we can assume that $f \le g$. It follows that the positive part f^+ of f satisfies $f^+ \le g$. Hence,

$$E^+ = f^+(G) \le g(G) = B$$

Now let $C \in \mathcal{W}(A, B)$ be a commuting witness. Then

$$A + B - I \le C \le A, B$$

and since C commutes with $A + B - I$ and $C \ge 0$, by our previous work we have

$$(A + B - I)^+ \le C \le A, B$$

We conclude that $(A + B - I)^+ \in \mathcal{W}(A, B)$. $\square$

Corollary 2.9. *If* $\mathcal{W}(A,B)$ *contains a projection, then* $(A+B-I)^+ \in \mathcal{W}(A,B)$.

Proof. If $P \in \mathcal{W}(A,B) \cap \mathcal{P}(H)$ then $P \le A, B$ implies that P is a commuting witness. $\square$

3 Measure of Sharpness

This section discusses a measure of sharpness for finite-dimensional effects. We call an effect in the two-dimensional Hilbert space $\mathbb{C}^2$ a **qubit effect**. Let $A \in \mathcal{E}(\mathbb{C}^n)$, $n \ge 2$, be an n-dimensional effect and suppose the eigenvalues of A (repeated according to multiplicity) are $\lambda_1, \dots, \lambda_n$. We write $\lambda_i' = 1 - \lambda_i$, $i = 1, \dots, n$. We define

$$T_n(A) = \sum_{i<j=1}^{n} \left[(\lambda_i\lambda_j)^{1/2} + (\lambda_i'\lambda_j')^{1/2}\right] - \sum_{i<j<k=1}^{n} \left[\lambda_i\lambda_j\lambda_k)^{1/3} + (\lambda_i'\lambda_j'\lambda_k')^{1/3}\right]$$
$$+ \cdots + (-1)^n \left[(\lambda_1 \cdots \lambda_n)^{1/n} + (\lambda_1' \cdots \lambda_n')^{1/n}\right]$$

The sharpness of A is defined by

$$\mathcal{S}_n(A) = \frac{1}{2n-3}\left\{(n-1)^2 - [T_n(A)]^2\right\}$$

For example, for qubit effects we have

$$\mathcal{S}_2(A) = 1 - \left[(\lambda_1\lambda_2)^{1/2} + (\lambda_1'\lambda_2')^{1/2}\right]^2$$

and for the three-dimensional effects we have

$$\mathcal{S}_3(A) = \frac{1}{3}\{4-$$
$$\left[(\lambda_1\lambda_2)^{1/2} + (\lambda_1\lambda_3)^{1/2} + (\lambda_2\lambda_3)^{1/2} + (\lambda_1'\lambda_2')^{1/2} + (\lambda_1'\lambda_3')^{1/2} + (\lambda_2'\lambda_3')^{1/2}\right.$$
$$\left.-(\lambda_1\lambda_2\lambda_3)^{1/3} - (\lambda_1'\lambda_2'\lambda_3')^{1/3}\right]^2\Big\}$$

If $A = \lambda I$ for $0 \le \lambda \le 1$, then A corresponds to a maximally unsharp effect. At the other extreme is a nontrivial projection which is an element $P \in \mathcal{P}(H)$ with $P \ne 0, I$. A measure of sharpness should map the former type of effect to 0 and the latter type of effect to 1 [1, 11]. As we shall see, this is indeed the case.

Theorem 3.1. *The function* $\mathcal{S}_2\colon \mathcal{E}(H) \to \mathbb{R}$, $H = \mathbb{C}^2$, *is a continuous function that satisfies the following conditions.*
(a) $0 \leq \mathcal{S}_2(A) \leq 1$ *for all* $A \in \mathcal{E}(H)$.
(b) $\mathcal{S}_2(A') = \mathcal{S}(A)$ *for all* $A \in \mathcal{E}(H)$.
(c) $\mathcal{S}_2(UAU^*) = \mathcal{S}_2(A)$ *for every unitary operator* U *on* H.
(d) $\mathcal{S}_2(A) = 0$ *if and only if* $A = \lambda I$, $0 \leq \lambda \leq 1$.
(e) $\mathcal{S}_2(A) = 1$ *if and only if* A *is a nontrivial projection*

Proof. That $\mathcal{S}_2$ is continuous and satisfies (b) and (c) are clear. To prove the rest, if x, y are the eigenvalues of A, then

$$T_2(A) = \sqrt{xy} + \sqrt{(1-x)(1-y)}$$

Clearly $T_2(A) \geq 0$ and to show that $T_2(A) \leq 1$ we have

$$x^2 + y^2 - 2xy = (x-y)^2 \geq 0$$

Hence, $x^2 + y^2 + 2xy \geq 4xy$ so that $x + y \geq 2\sqrt{xy}$. It follows that

$$(1-x)(1-y) \leq 1 - 2\sqrt{xy} + xy$$

Taking square roots gives $\sqrt{(1-x)(1-y)} \leq 1 - \sqrt{xy}$ so $T_2(A) \leq 1$. By the previous computation $T_2(A) = 1$ if and only if $(x-y)^2 = 0$ or $x = y$. In this case $A = xI$, $0 \leq x \leq 1$. Finally, $T_2(A) = 0$ if and only if

$$\sqrt{xy} = \sqrt{(1-x)(1-y)} = 0$$

But this is equivalent to $x = 0$, $y = 1$ or $x = 1$, $y = 0$. In this case A is a nontrivial projection. Since $\mathcal{S}_2(A) = 1 - [T_2(A)]^2$ the result follows. $\square$

The conditions in Theorem 3.1 were given in [1, 11]. We conjecture that $\mathcal{S}_n$ satisfies these conditions for all $n \geq 2$. It is clear that $\mathcal{S}_n$ is continuous and satisfies (b) and (c). Hence, the conjecture holds if

$$n - 2 \leq T_n(A) \leq n - 1$$

with equality on the right if and only if $A = \lambda I$, $0 \leq \lambda \leq 1$, and equality on the left if and only if A is a nontrivial projection. We now present a partial result in this direction.

Theorem 3.2. (a) *If* $A = \lambda I$, $0 \leq \lambda \leq 1$, *then* $T_n(A) = n - 1$. (b) *If* A *is a nontrivial projection, then* $T_n(A) = n - 2$.

Proof. (a) If $A = \lambda I$, $0 \leq \lambda \leq 1$. Then $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$ and we have that

$$T_n(A) = \sum_{i<j=1}^{n} 1 - \sum_{i<j<k=1}^{n} 1 + \cdots + (-1)^n = \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n}$$

But the binomial formula gives

$$0 = (1-1)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} = 1 - \binom{n}{1} + \sum_{i=2}^{n} (-1)^i \binom{n}{i} = 1 - n + T_n(A)$$

Hence, $T_n(A) = n - 1$. (b) Let A be an m-dimensional projection with $1 \leq m < n$. We can assume without loss of generality that $\lambda_1 = \cdots = \lambda_m = 1$, $\lambda_i = 0$, $i = m+1, \ldots, n$. Using the convention $\binom{j}{i} = 0$ if $i > j$ we have

$$\begin{aligned} T_n(A) = &\left[\binom{m}{2} + \binom{n-m}{2}\right] - \left[\binom{m}{3} + \binom{n-m}{3}\right] \\ &+ \cdots + (-1)^m \left[\binom{m}{m} + \binom{n-m}{m}\right] + (-1)^{m+1} \binom{n-m}{m+1} \\ &+ \cdots + (-1)^{n-m} \binom{n-m}{n-m} \\ = &\,(m-1) + (n-m-1) = n-2 \end{aligned}$$

□

Of course, it follows from Theorem 3.2 that if $A = \lambda I$, $0 \leq \lambda \leq 1$, then $\mathcal{S}_n(A) = 0$ and if A is a nontrivial projection, then $\mathcal{S}_n(A) = 1$.

Another measure of sharpness, which was recently introduced in [1], is given by

$$\mathcal{S}(A) = \|A\| + \|A'\| - \|AA'\| - \|(AA')'\|$$

It is clear that $\mathcal{S}$ is a norm-continuous function satisfying conditions (b) and (c) of Theorem 3.1. We now show that $\mathcal{S}$ satisfies the other conditions. This result was proved in [1], but our proof is considerably shorter.

Theorem 3.3. *For all $A \in \mathcal{E}(H)$, $0 \leq \mathcal{S}(A) \leq 1$, $\mathcal{S}(A) = 0$ if and only if $A = \lambda I$ for some $\lambda \in [0,1]$ and $\mathcal{S}(A) = 1$ if and only if A is a nontrivial projection.*

Proof. Since

$$AA' = \frac{1}{4}\,I - \left(A - \frac{1}{2}I\right)^2$$

and

$$(AA')' = \frac{3}{4}\,I + \left(A - \frac{1}{2}\,I\right)^2$$

we have by the spectral mapping theorem that

$$\begin{aligned}\mathcal{S}(A) =& \sup\{\lambda\colon \lambda \in \sigma(A)\} + \sup\{\lambda\colon \lambda \in 1 - \sigma(A)\}\\ &- \sup\left\{\lambda\colon \lambda \in \frac{1}{4} - \sigma\left(\left(A - \frac{1}{2}\,I\right)^2\right)\right\}\\ &- \sup\left\{\lambda\colon \lambda \in \frac{3}{4} + \sigma\left(\left(A - \frac{1}{2}\,I\right)^2\right)\right\}\\ =& \sup\{\lambda\colon \lambda \in \sigma(A)\} + 1 - \inf\{\lambda\colon \lambda \in \sigma(A)\}\\ &- \frac{1}{4} + \inf\left\{\left(\lambda - \frac{1}{2}\right)^2 : \lambda \in \sigma(A)\right\}\\ &- \frac{3}{4} - \sup\left\{\left(\lambda - \frac{1}{2}\right)^2 : \lambda \in \sigma(A)\right\}\end{aligned}$$

where $\sigma(A)$ denotes the spectrum of A. Let $\lambda_1 = \inf\{\lambda\colon \lambda \in \sigma(A)\}$ and let $\lambda_2 = \sup\{\lambda\colon \lambda \in \sigma(A)\}$. Define $\lambda_3, \lambda_4 \in \sigma(A)$ by

$$\inf\left\{\left(\lambda - \frac{1}{2}\right)^2 : \lambda \in \sigma(A)\right\} = \left(\lambda_3 - \frac{1}{2}\right)^2$$

$$\sup\left\{\left(\lambda - \frac{1}{2}\right)^2 : \lambda \in \sigma(A)\right\} = \left(\lambda_4 - \frac{1}{2}\right)^2$$

and notice that λ_3, λ_4 need not be unique. We then have

$$\mathcal{S}(A) = \lambda_2 - \lambda_1 + \left(\lambda_3 - \frac{1}{2}\right)^2 - \left(\lambda_4 - \frac{1}{2}\right)^2$$

Case 1. Suppose that $0 \leq \lambda_1 \leq \lambda_2 \leq 1/2$. In this case, $\lambda_3 = \lambda_2$, $\lambda_4 = \lambda_1$ and we have

$$\mathcal{S}(A) = \lambda_2 - \lambda_1 + \left(\lambda_2 - \frac{1}{2}\right)^2 - \left(\lambda_1 - \frac{1}{2}\right)^2 = \lambda_2^2 - \lambda_1^2$$

Now clearly $0 \leq \mathcal{S}(A) \leq 1$. Also, $\mathcal{S}(A) = 0$ if and only if $\lambda_1 = \lambda_2$ in which case $\sigma(A) = \{\lambda_1\}$ so $A = \lambda_1 I$. In this situation, $\mathcal{S}(A) \neq 1$.

Case 2. If $1/2 \leq \lambda_1 \leq \lambda_2 \leq 1$, then $0 \leq \lambda_1' \leq \lambda_1' \leq 1/2$ and by Case 1 we have that $0 \leq \mathcal{S}(A') \leq 1$ and $\mathcal{S}(A') = 0$ if and only if $A' = \lambda_1' I$. Since $\mathcal{S}(A) = \mathcal{S}(A')$ the result also holds for A.

Case 3. Suppose that $0 \leq \lambda_1 \leq 1/2 \leq \lambda_2 \leq 1$. In this case, $\lambda_4 = \lambda_1$ or $\lambda_4 = \lambda_2$ depending on which is further from $1/2$. Suppose that $\lambda_4 = \lambda_1$. Since $\left(\lambda_3 - \frac{1}{2}\right)^2 \leq \left(\lambda_1 - \frac{1}{2}\right)^2$ we have

$$\mathcal{S}(A) = \lambda_2 - \lambda_1 + \left(\lambda_3 - \frac{1}{2}\right)^2 - \left(\lambda_1 - \frac{1}{2}\right)^2 \leq \lambda_2 - \lambda_1 \leq 1$$

Since $\lambda_1 \leq \lambda_3 \leq \lambda_2$ we have

$$\mathcal{S}(A) = \lambda_2 + \lambda_3^2 - \lambda_3 - \lambda_1^2 \geq \lambda_3^2 - \lambda_1^2 \geq 0$$

Also, we see that $\mathcal{S}(A) = 1$ if and only if $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 0$ or 1. This is equivalent to $\sigma(A) = \{0, 1\}$ in which case A is a nontrivial projection. Moreover, $\mathcal{S}(A) = 0$ if and only if $\lambda_3 = \lambda_1$ and hence, $\lambda_2 = \lambda_3 = \lambda_1$. This is equivalent to $A = \lambda_1 I$. If $\lambda_4 = \lambda_2$ then by symmetry the result holds for A' and hence, it holds for A. $\square$

Two advantages of $\mathcal{S}$ are its simplicity and the fact that $\mathcal{S}$ is defined for infinite-dimensional Hilbert spaces. However, $\mathcal{S}$ is a very course measure of sharpness. For example, suppose that $0 \leq \lambda_1 \leq \lambda_2 \leq 1/2$. Then as we saw in Theorem 3.3, $\mathcal{S}(A) = \lambda_2^2 - \lambda_1^2$ depends only on λ_1 and λ_2 and ignores the spectrum between λ_1 and λ_2. For example, if $\lambda_1 = 0$, $\lambda_2 = 1/2$, then $\mathcal{S}(A) = 1/4$ no matter what the rest of the spectrum looks like. In contrast, although our proposed measure of sharpness $\mathcal{S}_n$ is only defined for finite-dimensional Hilbert spaces, it is sensitive to all of the eigenvalues of A.

4 Coexistence of Qubit Effects

The previous characterizations of the coexistence of qubit effects [2, 8, 11] have not been given in terms of operational parameters. Moreover, the parameters employed are particular to two-dimensional effects (three-dimensional Block sphere) and do not indicate a method for generalization to higher dimensions. In this section we translate the conditions given in [11] into conditions involving operational parameters. These conditions are in terms of the measure of sharpness discussed in Section 3 and other quantities that can be generalized to higher dimensions. We do not attempt a higher dimensional proof here and leave that to further research. As before, the eigenvalues λ_1, λ_2 of a qubit effect A are repeated according to multiplicity. We denote the one-dimensional projection onto the span of a unit vector x by P_x.

Lemma 4.1. *Any qubit effect A has a unique representation $A = \lambda I + \mu P_x$ where $\lambda, \mu \geq 0$ and $\lambda + \mu \leq 1$. In this representation, if λ_1, λ_2 are the eigenvalues of A, then $\lambda = \min(\lambda_1, \lambda_2)$, $\mu = \max(\lambda_1, \lambda_2) - \lambda$ and $Ax = \max(\lambda_1, \lambda_2)x$. Conversely, if $A = \lambda I + \mu P_x$ where $\lambda, \mu \geq 0$ and $\lambda + \mu \leq 1$, then $A \in \mathcal{E}(H)$.*

Proof. For uniqueness, suppose that

$$\lambda_1 I + \mu_1 P_x = \lambda_2 I + \mu_2 P_y$$

Without loss of generality we can assume that $\lambda_1 \leq \lambda_2$. Then

$$\mu_1 P_x = (\lambda_2 - \lambda_1)I + \mu_2 P_y$$

If $x' \perp x$ and $\|x'\| = 1$, then $\lambda_2 - \lambda_1 + \mu_2 \left|\langle y, x\rangle\right|^2 = 0$. Hence, $\lambda_1 = \lambda_2$ and $\langle y, x'\rangle = 0$. Therefore, $y = \alpha x$ with $|\alpha| = 1$ so $P_x = P_y$. It follows that $\mu_1 = \mu_2$. For existence, let $A = \lambda_1 P_y + \lambda_2 P_x$ where $x \perp y$. Assuming that $\lambda_1 \leq \lambda_2$ we have

$$A = \lambda_1(I - P_x) + \lambda_2 P_x = \lambda_1 I + (\lambda_2 - \lambda_1)P_x = \lambda I + \mu P_x$$

It is clear that λ, μ and x satisfy the conditions in the second sentence of the lemma. Conversely, if $A = \lambda I + \mu P_x$ where $\lambda, \mu \geq 0$ and $\lambda + \mu \leq 1$, then

$$0 \leq A = \lambda I + (1 - \lambda)P_x \leq \lambda I + (1 - \lambda)I = I$$

Hence, $A \in \mathcal{E}(H)$. □

Let $\sigma_1, \sigma_2, \sigma_3$ be the Pauli matrices and denote the vector of Pauli matrices by $\underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. For a three-dimensional vector $\underline{a}$ we use the notation $a = \|\underline{a}\|$. According to [2, 11], any two qubit effects A, B can be parameterized by vectors $(\alpha, \underline{a}), (\beta, \underline{b}) \in \mathbb{R}^4$ as follows:

$$A = \frac{1}{2}(\alpha I + \underline{a} \cdot \underline{\sigma}), \quad a \leq \alpha \leq 2 - a \tag{4.1}$$

$$B = \frac{1}{2}(\beta I + \underline{b} \cdot \underline{\sigma}), \quad b \leq \beta \leq 2 - b \tag{4.2}$$

The following quantities are now introduced [11]:

$$\begin{aligned}
\mathcal{S}(A) &= \frac{1}{2}\left(a^2 + \alpha(2-\alpha) - \sqrt{(\alpha^2 - a^2)\left[(2-\alpha)^2 - a^2\right]}\right) \\
b_0 &= \frac{1}{a}(1-\alpha)(1-\beta) \\
\omega &= \frac{1}{a}\sqrt{(1-\alpha)^2 - \beta\left[(1-\alpha)^2 + 1 - a^2\right] + \beta^2} \\
b_\perp^{\max} &= \frac{1}{2a}\sqrt{\left[(2-\alpha)^2 - a^2\right]\left\{a^2 - \left[a(b_{11} - b_0) + (1-\beta)\right]^2\right\}} \\
&\quad + \frac{1}{2a}\sqrt{\left[\alpha^2 - a^2\right]\left\{a^2 - \left[a(b_{11} - b_0) - (1-\beta)\right]^2\right\}}
\end{aligned}$$

where $b_{11} = \underline{b} \cdot \underline{a}/a$. Defining $b_\perp = \sqrt{b^2 - b_{11}^2}$, the following result is proved [11].

Theorem 4.2. *A qubit effect B coexists with a qubit effect A if and only if one of the following mutually exclusive conditions hold:*
(a) *If $\beta \leq 1 - \mathcal{S}(A)$ then $A\mathrm{co}B$ irrespective of b;*
(b) *If $\beta > 1 - \mathcal{S}(A)$ and $|b_{11} - b_0| \geq \omega$, then $A\mathrm{co}B$;*
(c) *If $\beta > 1 - \mathcal{S}(A)$ and $|b_{11} - b_0| < \omega$, then $A\mathrm{co}B$ if and only if $b_\perp \leq b_\perp^{\max}$.*

Instead of Equations (4.1) and (4.2), we write A and B in terms of their canonical representations given by Lemma 4.1:

$$A = \lambda I + (\mu - \lambda)P_x \tag{4.3}$$

$$B = \lambda_1 I + (\mu_1 - \lambda_1)P_y \tag{4.4}$$

According to Lemma 4.1, λ is the smaller and μ is the larger of the eigenvalues of A, while x is the unit eigenvector corresponding to eigenvalue μ. Similar interpretations apply to B. Theorem 4.4, to follow, shows that the conditions in Theorem 4.2 can be replaced by simpler conditions involving the operational elements $\lambda, \mu, \lambda_1, \mu_1$ and the transition probability $|\langle x, y\rangle|^2$ which is also operational.

Before proving Theorem 4.4, we shall need a preliminary lemma. This lemma is known [10] but we include its proof for the convenience of the reader. If $x \in \mathbb{C}^2$ is a unit vector, then we can represent the pure state P_x as

$$P_x = \frac{1}{2}\left(I + \underline{r}_x \cdot \underline{\sigma}\right)$$

where $\underline{r}_x \in \mathbb{R}^3$ is again a unit vector. We call $\underline{r}_x$ the **Bloch sphere** representation of P_x.

Lemma 4.3. *If $x, y \in \mathbb{C}^2$ are unit vectors, then*

$$\underline{r}_x \cdot \underline{r}_y = 2\left|\langle x, y\rangle\right|^2 - 1$$

Proof. We can represent $x, y \in \mathbb{C}^2$ by

$$x = \left(\cos\frac{\theta}{2}, e^{i\phi}\sin\frac{\theta}{2}\right)$$
$$y = \left(\cos\frac{\theta'}{2}, e^{i\phi'}\sin\frac{\theta'}{2}\right)$$

where $\theta, \theta', \phi, \phi' \in \mathbb{R}$. Then $\underline{r}_x, \underline{r}_y \in \mathbb{R}^3$ have the spherical coordinate form [10]

$$\underline{r}_x = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$$
$$\underline{r}_y = (\cos\phi'\sin\theta', \sin\phi'\sin\theta', \cos\theta')$$

We then have

$$\langle x, y\rangle = \cos\frac{\theta}{2}\cos\frac{\theta'}{2} + e^{i(\phi-\phi')}\sin\frac{\theta}{2}\sin\frac{\theta'}{2}$$

Since $\operatorname{Im}\langle x, y\rangle = \sin(\phi - \phi')\sin\frac{\theta}{2}\sin\frac{\theta'}{2}$ and

$$\operatorname{Re}\langle x, y\rangle = \cos\frac{\theta}{2}\cos\frac{\theta'}{2} + \cos(\phi-\phi')\sin\frac{\theta}{2}\sin\frac{\theta'}{2}$$

we have

$$
\begin{aligned}
|\langle x,y\rangle|^2 &= (\mathrm{Re}\langle x,y\rangle)^2 + (\mathrm{Im}\langle x,y\rangle)^2 \\
&= \cos^2\frac{\theta}{2}\cos^2\frac{\theta'}{2} + 2\cos(\phi-\phi')\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\cos\frac{\theta}{2}\cos\frac{\theta'}{2} \\
&\quad + \sin^2\frac{\theta}{2}\sin^2\frac{\theta'}{2} \\
&= \frac{1}{2}\cos(\phi-\phi')\sin\theta\sin\theta' + \cos\theta\cos\theta' + \sin^2\frac{\theta}{2}\cos^2\frac{\theta'}{2} \\
&\quad + \sin^2\frac{\theta'}{2}\cos^2\frac{\theta}{2} \\
&= \frac{1}{2}\cos(\phi-\phi')\sin\theta\sin\theta' + \cos\theta\cos\theta' + \frac{1}{4}(1-\cos\theta)(1+\cos\theta') \\
&\quad + \frac{1}{4}(1-\cos\theta')(1+\cos\theta) \\
&= \frac{1}{2}\cos(\phi-\phi')\sin\theta\sin\theta' + \frac{1}{2}\cos\theta\cos\theta' + \frac{1}{2} \\
&= \frac{1}{2}\underline{r}_x\cdot\underline{r}_y + \frac{1}{2}
\end{aligned}
$$

Hence, $\underline{r}_x\cdot\underline{r}_y = 2|\langle x,y\rangle|^2 - 1$. □

Theorem 4.4. *If A and B have representations* (4.3) *and* (4.4)*, then $\alpha = \lambda+\mu$, $a=\mu-\lambda$, $\beta=\lambda_1+\mu_1$, $b=\mu_1-\lambda_1$. Assuming $a\neq 0$ ($a=0$ is a trivial case) we have*

$$
\begin{aligned}
\mathcal{S}(A) = \mathcal{S}_2(A) &= 1 - \left(\sqrt{\lambda\mu} + \sqrt{\lambda'\mu'}\right)^2 \\
b_0 &= \frac{1}{2}(1-\alpha)(1-\beta) \\
\omega &= \frac{1}{4}\sqrt{[\beta-(\lambda\mu+\lambda'\mu')]^2 - 4\lambda\mu\lambda'\mu'} \\
b_{11} &= b\left(2|\langle x,y\rangle|^2 - 1\right) \\
b_\perp^{\max} &= \frac{1}{a}\sqrt{\lambda'\mu'\left\{a^2 - [a(b_{11}-b_0)+(1-\beta)]^2\right\}} \\
&\quad + \sqrt{\lambda\mu\left\{a^2\,[a(b_{11}-b_0)-(1-\beta)]^2\right\}}
\end{aligned}
$$

Proof. Representing P_x on the Block sphere, (4.3) gives

$$A = \lambda I + \frac{(\mu+\lambda)}{2}(I + \underline{r}_x \cdot \underline{\sigma}) = \frac{1}{2}\ [(\lambda+\mu)I + (\mu-\lambda)\underline{r}_x \cdot \underline{\sigma}]$$

where $\|\underline{r}_x\| = 1$. From (4.1) we obtain $\alpha = \lambda + \mu$ and

$$a = \|(\mu-\lambda)\underline{r}_x\| = \mu - \lambda$$

Similarly, $\beta = \lambda_1 + \mu_1$, $b = \mu_1 - \lambda_1$. Hence,

$$\begin{aligned}
(2-\alpha)^2 - a^2 &= [2-(\lambda+\mu)]^2 - (\mu-\lambda)^2 = 4(1-\lambda-\mu+\lambda\mu) \\
&= 4\lambda'\mu' && (4.5) \\
\alpha^2 - a^2 &= (\lambda+\mu)^2 - (\mu-\lambda)^2 = 4\lambda\mu && (4.6) \\
a^2 + \alpha(2-\alpha) &= (\mu-\lambda)^2 + (\lambda+\mu)(2-\lambda-\mu) = 2(\lambda+\mu) - 4\lambda\mu && (4.7) \\
(1-\alpha)^2 + 1 - a^2 &= [1-(\lambda+\mu)]^2 + 1 - (\mu-\lambda)^2 \\
&= 2\,[1+2\lambda\mu-(\lambda+\mu)] = 2(\lambda\mu + \lambda'\mu' && (4.8)
\end{aligned}$$

Applying (4.5), (4.6) and (4.7) gives

$$\begin{aligned}
\mathcal{S}(A) &= \frac{1}{2}\left[2(\lambda+\mu) - 4\lambda\mu - 4\sqrt{\lambda\mu\lambda'\mu'}\right] \\
&= (\lambda+\mu) - 2\lambda\mu - 2\sqrt{\lambda\mu}\ \sqrt{\lambda'\mu'} \\
&= 1 - \left(\sqrt{\lambda\mu} + \sqrt{\lambda'\mu'}\right)^2
\end{aligned}$$

Applying (4.8) we have

$$\begin{aligned}
\omega &= \frac{1}{a}\sqrt{[1-(\lambda+\mu)]^2 - 2\beta(\lambda\mu+\lambda'\mu') + \beta^2} \\
&= \frac{1}{a}\sqrt{(\lambda'\mu'-\lambda\mu)^2 - 2\beta(\lambda\mu+\lambda'\mu') + \beta^2} \\
&= \frac{1}{a}\sqrt{[\beta-(\lambda\mu+\lambda'\mu')]^2 - 4\lambda\mu\lambda'\mu'}
\end{aligned}$$

Applying (4.5) and (4.6) gives the expression for $b_\perp^{\max}$. Finally, applying Lemma 4.3 we have that

$$b_{11} = \tfrac{1}{a}\,\underline{a}\cdot\underline{b} = \tfrac{1}{a}\,a\underline{r}_x \cdot b\underline{r}_y = b\underline{r}_x \cdot \underline{r}_y = b\left(2\,|\langle x,y\rangle|^2 - 1\right) \qquad \square$$

References

[1] P. Busch, On the sharpness of an effect, arXiv: 0706.3532 (math-ph), 2007.

[2] P. Busch and H.-J. Schmidt, Coexistence of qubit effects, arXiv: 0802.4167 (quant-ph), 2008.

[3] P. Busch, P. J. Lahti and P. Mittlestaedt *The Quantum Theory of Measurement*, Springer-Verlag, Berlin, 1996.

[4] E. B. Davies, *Quantum Theory of Open Systems*, Academic Press, New York, 1976.

[5] S. Gudder, Lattice properties of quantum effects, *J. Math. Phys.* **37** (1996), 2637–2642.

[6] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland, Amsterdam, 1982.

[7] K. Kraus, *States, Effects and Operations*, Springer-Verlag, Berlin, 1983.

[8] N.-L. Liu, L. Li, S. Yu and Z.-B. Chen, Complementarity enforced by joint measurability of unsharp observables, arXiv: 0712.3653 (quant-ph), 2008.

[9] G. Ludwig, *Foundations of Quantum Mechanics*, vol. I, Springer, Berlin, 1983.

[10] M. Nielsen and J. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2001.

[11] P. Stano, D. Reitzner and T. Heinosaari, Coexistence of qubit effects, arXiv: 0802.4248 (quant-ph), 2008.