FIBONACCI NUMBERS, REDUCED DECOMPOSITIONS AND 321/3412 PATTERN CLASSES

DANIEL DALY

Abstract. We provide a bijection from the permutations in S_n that avoid 3412 and contain exactly one 321 pattern to the permutations in S_{n+1} that avoid 321 and contain exactly one 3412 pattern. The enumeration of these classes is obtained from their classification via reduced decompositions. The results are extended to involutions in the above pattern classes using reduced decompositions reproducing a result of Egge.

1. Permutation Patterns and Reduced Decompositions

Throughout this paper, permutations will be written in one-line notation, $\pi = \pi_1 \dots \pi_n$, where the image of i under π is π_i.

Definition 1.1. A permutation $\pi = \pi_1\pi_2 \dots \pi_n \in S_n$ is said to *contain* a permutation $\sigma = \sigma_1 \dots \sigma_m \in S_m$ if there exists a subsequence $1 \leq i_1 < i_2 < \dots i_m \leq n$ such that $\sigma_j < \sigma_k$ if and only if $\pi_{i_j} < \pi_{i_k}$. If π does not contain σ, then π *avoids* σ.

Let $\pi \in S_m$. We denote the set of permutations in S_n that avoid π by $\mathrm{Av}_n(\pi)$.

Definition 1.2. A *reduced decomposition* of $\pi \in S_n$ is a word $s_1 \dots s_k$ where each s_j is a transposition of the form $(i, i+1)$ for some i such that $\pi = s_1 \dots s_k$ and k is as small as possible. k is the *length* of the permutation denoted $l(\pi)$.

Reduced decompositions are not unique. For example, $(12)(23)(12) = 321 = (23)(12)(23)$. In order to simplify the notation, we will write i to represent the transposition $(i, i+1)$ and to distinguish expressions on the transpositions $(i, i+1)$ (particularly reduced decompositions) from permutations, we will put brackets around such expressions. For instance, $[121] = (12)(23)(12) = 321$.

Definition 1.3. A *factor* in a reduced decomposition $[i_1 \dots i_k]$ is a consecutive substring.

[321323] is a reduced decomposition for 4321. [213] is a factor, but [313] is not.

Any reduced decomposition for a fixed permutation π can be transformed into any other by the use of braid moves. The two braid moves are:

- (Short Braid Move) $[ij] = [ji]$ if $|i - j| > 1$.
- (Long Braid Move) $[i(i+1)i] = [(i+1)i(i+1)]$ for all i.

Reduced decompositions have some very special properties. The following holds more generally for any reduced decomposition in any Coxeter group, but we will only need it for reduced decompositions of permutations. The proof of this property can be found in [5] or [2].

Theorem 1.4. (Exchange Property) Let $[s_1 \dots s_k]$ be a reduced decomposition for $\pi \in S_n$ and let s be any transposition of the form $(i, i+1)$. If $l([s_1 \dots s_k s]) < l([s_1 \dots s_k s])$, then $[s_1 \dots s_k] = [s_1 \dots \hat{s_j} \dots s_k]$ for some $1 \leq j \leq k$.

Connections between permutation patterns and reduced decompositions have been studied in [1], [8], [7] and [3]. In [8], Tenner shows the following:

Theorem 1.5. (Tenner) $\pi \in S_n$ avoids 321 and 3412 if and only if π has a reduced decomposition that contains no repeated elements.

It should be noted that if π has one reduced decomposition with no repeated elements, then all reduced decompositions of π have no repeated elements. Motivated by the results in [8], the author proves in [3] that

Theorem 1.6. $\pi \in S_n$ has a reduced decomposition with exactly one element repeated if and only if either $\pi \in \mathrm{Av}_n(3412)$ and contains exactly one 321 pattern, or $\pi \in \mathrm{Av}_n(321)$ and contains exactly one 3412 pattern. More specifically,

- $\pi \in \mathrm{Av}_n(3412)$ and contains exactly one 321 pattern if and only if π has a reduced decomposition with $[i(i+1)i]$ as a factor for some $i \in \{1, \ldots, n-2\}$ and no other repetitions.
- $\pi \in \mathrm{Av}_n(321)$ and contains exactly one 3412 pattern if and only if π has a reduced decomposition with $[i(i-1)(i+1)i]$ as a factor for some $i \in \{2, \ldots, n-2\}$ and no other repetitions.

2. Bijection

As our goal is to count the number of permutations in $\mathrm{Av}_n(3412)$ that contain exactly one 321 pattern and the number of permutations in $\mathrm{Av}_n(321)$ that contain exactly one 3412 pattern, we first provide a bijection between the two classes. Let $\mathcal{A}_n = \{\pi \in \mathrm{Av}_n(3412) \mid \pi \text{ contains exactly one 321 pattern}\}$ and let $\mathcal{B}_n = \{\pi \in \mathrm{Av}_n(321) \mid \pi \text{ contains exactly one 3412 pattern}\}$. We will show $|\mathcal{A}_n| = |\mathcal{B}_{n+1}|$.

Before we create the bijection, some propositions concerning the properties of reduced decompositions with one repetition are required.

Proposition 2.1. Let $\pi \in S_n$ have a reduced decomposition $\mathbf{s} = [s_1 \ldots s_k]$ with $[i(i+1)i]$ as a factor and no other repetitions and let $\sigma \in S_n$ have a reduced decomposition $\mathbf{t} = [t_1 \ldots t_l]$ with $[j(j+1)j]$ as a factor and no other repetitions. If $i \neq j$, then $\pi \neq \sigma$.

Proof. Suppose $[i-1]$ does not occur in $[s_1 \ldots s_k]$ or $[i-1]$ occurs to the left of the factor $[i(i+1)i]$. Then the image of i under $[i(i+1)i \ldots s_k]$ is $i+2$. Since $[i(i+1)i]$ contains only the repetition in $[s_1 \ldots s_k]$, we must have $\pi_i \geq i+2$. Thus, $\pi_{i+2}, \ldots, \pi_n$ must contain some element of $\{1, 2, \ldots i\}$, so $\pi_{i+1} = i+1$ is the middle element of the 321 pattern. Now suppose $[i-1]$ occurs to the right of $[i(i+1)i]$ in $[s_1 \ldots s_k]$. If l is the greatest integer such that $i-1, i-2, \ldots i-l$ appear in that order to the right of $[i(i+1)i]$, then $\pi_{i-l} \geq i-2$. As in the previous case, this implies $\pi_{i+1} = i+1$ is the middle element of the 321 pattern. By a similar argument $j+1$ is the middle element of the 321 pattern in σ. Since $i \neq j$, we also have $\pi \neq \sigma$. ☐

Proposition 2.2. Let $\pi \in S_n$ have a reduced decomposition $\mathbf{s} = [s_1 \ldots s_k]$ with $[i(i+1)(i-1)i]$ as a factor with no other repetitions, and let $\sigma \in S_n$ have a reduced decomposition $\mathbf{t} = [t_1 \ldots t_l]$ with $[j(j+1)(j-1)j]$ as a factor and no other repetitions. If $i \neq j$, then $\pi \neq \sigma$.

Proof. The occurrence of $[i(i+1)(i-1)i]$ in a reduced decomposition with no other repetitions means the following must occur by methods similar to the proof of Proposition 2.1:

- there exists an element $a \leq i-1$ such that $\pi_a = i+1$. Note $a = i-1$ if $[i-2]$ does not exist to the right of the factor.
- $\pi_i \geq i+2$.
- $\pi_{i+1} \leq i-1$.
- there exists an element $b \geq i+2$ such that $\pi_b = i$. Note $b = i+2$ if $[i+2]$ does not exist to the right of the factor.

Hence, π_i and π_{i+1} are the middle elements of the 3412 pattern of π. We can similarly conclude that σ_j and σ_{j+1} are the middle elements of the 3412 pattern of σ. Since $i \neq j$, $\pi \neq \sigma$. □

Proposition 2.3. Let $\pi \in \mathcal{A}_n$ and let $\mathbf{s} = [s_1 \dots s_k]$ and $\mathbf{t} = [t_1 \dots t_k]$ be reduced decompositions for π. Let i be the element such that $[i(i+1)i]$ appears in a reduced decomposition of π. $\mathbf{s}$ can be transformed into $\mathbf{t}$ using only short braid moves and the long braid move $[i(i+1)i] = [(i+1)i(i+1)]$ for the specific element i.

Proof. Since any reduced decomposition can be obtained from any other through the use of braid moves, it is sufficient to assume that $\mathbf{s}$ is a reduced decomposition that contains $[i(i+1)i]$ as a factor with no other repetitions. Recall such a reduced decomposition is guaranteed to exist by Theorem 1.6. Let $\tau_1 \dots \tau_m$ be the sequence of braid moves that transform $\mathbf{s}$ into $\mathbf{t}$. Assume there exists j such that τ_j is a long braid move, but not $[i(i+1)i] = [(i+1)i(i+1)]$. Let j be the smallest index value such that τ_j is a long braid move, but not $[i(i+1)i] = [(i+1)i(i+1)]$. This implies $\tau_1 \dots \tau_{j-1}$ is a sequence of short braid moves and long braid moves of the form $[i(i+1)i] = [(i+1)i(i+1)]$. Let $(\tau_1 \dots \tau_{j-1})\mathbf{s}$ be the reduced decomposition obtained after applying $\tau_1 \dots \tau_{j-1}$. Either $[i]$ occurs twice in $(\tau_1 \dots \tau_{j-1})\mathbf{s}$ or $[i+1]$ occurs twice, but not both. If $[i]$ occurs twice, then since τ_j is not $[i(i+1)i] = [(i+1)i(i+1)]$, τ_j must be the long braid move $[i(i-1)i] = [(i-1)i(i-1)]$. If $(i-1)$ does not occur in $\mathbf{s}$, we have a contradiction. If $[i-1]$ does occur in $\mathbf{s}$, then let τ_m, $m < j$, be the last occurrence of the long braid move $[i(i+1)i] = [(i+1)i(i+1)]$ before τ_j. This implies $[i-1]$ occurs either to the left or to the right of both occurrences of $[i]$ in $(\tau_1 \dots \tau_m)\mathbf{s}$. In order to apply τ_j, i and $i-1$ must commute by one of $\tau_{m+1} \dots \tau_{j-1}$. $[i]$ and $[i-1]$ cannot commute by a short braid move as $|i-(i-1)| \not> 1$. The only other possibility is for i and $[i-1]$ to commute by a long braid move, but τ_j is the first long braid move that does not involve $[i]$ and $[i+1]$ which is a contradiction. If $[i+1]$ occurs twice, similar reasoning also leads to a contradiction. □

Proposition 2.4. Let $\pi \in \mathcal{B}_n$ and let $\mathbf{s} = [s_1 \dots s_k]$ and $\mathbf{t} = [t_1 \dots t_k]$ be reduced decompositions for π. $\mathbf{s}$ can be transformed into $\mathbf{t}$ using only short braid moves.

Proof. Since π avoids 321, this is a specific instance of Proposition 2.2.15 in [6]. □

Proposition 2.4 implies that each element s_j occurs the same number of times in every reduced decomposition of π as short braid moves do not change the number of times an element occurs in $\mathbf{s}$.

The bijection from $\mathcal{A}_n$ to $\mathcal{B}_n$ can now be constructed. By Theorem 1.6, $\pi \in \mathcal{A}$ implies that π has a reduced decomposition with $[i(i+1)i]$ as a factor for some i and no other repetitions. Hence, by an application of a long braid move, π has a reduced decomposition with $[(i+1)i(i+1)]$ as a factor and no other repetitions.

Let $\mathcal{R}_k^i$ be the set of all reduced decompositions of length k with $[(i+1)i(i+1)]$ as a factor and no other repetitions. Note these are all such reduced decompositions, not just those for a particular permutation π. Let $\mathcal{S}_k^i$ be the set of all reduced decompositions of length k with $[i(i-1)(i+1)i]$ as a factor and no other repetitions. Define a map $g_i^k : \mathcal{R}_k^i \to \mathcal{S}_{k+1}^{i+1}$ by the following method. Let $\mathbf{s} = [s_1 \dots s_k] \in \mathcal{R}_k^i$. Replace each s_j in $\mathbf{s}$ with s'_j where $s'_j = \begin{cases} s_j & \text{if } s_j \leq i+1 \\ s_j + 1 & \text{if } s_j > i+1 \end{cases}$. Lastly, insert the element $[i+2]$ into the factor $[(i+1)i(i+1)]$ giving $[(i+1)i(i+2)(i+1)]$. Call this new expression $\mathbf{s}'$. Define $g_i^k(\mathbf{s}) := \mathbf{s}'$.

Now consider the lexicographic ordering on $\mathcal{R}_k^i$. It is clear that if $\mathbf{s} < \mathbf{t}$ in lexicographic ordering, then $g_i^k(\mathbf{s}) < g_i^k(\mathbf{t})$ in the lexicographic ordering on $\mathcal{S}_{k+1}^{i+1}$. This gives the following lemma.

Lemma 2.5. Assume $\mathbf{s}$ is the lexicographically smallest reduced decomposition for $\pi \in S_n$ with $[(i+1)i(i+1)]$ as a factor and that $l(\pi) = k$. The reduced decomposition $g_i^k(\mathbf{s})$ is the lexicographically smallest reduced decomposition among all reduced decompositions of π' with $[(i+1)i(i+2)(i+1)]$ as a factor.

Table 1. $E^i_j(4)$ (i - rows; j - cols)

	2	3
1	$\{[121]\}$	$\{[121],[3121],[1213]\}$
2	$\emptyset$	$\{[232],[1232],[2321]\}$

In addition, it is clear that g^k_i is a bijection. The bijection $f : \mathcal{A}_n \to \mathcal{B}_{n+1}$ may now be induced from g^k_i. Let $\pi \in \mathcal{A}_n$ such that $l(\pi) = k$. Let $\mathbf{s}$ be the lexicographically smallest reduced decomposition for π with $[i(i+1)i]$ as a factor. Define $f(\pi) := g^i_k(\mathbf{s})$.

Example 2.6. Consider the permutation $\pi = 243165 \in S_6$. The set of reduced decompositions with a factor of the form $[(i+1)i(i+1)]$ of π is $\{[15323],[51323],[13235]\}$. The least element under lexicographic ordering is $\mathbf{s} = [13235]$. Transforming $\mathbf{s}$ into $\mathbf{s}'$ gives $\mathbf{s}' = [132436]$. 2451376 is the permutation represented by [132436]. Therefore $f(243165) = 2451376$.

Restricting to the lexicographically smallest reduced decomposition for $\pi \in S_n$ and using the fact that g^i_k is a bijection gives the desired bijection.

Lemma 2.7. $f : \mathcal{A}_n \to \mathcal{B}_{n+1}$ as defined above is a bijection.

Theorem 2.8. The number of permutations in $\mathrm{Av}_n(3412)$ that contain exactly one 321 pattern is equal to the number of permutations in $\mathrm{Av}_{n+1}(321)$ that contain exactly one 3412 pattern.

3. Counting $\pi \in \mathrm{Av}_n(3412)$ that contain exactly one 321

By Theorem 2.8, in order to count the number of permutations in $\mathrm{Av}_n(3412)$ that contain exactly one 321 and the number of permutations in $\mathrm{Av}_n(321)$ that contain exactly one 3412, it suffices to count the former. The strategy to count such permutations is to count equivalence classes of reduced decompositions. Two reduced decompositions $\mathbf{s}$ and $\mathbf{t}$ are considered equivalent if and only if $\mathbf{s}$ and $\mathbf{t}$ represent the same permutation. For a fixed n and $i \in \{1 \dots n-2\}$, we will count the number of equivalence classes having $[i(i+1)i]$ as a factor and then sum over all i.

To accomplish the count, we will construct sets $E^i_j(n)$ of reduced decompositions such that the elements of $E^i_j(n)$ are representatives of distinct equivalence classes of reduced decompositions with the property that every permutation in $\mathcal{A}_n$ which has a reduced decomposition $[s_1 \dots s_k]$ in which $[i(i+1)i]$ is a factor and $s_m \le j$ for all $1 \le m \le k$ has a representative in $E^i_j(n)$.

Note $E^i_j(n)$ is empty when $j < i+1$. Tables 1 and 2 give the sets $E^i_j(4)$ and $E^i_j(5)$ in terms of their reduced decompositions. Note, only one reduced decomposition from each equivalence class is listed.

It is clear $|\mathcal{A}_n| = \sum_{i=1}^{n-2} |E^i_{n-1}(n)|$. To compute the cardinalities of the sets $E^i_{n-1}(n)$, we will show how to construct each set. This procedure is broken into two parts: i.) how to construct the set $E^i_{i+1}(n)$ and ii.) how to construct $E^i_{j+1}(n)$ given $E^i_j(n)$. Assuming we know how to construct $E^i_{i+1}(n)$, we will first show how to construct $E^i_{j+1}(n)$ and then we will go back and show how to construct $E^i_{i+1}(n)$.

TABLE 2. $E^i_j(5)$ (i - rows; j - cols)

	2	3	4
1	$\{[121]\}$	$\{[121],[3121],[1213]\}$	$\{[121],[3121],[1213],[4121],[43121],[31214],[41213],[12134]\}$
2	$\emptyset$	$\{[232],[1232],[2321]\}$	$\{[232],[1232],[2321],[4232],[2324],[41232],[12324],[42321],[23214]\}$
3	$\emptyset$	$\emptyset$	$\{[343],[2343],[3432],[1343],[12343],[23431],[13432],[34321]\}$

3.1. **Constructing the set $E^i_j(n)$ from $E^i_{j-1}(n)$.** The smallest j for which $E^i_j(n)$ is nonempty is $i+1$, so let us assume we have $E^i_{i+1}(n)$ and show how to construct $E^i_{i+2}(n)$. For all reduced decompositions $\mathbf{s} \in E^i_{i+1}(n)$, construct a set X of reduced decompositions as follows:

(1) Add $\mathbf{s}$ to X.
(2) Concatenate (i+2) to $\mathbf{s}$ on both sides, giving $[(i+2)\mathbf{s}]$ and $[\mathbf{s}(i+2)]$, and add them to X.

$\mathbf{s}$ only contains elements in $\{1,\dots i+1\}$ so both $[(i+2)\mathbf{s}]$ and $[\mathbf{s}(i+2)]$ are reduced.

Lemma 3.1. The set X is $E^i_{i+2}(n)$.

Proof. First, it must be shown why all reduced decompositions in X represent distinct permutations. No reduced decomposition from step 1 can be equivalent to any reduced decomposition from step 2, as the set of elements in a reduced decomposition is invariant among all equivalent reduced decompositions. The set of elements created in step 1 are all distinct as they are assumed to be distinct from being in the set $E^i_{i+1}(n)$. Assume $\mathbf{s}$ and $\mathbf{t}$ are distinct elements of $E^i_{i+1}(n)$ and that $[(i+2)\mathbf{s}] = [(i+2)\mathbf{t}]$. Multiplying both sides by $(i+2)$ gives $\mathbf{s} = \mathbf{t}$ which is a contradiction. Now assume $[(i+2)\mathbf{s}] = [\mathbf{t}(i+2)]$. This implies $(i+2)$ must commute with every element of $\mathbf{s}$. This can happen only by short braid moves or by the use of long braid moves of the form $[(i+2)(i+1)(i+2)] = [(i+1)(i+2)(i+1)]$ or $[(i+2)(i+3)(i+2)] = [(i+3)(i+2)(i+3)]$. Such long braid moves are impossible by Proposition 2.3. This implies $(i+2)$ commutes with every element. $(i+1)$ is one of the elements in $\mathbf{s}$ and $\mathbf{t}$, so $(i+2)$ does not commute with every element which is a contradiction.

Second, it must be shown why every permutation is represented by an element in X. Let π be a permutation with a reduced decomposition $\mathbf{s}$ with $[i(i+1)i]$ as a factor and no other repetitions such that the elements of $\mathbf{s}$ are a subset of $\{1,\dots,i+2\}$. If the element $(i+2)$ does not appear in $\mathbf{s}$, then $\mathbf{s} \in E^i_{i+1}(n)$ by assumption and so must be in $E^i_{i+2}(n)$. If the element $(i+2)$ does appear in $\mathbf{s}$, then $\mathbf{s} = [s_1 \dots (i+2) \dots s_k]$. The element $(i+1)$ either occurs to the left or to the right of $(i+2)$. If $(i+1)$ occurs to the left of $(i+2)$, then applying short braid moves produces $[s_1 \dots s_k(i+2)]$ which is equivalent to $\mathbf{s}$. $[s_1 \dots s_k]$ is a reduced decomposition on the elements $\{1,\dots,i+1\}$ and so is a member of $E^i_{i+1}(n)$ and so by step 2 of the construction $[s_1 \dots s_k(i+2)]$ is an element of E^i_{i+2}. The argument is similar if $(i+1)$ occurs to the right of $(i+2)$.

□

Once the set $E^i_{i+2}(n)$ is built, we can generalize the procedure for building inductively $E^i_{j+1}(n)$ from $E^i_j(n)$ and $E^i_{j-1}(n)$ as follows for all $\mathbf{s} \in E^i_j(n)$.

(1) Add $\mathbf{s}$ to $E^i_{j+1}(n)$.
(2) If $\mathbf{s} \in E^i_j(n) \cap E^i_{j-1}(n)$, then add $[(j+1)\mathbf{s}]$ to $E^i_{j+1}(n)$.
(3) If $\mathbf{s} \in E^i_j(n) \setminus E^i_{j-1}(n)$, then add $[(j+1)\mathbf{s}]$ and $[\mathbf{s}(j+1)]$ to $E^i_{j+1}(n)$.

Example 3.2. Here is an example of this procedure to produce $E^1_5(6)$ from
$E^1_4(6) = \{[121], [3121], [1213], [4121], [43121], [31214], [41213], [12134]\}$ and $E^1_3(6) = \{[121], [3121], [1213]\}$.

Step 1 adds $\{[121], [3121], [1213], [4121], [43121], [31214], [41213], [12134]\}$.
Step 2 adds $\{[5121], [53121], [51213]\}$.
Step 3 adds $\{[54121], [41215], [543121], [431215], [531214], [312145], [541213], [412135], [512134], [121345]\}$.

Lemma 3.3. The procedure outlined above correctly produces the set $E^i_{j+1}(n)$ for all $i + 1 \leq j < n - 1$.

Proof. The proof is similar to that of Lemma 3.2 and is omitted. □

As stated previously, the set $E^i_{n-1}(n)$ will give one reduced decomposition with factor $[i(i + 1)i]$ and no other repetitions for each permutation in $\mathcal{A}_n$. Define $a_i(k) := |E^i_{i+k}(n)|$ for $k \geq 0$ and note $a_i(0) = 0$. By our construction of these sets $a_i(k)$ satisfies the following recurrence: $a_i(k) = a_i(k-1)+a_i(k-2)+2(a_i(k-1)-a_i(k-2)) = 3a_i(k-1)-a_i(k-2)$. It is a well-known result that when a recurrence of the form $b(k) = 3b(k-1)-b(k-2)$ with $b(0) = 0$ and $b(1) = 1$, then $b(k) = F_{2k}$ where F_j is the j^{th} Fibonacci number. Note that $a_i(1) = |E^i_{i+1}(n)|$. Note, in particular, that for each i, the value of k that makes $i + k = n - 1$ is $k = n - i - 1$. Therefore $|E^i_{n-1}(n)| = a_i(n - i - 1) = |E^i_{i+1}(n)| \cdot F_{2(n-i-1)}$.

3.2. **Constructing the sets $E^i_{i+1}(n)$.** Note that $E^1_2(n) = [121]$ for all n. The sets $E^i_{i+1}(n)$ are constructed inductively from $E^{i-1}_i(n)$ by the following procedure for all $\mathbf{s}$ in $E^{i-1}_i(n)$:

1. If $\mathbf{s} = [s_1 \dots s_k]$, then add $[(s_1 + 1)(s_2 + 1) \dots (s_k + 1)]$ to $E^i_{i+1}(n)$.
2. If $\mathbf{t}$ was added in step 1 and only contains elements in $\{3, \dots, i+1\}$, then add the reduced decomposition $[1\mathbf{t}]$ to $E^i_{i+1}(n)$.
3. If $\mathbf{t}$ was added in step 1 and contains the element 2, then add the reduced decompositions $[1\mathbf{t}]$ and $[\mathbf{t}1]$ to $E^i_{i+1}(n)$.

Example 3.4. Here is an example of the procedure creating $E^4_5(6)$ from
$E^3_4(6) = \{[343], [2343], [3432], [1343], [12343], [23431], [13432], [34321]\}$.

Step 1 adds $\{[454], [3454], [4543], [2454], [23454], [34542], [24543], [45432]\}$.
Step 2 adds $\{[1454], [13454], [14543]\}$.
Step 3 adds $\{[12454], [24541], [123454], [234541], [134542], [345421], [124543], [245431], [145432], [454321]\}$.

By an induction similar to Lemma 3.3, we have the following lemma.

Lemma 3.5. The procedure above correctly produces the sets $E^i_{i+1}(n)$.

Define $b(i) := |E^i_{i+1}(n)|$. The construction of the sets implies $b(i)$ satisfies the recurrence: $b(i) = b(i - 1) + b(i - 2) + 2(b(i - 1) - b(i - 2)) = 3b(i - 1) - b(i - 2)$ where $b(0) = 0$ and $b(1) = 1$. Similar to the previous construction, we have $b(i) = F_{2i}$ where F_i is the i^{th} Fibonacci number.

3.3. **Completing the Count.** Combining all the lemmas above gives

$$|\mathcal{A}_n| = \sum_{i=1}^{n-2} |E^i_{n-1}(n)| = \sum_{i=1}^{n-2} |E^i_{i+1}(n)| F_{2(n-i-1)} = \sum_{i=1}^{n-2} F_{2i} F_{2(n-i-1)}$$

Theorem 3.6. The number of permutations in $\mathrm{Av}_n(3412)$ that contain exactly one 321 is $\sum_{i=1}^{n-2} F_{2i}F_{2(n-i-1)}$.

TABLE 3. $|A_n|$ for small n

n	3	4	5	6	7	8	9	10	11	12
$\|\mathcal{A}_n\|$	1	6	25	90	300	954	2939	8850	26195	76500

Table 3 shows these numbers for the first few n.

Using the Online Encyclopedia of Integer Sequences (http://www.research.att.com/˜njas/sequences/), one sees for small values of n that this sequence matches sequence A001871 which count the number of ordered trees of height at most 4 where only the right-most branch at the root achieves this height. Using Maple, one can verify that the sum $\sum_{i=1}^{n-2} F_{2i}F_{2(n-i-1)}$ satisfies the recurrence for this sequence

$$a(n) = \frac{2a(n-1) + (n+1)F_{2n+4}}{3}$$

giving the following theorem.

Theorem 3.7. $|\mathcal{A}_n|$ is counted by sequence A001871 and so

$$a(n) = \sum_{i=1}^{n-2} F_{2i}F_{2(n-i-1)} = \frac{2(2n-5)F_{2n-6} + (7n-16)F_{2n-5}}{5}$$

satisfies the recurrence

$$a(n) = \frac{2a(n-1) + (n+1)F_{2n+4}}{3}$$

and has generating function

$$\frac{x^3}{(1-3x+x^2)^2}$$

4. INVOLUTIONS

In [4], Egge enumerated the number of 3412-avoiding involutions that contained exactly one decreasing sequence of length k using lattice paths and Chebyshev polynomials. Here we reproduce that result for $k = 3$ using reduced decompositions. To enumerate the involutions requires characterizing the reduced decompositions of the involutions in $\mathcal{A}_n$.

Lemma 4.1. Let $[i(i+1)is_1 \dots s_k]$ be a reduced decomposition such that $s_a \neq s_b$ when $a \neq b$ and for all a, $s_a \neq i$ and $s_a \neq i+1$. Assume $[i(i+1)is_1 \dots s_k i]$ is not reduced. Then $[i(i+1)s_1 \dots s_k]$ is a reduced decomposition for $[i(i+1)is_1 \dots s_k i]$ and $[is_j] = [s_j i]$ for all $1 \leq j \leq k$.

Proof. To show $[is_j] = [s_j i]$ for all $1 \leq j \leq k$, it suffices to show that $s_j \neq [i-1]$ for all j. Assume for purposes of a contradiction that there exists j (and this j must be unique by assumption $s_a \neq s_b$ for all $a \neq b$) such that $s_j = [i-1]$. Therefore $[i(i+1)is_1 \dots s_k] = [i(i+1)is_i \dots s_{j-1}(i-1)s_{j+1} \dots s_k]$. Let π be the permutation whose reduced decomposition is $[i(i+1)is_1 \dots s_k i]$ where $s_j = [i-1]$. Consider π_i. By the right-most transposition i is mapped to $i+1$. By the assumptions, none of the s_a for $1 \leq a \leq k$ move $i+1$. The factor $[i(i+1)i]$ leaves $i+1$ fixed, so $\pi_i = i+1$. It is a well-known result (see [2] for more details) that if $\mathbf{s}$ is a reduced decomposition and s is a transposition not necessarily appearing in $\mathbf{s}$ then $l([\mathbf{s}s]) = l([\mathbf{s}]) \pm 1$. $l([i(i+1)is_1 \dots s_k i]) = l([i(i+1)is_1 \dots s_k]) - 1$ because by assumption $[i(i+1)is_1 \dots s_k i]$ is not reduced. By the

Exchange property, theorem 1.4, $[i(i+1)is_1\dots s_k i]$ is equivalent to one of the following reduced decompositions: $[i(i+1)s_1\dots s_k]$, $[(i+1)is_1\dots s_k]$, or $[i(i+1)is_1\dots \hat{s_a}\dots s_k]$ for some $1\le a\le k$.

Consider the cases:

(1) $[i(i+1)s_1\dots s_k]$. i is mapped to $i-1$ by the occurrence of $[i-1]$ in position j and so i must be mapped to $i-a$ where $a\ge 1$. Therefore $[i(i+1)s_1\dots s_k]\ne[i(i+1)is_1\dots s_k i]$ since $\pi_i=i+1$.
(2) $[(i+1)is_1\dots s_k]$. i is mapped to $i-1$ by the transposition $[i-1]$ and so again must be mapped to $i-a$ for some $a\ge 1$. Therefore, $[(i+1)is_1\dots s_k]\ne[i(i+1)is_1\dots s_k i]$.
(3) $[i(i+1)is_1\dots\hat{s_b}\dots s_k]$ for some $1\le b\le k$. If $b\ne j$, then i is mapped to $i-a$ for some $a\ge 1$. If $b=j$, then the factor $[i(i+1)i]$ sends i to $i+2$. So, $[i(i+1)is_1\dots\hat{s_b}\dots s_k]\ne[i(i+1)is_1\dots s_k i]$.

Since all three possibilities lead to a contradiction we must have $s_j\ne[i-1]$ for all j and so $[is_j]=[s_j i]$ for all j. Therefore $[i(i+1)is_1\dots s_k i]=[i(i+1)iis_1\dots s_k]=[i(i+1)s_1\dots s_k]$. $[i(i+1)s_1\dots s_k]$ must be reduced because of the length.

□

Theorem 4.2. Suppose $\pi\in\mathcal{A}_n$. π is an involution if and only if π has a reduced decomposition $[s_1\dots s_k]$ for which the following hold.

(1) $[s_1\dots s_k]$ has a factor of the form $[i(i+1)i]$ for some i and no other repetitions.
(2) If $|s_j-s_m|=1$ then $\{s_j,s_m\}=\{i,i+1\}$.

Proof. ($\Leftarrow$) By Theorem 1.6, such a reduced decomposition for π implies $\pi\in\mathcal{A}_n$. By the structure of the reduced decomposition and since $[i(i+1)i]^2=[\emptyset]$, π is easily seen to be an involution.

($\Rightarrow$) Assume π is an involution. $\pi\in\mathcal{A}_n$ implies that π has a reduced decomposition $\mathbf{s}=[s_1\dots s_k]$ with $[i(i+1)i]$ as a factor with no other repetitions. Assume there exist s_j, s_m such that $|s_j-s_m|=1$ and $\{s_j,s_m\}\ne\{i,i+1\}$. Without loss of generality, we may assume $s_m=s_j+1$.

If $[s_m]$ occurs to the left of $[s_j]$, then consider the image of s_j under π.

If $[s_j-1]$ occurs as an element of $\mathbf{s}$ to the right of $[s_j]$, then s_j must be mapped to s_j-k for some $k\ge 1$. This implies $[s_j-(k-1)]$, $[s_j-(k-2)]$, …, $[s_j-2]$ occur in that order to the left of $[s_j-1]$. Now, if $[s_j-k]$ occurs to the right of $[s_j-(k-1)]$, then s_j-k must be mapped to $s_j-(k+l)$ for some $l\ge 1$ and therefore π is not an involution. If $[s_j-k]$ does not occur to the right of $[s_j-(k-1)]$, then s_j-k is mapped to $s_j-(k-1)$ and again π is not an involution.

If $[s_j-1]$ does not occur as an element of $\mathbf{s}$ to the right of $[s_j]$, then s_j must be mapped to s_j+k for some $k\ge 2$. This implies $[s_j+k-1]$, $[s_j+k-2]$, …, $[s_j+2]$ occur to the left of $[s_j+1]$. If $[s_j+k]$ occurs to the right of $[s_j+k-1]$, then s_j+k must be mapped to s_j+k+l for some $l\ge 1$ and hence π is not an involution. If not, s_j+k must be mapped to s_j+k-1 and again π is not an involution.

The argument for the case $[s_m]$ occurring to the right of $[s_j]$ is similar to the above argument.

□

In order to count the number of involutions, we define sets similar to those constructed in section 3.

Let $I^i_j(n)$ be the sets of reduced decompositions as in the construction of $E^i_j(n)$ in the previous section with the additional property that the reduced decompositions represent involutions.

Table 4 gives the sets $I^i_j(7)$.

4.1. **Constructing the sets $I^i_j(n)$.** The sets $I^i_{j+1}(n)$ are constructed inductively from $\mathbf{s}\in I^i_j(n)$ and $\mathbf{s}\in I^i_{j-1}(n)$ as follows:

TABLE 4. $I_j^i(7)$ (i - rows; j - cols)

	2	3	4	5	6
1	{[121]}	{[121]}	{[121], [4121]}	{[121], [4121], [5121]}	{[121], [4121], [5121], [6121], [64121]}
2	∅	{[232]}	{[232]}	{[232], [5232]}	{[232], [5232], [6232]}
3	∅	∅	{[343], [1343]}	{[343], [1343]}	{[343], [1343], [6343], [61343]}
4	∅	∅	∅	{[454], [1454], [2454]}	{[454], [1454], [2454]}
5	∅	∅	∅	∅	{[565], [1565], [2565], [3565], [13565]}

(1) If $\mathbf{s} \in I_j^i(n)$, then add $\mathbf{s}$ to $I_{j+1}^i(n)$
(2) If $\mathbf{s} \in I_j^i(n) \cap I_{j-1}^i(n)$, then add $[j\mathbf{s}]$ to $I_{j+1}^i(n)$

Similarly to the case of general permutations, $I_2^1(n) = \{[121]\}$ for all n. The set $I_{i+1}^i(n)$ is constructed from $I_i^{i-1}(n)$ as follows:

(1) If $\mathbf{s} = [s_1 \dots s_k]$ then add $[(s_1+1)(s_2+1)\dots(s_k+1)]$ to $I_{i+1}^i(n)$.
(2) If $\mathbf{t}$ was added in step 1 and the element 2 does not occur in $\mathbf{t}$ then add $[1\mathbf{t}]$ to $I_{i+1}^i(n)$.

Lemma 4.3. The above procedures correctly produce the sets $I_j^i(n)$ for all i and j.

Proof. Similar to the proofs of Lemmas 3.1 and 3.3. □

4.2. **Counting the involutions in $\mathcal{A}_n$.** The number of inversions in $\mathcal{A}_n$ is given by $\sum_{i=1}^{n-2} |I_{n-1}^i(n)|$. Counting similarly to the case for permutations, define $c_i(k) := |I_{i+k}^i(n)|$ for $k \geq 0$. Note, $c_i(0) = 0$. The $c_i(k)$ satisfy the recurrence $c_i(k) = c_i(k-1) + c_i(k-2)$. Such a recurrence generates the Fibonacci numbers when the initial conditions are 0 and 1. Therefore, $|I_{n-1}^i(n)| = |I_{i+1}^i| F_{n-i-1}$.

Now define $d(i) := |I_{i+1}^i(n)|$. By the second procedure $d(i) = d(i-1) + d(i-2)$ and so $d(i) = F_i$. Therefore, the number of involutions in $\mathcal{A}_n$ is:

$$\sum_{i=1}^{n-2} |I_{n-1}^i(n)| = \sum_{i=1}^{n-2} |I_{i+1}^i(n)| \cdot F_{n-i-1} = \sum_{i=1}^{n-2} F_i F_{n-i-1}$$

.

The closed form of the above sum corresponds to Egge's result from [4] cited below.

Theorem 4.4. (Egge) The number of involutions in $\mathrm{Av}_n(3412)$ that contain exactly one 321 pattern is

$$\frac{2(n-1)F_n - nF_{n-1}}{5}$$

.

In the Online Encyclopedia of Integer Sequences this is sequence A001629 and is a very well studied sequence. Table 5 gives these numbers for the first few n.

5. Acknowledgements

Many thanks go to Petr Vojtěchovský for his suggestions about this project and to the anonymous referee whose comments clarified and greatly improved the presentation.

Table 5. Number of Involutions in $\mathcal{A}_n$ for small n.

n	3	4	5	6	7	8	9	10	11	12
Involutions in $\mathcal{A}_n$	1	2	5	10	20	38	71	130	235	420

References

[1] Sara C. Billey, William Jockusch, and Richard P. Stanley. Some combinatorial properties of Schubert polynomials. *J. Algebraic Combin.*, 2(4):345–374, 1993.

[2] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.

[3] Daniel Daly. Reduced decompositions with one repetition and permutation pattern avoidance. Available www.math.du.edu/˜ddaly. Submitted to Order, Springer-Verlag, March 2008.

[4] Eric S. Egge. Restricted 3412-avoiding involutions, continued fractions, and Chebyshev polynomials. *Adv. in Appl. Math.*, 33(3):451–475, 2004.

[5] James E. Humphries. *Reflection Groups and Coxeter Groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1990.

[6] Laurent Manivel. *Symmetric Functions, Schubert Polynomials and Degeneracy Loci*, volume 6 of *SMF/AMS Texts and Monographs*. SMF/AMS, 2001.

[7] Bridget Eileen Tenner. Reduced decompositions and permutation patterns. *J. Algebraic Combin.*, 24(3):263–284, 2006.

[8] Bridget Eileen Tenner. Pattern avoidance and the bruhat order. *Journal of Combinatorial Theory, Series A*, 114(5):888 – 905, 2007.

E-mail address: ddaly@du.edu