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AN EXAMPLE OF KAKUTANI EQUIVALENT AND STRONG ORBIT EQUIVALENT SUBSTITUTION SYSTEMS THAT ARE NOT CONJUGATE

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Abstract. We present an example of Kakutani equivalent and strong orbit equivalent substitution systems that are not conjugate.

Introduction. The motivation for this example came from [2], in which Dartnell, Durand, and Maass show that a minimal Cantor system and a Sturmian subshift are conjugate if and only if they are Kakutani equivalent and orbit equivalent (or equivalently strong orbit equivalent for Sturmian subshifts). In their paper, they posed the question if this is true for general minimal Cantor systems or even for substitution systems. Kosek, Ormes, and Rudolph [7] answered this question negatively by giving an example of orbit equivalent and Kakutani equivalent substitution systems that are not conjugate. Furthermore, in [7] it is shown that if two minimal Cantor systems are Kakutani equivalent by map that extends to a strong orbit equivalence, then the systems are conjugate. The question that we then considered is if two minimal Cantor systems are Kakutani equivalent and strong orbit equivalent, does this mean that the systems are conjugate? The answer to this question is again answered negatively as the substitution systems in this paper provide a counterexample.

Background & Definitions. We begin with a minimal Cantor system, i.e. an ordered pair (X, T) where X is a Cantor space and $T : X \to X$ is a minimal homeomorphism. The minimality of T means that every T-orbit is dense in X, i.e. $\forall\, x \in X$, the set $\{T^n(x) \mid n \in \mathbb{Z}\}$ is dense in X. There are several notions of equivalence in dynamical systems. The strongest of these is conjugacy. Two dynamical systems (X, T) and (Y, S) are *conjugate* if there exists a homeomorphism $h : X \to Y$ such that $h \circ T = S \circ h$.

A weaker notion of equivalence is orbit equivalance. With orbit equivalence, the spaces still must be homeomorphic, but the homeomorphism need only preserve the orbits within each system, i.e. (X, T) and (Y, S) are *orbit equivalent* if there exists a homeomorphism $h : X \to Y$ and functions $n, m : X \to \mathbb{Z}$ such that for all $x \in X$, $h \circ T(x) = S^{n(x)} \circ h(x)$ and $h \circ T^{m(x)}(x) = S \circ h(x)$. We refer to m and n as the *orbit cocycles* associated to h. We say that the systems are *strong orbit equivalent* if the cocycles have at most one point of discontinuity each.

The last notion of equivalence we will consider is Kakutani equivalence. If we let (X,T) be a minimal Cantor system and A a clopen set in X, because T is minimal and X compact, each $a \in A$ returns to A in a finite number of T iterations. This allows us to define a continuous map $r_A : A \to A$ by $r_A(a) = \min\{n \geq 1 \mid T^n(a) \in A\}$. If we define the map $T_A : A \to A$ by $T_A(a) = T^{r_A}(a)$, the system (A, T_A) is again a minimal Cantor system and we say that (A, T_A) is an *induced system* of (X,T). We say that two minimal Cantor systems are *Kakutani equivalent* if they have conjugate induced systems.

In this paper, we will be looking at substitution systems in two ways, as Bratteli diagrams and as typical substitutions with the shift map. We will introduce these here.

Bratteli Diagrams

This will be a brief introduction to Bratteli diagrams. We refer you to [6] for more details. A *Bratteli diagram* (V,E) consists of a vertex set V and an edge set E, where V and E can be written as the countable union of finite disjoint sets:

$$V = V_0 \cup V_1 \cup V_2 \cup \dots \quad \text{and} \quad E = E_1 \cup E_2 \cup \dots,$$

where we think of the V_k as representing the vertices at level k and E_k as representing the edges between the vertices at levels $k-1$ and k. Furthermore, the following properties hold:

1. $V_0 = \{v_0\}$ is a one point set;
2. There is a range map r and a source map s each going from E into V such that $r(E_k) \subset V_k$ and $s(E_k) \subset V_{k-1}$. We also require that $s^{-1}(v) \neq \emptyset \ \forall \ v \in V$ and $r^{-1}(v) \neq \emptyset \ \forall \ v \in V \setminus V_0$.

An *ordered Bratteli diagram* is a Bratteli diagram $B = (V, E, \leq)$ along with a partial order $\leq$ on E such that two edges are comparable if and only if they have the same range. We can extend this to a reverse lexicographical ordering on paths. So for $k, l \in \mathbb{Z}^+$ with $k < l$, we denote all of the edge paths between V_{k-1} and V_l by $E[k,l]$, and the ordering $\leq'$ induced on $E[k,l]$ is given by $(e_k, \dots, e_l) \leq' (f_k, \dots, f_l)$ if and only if there is a j with $k \leq j \leq l$ such that $e_i = f_i$ for $j < i \leq k$ and $e_j \leq f_j$. There are also natural extensions of the range and source maps to $E[k,l]$ by defining $s(e_k, \dots, e_l) = s(e_k)$ and $r(e_k, \dots, e_l) = r(e_l)$.

Given a Bratteli Diagram, it is possible to create a new Bratteli Diagram by a process called telescoping. Let $B = (V, E, \leq)$ be a Bratteli Diagram and remove $E[k,l]$ and $V_{k+1}, V_{k+2}, \dots, V_{l-1}$. If we then reconnect levels V_k and V_l by single edges, one for each of the paths in $E[k,l]$ beginning and ending at its corresponding source and range, respectively, and order the edges by $\leq'$, we call this process *telescoping* between levels k and l. If we let $\{n_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{Z}^+$ such that $n_1 = 0$ and $n_k < n_{k+1} \ \forall \ k$, and we telescope B between levels n_k and n_{k+1} for each k ordering the edges as described above, we have a new ordered Bratteli diagram $B' = (V', E', \leq')$. We say that B' is a *telescoping* of B. If the telescoping is done by telescoping a finite number of levels, i.e. there exists $K \in \mathbb{Z}^+$ such that $\forall \ j \in \mathbb{Z}^+$, $n_{K+j} = n_K + j$, we say that B' is a *finite telescoping* of B.

Definition. *An ordered Bratelli diagram $B = (V, E, \leq)$ is* properly ordered *if*

1. *there is a telescoping (not necessarily finite) B' of B such that any two verticies at consecutive levels in B' are connected;*
2. *there are unique infinite edge paths $x_{\max}$ and $x_{\min}$ in B such that each edge of $x_{\max}$ is maximal in $\leq$ and each edge of $x_{\min}$ is minimal in $\leq$.*

Now, given a properly ordered Brattelli diagram $B = (V, E, \leq)$, we define X_B to be the set of all infinite paths in B. We topologize X_B by making the family of cylinder sets a basis for the topology. By a *cylinder set*, we mean the sets of paths that begin with a particular path, i.e. a cylinder set denoted by $[e_1, \ldots, e_k] = \{(x_1, x_2, \ldots) \in X_B : x_i = e_i \ \forall\ i \leq k\}$. X_B along with this topology is a Cantor space. We define the *Vershik map* $V_B : X_B \to X_B$ in the following way. Let $(e_1, e_2, \ldots) \in X_B \setminus \{x_{\max}\}$. There is smallest k such that e_k is not maximal. If we let f_k be the successor of e_k and let $(f_1, \ldots, f_{k-1})$ be the minimal path from the v_0 to f_k, we then define $V_B(e_1, e_2, \ldots) = (f_1, f_2, \ldots, f_k, e_{k+1}, e_{k+2}, \ldots)$. We define $V_B(x_{\max}) = x_{\min}$. V_B acting on X_B is a minimal homeomorphism, and therefore (X_B, V_B) is a minimal Cantor system and we refer to it as a *Bratteli-Vershik system.* As shown in [6], any minimal Cantor system is conjugate to a Bratelli-Vershik system.

For a given Bratteli diagram $B = (V, E)$ denote the vertices in V at level k by $\{V(k, j) \mid 1 \leq j \leq |V_k|\}$. For each $k \geq 0$, there is an associated incidence matrix specifying the number of edges between vertices, i.e. for each $k \geq 0$, we define the incidence matrix $M_k = (m_{ij})$, $i = 1, \ldots, |V_k|$, $j = 1, \ldots, |V_{k+1}|$ where m_{ij} is the number of edges between $V(k, j)$ and $V(k+1, i)$. Then, we can associate a dimension group $K_0(V, E)$ to the Bratteli diagram by taking the inductive limit of groups $\varinjlim(\mathbb{Z}^{|V_k|}, M_k)$. We can make this an ordered group by declaring that any $[v] \in K_0(V, E)^+$ if there is a $v \in [v]$ such that each coordinate of v is nonnegative. We distinguish an order unit to be the element in $K_0(V, E)$ associated to $1 \in \mathbb{Z}^{|V_0|} = \mathbb{Z}$.

Substitution Systems

Again as this will be a brief introduction, we refer you to [3] for more details. We start with a finite nonempty alphabet $\mathcal{A} = \{1, 2, \ldots, d\}$. If we let $\mathcal{A}^*$ be the set of finite nonempty words in $\mathcal{A}$, a *substitution* is a map $\sigma : \mathcal{A} \to \mathcal{A}^*$. There is a natural extension of σ to $\mathcal{A}^*$ by concatenation. We say that σ is *primitive* if there is a $k > 0$ such that for each $i, j \in \mathcal{A}$, j appears in $\sigma^k(i)$, and there is some $i \in \mathcal{A}$ such that $\lim_{n\to\infty} |\sigma^n(i)| = \infty$, where $|\sigma^n(i)|$ represents the length of the word. We say σ is *proper* if there exists $p > 0$ and two letters $r, l \in \mathcal{A}$ such that

1. $\forall\ i \in \mathcal{A}$, r is the last letter of $\sigma^p(i)$;
2. $\forall\ i \in \mathcal{A}$, l is the first letter of $\sigma^p(i)$.

We say that a word (not necessarily finite) w is σ*-allowed* if and only if each finite subword of w is a subword of some $\sigma^n(i)$ for some $i \in \mathcal{A}$, and we define X_σ to be the set of all σ-allowed bi-infinite words in $\mathcal{A}$. There are substitutions σ for which X_σ will be finite. We are only interested in substitutions where X_σ is infinite, so we will say that σ is *aperiodic* if X_σ is infinite.

If we take X_σ with the shift map, say S_σ, i.e. if $x = (\dots x_{-2}x_{-1}.x_0x_1x_2\dots)$, $S_\sigma(x) = (\dots x_{-2}x_{-1}x_0.x_1x_2\dots)$, we say the (X_σ, S_σ) is the *substitution system* associated to σ. For $x \in X_\sigma$, we define $[x]$ to be the set of all backward and forward shifts of x or equivalently the orbit of x under S_σ. We say that an orbit $[x]$ is *left asymptotic* if there is another orbit $[x']$ with $[x] \cap [x'] = \emptyset$ and $y \in [x]$, $y' \in [x']$, $k \in \mathbb{Z}$ such that for all $i \le k$, $y_i = y'_i$. Right asymptotic orbits are defined analagously, and we say an orbit is *asymptotic* if it is either left or right asymptotic.

If we let (X_σ, S_σ) be a substitution system associated to a primitive, aperiodic substitution σ, this is a minimal Cantor system and has a natural representation as a Bratteli-Vershik system. In the case that σ is proper, which is what we are concerned with, the Bratteli diagram as done in [3] is constructed by first making $|V_k| = |\mathcal{A}|\ \forall\ k \ge 1$ and we associate each vertex at these levels to a letter in $\mathcal{A}$, i.e. we denote vertices at level k by $\{V(k,j) \mid j \in \mathcal{A}\}$. For each $j \in \mathcal{A}$, $V(1,j)$ is connected by a single edge to the top vertex. Then, for a fixed $q \in \mathcal{A}$, we connect $V(2,q)$ with an edge from each $V(1,j)$ for each time j appears in $\sigma(q)$ and the edges are ordered by the order they appear in $\sigma(q)$. We do this process for each $q \in \mathcal{A}$. We repeat these edge connections for all consecutive edge sets farther down in the diagram. Therefore, the diagram repeats after level 1, so we refer to this as a *stationary Bratteli diagram.*

Then, there is a correspondence between each bi-infinite word in X_σ and infinite paths in the Bratteli diagram. Let $x \in X_\sigma$ and let z be the corresponding infinite path in the Brattelli diagram. The correspondence is given by the following. For each $k \ge 0$, there is a word in x around the origin, say $w = x_{-n}\dots x_{-1}.x_0\dots x_m$ such that for some $a \in \mathcal{A}$, $\sigma^k(a) = w$. Then the path that z follows from the top of the diagram to level k is the $(n+1)$st ordered path in the set of paths that terminate at the vertex that corresponds to a at level k.

The Counterexample. The substitutions for these two systems are defined accordingly. First, we define two substitutions σ_1 and σ_2 on an alphabet $\mathcal{A} = \{a, b\}$ as follows:

$$\sigma_1 : \begin{cases} a \to aabb \\ b \to abb \end{cases}$$

$$\sigma_2 : \begin{cases} a \to abab \\ b \to abb. \end{cases}$$

We define $\sigma = \sigma_1 \circ \sigma_2$ and $\tau = \sigma_2 \circ \sigma_1$. So, we have

$$\sigma : \begin{cases} a \to aabbabbaabbabb \\ b \to aabbabbabb \end{cases}$$

$$\tau : \begin{cases} a \to abababababbabb \\ b \to abababbabb. \end{cases}$$

We let (X, T) be the substitution system associated to σ and (Y, S) be the substitution system associated to τ. The Bratteli diagrams associated to these systems

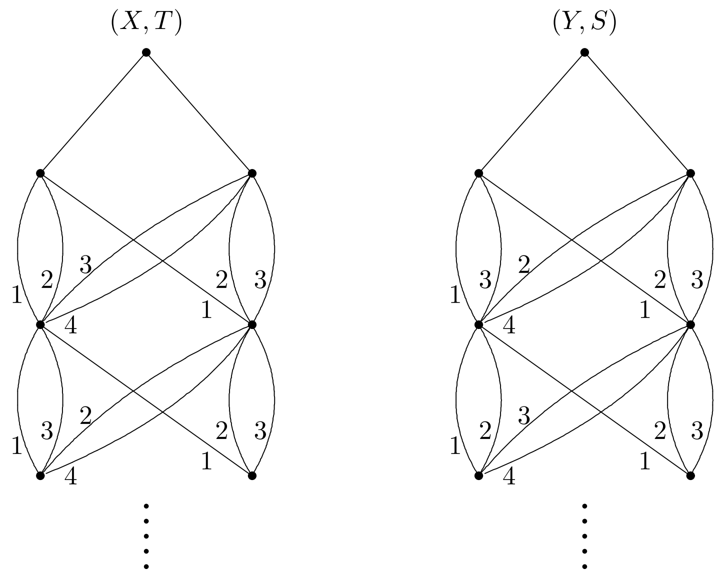

Figure 1. (X,T) & (Y,S) as Bratteli diagrams

are shown in Figure 1. If you telescope these diagrams between odd levels, you get exactly the stationary Bratteli diagrams associated to the substitution systems described previously. However, since the substitutions here are given by the composition of two substitutions, it is more convenient to look at them in their untelescoped form.

Theorem 1. *The systems (X,T) and (Y,S) defined above are Kakutani equivalent and strong orbit equivalent, but not conjugate.*

In order to prove this theorem we need the following:

Theorem 2 (Durand, Host, and Skau [3]). *Two Bratteli-Vershik systems associated to properly ordered Bratteli diagrams are Kakutani equivalent if and only if one diagram can be obtained from the other by a* finite change*, i.e. doing a finite number of finite telescopings and adding and/or removing a finite number of edges.*

Theorem 3 (Giordano, Putnam, and Skau [5]). *Two minimal Cantor systems are strong orbit equivalent if and only if their associated order groups are order isomorphic by a map preserving the distinguished order unit.*

Theorem 4 (Barge, Diamond, and Holton [1]). *A primitive, aperiodic, substitution σ on d letters has at most d^2 asymptotic orbits. If σ is proper, then σ has at most $4(d-1)$ asymptotic orbits.*

Theorem 5 (Gottschalk and Hedlund [4]). *Any infinite minimal substitutions system must have at least one pair each of left and right asymptotic orbits.*

We will prove Theorem 1 by a series of propositions.

Propostion 1. *The systems (X, T) and (Y, S) defined above are Kakutani equivalent.*

Proof. By Theorem 2, two Bratteli-Vershik systems are Kakutani equivalent to one another if one can be obtained from the other by doing a finite change. Looking at the diagrams in Figure 1, if we telescope between the top vertex and level 2 of (X, T) and then remove edges between the top vertex and the new level 1 so there is exactly one edge between the top vertex and each of the two vertices at the new level 1, we get precisely the ordered Bratteli diagram representing (Y, S). Hence, by Theorem 2 the systems are Kakutani equivalent. □

Propostion 2. *The systems (X, T) and (Y, S) defined above are strong orbit equivalent.*

Proof. To see that the substitution systems are strong orbit equivalent, we again refer to the diagrams in Figure 1. If we consider the diagrams as being unordered, they are identical. Consequently, their associated dimension groups are order isomorphic by a map preserving the distinguished order unit. By Theorem 3, the systems are strong orbit equivalent. □

Showing that these two systems are not conjugate is a more subtle problem as almost any invariants of the two systems are the same. By Theorem 4, since our substitution systems are primitive, aperiodic, and proper on two symbols, they can have at most four asymptotic orbits. Furthermore, from Theorem 5, we know that each of our systems has at least one pair each of left and right aymptotic orbits, so each of our systems must have exactly two left asymptotic orbits and exactly two right asymptotic orbits.

As shown in Lemma 2 of [1], left asymptotic orbits can arise in only one of two ways. As it turns out in our systems, the left asymptotic orbits in (X, T) are the orbits of

$$\alpha = \dots \sigma^2(u)\sigma(u)u.ax\sigma(x)\sigma^2(x)\dots \text{ and } A = \dots \sigma^2(u)\sigma(u)u.bb\sigma(b)\sigma^2(b)\dots$$

where $u = aabbabba$ and $x = bbabb$, and the left asymptotic orbits in (Y, S) are the orbits of

$$\beta = \dots \tau^2(v)\tau(v)v.a\tau(z)\tau^2(z)\dots \text{ and } B = \dots \tau^2(v)\tau(v)v.b\tau(w)\tau^2(w)\dots$$

where $v = ababab$, $z = babbabb$, and $w = abb$.

To see that these are allowable sequences in the system, note that for all $n \in \mathbb{N}$,

$$\sigma^n(u)\dots\sigma^2(u)\sigma(u)uax\sigma(x)\sigma^2(x)\dots\sigma^n(x) = \sigma^{n+1}(a),$$

$$\sigma^n(u)\dots\sigma^2(u)\sigma(u)ubb\sigma(b)\sigma^2(b)\dots\sigma^n(x) = \sigma^{n+1}(b),$$

$$\tau^n(v)\dots\tau^2(v)\tau(v)vaz\tau(z)\tau^2(z)\dots\tau^n(z) = \tau^{n+1}(a), \text{ and}$$

$$\tau^n(v)\dots\tau^2(v)\tau(v)vbw\tau(w)\tau^2(w)\dots\tau^n(w) = \tau^{n+1}(b).$$

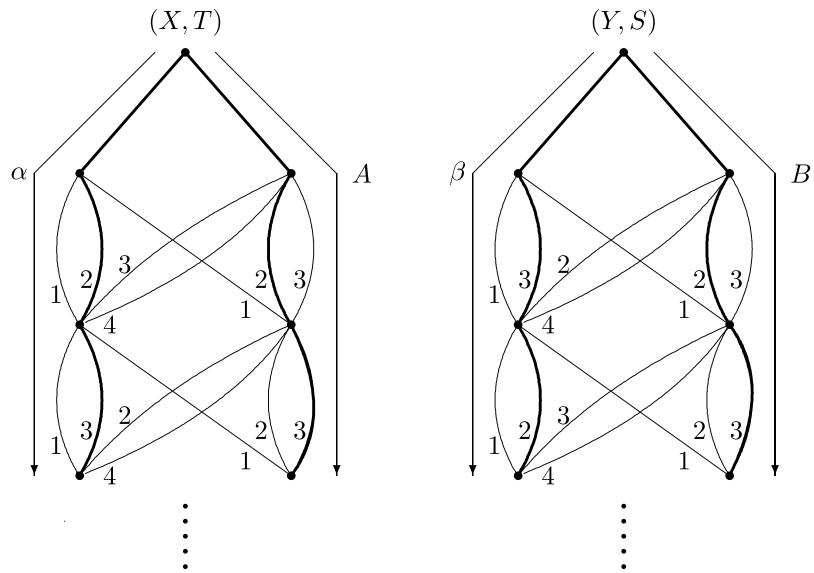

FIGURE 2. Left asymptotic points shown in bold

So, α and A are allowable in (X,T), and β and B are allowable sequences in (Y,S). The representations of these points in the Bratteli diagrams are shown in Figure 2.

To see that α and A correspond to the paths as shown in Figure 2, we first introduce some notation. If $x = (x_1, x_2, \ldots)$ is an infinite path in a Bratteli diagram and $l < k$, let $x[l,k]$ denote the path $(x_{l+1}, \ldots, x_k)$, i.e. the edge path that x follows from level l to level k. Also, we will denote the vertices in the Bratteli diagram for (X,T) in the following way: L_k and R_k will represent the vertices on the left and right side, respectively, at level k of the diagram. Furthermore, $P(v)$ will represent the set of paths whose range is v and whose source is v_0, i.e. the set of paths that start from the top vertex and terminate at v. Given a path in $P(v)$, if it is the nth path in the ordering, we will refer to n as its *order index* in $P(v)$.

By the characterization of α above, $\forall\ k \geq 1$, α passes through L_{2k+1} and the order index of $\alpha[0, 2k+1]$ in $P(L_{2k+1})$ is $\sum_{j=0}^{k-1} |\sigma^j(u)| + 1$. The path of order index $|u|+1$ in $P(L_3)$ is the $\alpha[0,3]$ path shown in Figure 2, and in general $\forall\ k \geq 1$, the path of order index $\sum_{j=0}^{k-1} |\sigma^j(u)| + 1$ in $P(L_{2k+1})$, is the $\alpha[0, 2k+1]$ path shown in Figure 2. Therefore, the representation of α in the Bratelli diagram is as shown in Figure 1. Moreover, by equation (2) above $\forall\ k \geq 1$, A passes through R_{2k+1} and the order index of $A[0, 2k+1]$ in $P(R_{2k+1})$ is $\sum_{j=0}^{k-1} |\sigma^j(u)| + 1$ which corresponds to the $A[0, 2k+1]$ path as shown in Figure 2. So, A corresponds to the path shown in Figure 2. Similarly, we can conclude that β and B also coincide with the paths shown in Figure 2.

Now, suppose there is a conjugacy h between (X, T) and (Y, S). The conjugacy must map left (right) asymptotic orbits to left (right) asymptotic orbits. To see this, note that if $[x]$ and $[x']$ are left asymptotic orbits in X, for each point $y \in [x]$, there is unique point $y' \in [x']$ such that $\lim_{k\to\infty} d_X(T^{-k}y, T^{-k}y') = 0$, where d_X represents a metric on X that gives rise to the cylinder set topology. If we let d_Y be the corresponding metric on Y, by the uniform continuity of h, we must have that $\lim_{k\to\infty} d_Y(h(T^{-k}y), h(T^{-k}y')) = 0$. But then since h is a conjugacy, $h(T^{-k}y) = S^{-k}(h(y))$ and $h(T^{-k}y') = S^{-k}(h(y'))$ meaning that the orbits of $h(y)$ and $h(y')$ are left asymptotic and $h(y')$ is the unique point in Y such that $\lim_{k\to\infty} d_Y(S^{-k}(h(y)), S^{-k}(h(y'))) = 0$. Therefore, if h is a conjugacy, it must map α into the orbit of β and A into the orbit of B or vice versa. Since a conjugacy can always be modified to map a point to anything in the orbit of its image, without loss of generality, we can assume that h maps α to either β or B. Then, since A is the unique point in X such that $\lim_{k\to\infty} d_X(T^{-k}(\alpha), T^{-k}(A)) = 0$ and B is the unique point in Y such that $\lim_{k\to\infty} d_Y(S^{-k}(\beta), S^{-k}(B)) = 0$, it must be true that if $h(\alpha) = \beta$, then $h(A) = B$. Similarly if $h(\alpha) = B$, then $h(A) = \beta$. If we can show that neither of these cases are possible, we will have proven that these systems are not conjugate.

Consider the sequence $\{A_k\}_{k=1}^{\infty}$ in X where A_k is the path in the diagram in Figure 1 that agrees with A until level k, crosses over to L_{k+1} on the order 4 path and agrees with α past level $k+1$ as is shown in Figure 3 for an even value of k. Note that $\lim_{k\to\infty} A_k = A$, and since each A_k is cofinal with α, for each k there is an n_k such that $T^{n_k}(\alpha) = A_k$. So, if there is a conjugacy h between (X, T) and (Y, S), the following must hold:

$$h(A) = h(\lim_{k\to\infty} T^{n_k}(\alpha)) = \lim_{k\to\infty} h(T^{n_k}(\alpha)) = \lim_{k\to\infty} S^{n_k}(h(\alpha)).$$

Since $h(A)$ must be either β or B and $h(\alpha)$ is the other, then either

$$\lim_{k\to\infty} S^{n_k}(\beta) = B \text{ or} \tag{1}$$

$$\lim_{k\to\infty} S^{n_k}(B) = \beta \tag{2}$$

and if neither (1) nor (2) hold, h cannot be a conjugacy.

Propostion 3. *The number n_k such that $T^{n_k}(\alpha) = A_k$ is given by*

$$n_k = \begin{cases} |P(L_k)| + |P(R_k)| & \text{if } k \text{ is odd} \\ |P(L_k)| & \text{if } k \text{ is is even} \end{cases}$$

Proof. We let Δ_k denote the order index of $\alpha[0, k]$ in $P(L_k)$ and Γ_k denote the order index of $A_k[0, k+1]$ in $P(L_{k+1})$. Note that Δ_k is also the order index of $A[0, k]$ in $P(R_k)$. We have the following:

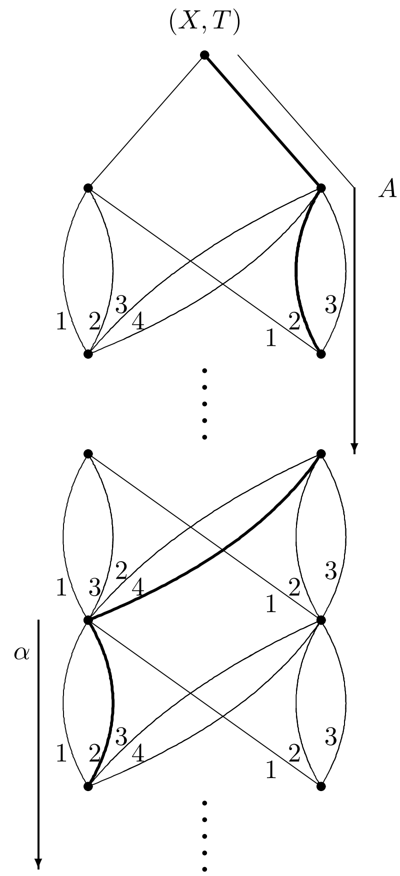

FIGURE 3. A_k shown in bold for an even value of k

$$\Delta_1 = 1 \text{ and } \forall\ k \geq 1,\ \Delta_{k+1} = \begin{cases} |P(L_k)| + \Delta_k \text{ if } k \text{ is odd} \\ |P(L_k)| + |P(R_k)| + \Delta_k \text{ if } k \text{ is even;} \end{cases}$$

$$\forall\ k > 1,\ \Gamma_k = 2|P(L_k)| + |P(R_k)| + \Delta_k.$$

Since both α and A_k pass through L_{k+1}, n_k is given by the difference in the order indicies of $A_k[0, k+1]$ and $\alpha[0, k+1]$. So, $n_k = \Gamma_k - \Delta_{k+1}$ proving the proposition. □

Propostion 4. *For odd values of* k*,* $\lim_{k\to\infty} S^{n_k}(\beta) \to \beta$ *and* $\lim_{k\to\infty} S^{n_k}(B) = B$*.*

Before we begin the proof, we introduce some notation. Denote the left and right vertices at level k of (Y, S), respectively, as L'_k and R'_k. Let Δ'_k denote the order index of $\beta[0, k]$ in $P(L'_k)$ and Γ'_k the order index of $B[0, k]$ in $P(R'_k)$. For all k, note that the recursion $|P(L'_{k+1})| = 2|P(L'_k)| + 2|P(R'_k)|$ is satisfied.

Now, we need a way to identify paths and edges in the diagram. We will denote the maximal path from the top of the diagram to vertex v by $M(v)$ and the minimal path by $m(v)$. Also, we will denote the edge of order index j that terminates at vertex v by j_v. We also need to identify compositions of paths in the diagram, so for example, in our notation $M(R'_k)3_{L'_{k+1}}\beta[k+1,k+3]$ represents the path that is maximal down to R'_k, takes the order 3 path to L'_{k+1}, and follows β from level $k+1$ to $k+3$.

Proof of Proposition 4. Consider $S^{n_k}(\beta) = S^{|P(L'_k)|+|P(R'_k)|}(\beta)$ for a fixed odd value of k. We determine what this is by comparing order indicies of paths in $P(L'_{k+2})$. We would like to know the path whose order index in $P(L'_{k+2})$ is greater than the order index of $\beta[0,k+2]$ by $|P(L'_k)|+|P(R'_k)|$. We do the computation in a series of steps which are easily checked.

1. The path $M(L'_k)\beta[k,k+2] > \beta[0,k+2]$ and the difference in the order indicies is $|P(L'_k)| - \Delta'_k$.
2. The path $m(R'_k)4_{L'_{k+1}}\beta[k+1,k+2] > M(L'_k)\beta[k,k+2]$ and the difference in order indicies is 1.
3. The path $M(R'_k)4_{L'_{k+1}}\beta[k+1,k+2] > m(R'_k)4_{L'_{k+1}}\beta[k+1,k+2]$ and the difference in order indicies is $|P(R'_k)| - 1$.
4. The path $m(R'_{k+1})3_{L'_{k+2}} = m(L'_k)1_{R'_{k+1}}3_{L'_{k+2}} > M(R'_k)4_{L'_{k+1}}\beta[k+1,k+2]$ and the difference in order indicies is 1.
5. The path $\beta[0,k]1_{R'_{k+1}}3_{L'_{k+2}} > M(R'_k)4_{L'_{k+1}}\beta[k+1,k+2]$ and the difference in order indicies is $\Delta'_k - 1$.

The difference in order indicies applied above add to $|P(L'_k)|+|P(R'_k)|$, and the last path in our computation begins with $\beta[0,k]$, so $S^{n_k}(\beta)$ agrees with β down to level k showing $S^{n_k}(\beta) \to \beta$ for odd values of k.

We now consider $S^{n_k}(B) = S^{|P(L'_k|+|P(R'_k)|}(B)$ for an odd value of k. We calculate this by comparing order indicies of paths in $P(R'_{k+3})$. We would like to know the path whose order index in $P(R'_{k+3})$ is greater than the order index of $B[0,k+3]$ by $|P(L'_k)|+|P(R'_k)|$. Again, we compute this is in a series of steps which can easily be checked.

1. The path $B[0,k]3_{R'_{k+1}}B[k+1,k+3] > B[0,k+3]$ and the difference in order indicies is $|P(R'_k)|$.
2. The path $M(R'_{k+2})B[k+2,k+3] > B[0,k]3_{R'_{k+1}}B[k+1,k+3]$ and the difference in order indicies is $|P(R'_{k-1})| - \Gamma'_{k-1}$.
3. The path $m(R'_{k+2})3_{R'_{k+3}} = m(L'_k)1_{L'_{k+1}}1_{R'_{k+2}}3_{R'_{k+3}} > M(R'_{k+2})B[k+2,k+3]$ and the difference in order indicies is 1.
4. The path $m(R'_{k-1})4_{L'_k}1_{L'_{k+1}}1_{R'_{k+2}}3_{R'_{k+3}} > m(L'_k)1_{L'_{k+1}}1_{R'_{k+2}}3_{R'_{k+3}}$ and the difference in order indicies is $2|P(L'_{k-1})|+|P(R'_{k-1})|$.
5. The path $B[0,k-1]4_{L'_k}1_{L'_{k+1}}1_{R'_{k+2}}3_{R'_{k+3}} > m(R'_{k-1})4_{L'_k}1_{L'_{k+1}}1_{R'_{k+2}}3_{R'_{k+3}}$ and the difference in order indicies is $\Gamma'_{k-1} - 1$.

Using the recursion stated above, we get that the sum of the differences in order indicies above is $|P(L'_k)|+|P(R'_k)|$. The last path in our computation begins with $B[0,k-1]$, so $S^{n_k}(B)$ agrees with B down to level $k-1$ finishing the proof. $\square$

Proof of Theorem 1. By Proposition 4, neither (1) nor (2) can hold. This statement along with Propositions 1 and 2 prove the theorem. □

Remark. *Using similar techniques to those used in Proposition 4, it can also be shown that these two systems are not* flip conjugate*, i.e* (X, T) *is not conjugate to* (Y, S^{-1}).

REFERENCES

[1] M. Barge, B. Diamond, and C. Holton, *Asymptotic orbits of primitive substitutions*, Theoretical Computer Science **301** (2003), 439-450.
[2] P. Dartnell, F. Durand, and A. Maass, *Orbit equivalence and Kakutani equivalence with Sturmian subshifts*, Studia Mathematica **142**(1) (2000), 25-45.
[3] F. Durand, B. Host, and C. Skau, *Substitutional dynamical systems, Bratteli diagrams and dimension groups*, Ergodic Theory & Dynamical Systems **19** (1999), 953-993.
[4] W. Gottschalk and G. A. Hedlund, *Topological Dynamics*, American Mathematical Society Colloquium Publications **36**, American Mathematical Society, Providence, RI, 1955.
[5] T. Giordano, I. F. Putnam, and C. Skau, *Topological orbit equivalence and* C^**-crossed products*, Journal für die reine und angewandte Mathemetik **469** (1995), 51-111.
[6] R. H. Herman, I. F. Putnam, and C. Skau, *Ordered Bratteli diagrams, dimension groups and topological dynamics*, International Journal of Mathematics **3**(1) (1992), 827-864.
[7] W. Kosek, N. Ormes, and D. J. Rudolph, *Flow-orbit equivalence for minimal Cantor systems*, Ergodic & Theory Dynamical Systems **28**(2) (2008), 481-500.

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