TUKEY TYPES OF ULTRAFILTERS

NATASHA DOBRINEN AND STEVO TODORCEVIC

Abstract. We investigate the structure of the Tukey types of ultrafilters on countable sets partially ordered by reverse inclusion. A canonization of cofinal maps from a p-point into another ultrafilter is obtained. This is used in particular to study the Tukey types of p-points and selective ultrafilters. Results fall into three main categories: comparison to a basis element for selective ultrafilters, embeddings of chains and antichains into the Tukey types, and Tukey types generated by block-basic ultrafilters on FIN.

1. Introduction

Let D and E be partial orderings. We say that a function $f : E \to D$ is *cofinal* if the image of each cofinal subset of E is cofinal in D. We say that D is *Tukey reducible* to E, and write $D \le_T E$, if there is a cofinal map from E to D. An equivalent formulation of Tukey reducibility was noticed by Schmidt in [25]. Given partial orderings D and E, a map $g : D \to E$ such that the image of each unbounded subset of D is an unbounded subset of E is called a *Tukey map* or an *unbounded map*. $E \ge_T D$ iff there is a Tukey map from D into E. If both $D \le_T E$ and $E \le_T D$, then we write $D \equiv_T E$ and say that D and E are Tukey equivalent. $\equiv_T$ is an equivalence relation, and $\le_T$ on the equivalence classes forms a partial ordering. The equivalence classes can be called *Tukey types* or *Tukey degrees*.

In [33], Tukey introduced the Tukey ordering to develop the notion of Moore-Smith convergence in topology to the more general setting of directed partial orderings. The study of cofinal types and Tukey types of partial orderings often reveals useful information for the comparison of different partial orderings. For example, Tukey reducibility downward preserves calibre-like properties, such as c.c.c., property K, precalibre $\aleph_1$, σ-linked, and σ-centered (see [31]).

Satisfactory classification theories of Tukey degrees have been developed for several classes of ordered sets. The cofinal types of countable directed systems are 1 and ω (see [33]). Day found a classification of countable oriented systems (partially ordered sets) in [8] in terms of a three element basis. Assuming PFA, Todorcevic in [30] classified the Tukey degrees of directed partial orderings of cardinality $\aleph_1$ by showing that there are exactly five cofinal types, and in [31] classified the Tukey degrees of oriented systems (partially ordered sets) of size $\aleph_1$ in terms of a basis consisting of five forms of partial orderings. However, he also showed in [31] that there are at least $2^{\aleph_1}$ many Tukey incomparable separative σ-centered partial orderings of size $\mathfrak{c}$. This would preclude a satisfactory classification theory of all partial orderings of size continuum.

However, the structure of the Tukey types of particular classes of partial orderings of size continuum can yield useful information. This has been fully stressed first in the paper [10] by Fremlin who considered partially ordered sets occurring in analysis. After this, several papers appeared dealing with different classes of posets such as, for example, the paper [27] of Solecki and Todorcevic which makes a systematic study of the structure of the Tukey degrees of topological directed sets. The paper

1991 *Mathematics Subject Classification.* 03E05, 03E17 03E35, 06A07.

[22] of Milovich is the first paper after Isbell [13] to study Tukey degrees of ultrafilters on ω.

In this paper, we investigate the structure of the Tukey degrees of ultrafilters on ω ordered by reverse inclusion. For any ultrafilter $\mathcal{U}$ on ω, $(\mathcal{U}, \supseteq)$ is a directed partial ordering. We remark that for any two directed partial orderings D and E, $D \equiv_T E$ iff D and E are *cofinally similar*; that is, there is a partial ordering into which both D and E embed as cofinal subsets [33]. So for ultrafilters, Tukey equivalence is the same as cofinal similarity.

Another motivation for this study is that Tukey reducibility is a generalization of Rudin-Keisler reducibility.

Fact 1. *Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on ω. If $\mathcal{U} \geq_{RK} \mathcal{V}$, then $\mathcal{U} \geq_T \mathcal{V}$.*

Proof. Take a function $h : \omega \to \omega$ satisfying $\mathcal{V} = h(\mathcal{U}) := \{X \subseteq \omega : h^{-1}(X) \in \mathcal{U}\}$. Define $f : \mathcal{U} \to \mathcal{V}$ by $f(X) = \{h(n) : n \in X\}$, for each $X \in \mathcal{U}$. Then f is a cofinal map. □

Thus arises the question: How different are Tukey and Rudin-Keisler reducibility? We shall study this question particularly for p-points.

2. Notation and basic facts

In this section, we fix notation and provide some basic facts. All ultrafilters in this paper have a base set which is countable. The base set will usually be ω, but in Section 6 we also investigate ultrafilters on FIN, the family of finite, nonempty subsets of ω.

Definition 2. Let $(P, \leq)$ be a partial ordering. We say that a subset $C \subseteq P$ is *cofinal* in P if for each $p \in P$ there is a $c \in C$ such that $p \leq c$. We say that $(P, \leq)$ is *directed* if for any $p, r \in P$, there is an $s \in P$ such that $p \leq s$ and $r \leq s$.

Fact 3. *If C is a cofinal subset of a partial ordering $(P, \leq)$, then $(C, \leq) \equiv_T (P, \leq)$.*

Proof. Let C be a cofinal subset of P and let $id_C : C \to P$ be the identity map. Then id_C is both a cofinal map and a Tukey map. For if $D \subseteq C$ is cofinal in $(C, \leq)$, then $id_C'' D = D$ is also cofinal in $(P, \leq)$. If $B \subseteq P$ is bounded in $(P, \leq)$, then there is a $p \in P$ bounding each element of B from above. Take a $c \in C$ such that $p \leq c$. Then c bounds $id_C^{-1}(B)$. Thus, id_C maps each unbounded subset of C to an unbounded subset of P, hence is a Tukey map. □

The partial ordering $\leq$ on an ultrafilter $\mathcal{U}$ is $\supseteq$; that is, for $X, Y \in \mathcal{U}$, $X \leq Y$ iff $X \supseteq Y$. Note that $(\mathcal{U}, \supseteq)$ is a directed partial ordering.

We now show that, for ultrafilters, there is a nice subclass of cofinal maps, namely the monotone cofinal maps, to which we may restrict our attention.

Definition 4. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be partial orderings. A map $f : P \to Q$ is *monotone* if whenever p, r are in P and $p \leq_P r$, then $f(p) \leq_Q f(r)$. For the special case of ultrafilters $\mathcal{U}, \mathcal{V}$, this translates to the following: a map $f : \mathcal{U} \to \mathcal{V}$ is *monotone* if whenever $W, X \in \mathcal{U}$ and $W \supseteq X$, then $f(W) \supseteq f(X)$.

Fact 5. *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be partial orderings. A monotone map $f : P \to Q$ is a cofinal map if and only if its image $f''P$ is a cofinal subset of Q.*

Proof. Let $f : P \to Q$ be a monotone map. If f is a cofinal map, then certainly the image of P under f is a cofinal subset of Q.

Conversely, suppose the image $f''P$ is cofinal in Q. Let $C \subseteq P$ be a cofinal subset of P and let $q \in Q$ be given. Since $f''P$ is cofinal in Q, there is a $p \in P$ such that $q \leq_Q f(p)$. Since C is cofinal in P, there is a $c \in C$ such that $p \leq_P c$. Since f is monotone, $q \leq_Q f(p) \leq_Q f(c)$. Therefore, $f''C$ is cofinal in Q. □

Fact 6. *Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters. If $\mathcal{U} \geq_T \mathcal{V}$, then this is witnessed by a monotone cofinal map.*

Proof. Suppose $\mathcal{U} \geq_T \mathcal{V}$. Then there is a Tukey map $g : \mathcal{V} \to \mathcal{U}$ witnessing this. Define $f : \mathcal{U} \to \mathcal{V}$ by $f(U) = \bigcap\{V \in \mathcal{V} : g(V) \supseteq U\}$.

First we check that f is a function from $\mathcal{U}$ into $\mathcal{V}$. Let $U \in \mathcal{U}$. Note that $\{V \in \mathcal{V} : g(V) \supseteq U\} = g^{-1}(\{U' \in \mathcal{U} : U' \supseteq U\})$. Since the set $\{U' \in \mathcal{U} : U' \supseteq U\}$ is bounded in $\mathcal{U}$ and g is a Tukey map, it follows that $\{V \in \mathcal{V} : g(V) \supseteq U\}$ is bounded in $\mathcal{V}$. Thus, $\bigcap\{V \in \mathcal{V} : g(V) \supseteq U\}$ is a member of $\mathcal{V}$.

Next we check that f is monotone. Let $U \supseteq U'$ be elements of $\mathcal{U}$. Then it is the case that $\{V \in \mathcal{V} : g(V) \supseteq U\} \subseteq \{V \in \mathcal{V} : g(V) \supseteq U'\}$. Thus, $f(U) = \bigcap\{V \in \mathcal{V} : g(V) \supseteq U\} \supseteq \bigcap\{V \in \mathcal{V} : g(V) \supseteq U'\} = f(U')$.

Finally, we show that $f''\mathcal{U}$ is cofinal in $\mathcal{V}$. Let $V' \in \mathcal{V}$. Then $g(V')$ is in $\mathcal{U}$; let U denote $g(V')$. By definition, $f(U) = \bigcap\{V \in \mathcal{V} : g(V) \supseteq g(V')\} \subseteq V'$. Thus, by Fact 5, f is a monotone cofinal map from $\mathcal{U}$ into $\mathcal{V}$. □

Thus, for ultrafilters, we can restrict ourselves to using monotone cofinal maps.

We now fix some notation for the duration of the paper. Recall that the partial ordering on a (finite or infinite) cartesian product of partially ordered sets is the coordinate-wise ordering. Thus, the partial ordering on a cartesian product of directed partial orderings is again a directed partial ordering.

Notation. Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{U}_n$ $(n < \omega)$ be ultrafilters. We define the notation for the following ultrafilters.

1. $\mathcal{U} \cdot \mathcal{V} = \{A \subseteq \omega \times \omega : \{i \in \omega : \{j \in \omega : (i, j) \in A\} \in \mathcal{V}\} \in \mathcal{U}\}$.
2. $\lim_{n \to \mathcal{U}} \mathcal{U}_n = \{A \subseteq \omega \times \omega : \{n \in \omega : \{j \in \omega : (n, j) \in A\} \in \mathcal{U}_n\} \in \mathcal{U}\}$.
3. We shall use $\mathcal{U}^2$ to denote $\mathcal{U} \cdot \mathcal{U}$; and more generally, $\mathcal{U}^{n+1}$ shall denote $\mathcal{U} \cdot \mathcal{U}^n$. We shall use $\mathcal{U}^\omega$ to denote $\lim_{n \to \mathcal{U}} \mathcal{U}^{k_n}$, where $(k_n)_{n<\omega}$ is any strictly increasing sequence of natural numbers. More generally, for any ordinal $\alpha < \omega_1$, $\mathcal{U}^{\alpha+1}$ denotes $\lim_{n \to \mathcal{U}} \mathcal{U}^\alpha$. For α a limit ordinal, $\mathcal{U}^\alpha$ is used to denote any ultrafilter of the form $\lim_{n \to \mathcal{U}} \mathcal{U}^{\beta_n}$, where $(\beta_n)_{n<\omega}$ is a strictly increasing sequence of ordinals such that $\sup_{n<\omega} \beta_n = \alpha$. (So for $\omega \leq \alpha < \omega_1$, $\mathcal{U}^\alpha$ does not denote a unique ultrafilter, but rather any ultrafilter formed in the way described above.)
4. $\mathcal{U} \times \mathcal{V}$ is defined to be the ordinary cartesian product of $\mathcal{U}$ and $\mathcal{V}$ with the coordinate-wise ordering $\langle \supseteq, \supseteq \rangle$.
5. $\Pi_{n<\omega} \mathcal{U}_n$ is the cartesian product of the $\mathcal{U}_n$ with its natural coordinate-wise product ordering. We will let $\Pi_{n<\omega} \mathcal{U}$ denote the cartesian product of ω many copies of $\mathcal{U}$.

The following basic facts are used throughout the paper.

Fact 7. *Let $\mathcal{U}, \mathcal{U}_0, \mathcal{U}_1, \mathcal{V}, \mathcal{V}_0$, and $\mathcal{V}_1$ be ultrafilters.*

1. $\mathcal{U} \times \mathcal{U} \equiv_T \mathcal{U}$.
2. $\mathcal{U} \times \mathcal{V} \geq_T \mathcal{U}$ *and* $\mathcal{U} \times \mathcal{V} \geq_T \mathcal{V}$.
3. *If* $\mathcal{U}_1 \geq_T \mathcal{U}_0$ *and* $\mathcal{V}_1 \geq_T \mathcal{V}_0$, *then* $\mathcal{U}_1 \times \mathcal{V}_1 \geq_T \mathcal{U}_0 \times \mathcal{V}_0$.
4. *If* $\mathcal{W} \geq_T \mathcal{U}$ *and* $\mathcal{W} \geq_T \mathcal{V}$, *then* $\mathcal{W} \geq_T \mathcal{U} \times \mathcal{V}$. *Thus,* $\mathcal{U} \times \mathcal{V}$ *is the minimal Tukey type which is Tukey greater than or equal to both* $\mathcal{U}$ *and* $\mathcal{V}$.
5. $\mathcal{U} \cdot \mathcal{V} \geq_T \mathcal{U}$ *and* $\mathcal{U} \cdot \mathcal{V} \geq_T \mathcal{V}$, *and therefore* $\mathcal{U} \cdot \mathcal{V} \geq_T \mathcal{U} \times \mathcal{V}$.

Proof. Let π_1, π_2 denote the projection maps $\pi_i : \omega \times \omega \to \omega$ $(i = 1, 2)$ given by $\pi_1(m, n) = m$, and $\pi_2(m, n) = n$.

(1) π_1 induces the map $\bar{\pi}_1 : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$, given by $\bar{\pi}_1(U, U') = U$, which is a cofinal map. Conversely, the map $f(U) = (U, U)$ is a cofinal map from $\mathcal{U}$ into $\mathcal{U} \times \mathcal{U}$.

(2) Again, the induced map $\bar{\pi}_1 : \mathcal{U} \times \mathcal{V} \to \mathcal{U}$ given by $\bar{\pi}_1(U, V) = U$ is a cofinal map. The second part follows since $\mathcal{U} \times \mathcal{V} \equiv_T \mathcal{V} \times \mathcal{U}$.

(3) Given monotone cofinal maps $f : \mathcal{U}_1 \to \mathcal{U}_0$ and $g : \mathcal{V}_1 \to \mathcal{V}_0$, define the map $h : \mathcal{U}_1 \times \mathcal{V}_1 \to \mathcal{U}_0 \times \mathcal{V}_0$ by $h(U, V) = (f(U), g(V))$. Let $\mathcal{X}$ be a cofinal subset of $\mathcal{U}_1 \times \mathcal{V}_1$ and let $(A, B) \in \mathcal{U}_0 \times \mathcal{V}_0$. There are $U \in \mathcal{U}_1$ and $V \in \mathcal{V}_1$ such that $f(U) \subseteq A$ and $g(V) \subseteq B$. Since $\mathcal{X}$ is cofinal in $\mathcal{U}_1 \times \mathcal{V}_1$, there is some $(U', V') \in \mathcal{X}$ such that $U' \subseteq U$ and $V' \subseteq V$. Since f and g are monotone, $h(U', V') = (f(U'), g(V')) \geq (f(U), g(V)) \geq (A, B)$. Thus, $h''\mathcal{X}$ is cofinal in $\mathcal{U}_0 \times \mathcal{V}_0$.

(4) follows immediately from (1) - (3).

(5) Define $f : \mathcal{U} \cdot \mathcal{V} \to \mathcal{U}$ by $f(A) = \{\pi_1(m, n) : (m, n) \in A\}$, for each $A \in \mathcal{U} \cdot \mathcal{V}$. Then f is monotone, and has cofinal range in $\mathcal{U}$. Hence, by Fact 6, $\mathcal{U} \cdot \mathcal{V} \geq_T \mathcal{U}$. (Alternatively, one can just note that the map π_1 is a Rudin-Keisler map from $\mathcal{U} \cdot \mathcal{V}$ to $\mathcal{U}$; and hence $\mathcal{U} \cdot \mathcal{V} \geq_T \mathcal{U}$.)

Let $g : \mathcal{U} \cdot \mathcal{V} \geq_T \mathcal{V}$ be defined by $g(A) = \{\pi_2(m, n) : (m, n) \in A\}$, for each $A \in \mathcal{U} \cdot \mathcal{V}$. Then g is monotone and has cofinal range in $\mathcal{V}$, hence is a cofinal map. □

Remark. One cannot conclude from the above that $\mathcal{U} \cdot \mathcal{V} \equiv_T \mathcal{U} \times \mathcal{V}$. Section 4 contains an investigation into this matter.

At this point, we recall the definitions of the following special ultrafilters. All these definitions can found in [2]. Recall the standard notation $\subseteq^*$, where for X, Y in an ultrafilter $\mathcal{U}$, we write $X \subseteq^* Y$ to denote that $|X \setminus Y| < \omega$.

Definition 8. Let $\mathcal{U}$ be an ultrafilter.

(1) $\mathcal{U}$ is *selective* if for every function $f : \omega \to \omega$, there is an $X \in \mathcal{U}$ such that either $f \restriction X$ is constant or $f \restriction X$ is one-to-one.
(2) $\mathcal{U}$ is a *p-point* if for every family $\{X_n : n < \omega\} \subseteq \mathcal{U}$ there is an $X \in \mathcal{U}$ such that $X \subseteq^* X_n$ for each $n < \omega$.
(3) $\mathcal{U}$ is a *q-point* if for each partition of ω into finite pieces $\{I_n : n < \omega\}$, there is an $X \in \mathcal{U}$ such that $|X \cap I_n| \leq 1$ for each $n < \omega$.
(4) $\mathcal{U}$ is *rapid* if for each function $f : \omega \to \omega$, there exists an $X \in \mathcal{U}$ such that $|X \cap f(n)| \leq n$ for each $n < \omega$.

The following well-known implications can be found in [2].

Theorem 9. (1) *An ultrafilter is selective if and only if it is both a p-point and is a q-point.*
(2) *Every q-point is rapid.*

We point out that all of these special ultrafilters exist under CH, under MA, and even under weaker assumptions involving cardinal invariants. However, the existence of selective ultrafilters, p-points, q-points, or even rapid ultrafilters does not follow from ZFC. We refer the interested reader to [2] for further exposition on these topics.

We point out the next fact, since it is useful to know, especially in Section 4.

Fact 10. *For any ultrafilter $\mathcal{U}$, $\mathcal{U} \cdot \mathcal{U}$ is not a p-point.*

Proof. If $\mathcal{U}$ is principle, generated by $\{n\}$, then $\mathcal{U} \cdot \mathcal{U}$ is also principle, generated by $\{(n, n)\}$.

If $\mathcal{U}$ is not principle, then it contains the Fréchet filter. For each $n < \omega$, let $A_n = [n, \omega) \times \omega$. Then each A_n is in $\mathcal{U}$. However, there is no $B \in \mathcal{U} \cdot \mathcal{U}$ such that $B \subseteq^* A_n$ for all $n < \omega$; for if $B \subseteq^* A_n$ for all $n < \omega$, then for each n there could only be finitely many j such that $(n, j) \in B$. □

A word about the top Tukey type for ultrafilters. The directed set $([\mathfrak{c}]^{<\omega}, \subseteq)$ is the maximal Tukey type among all directed partial orderings of cardinality $\mathfrak{c}$.

Fact 11. *Let $(X, \leq)$ be any directed partial ordering of cardinality $\mathfrak{c}$. Then $(X, \leq) \leq_T ([\mathfrak{c}]^{<\omega}, \subseteq)$.*

Proof. Let $g : X \to [\mathfrak{c}]^{<\omega}$ be any one-to-one function. Then g is a Tukey map. To see this, let W be any unbounded subset of X. Then in particular, W must be infinite, since every finite subset of X is bounded since X is directed. Since g is one-to-one, the image $g''W$ is also infinite. Every infinite subset of $[\mathfrak{c}]^{<\omega}$ is unbounded, so $g''W$ is unbounded. □

The following combinatorial characterization of when an ultrafilter has top Tukey type is useful.

Fact 12. *Let $\mathcal{U}$ be an ultrafilter. $(\mathcal{U}, \supseteq) \equiv_T ([\mathfrak{c}]^{<\omega}, \subseteq)$ if and only if there is a subset $\mathcal{X} \subseteq \mathcal{U}$ such that $|\mathcal{X}| = \mathfrak{c}$ and for each infinite $\mathcal{Y} \subseteq \mathcal{X}$, $\bigcap \mathcal{Y} \notin \mathcal{U}$.*

Proof. We first show the foreword direction by contrapositive. Suppose that there is no subset $\mathcal{X} \subseteq \mathcal{U}$ such that $|\mathcal{X}| = \mathfrak{c}$ and for each infinite $\mathcal{Y} \subseteq \mathcal{X}$, $\bigcap \mathcal{Y} \notin \mathcal{U}$. Then for each subset $\mathcal{X} \subseteq \mathcal{U}$ such that $|\mathcal{X}| = \mathfrak{c}$, there is an infinite $\mathcal{Y} \subseteq \mathcal{X}$ such that $\bigcap \mathcal{Y} \in \mathcal{U}$. We shall show that there is no Tukey map from $([\mathfrak{c}]^{<\omega}, \subseteq)$ into $(\mathcal{U}, \supseteq)$.

Let $g : ([\mathfrak{c}]^{<\omega}, \subseteq) \to (\mathcal{U}, \supseteq)$ be given. If the range of g is countable, then there is an uncountable subset $\mathcal{C} \subseteq [\mathfrak{c}]^{<\omega}$ and a $U \in \mathcal{U}$ such that $g''\mathcal{C} = \{U\}$. So g maps an unbounded set to a bounded set, hence is not a Tukey map. Otherwise, the range of g is uncountable. By our hypothesis, there is an infinite set $\mathcal{Y} \subseteq g''[\mathfrak{c}]^{<\omega}$ such that $\bigcap \mathcal{Y} \in \mathcal{U}$. Letting $\mathcal{C}$ be the g-preimage of $\mathcal{Y}$, we see that $\mathcal{C}$ is infinite, hence unbounded. Thus, g is not a Tukey map. Therefore, $([\mathfrak{c}]^{<\omega}, \subseteq) \not\leq_T (\mathcal{U}, \supseteq)$.

Suppose there is a subset $\mathcal{X} \subseteq \mathcal{U}$ such that $|\mathcal{X}| = \mathfrak{c}$ and for each infinite $\mathcal{Y} \subseteq \mathcal{X}$, $\bigcap \mathcal{Y} \notin \mathcal{U}$. By Fact 11, we know that $(\mathcal{U}, \supseteq) \leq_T ([\mathfrak{c}]^{<\omega}, \subseteq)$, so it remains to show that $(\mathcal{U}, \supseteq) \geq_T ([\mathfrak{c}]^{<\omega}, \subseteq)$. Let $g : [\mathfrak{c}]^{<\omega} \to \mathcal{X}$ be any one-to-one function. Let $Z \subseteq [\mathfrak{c}]^{<\omega}$ be unbounded. Then Z must be infinite, since $([\mathfrak{c}]^{<\omega}, \subseteq)$ is directed. Since g is one-to-one, $g''Z$ is an infinite subset of $\mathcal{X}$. Thus, $\bigcap g''Z$ is not in $\mathcal{U}$, so $g''Z$ is unbounded in $(\mathcal{U}, \supseteq)$. Therefore, g is a Tukey map. □

3. Basic and basically generated ultrafilters

The following type of partial ordering was introduced by Solecki and Todorcevic in [27].

Definition 13 ([27])**.** Let D be a separable metric space and let $\leq$ be a partial ordering on D. We say that $(D, \leq)$ is *basic* if

(1) each pair of elements of D has the least upper bound with respect to $\leq$ and the binary operation of least upper bound from $D \times D$ to D is continuous;
(2) each bounded sequence has a converging subsequence;
(3) each converging sequence has a bounded subsequence.

Each ultrafilter is a separable metric space using the metric inherited from $\mathcal{P}(\omega)$ viewed as the Cantor space, and recall that we define $\leq$ on an ultrafilter to be $\supseteq$. In this context, a sequence $(W_n)_{n<\omega}$ of elements of $\mathcal{P}(\omega)$ converges to $W \in \mathcal{P}(\omega)$ iff for each m there is some k such that for each $n \geq k$, $W_n \cap m = W \cap m$. It is not hard to see that every bounded subset of an ultrafilter has a convergent subsequence. Thus, an ultrafilter is basic iff (3) holds.

The next theorem shows that the basic ultrafilters are exactly the p-points. We recall the following characterization of non-meager ideals, which can be found in [14] or [28]. An ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is called *unbounded* if for each strictly increasing sequence of natural numbers $(n_i)_{i<\omega}$, there is an $X \in \mathcal{I}$ such that $[n_i, n_{i+1}) \subseteq X$ for infinitely many $i < \omega$. It was shown in [14] that an ideal is unbounded if and only if it is nonmeager (as a subset of $\mathcal{P}(\omega)$ with the topology inherited from the Cantor space).

Theorem 14. *An ideal $\mathcal{I}$ on $\mathcal{P}(\omega)$ containing all finite subsets of ω is basic relative to the Cantor topology iff $\mathcal{I}$ is a non-meager p-ideal. Hence, an ultrafilter is basic iff it is a p-point.*

Proof. Let $\mathcal{I}$ be an ideal on $\mathcal{P}(\omega)$ containing all finite subsets of ω.

Assume $\mathcal{I}$ is basic. Let $\langle n_k : k < \omega\rangle$ be an increasing sequence of integers. Note that each $[n_k, n_{k+1}) \in \mathcal{I}$, since Fin $\subseteq \mathcal{I}$. $[n_k, n_{k+1}) \to \emptyset$; so by basicness, there is a subsequence whose union is in $\mathcal{I}$. Hence, $\mathcal{I}$ unbounded, and thus is nonmeager.

Let $\{A_n : n < \omega\} \subseteq \mathcal{I}$. We can assume that for each $n < \omega$, $A_n \subseteq A_{n+1}$. Let $A'_n = A_n \setminus n$. Then $A'_n \subseteq A_n$, so $A'_n \in \mathcal{I}$. $A'_n \to \emptyset$ in the Cantor topology, so since $\mathcal{I}$ is basic, there is a subsequence n_k such that $\bigcup_{k<\omega} A'_{n_k} \in \mathcal{I}$. Let $A = \bigcup_{k<\omega} A_{n_k}$. Then for each $n < \omega$, $A_n \subseteq^* A$, since for each n there is an $n_k > n$ such that $A_n \subseteq A_{n_k} \subseteq^* A'_{n_k} \subseteq A$. Thus, $\mathcal{I}$ is a p-ideal.

Now suppose $\mathcal{I}$ is a nonmeager p-ideal. Suppose $A_n, A \in \mathcal{I}$ and $A_n \to A$ in the Cantor topology. Take $B \in \mathcal{I}$ such that for each n, $A_n \subseteq^* B$. Let m_k be a strictly increasing sequence such that $m_0 = 0$ and

(1) $n \geq m_{k+1}$ implies $A_n \cap m_k = A \cap m_k$, and
(2) $n \leq m_k$ implies $A_n \setminus m_{k+1} \subseteq B$.

Since $\mathcal{I}$ is nonmeager, there is a subsequence $(m_{k_i})_{i<\omega}$ of $(m_k)_{k<\omega}$ such that $C := \bigcup_{i<\omega}[m_{k_i}, m_{k_i+2}) \in \mathcal{I}$. Let $X = \bigcup_{i<\omega} A_{m_{k_i}+1}$.

We claim that $X \subseteq A \cup B \cup C$. Let $i < \omega$ be given. Then $A_{m_{k_i}+1} \cap m_{k_i} = A \cap m_{k_i}$, by (1). $A_{m_{k_i}+1} \cap [m_{k_i}, m_{k_i+2}) \subseteq C$, since C contains the interval $[m_{k_i}, m_{k_i+2})$. Finally, $A_{m_{k_i}+1} \setminus m_{k_i+2} \subseteq B$, by (2). Thus, $A_{m_{k_i}+1} \subseteq A \cup B \cup C$. Since i was arbitrary, we have the desired conclusion that $X \subseteq A \cup B \cup C$, and hence $X \in \mathcal{I}$. Therefore, $\mathcal{I}$ is basic, since every convergent sequence of elements of $\mathcal{I}$ has a bounded subsequence. □

Remark. From the proof, we can see that an ultrafilter is basic iff every sequence which converges to ω has a bounded subsequence.

The next definition gives a notion of ultrafilters which is weaker than p-point.

Definition 15. We say that an ultrafilter $\mathcal{U}$ on $\mathcal{P}(\omega)$ is *basically generated* if it has a filter basis $\mathcal{B} \subseteq \mathcal{U}$ (i.e. $\forall A \in \mathcal{U}\ \exists B \in \mathcal{B}\ B \subseteq A$) with the property that each sequence $\{A_n : n < \omega\} \subseteq \mathcal{B}$ converging to an element of $\mathcal{B}$ has a subsequence $\{A_{n_k} : k < \omega\}$ such that $\bigcap_{k<\omega} A_{n_k} \in \mathcal{U}$.

Theorem 16. *Suppose that $\mathcal{U}$ and $\mathcal{U}_n$ $(n < \omega)$ are basically generated ultrafilters on $\mathcal{P}(\omega)$ by filter bases which are closed under finite intersection. Then $\mathcal{V} = \lim_{n\to\mathcal{U}} \mathcal{U}_n$ is basically generated by a filter basis which is closed under finite intersections. It follows that whenever $\mathcal{U}$ is basically generated by a filter basis closed under finite intersection, $\mathcal{U}^\alpha$ is basically generated by a filter basis closed under finite intersections, for each $\alpha < \omega_1$.*

Proof. Let $\mathcal{B}, \mathcal{B}_n$ be filter bases of $\mathcal{U}, \mathcal{U}_n$ $(n < \omega)$ which are closed under finite intersection and which witness the fact that $\mathcal{U}, \mathcal{U}_n$ are basically generated, respectively. Let $p_1 : \omega \times \omega \to \omega$ be the projection map onto the first coordinate. For $A \subseteq \omega \times \omega$ and $n < \omega$, let $(A)_n$ denote $\{j < \omega : (n, j) \in A\}$. Let $\mathcal{C} = \{A \in \mathcal{V} : p_1[A] \in \mathcal{B}$ and for each $n < \omega$, either $(A)_n = \emptyset$ or $(A)_n \in \mathcal{B}_n\}$. Then $\mathcal{C}$ is a filter basis for $\mathcal{V}$ which is closed under finite intersections.

Consider a converging sequence $A_n \to B$ in $\mathcal{C}$. Note that $p_1[A_n] \to U$ for some $U \in \mathcal{U}$ containing $p_1[B]$. U might not be in $\mathcal{B}$, but $p_1[B]$ is in $\mathcal{B}$, since $B \in \mathcal{C}$. So for each $n < \omega$, let $A'_n = A_n \cap (p_1[B] \times \omega)$, so that $A'_n \in \mathcal{C}$. Note that $A'_n \to B$, $p_1[A'_n] \to p_1[B]$, and all $p_1[A'_n] \in \mathcal{B}$, since $\mathcal{B}$ is closed under finite intersections. Since $\mathcal{B}$ witnesses that $\mathcal{U}$ is basically generated, there is a subsequence of $(p_1[A'_n])_{n<\omega}$ whose intersection is in $\mathcal{U}$. Take such a subsequence and reindex it, so that we have $\bigcap_{n<\omega} p_1[A'_n] \in \mathcal{U}$. Let U denote $\bigcap_{n<\omega} p_1[A'_n]$. Note that $U \subseteq \bigcap_{n<\omega} p_1[A_n]$. Enumerate U as $(n_k)_{k<\omega}$. Then for each $k < \omega$ and each $m < \omega$, $(A'_m)_{n_k} = (A_m)_{n_k}$ since $n_k \in U \subseteq p_1[B]$. So for each $k < \omega$, we have that $(A_m)_{n_k} \to (B)_{n_k}$ as $m \to \infty$. Take a decreasing sequence $M_0 \supseteq M_1 \supseteq \ldots \supseteq M_k \supseteq \ldots$ of infinite subsets of ω

such that for each k, $\bigcap_{m\in M_k}(A_m)_{n_k} \in \mathcal{U}_{n_k}$. We may assume that $m_k = \min M_k$ is a strictly increasing sequence.

Let $C = \bigcap_{l<\omega} A_{m_l}$. We claim that $C \in \mathcal{V}$. Note that $U = \{n_k : k < \omega\} \subseteq p_1[A_{m_l}]$ for all l, so $U \subseteq p_1[C]$. Thus, $p_1[C] \in \mathcal{U}$. For each k, $\bigcap_{l\ge k}(A_{m_l})_{n_k} \supseteq \bigcap_{m\in M_k}(A_m)_{n_k}$ which is in $\mathcal{U}_{n_k}$. Hence, intersecting $\bigcap_{l\ge k}(A_{m_l})_{n_k}$ with finitely more members $(A_{m_l})_{n_k}$, $l < k$, of $\mathcal{U}_{n_k}$ still yields a member of $\mathcal{U}_{n_k}$. Thus, $(C)_{n_k} = \bigcap_{l<\omega}(A_{m_l})_{n_k}$, which is in $\mathcal{U}_{n_k}$. Therefore, $C \in \mathcal{V}$. □

Remark. For any ultrafilter $\mathcal{U}$, $\mathcal{U}\cdot\mathcal{U}$ is not a p-point. Thus, there are basically generated ultrafilters which are not p-points.

Recall Fact 11 which says that for every ultrafilter $\mathcal{U}$, $(\mathcal{U},\supseteq) \le_T ([\mathfrak{c}]^{<\omega},\subseteq)$. We say that an ultrafilter $\mathcal{U}$ *has top Tukey type* if $(\mathcal{U},\supseteq) \equiv_T ([\mathfrak{c}]^{<\omega},\subseteq)$. The following theorem of Isbell shows that, in ZFC, there is always an ultrafilter which has top Tukey type.

Theorem 17 (Isbell [13])**.** *There is an ultrafilter $\mathcal{U}_{\rm top}$ on ω realizing the maximal cofinal type among all directed sets of cardinality continuum, i.e. $\mathcal{U}_{\rm top} \equiv_T [\mathfrak{c}]^{<\omega}$.*

We remark here that the same construction in Isbell's proof was done independently by Juhász in [15] (stated in [16]) in connection with strengthening a theorem of Pospíšil [23], though without the Tukey terminology.

There are in fact $2^{\mathfrak{c}}$ many ultrafilters on ω having Tukey type exactly $([\mathfrak{c}]^{<\omega},\subseteq)$, since any collection of independent sets can be used in a canonical way to construct an ultrafilter with top Tukey type. Thus, already we see that for the case of the top Tukey type, the Rudin-Keisler equivalence relation is strictly finer than the Tukey equivalence relation, since every Rudin-Keisler equivalence class has cardinality $\mathfrak{c}$.

Note also that $\mathcal{U}_{\rm top}$ is not basically representable, or in other words,

Theorem 18. *If $\mathcal{U}$ is a basically generated ultrafilter on ω, then $\mathcal{U} <_T [\mathfrak{c}]^{<\omega}$.*

Proof. Let $\mathcal{U}$ be basically generated. Then there is a filter basis $\mathcal{B} \subseteq \mathcal{U}$ with the property that each sequence $(A_n)_{n<\omega} \subseteq \mathcal{B}$ converging to an element of $\mathcal{B}$ has a subsequence $(A_{n_k})_{k<\omega}$ such that $\bigcap_{k<\omega} A_{n_k} \in \mathcal{U}$.

Let $\mathcal{X}$ be any subset of $\mathcal{U}$ of cardinality $\mathfrak{c}$. For each $X \in \mathcal{X}$, choose one $B_X \in \mathcal{B}$ such that $B_X \subseteq X$. If there is an infinite $\mathcal{Y} \subseteq \mathcal{X}$ and a $B \in \mathcal{B}$ such that all $X \in \mathcal{Y}$ have $B_X = B$, then this $B \subseteq \bigcap \mathcal{Y}$. Otherwise, $\{B_X : X \in \mathcal{X}\}$ is uncountable, so there is a sequence $(A_n)_{n<\omega} \subseteq \{B_X : X \in \mathcal{X}\}$ which converges to some $B \in \{B_X : X \in \mathcal{X}\}$, and such that all A_n are distinct. Since $\mathcal{B}$ witnesses that $\mathcal{U}$ is basically generated, there is a subsequence $(A_{n_k})_{k<\omega}$ such that $\bigcap_{k<\omega} A_{n_k} \in \mathcal{U}$. Taking $\mathcal{Y}$ to be the collection of $X \in \mathcal{X}$ such that $B_X = A_{n_k}$ for some k, we have that $\mathcal{Y}$ is infinite and $\bigcap \mathcal{Y} \supseteq \bigcap_{k<\omega} A_{n_k}$ which is in $\mathcal{U}$. By Fact 12, $(\mathcal{B},\supseteq) <_T ([\mathfrak{c}]^{<\omega},\subseteq)$. □

Corollary 19. *Every p-point has Tukey type strictly below the top Tukey type.*

Proof. Since every basic ultrafilter is basically generated, it follows from Theorems 14 and 18 that every p-point has Tukey type strictly below $[\mathfrak{c}]^{<\omega}$. □

The next theorem gives a canonical form for cofinal maps from p-points to any other ultrafilter. This theorem or similar ideas will be used in the majority of proofs in the rest of this paper.

Recall that any subset of $\mathcal{P}(\omega)$ is a topological space, with the subspace topology inherited from the Cantor space. Thus, given any $\mathcal{X},\mathcal{Y} \subseteq \mathcal{P}(\omega)$, a function $f : \mathcal{X} \to \mathcal{Y}$ is continuous if it is continuous with respect to the subspace topologies on $\mathcal{X}$ and $\mathcal{Y}$. Equivalently, a function $f : \mathcal{X} \to \mathcal{Y}$ is continuous if for each sequence $(X_n)_{n<\omega} \subseteq \mathcal{X}$ which converges to some $X \in \mathcal{X}$, the sequence $(f(X_n))_{n<\omega}$ converges to $f(X)$.

If $X \in \mathcal{U}$, then we use $\mathcal{U} \restriction X$ to denote $\{Y \in \mathcal{U} : Y \subseteq X\}$. Note that $\mathcal{U} \restriction X$ is a filter base for $\mathcal{U}$, and hence $(\mathcal{U},\supseteq) \equiv_T (\mathcal{U} \restriction X,\supseteq)$.

Theorem 20. *Suppose $\mathcal{U}$ is a p-point on ω and that $\mathcal{V}$ is an arbitrary ultrafilter on ω such that $\mathcal{U} \geq_T \mathcal{V}$. Then there is a continuous monotone map $f^* : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ whose restriction to $\mathcal{U}$ is continuous and has cofinal range in $\mathcal{V}$. Hence, there is a continuous monotone cofinal map from $\mathcal{U}$ into $\mathcal{V}$ witnessing that $\mathcal{U} \geq_T \mathcal{V}$.*

Proof. Let $\mathcal{U}$ be a p-point, $\mathcal{V}$ be an ultrafilter, and suppose that $\mathcal{U} \geq_T \mathcal{V}$. By Fact 6, there is a monotone cofinal map $f : \mathcal{U} \to \mathcal{V}$. We claim that there is an $\tilde{X} \in \mathcal{U}$ such that $f : \mathcal{U} \restriction \tilde{X} \to \mathcal{V}$ is continuous.

Construct a decreasing sequence $X_0 \supseteq X_1 \supseteq \dots$ of elements of $\mathcal{U}$ as follows. Let $X_0 = \omega$. For $1 \leq n < \omega$, given X_{n-1}, we take an $X_n \in \mathcal{U}$ with the following properties:

1. $X_n \subseteq X_{n-1}$;
2. $X_n \cap n = \emptyset$;
3. for each $s \subseteq n$, for each $k \leq n$, if there is a $Y' \in \mathcal{U}$ such that $s = Y' \cap (n+1)$ and $k \notin f(Y')$, then $k \notin f(s \cup X_n)$.

That there is such a sequence of X_n follows from f being monotone, as we shall see now. Suppose we already have X_{n-1}. Fix a $W_0 \in \mathcal{U}$ such that $W_0 \subseteq X_{n-1}$ and $W_0 \cap n = \emptyset$. List out all subsets of n as $s_1, \dots, s_{2^n}$. Suppose there is a $Y' \in \mathcal{U}$ such that $s_1 = Y' \cap (n+1)$ and $k \notin f(Y')$. Then take some such Y'_1 and let $W_1 = W_0 \cap Y'_1$. If there is no such $Y' \in \mathcal{U}$, then let $W_1 = W_0$. For $1 \leq l < 2^n$, given $W_0 \supseteq \dots \supseteq W_l$, if there is a $Y' \in \mathcal{U}$ such that $s_{l+1} = Y' \cap (n+1)$ and $k \notin f(Y')$, then take some such Y'_{l+1} and let $W_{l+1} = W_l \cap Y'_{l+1}$. If there is no such $Y' \in \mathcal{U}$, then let $W_{l+1} = W_l$. After the 2^n many steps of this process, we let $X_n = W_{2^n}$.

Note the following for each $1 \leq l \leq 2^n$. If there is a $Y' \in \mathcal{U}$ such that $s_l = Y' \cap (n+1)$ and $k \notin f(Y')$, then W_l was taken to be $W_{l-1} \cap Y'_l$. So for any $U \in \mathcal{U} \restriction X_n$, we have $s_l \cup U \subseteq Y'_l$. Since f is monotone, we have $f(s_l \cup U) \subseteq f(Y'_l)$. Thus, $k \notin f(s_l \cup U)$, since $k \notin f(Y'_l)$.

We check that X_n has the desired properties. By construction, (1) holds. Since $X_n \subseteq W_0$, we have that $X_n \cap n = \emptyset$, so (2) holds. Let s be any subset of n. Then there is some $1 \leq l \leq 2^n$ such that $s = s_l$. Suppose there is a $Y' \in \mathcal{U}$ such that $s_l = Y' \cap (n+1)$ and $k \notin f(Y')$. Then by the preceding paragraph, k is not in $f(s \cup X_n)$.

Since $\mathcal{U}$ is a p-point, fix some $Y \in \mathcal{U}$ be such that for each $n < \omega$, $Y \subseteq^* X_n$. Let $0 = n_0 < n_1 < \dots$ be such that for each $i < \omega$, for each $n \leq n_i$, $Y \setminus n_{i+1} \subseteq X_n$. Let $Z = \bigcup_{i=0}^{\infty} [n_{2i+1}, n_{2i+2})$. Without loss of generality, assume that $Z \notin \mathcal{U}$. (If Z is in $\mathcal{U}$, then let $\tilde{X}$ be $Y \cap Z$. The proof for this case goes through exactly as the one we give below, with the minor modification of readjusting the indexes by 1 at the outset.) Let $\tilde{X} = Y \setminus Z$. We show that $f \restriction (\mathcal{U} \restriction \tilde{X})$ is continuous. Precisely, we shall show that there is a strictly increasing sequence $(m_k)_{k<\omega}$ such that for each $W \in \mathcal{U} \restriction \tilde{X}$, the initial segment $f(W) \cap (k+1)$ of $f(W)$ is determined by $W \cap m_k$.

Given $k < \omega$, let i_k denote the least i for which $n_{2i_k+1} \geq k$. Let $W \in \mathcal{U} \restriction \tilde{X}$ be given and let $s = W \cap n_{2i_k+1}$. Recalling that $\tilde{X} \cap [n_{2i_k+1}, n_{2i_k+2}) = \emptyset$, we have that $W \setminus n_{2i_k+1} \subseteq \tilde{X} \setminus n_{2i_k+1} = \tilde{X} \setminus n_{2i_k+2} \subseteq Y \setminus n_{2i_k+2} \subseteq X_{n_{2i_k+1}}$. Therefore, $k \in f(W)$ iff $k \in f(s \cup X_{n_{2i_k+1}})$ iff $k \in f(s \cup (\tilde{X} \setminus n_{2i_k+1}))$. Letting $m_k = n_{2i_k+1}$, we see that $f \restriction (\mathcal{U} \restriction \tilde{X})$ is continuous, since the question of whether or not $k \in f(W)$ is determined by the finite initial segment $W \cap m_k$ along with $\tilde{X} \setminus m_k$.

Next, we extend f on $\mathcal{U} \restriction \tilde{X}$ to all of $\mathcal{U}$ by defining $f'(X) = f(X \cap \tilde{X})$, for $X \in \mathcal{U}$. Then $f' : \mathcal{U} \to \mathcal{V}$ is again monotone. Moreover, for each $X \in \mathcal{U}$ and $k < \omega$, $k \in f'(X)$ iff $k \in f(X \cap \tilde{X})$ iff $k \in f(s \cup (\tilde{X} \setminus m_k))$, where $s = X \cap \tilde{X} \cap m_k$. So whether or not k is in $f'(X)$ is determined by the initial segment $X \cap \tilde{X} \cap m_k$ of $X \cap \tilde{X}$; hence f' is continuous.

Finally, we extend f' to a monotone continuous map f^* defined on all of $\mathcal{P}(\omega)$. For an arbitrary $Z \subseteq \omega$ set

$$f^*(Z) = \bigcap\{f'(Z) : X \supseteq Z \text{ and } X \text{ is cofinite}\}. \tag{1}$$

Note that since f' is monotone, $f^*(Z)$ is exactly $\bigcap\{f'((Z \cap n) \cup [n, \omega)) : n < \omega\}$, since every cofinite X containing Z contains $(Z \cap n) \cup [n, \omega)$ for some n. From the definition of f^* and the fact that f' is monotone, it follows that f^* is monotone.

First, we show that $f^* \restriction \mathcal{U} = f'$. Let $Z \in \mathcal{U}$ be given. Let $Z_n = (Z \cap n) \cup [n, \omega)$, for each $n < \omega$. Then $f^*(Z) = \bigcap\{f'(Z_n) : n < \omega\}$. Since $Z_n \to Z$ and f' is continuous on $\mathcal{U}$, it follows that $f'(Z_n) \to f'(Z)$. This, along with the fact that each $f'(Z_n) \supseteq f'(Z)$ imply that $\bigcap_{n<\omega} f'(Z_n)$ equals $f'(Z)$. Hence, $f^*(Z) = \bigcap_{n<\omega} f'(Z_n) = f'(Z)$. Thus, $f^* \restriction \mathcal{U} = f'$.

To see that f^* is continuous, we show that for each $k < \omega$ and $Z \subseteq \omega$, whether or not k is in $f^*(Z)$ is determined by the initial segment $Z \cap \tilde{X} \cap m_k$ of $Z \cap \tilde{X}$, along with $\tilde{X} \setminus m_k$. Let $Z \subseteq \omega$ and $k < \omega$, and let $Z_n = (Z \cap n) \cup [n, \omega)$ for each $n < \omega$. Then $k \in f^*(Z)$ iff for each $n < \omega$, $k \in f'(Z_n)$ iff for each $n \geq m_k$, $k \in f'(Z_n)$ iff for each $n \geq m_k$, $k \in f'(Z_n \cap \tilde{X})$ iff $k \in f(s \cup (\tilde{X} \setminus m_k))$, where $s = Z \cap \tilde{X} \cap m_k$. □

Remark. Note that Theorem 20 gives the canonical form of cofinal maps that is likely going to be the main object of study in this area from now on: Every Tukey reduction $\mathcal{U} \geq_T \mathcal{V}$ for $\mathcal{U}$ a p-point is witnessed by some monotone continuous $f^* : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $f^* \restriction \mathcal{U}$ is a cofinal map from $\mathcal{U}$ into $\mathcal{V}$. Moreover, for any monotone cofinal map $f : \mathcal{U} \to \mathcal{V}$, (where $\mathcal{U}$ is a p-point), there is a a cofinal subset of the form $\mathcal{U} \restriction \tilde{X}$ for some $\tilde{X} \in \mathcal{U}$ such that $f \restriction (\mathcal{U} \restriction \tilde{X})$ is continuous. Note that the restriction of f to any cofinal subset of $\mathcal{U} \restriction \tilde{X}$ retains continuity, justifying the use of the word *canonical*.

Remark. Whereas the top Tukey type has cardinality $2^{\mathfrak{c}}$, the previous theorem implies that the Tukey type of any p-point has cardinality $\mathfrak{c}$.

Corollary 21. *Every $\leq_T$-chain of p-points on ω has cardinality $\leq \mathfrak{c}^+$.*

Proof. Theorem 20 shows that every Tukey chain $\mathcal{F} \subseteq \{\text{p-points}\}$ is $\mathfrak{c}^+$-like, that is, $|\{\mathcal{V} \in \mathcal{F} : \mathcal{V} \leq_T \mathcal{U}\}| \leq \mathfrak{c}$ for all $\mathcal{U} \in \mathcal{F}$. □

Recall the Free Set Lemma of Hajnal.

Lemma 22 (Free Set Lemma of Hajnal [17])**.** *If $|X| = \kappa$ and $\lambda < \kappa$ and $F : X \to \mathcal{P}(X)$ satisfies $x \notin F(x)$ and $|F(x)| < \lambda$, for all $x \in X$, then there is a $Y \subseteq X$ with $x \notin F(y)$ and $y \notin F(x)$ for all $x, y \in Y$ and $|Y| = \kappa$.*

Corollary 23. *Every family $\mathcal{X}$ of p-points on ω of cardinality $> \mathfrak{c}^+$ contains a subfamily $\mathcal{Z} \subseteq \mathcal{X}$ of equal size such that $\mathcal{U} \not\leq_T \mathcal{V}$ whenever $\mathcal{U} \neq \mathcal{V}$ are in $\mathcal{Z}$.*

Proof. Let $\mathcal{X}$ be a family of p-points such that $\kappa := |\mathcal{X}| > \mathfrak{c}^+$. Define $F : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ by $F(\mathcal{U}) = \{\mathcal{V} \in \mathcal{X} : \mathcal{V} <_T \mathcal{U}\}$. By Theorem 20, for each $\mathcal{U} \in \mathcal{X}$, $|F(\mathcal{U})| < \mathfrak{c}^+$. So, by the Free Set Lemma 22, there is a family $\mathcal{Y} \subseteq \mathcal{X}$ such that $|\mathcal{Y}| = \kappa$ and for each $\mathcal{U}, \mathcal{V} \in \mathcal{Y}$, $\mathcal{U} \notin F(\mathcal{V})$ and $\mathcal{V} \notin F(\mathcal{U})$; that is, $\mathcal{U} \not<_T \mathcal{V}$ and $\mathcal{V} \not<_T \mathcal{U}$. By Theorem 20, there are at most $\mathfrak{c}$ many ultrafilters Tukey equivalent to any given p-point. Thus, there is a subfamily $\mathcal{Z} \subseteq \mathcal{Y}$ also of cardinality α such that every two p-points in $\mathcal{Z}$ are Tukey incomparable. □

Remark. A similar trick was used by Rudin and Shelah in [26] in part of their proof that there are always $2^{\mathfrak{c}}$ many Rudin-Keisler incomparable ultrafilters.

Next, we use Theorem 20 to see that some strength of selective ultrafilters is preserved downward in the Tukey ordering.

Theorem 24. *Suppose $\mathcal{U}$ is selective and $\mathcal{U} \geq_T \mathcal{V}$. Then $\mathcal{V}$ is basically generated.*

Proof. By Theorem 20, there is a continuous monotone map $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $f''\mathcal{U} \subseteq \mathcal{V}$ and $f''\mathcal{U}$ generates $\mathcal{V}$. By the selective version of the Prömel-Voight canonical form of the Galvin-Prikry Theorem, there is an $M \in \mathcal{U}$, a Lipschitz map $\varphi : [\omega]^\omega \to \mathcal{P}(\omega)$ such that $\varphi(X) \subseteq X$ for each $X \in [\omega]^\omega$, and a 1-1 homeomorphism $\psi : \text{range}(\varphi) \to \mathcal{P}(\omega)$ such that $f = \psi \circ \varphi$.

Let $\mathcal{B} = f''\mathcal{U} \upharpoonright M$. Note that $\mathcal{B}$ is a cofinal subset of $\mathcal{V}$. We claim that every converging sequence $X_n \to X$ of elements of $\mathcal{B}$ has a subsequence X_{n_k} such that $\bigcap_{k<\omega} X_{n_k} \in \mathcal{V}$. Let X_n, $n < \omega$, and X be elements of $\mathcal{B}$ such that $X_n \to X$. Let $Y = \psi^{-1}(X)$ and $Y_n = \psi^{-1}(X_n)$. Then $Y_n \to Y$, since ψ is a 1-1 homeomorphism. Let $K = \{A \in \mathcal{U} : \varphi(A) = Y\}$ and $K_n = \{A \in \mathcal{U} : \varphi(A) = Y_n\}$ $(n \in \omega)$. Then K and K_n are compact subsets of $\mathcal{U}$ such that $K_n \to K$. So in particular for an arbitrary choice $A_n \in K_n$, $(n \in \omega)$ we can find a subsequence A_{n_k} converging to a member B in K. Note that A_{n_k} is a sequence in $\mathcal{U}$ converging to the member B, which is in $\mathcal{U}$. Since $\mathcal{U}$ is basic there is a further subsequence $A_{n_{k_i}}$ such that

$$A = \bigcap_{i<\omega} A_{n_{k_i}} \in \mathcal{U}.$$

It follows that $X_{n_{k_i}} = f(A_{n_{k_i}}) \supseteq f(A)$ for all $i < \omega$ and so in particular, $f(A) \in \mathcal{V}$ and $f(A) \subseteq \bigcap_{i<\omega} X_{n_{k_i}}$. Thus, $\mathcal{B}$ witnesses that $\mathcal{V}$ is basically generated. □

It will be shown in Section 4 that for each selective ultrafilter $\mathcal{U}$, $\mathcal{U} \cdot \mathcal{U} \equiv_T \mathcal{U}$; hence $\mathcal{U} \equiv_T \mathcal{V}$ does not imply that $\mathcal{V}$ is selective.

Question 25. If $\mathcal{U}$ is a p-point and $\mathcal{U} \geq_T \mathcal{V}$, does it follow that $\mathcal{V}$ is basically generated?

Question 26. From Theorem 16, we know that every iteration of Fubini products of p-points is basically generated. Is there an ultrafilter which is basically generated but is not a Fubini limit of p-points?

Question 27. Can Theorem 20 be improved to show that if $\mathcal{U}$ is basically generated and $\mathcal{U} \geq_T \mathcal{V}$, then there is a continuous (or definable) monotone cofinal map $f : \mathcal{U} \to \mathcal{V}$ witnessing this?

More generally,

Question 28. If $\mathcal{V} \leq_T \mathcal{U} <_T [\mathfrak{c}]^{<\omega}$, then is there a continuous (or definable) monotone cofinal map $f : \mathcal{U} \to \mathcal{V}$ witnessing this?

One might first try to show that the existence of a continuous cofinal map propagates Tukey downwards, or in other words,

Question 29. Suppose that $\mathcal{U}$ is such that whenever $\mathcal{U} \geq_T \mathcal{V}$ then there is a continuous monotone cofinal map from $\mathcal{U}$ to $\mathcal{V}$. If $\mathcal{U} \geq_T \mathcal{W}$, then does it follow that for each $\mathcal{V} \leq_T \mathcal{W}$ there is a continuous monotone cofinal map from $\mathcal{W}$ into $\mathcal{V}$?

4. Comparing Tukey types of ultrafilters with $(\omega^\omega, \leq)$

In this section we investigate which ultrafilters are above $(\omega^\omega, \leq)$, where $h \leq g$ iff for each $n < \omega$, $h(n) \leq g(n)$.

Fact 30. *If $\mathcal{U}$ is a rapid ultrafilter, then $\mathcal{U} \geq_T \omega^\omega$.*

Proof. Define $f : \mathcal{U} \to \omega^\omega$ by letting $f(X)$ be the function which enumerates all but the least element of X in strictly increasing order. It is not hard to check that f is a cofinal map. □

Hence each selective ultrafilter and each q-point is Tukey above ω^ω.

Fact 31. *For each ultrafilter $\mathcal{U}$, $\mathcal{U} \cdot \mathcal{U} \geq_T \omega^\omega$.*

Proof. Define $f : \mathcal{U}\cdot\mathcal{U} \to \omega^\omega$ by letting $f(A)$ be the function $g_A : \omega \to \omega$ defined by $g_A(n) = \min(A)_{n_k}$, where $(n_k)_{k<\omega}$ enumerates those n for which $(A)_n \in \mathcal{U}$. We shall show that f is a cofinal map.

Let $\mathcal{X}$ consist of those $A \in \mathcal{U}\cdot\mathcal{U}$ with the properties that (a) whenever $(A)_n \neq \emptyset$, then $(A)_n \in \mathcal{U}$, and (b) whenever $m < n$ and $(A)_m, (A)_n \in \mathcal{U}$, then $\min(A)_m \leq \min(A)_n$. Note that $\mathcal{X}$ is a base for $\mathcal{U}\cdot\mathcal{U}$, so it suffices to show that $f \upharpoonright \mathcal{X}$ is a cofinal map from $\mathcal{X}$ into ω^ω. We show that $f \upharpoonright \mathcal{X}$ is monotone and has range which is cofinal in ω^ω, hence by Fact 5, $f \upharpoonright \mathcal{X}$ is a cofinal map from $\mathcal{X}$ into ω^ω.

Let $A, B \in \mathcal{X}$ be given such that $A \supseteq B$. Then the sequence $(i_k)_{k<\omega}$ enumerating those n for which $(B)_n \in \mathcal{U}$ is a subsequence of the sequence $(n_k)_{k<\omega}$ enumerating those n for which $(A)_n \in \mathcal{U}$. Hence, for each k, $n_k \leq i_k$. Since A, B are in $\mathcal{X}$ and $A \supseteq B$, we have that $\min(A)_{n_k} \leq \min(A)_{i_k} \leq \min(B)_{i_k}$; hence $g_A(k) \leq g_B(k)$ for all $k < \omega$. Therefore, $f \upharpoonright \mathcal{X}$ is monotone.

Next, let $h : \omega \to \omega$ be given. Define A to be the collection of pairs (n, l) such that $l > \max\{h(i) : i \leq n\}$. Then $A \in \mathcal{X}$ and $g_A(n) \geq h(n)$ for all $n < \omega$. Thus, $f \upharpoonright \mathcal{X}$ has cofinal range in ω^ω. □

Theorem 32. *For any ultrafilters $\mathcal{U}$,$\mathcal{U}_n$ $(n < \omega)$, $\lim_{n\to\mathcal{U}} \mathcal{U}_n \leq_T \mathcal{U} \times \Pi_{n<\omega}\mathcal{U}_n$, where $\mathcal{U} \times \Pi_{n<\omega}\mathcal{U}_n$ is given its natural product ordering. In particular, $\mathcal{U}\cdot\mathcal{U} \leq_T \Pi_{n<\omega}\mathcal{U}$.*

Proof. Let $\mathcal{V}$ denote $\lim_{n\to\mathcal{U}} \mathcal{U}_n$. Let $\mathcal{B} = \{A \in \mathcal{V} :$ for each $n < \omega$, either $(A)_n = \emptyset$ or $(A)_n \in \mathcal{U}_n\}$. Note that $\mathcal{B}$ is a basis for $\mathcal{V}$; hence it suffices to construct a Tukey map $g : \mathcal{B} \to \mathcal{U} \times \Pi_{n<\omega}\mathcal{U}_n$. Given $A \in \mathcal{B}$ let $g(A) = (p_1[A], (q_n(A))_{n<\omega})$, where $q_n(A) = (A)_n$ if $n \in p_1[A]$ and $q_n(A) = \omega$ otherwise.

To verify g is a Tukey map let $\mathcal{Y}$ be a bounded subset of $\mathcal{V}$. Then there is some $(C, (D_n : n < \omega)) \in \mathcal{U} \times \Pi_{n<\omega}\mathcal{U}_n$ which bounds $\mathcal{Y}$. Let $\mathcal{X} = \{A \in \mathcal{B} : p_1[A] \supseteq C$ and $\forall n < \omega,\ q_n(A) \supseteq D_n\}$. Note that $\mathcal{X}$ contains the g-preimage of $\mathcal{Y}$. Let $B = \bigcap \mathcal{X}$. Then $p_1[B] \supseteq C$ and for each $n \in C$, $(B)_n \supseteq D_n$, so $B \in \mathcal{V}$. Moreover, by its definition, B bounds $\mathcal{X}$. Hence B also bounds the g-preimage of $\mathcal{Y}$. □

Theorem 33. *If $\mathcal{U}$ is a p-point, then $\Pi_{n<\omega}\mathcal{U} \equiv_T \mathcal{U} \times \omega^\omega$ and therefore $\Pi_{n<\omega}\mathcal{U} \equiv_T \mathcal{U}\cdot\mathcal{U}\cdot\mathcal{U}$.*

Proof. First, we show that $\Pi_{n<\omega}\mathcal{U} \leq_T \mathcal{U}\times\omega^\omega$. Given a sequence $(A_n)_{n<\omega} \in \Pi_{n<\omega}\mathcal{U}$, choose a $B \in \mathcal{U}$ and an $h : \omega \to \omega$ such that $B \setminus h(n) \subseteq A_n$ for each n. (Since $\mathcal{U}$ is a p-point, there is a $B \in \mathcal{U}$ such that $B \subseteq^* A_n$ for each n. Let $h(n)$ be the least m such that $B \setminus m \subseteq A_n$.) Set $g((A_n)_{n<\omega}) = (B, h)$.

g is a Tukey map. To see this, let $\mathcal{Y}$ be a bounded subset of $\mathcal{U} \times \omega^\omega$. Then there is some $(B_*, h_*) \in \mathcal{U} \times \omega^\omega$ which bounds $\mathcal{Y}$. Let $\mathcal{X} = \{(A_n)_{n<\omega} : g((A_n)_{n<\omega}) \leq (B_*, h_*)\}$. Note that $\mathcal{X}$ set contains the g-preimage of $\mathcal{Y}$. We claim that $\mathcal{X}$ is bounded by $(B_* \setminus h_*(n))_{n<\omega}$. For given any $(A_n)_{n<\omega} \in \mathcal{X}$, letting (B, h) denote $g((A_n)_{n<\omega})$, we have that $(B, h) \leq (B_*, h_*)$, which means that $B \supseteq B_*$ and $h(n) \leq h_*(n)$ for all n. So for each n, $B_* \setminus h_*(n) \subseteq B \setminus h(n) \subseteq A_n$. Thus, $(B_* \setminus h_*(n))_{n<\omega}$ is a bound for $\mathcal{X}$.

On the other hand, $\omega^\omega \leq_T \mathcal{U}\cdot\mathcal{U} \leq_T \Pi_{n<\omega}\mathcal{U}$, by Fact 31 and Theorem 32. So $\mathcal{U} \times \omega^\omega \leq_T \mathcal{U} \times \Pi_{n<\omega}\mathcal{U} = \Pi_{n<\omega}\mathcal{U}$.

Finally, $\mathcal{U}\cdot\mathcal{U} \leq_T \mathcal{U}\cdot\mathcal{U}\cdot\mathcal{U}$ and Fact 31 imply that $\mathcal{U} \times \omega^\omega \leq_T \mathcal{U} \times (\mathcal{U}\cdot\mathcal{U}) \equiv_T \mathcal{U}\cdot\mathcal{U} \leq_T \mathcal{U}\cdot\mathcal{U}\cdot\mathcal{U}$. On the other hand, applying Theorem 32 twice, we have $\mathcal{U}\cdot\mathcal{U}\cdot\mathcal{U} \equiv_T \lim_{n\to\mathcal{U}\cdot\mathcal{U}} \mathcal{U} \leq_T (\mathcal{U}\cdot\mathcal{U}) \times \Pi_{n<\omega}\mathcal{U} \leq_T \Pi_{n<\omega}\mathcal{U} \times \Pi_{n<\omega}\mathcal{U} = \Pi_{n<\omega}\mathcal{U}$. Thus, $\mathcal{U}\cdot\mathcal{U}\cdot\mathcal{U} \equiv_T \Pi_{n<\omega}\mathcal{U}$. □

Corollary 34. *If $\mathcal{V}$ is a p-point, $\mathcal{V} \geq_T \omega^\omega$, and $\mathcal{U}$ is any ultrafilter, then $\mathcal{U}\cdot\mathcal{V} \equiv_T \mathcal{U} \times \mathcal{V}$.*

Proof. By Theorem 32, $\mathcal{U}\cdot\mathcal{V} \le_T \mathcal{U}\times\Pi_{n<\omega}\mathcal{V}$. Since $\mathcal{V}$ is a p-point, $\Pi_{n<\omega}\mathcal{V} \equiv_T \mathcal{V}\times\omega^\omega$, by Theorem 33. $\mathcal{V} \ge_T \omega^\omega$ implies that $\mathcal{V}\times\omega^\omega \equiv_T \mathcal{V}$. Therefore, $\mathcal{U}\cdot\mathcal{V} \le_T \mathcal{U}\times\Pi_{n<\omega}\mathcal{V} \equiv_T \mathcal{U}\times\mathcal{V} \le_T \mathcal{U}\cdot\mathcal{V}$. □

Theorem 35. *The following are equivalent for a p-point $\mathcal{U}$.*

(1) $\mathcal{U} \ge_T \omega^\omega$;
(2) $\mathcal{U} \equiv_T \Pi_{n<\omega}\mathcal{U}$;
(3) $\mathcal{U} \equiv_T \mathcal{U}\cdot\mathcal{U}$.

Proof. Suppose $\mathcal{U} \ge_T \omega^\omega$. By Theorem 33, $\Pi_{n<\omega}\mathcal{U} \equiv_T \mathcal{U}\times\omega^\omega \equiv_T \mathcal{U} \le_T \Pi_{n<\omega}\mathcal{U}$. Suppose $\mathcal{U} \equiv_T \Pi_{n<\omega}\mathcal{U}$. Since always $\mathcal{U} \le_T \mathcal{U}\cdot\mathcal{U}$, and $\mathcal{U}\cdot\mathcal{U} \le_T \Pi_{n<\omega}\mathcal{U}$ by Theorem 32, we have that $\mathcal{U} \equiv_T \mathcal{U}\cdot\mathcal{U}$. If $\mathcal{U} \equiv_T \mathcal{U}\cdot\mathcal{U}$, then since $\mathcal{U}\cdot\mathcal{U} \ge_T \omega^\omega$, we have that $\mathcal{U} \ge_T \omega^\omega$. □

We remark that the p-point property was only used for (1) implies (2).

Corollary 36. *If $\mathcal{U}$ is a p-point of cofinality $< \mathfrak{d}$, then $\mathcal{U} \not\ge_T \omega^\omega$ and therefore $\mathcal{U} <_T \mathcal{U}\cdot\mathcal{U}$.*

Remark. Such an ultrafilter $\mathcal{U}$ exists in any extension of a model of CH by a countable support iteration of length ω_2 of superperfect-set forcing since by a result of Shelah such an iteration preserves p-points.

Corollary 37. *If $\mathcal{U}$ is a rapid p-point then $\Pi_{n<\omega}\mathcal{U} \equiv_T \mathcal{U}\cdot\mathcal{U} \equiv_T \mathcal{U}$.*

Remark. By Corollary 37, for each selective ultrafilter $\mathcal{U}$, the Tukey type of $\mathcal{U}$ is strictly coarser than the Rudin-Keisler type of $\mathcal{U}$, even though they both have cardinality $\mathfrak{c}$. That is, if $\mathcal{U}$ is selective, then $\mathcal{U}\cdot\mathcal{U}$ is not a p-point yet $\mathcal{U} \equiv_T \mathcal{U}\cdot\mathcal{U}$. However, if $\mathcal{U} \equiv_{RK} \mathcal{V}$ then $\mathcal{V}$ is selective. We remark here that Todorcevic has more recently shown that if $\mathcal{U}$ is selective and $\mathcal{U} \ge_T \mathcal{V}$, then $\mathcal{U} \ge_{RK} \mathcal{V}$, and hence, $\mathcal{V} \equiv_{RK} \mathcal{U}$. (See [24].) Hence, although the Tukey type of a selective ultrafilter includes non-p-points, any two selective ultrafilters with the same Tukey type are isomorphic.

Theorem 38. *Assuming $\mathfrak{p} = \mathfrak{c}$, there is a p-point $\mathcal{U}$ such that $\mathcal{U} \not\ge_T \omega^\omega$ and therefore $\mathcal{U} <_T \mathcal{U}\cdot\mathcal{U} <_T \mathcal{U}_{top}$.*

Proof. Let $\{f_\alpha : 0 < \alpha < \mathfrak{c}\}$ be an enumeration of all Souslin-measurable mappings from ω^ω into $[\omega]^\omega$, and let $\{X_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathcal{P}(\omega)$. We build an ultrafilter $\mathcal{U}$ to be generated by a $\supseteq^*$ chain $\langle A_\alpha : \alpha < \mathfrak{c}\rangle$ of infinite subsets of ω, while diagonalizing over all Souslin-measurable mappings of the form $f_\alpha : \omega^\omega \to [\omega]^\omega$ ($\alpha < \mathfrak{c}$).

Let $A_0 = \omega$. Given $\alpha < \mathfrak{c}$ and $\{A_\xi : \xi < \alpha\}$, using the fact that $\mathfrak{p} = \mathfrak{c}$, there is an $A'_\alpha \in [\omega]^\omega$ such that $A'_\alpha \subseteq^* A_\xi$ for all $\xi < \alpha$. Let $A''_\alpha = A'_\alpha \cap X_\alpha$ if this is infinite, otherwise, let $A''_\alpha = A'_\alpha \setminus X_\alpha$. If there is an $x \in \omega^\omega$ such that $A''_\alpha \setminus f_\alpha(x)$ is infinite, then let $A_\alpha = A''_\alpha \setminus f_\alpha(x)$. Otherwise, we let $A_\alpha = A''_\alpha$.

Let $\mathcal{U}$ be the p-point generated by the tower $\{A_\alpha : \alpha < \mathfrak{c}\}$. We need to show that $\mathcal{U} \not\ge_T \omega^\omega$. Suppose toward a contradiction that $\mathcal{U} \ge_T \omega^\omega$. Then applying [Theorem 5.3 (i), [27]], there is a Souslin measurable map $f : \omega^\omega \to \mathcal{U}$ such that f is a Tukey map. Since we listed all Souslin measurable maps from ω^ω into $[\omega]^\omega$, there is an $\alpha < \mathfrak{c}$ such that $f_\alpha = f$. Since the range of f is contained in $\mathcal{U}$, A_α is not $A'_\alpha \setminus f_\alpha(x)$ for any $x \in \omega^\omega$. Hence, $A_\alpha = A''_\alpha$ and $A_\alpha \subseteq^* f_\alpha(x)$ for all $x \in \omega^\omega$.

Define P_n to be $\{x \in \omega^\omega : A_\alpha \setminus n \subseteq f_\alpha(x)\}$. There is an $n_0 \in \omega$ such that P_{n_0}, is not bounded in ω^ω relative to the ordering of eventual domination. (For if not, then for each n, there is some $g_n \in \omega^\omega$ which eventually dominates every element of P_n. Let g be a function which eventually dominates each g_n. Then $g \ge^* x$ for each x such that for some n, $A_\alpha \setminus n \subseteq f_\alpha(x)$. But $A_\alpha \subseteq^* f_\alpha(x)$ for all $x \in \omega^\omega$, and hence g eventually dominates every member of ω^ω, contradiction.) In particular, there is a $k \in \omega$ and an infinite subset $\{x_i : i < \omega\} \subseteq P_{n_0}$ such that $x_i(k) \geq i$ for all $i < \omega$. It follows that $\{x_i : i < \omega\}$ is unbounded in $(\omega^\omega, \leq)$ but its image $\{f_\alpha(x_i) : i < \omega\}$ is bounded by $A_\alpha \setminus n$, which is in $\mathcal{U}$. Thus, f_α is not a Tukey map from ω^ω into $\mathcal{U}$. $\square$

Question 39. Is there an ultrafilter $\mathcal{U}$ on ω such that $\mathcal{U} <_T \mathcal{U} \cdot \mathcal{U} <_T \mathcal{U} \cdot \mathcal{U} \cdot \mathcal{U} <_T \mathcal{U}_{top}$?

Remark. Using some assumptions like $\mathfrak{p} = \mathfrak{c}$, it seems possible to get Tukey chains of p-points of order-type $\mathfrak{c}^+$ which is, as we know, maximal possible. By Corollary 53 below, CH implies there are Tukey chains of p-points of length $\mathfrak{c}$. Dilip Raghavan has shown that, assuming CH, there is a Tukey chain of p-points isomorphic to the reals (see [24]).

Question 40. Is there an ultrafilter $\mathcal{U} <_T \mathcal{U}_{\mathrm{top}}$ which is not Tukey reducible to any p-point?

Question 41. Is every basically generated ultrafilter Tukey reducible to a p-point?

Both of the preceding two questions are answered using the assumption $\mathcal{U} \not\geq_T \omega^\omega$ for any p-point $\mathcal{U}$ (which is true in the iterated superperfect extension). Namely, then $\mathcal{U} \cdot \mathcal{U} \not\leq_T \mathcal{V}$ for every ultrafilter $\mathcal{U}$ and every p-point $\mathcal{V}$.

Question 42. Is there a p-ideal I on ω which is not countably generated but $I \not\geq_T \omega^\omega$?

Remark. If $\mathfrak{b} \neq \mathfrak{d}$ there is such a p-ideal, so the question is whether we can get one with no extra set-theoretic assumptions.

Question 43. Does $\mathcal{U} \cdot \mathcal{U} \equiv_T \mathcal{U} <_T \mathcal{U}_{\mathrm{top}}$ imply $\mathcal{U}$ is basically generated?

5. Antichains, chains, and incomparable predecessors

We now investigate the structure of the Tukey types of p-points and selective ultrafilters in terms of which chains, antichains, and incomparable ultrafilters with a common upper bound embed into the Tukey types.

Theorem 44.
(1) *Assume $cov(\mathscr{M}) = \mathfrak{c}$. Then there are $2^{\mathfrak{c}}$ pairwise Tukey incomparable selective ultrafilters.*
(2) *Assume $\mathfrak{d} = \mathfrak{u} = \mathfrak{c}$. Then there are $2^{\mathfrak{c}}$ pairwise Tukey incomparable p-points.*

We prove Theorem 44 by proving it first in the case that $2^{\mathfrak{c}} > \mathfrak{c}^+$ (see Theorem 47), and then proving it in the case that $2^{\mathfrak{c}} = \mathfrak{c}^+$ (see Theorem 49). Of use will be two propositions of Ketonen. Recall [Theorem 1.7, [18]] of Ketonen: If $\mathrm{cov}(\mathscr{M}) = \mathfrak{c}$ then every filter with a filter base of size less than $\mathfrak{c}$ can be extended to a selective ultrafilter. The key part of his proof uses the following proposition.

Proposition 45 (Ketonen, Proposition 1.8 [18])**.** *If $cov(\mathscr{M}) = \mathfrak{c}$ and $\mathcal{F}$ is a filter generated by less than $\mathfrak{c}$ many sets, and $\{P_i : i < \omega\}$ is a partition of ω so that for each $i < \omega$, $\bigcup\{P_j : j > i\} \in \mathcal{F}$, then there exists a set $X \subseteq \omega$ such that $\{X\} \cup \mathcal{F}$ has the finite intersection property, and for every $i < \omega$, $|X \cap P_i| \leq 1$.*

The following proposition of Ketonen was used in his proof of [Theorem 1.2, [18]]: $\mathfrak{d} = \mathfrak{c}$ if and only if any filter generated by a base of cardinality less than $\mathfrak{c}$ can be extended to a p-point.

Proposition 46 (Ketonen, Proposition 1.3 [18])**.** *If $\mathfrak{d} = \mathfrak{c}$, then given any filter $\mathcal{F}$ generated by less than $\mathfrak{c}$ elements and a sequence $\langle A_i : i < \omega\rangle$ of elements of $\mathcal{F}$, there exists a set $A \subseteq \omega$ so that $\mathcal{F} \cup \{A\}$ has the finite intersection property, and for each $i < \omega$, $A \subseteq^* A_i$.*

We are now equipped to prove Theorem 44 in the case that $2^{\mathfrak{c}} > \mathfrak{c}^+$.

Theorem 47. *Assume $2^{\mathfrak{c}} > \mathfrak{c}^+$.*

(1) *Assume $cov(\mathscr{M}) = \mathfrak{c}$. Then there are $2^{\mathfrak{c}}$ pairwise Tukey incomparable selective ultrafilters.*
(2) *Assume $\mathfrak{d} = \mathfrak{u} = \mathfrak{c}$. Then there are $2^{\mathfrak{c}}$ pairwise Tukey incomparable p-points.*

Proof. We prove (1) first. Recall that $\mathrm{cov}(\mathscr{M}) = \mathfrak{c}$ implies $\mathfrak{u} = \mathfrak{c}$, so every filter base of cardinality less than $\mathfrak{c}$ does not generate an ultrafilter. We fix some notation used throughout the proof. Fix a listing $\langle D_\alpha : \alpha < \mathfrak{c}\rangle$ of all the infinite subsets of ω. There are $\mathfrak{c}$ many partitions of ω, so we fix a sequence $\langle \vec{P}_\alpha : \alpha < \mathfrak{c}\rangle$ such that each $\vec{P}_\alpha = \langle P^n_\alpha : n < \omega\rangle$ is a partition of ω (that is, $\bigcup_{n<\omega} P^n_\alpha = \omega$ and for each $m \neq n$, $P^m_\alpha \cap P^n_\alpha = \emptyset$) and each partition of ω appears in the listing. We shall say that a filter $\mathcal{U}$ is *selective for the partition* $\vec{P}_\alpha$ if either there is some $n < \omega$ such that $P^n_\alpha \in \mathcal{U}$ or else there is some $X \in \mathcal{U}$ such that $|X \cap P^n_\alpha| \le 1$ for each $n < \omega$.

We now begin the construction. In a very similar manner to the proof of [Theorem 2, [3]] of Blass, we will construct selective ultrafilters $\mathcal{U}_x$, $x \in 2^{\mathfrak{c}}$, such that for $x \neq y$, $\mathcal{U}_x \neq \mathcal{U}_y$. Let $\mathcal{U}_{\langle\rangle}$ be the Fréchet filter. If there is an $i < \omega$ such that P^i_0 is infinite, then let $\mathcal{U}'_{\langle\rangle}$ be the filter generated by $\mathcal{U}_{\langle\rangle} \cup \{P^i_0\}$. Otherwise, for each $i < \omega$, P^i_0 is finite. Then take some infinite X such that for each i, $|X \cap P^i_0| \le 1$ and let $\mathcal{U}'_{\langle\rangle}$ be the filter generated by $\mathcal{U}_{\langle\rangle} \cup \{X\}$. (This is possible since $\mathcal{U}_{\langle\rangle}$ is the Fréchet filter.) Take α_0 minimal such that both D_{α_0} and $D^c_{\alpha_0}$ are in $(\mathcal{U}'_{\langle\rangle})^+$. Let $\mathcal{U}_{\langle 0\rangle}$ be the filter generated by $\mathcal{U}'_{\langle\rangle} \cup \{D_{\alpha_0}\}$ and let $\mathcal{U}_{\langle 1\rangle}$ be the filter generated by $\mathcal{U}'_{\langle\rangle} \cup \{D^c_{\alpha_0}\}$. Note that both $\mathcal{U}_{\langle 0\rangle}$ and $\mathcal{U}_{\langle 1\rangle}$ have countable filter bases, are selective for $\vec{P}_0$, and any ultrafilter extending $\mathcal{U}_{\langle i\rangle}$ does not extend $\mathcal{U}_{\langle 1-i\rangle}$, for each $i \le 1$.

Suppose for $t \in 2^{<\mathfrak{c}}$, the filter $\mathcal{U}_t$ has been constructed and has a filter base of cardinality less than $\mathfrak{c}$. Let β be the length of t. The partition of ω under consideration is $\vec{P}_\beta = \langle P^n_\beta : n < \omega\rangle$. If there is an $n < \omega$ such that $P^n_\beta \in \mathcal{U}_t$, then let $\mathcal{U}'_t = \mathcal{U}_t$. Otherwise, for each $n < \omega$, $\bigcup_{j>n} P^j_\beta \in \mathcal{U}_t$. Apply Proposition 45 to find an $X \in [\omega]^\omega$ such that $\{X\} \cup \mathcal{U}_t$ has the finite intersection property, and such that for each $n < \omega$, $|X \cap P^n_\beta| \le 1$. Let $\mathcal{U}'_t$ be the filter generated by $\{X\} \cup \mathcal{U}_t$. Take α_β minimal such that both D_{α_β} and $D^c_{\alpha_\beta}$ are in $(\mathcal{U}'_t)^+$. (Note that $\alpha_\beta \ge \beta$.) Let $\mathcal{U}_{t^\frown 0}$ be the filter generated by $\mathcal{U}'_t \cup \{D_{\alpha_\beta}\}$ and let $\mathcal{U}_{t^\frown 1}$ be the filter generated by $\mathcal{U}'_t \cup \{D^c_{\alpha_\beta}\}$. Note that for each $i \le 1$, both $\mathcal{U}_{t^\frown i}$ have filter bases of cardinality less than $\mathfrak{c}$, are selective for $\vec{P}_\beta$, and any ultrafilter extending $\mathcal{U}_{t^\frown i}$ does not extend $\mathcal{U}_{t^\frown(1-i)}$.

For $t \in 2^{<\mathfrak{c}}$ with length of t some limit ordinal γ, if for all $\beta < \gamma$, $\mathcal{U}_{t\restriction\beta}$ has been constructed, then we let $\mathcal{U} = \bigcup_{\beta<\gamma} \mathcal{U}_{t\restriction\beta}$.

This constructs filters $\mathcal{U}_t$, $t \in 2^{<\mathfrak{c}}$, satisfying the following. For each $t \in 2^{<\mathfrak{c}}$,

(1) $\mathcal{U}_t$ is a filter with a filter base of cardinality less than $\mathfrak{c}$;
(2) If s is an initial segment of t, then $\mathcal{U}_s \subseteq \mathcal{U}_t$;
(3) If the length of t is $\alpha + 1$ for some $\alpha < \mathfrak{c}$, then for all $\beta \le \alpha$, $\mathcal{U}_t$ is selective for $\vec{P}_\beta$, and either B_β or B^c_β is in $\mathcal{U}_t$;
(4) No ultrafilter can extend both $\mathcal{U}_{t^\frown 0}$ and $\mathcal{U}_{t^\frown 1}$.

For each $x \in 2^{\mathfrak{c}}$, let $\mathcal{U}_x = \bigcup_{\beta<\mathfrak{c}} \mathcal{U}_{x\restriction\beta}$. Then by (2) and (3), each $\mathcal{U}_x$ is a selective ultrafilter. Furthermore, (4) implies that for $x, y \in 2^{\mathfrak{c}}$, if $x \neq y$, then $\mathcal{U}_x \neq \mathcal{U}_y$. Thus, we have $2^{\mathfrak{c}}$ selective ultrafilters. By Theorem 20, each $\mathcal{U}_x$ has Tukey type of cardinality at most $\mathfrak{c}$. Thus, there are $2^{\mathfrak{c}}$ Tukey types among the collection of Tukey types of the $\mathcal{U}_x$, $x \in 2^{\mathfrak{c}}$. Since $2^{\mathfrak{c}} > \mathfrak{c}^+$, Corollary 23 yields $2^{\mathfrak{c}}$ Tukey incomparable selective ultrafilters.

The proof of (2) of the Theorem follows exactly the same steps as for (1) with only the following modification which ensures that we build p-points (instead of selective ultrafilters). Before starting the construction, fix an enumeration $\langle \vec{A}_\alpha : \alpha < \mathfrak{c}\rangle$, where $\vec{A}_\alpha = \langle A^n_\alpha : n < \omega\rangle$, such that for each countable collection $\vec{B} = \langle B_n : n < \omega\rangle$ of infinite subsets of ω, $\vec{B} = \vec{A}_\alpha$ for cofinally many $\alpha < \mathfrak{c}$.

We now begin the construction for (2). Let $\mathcal{U}_{\langle\rangle}$ be the Fréchet filter. If the sequence $\langle A^n_0 : n < \omega\rangle$ is contained in $\mathcal{U}_{\langle\rangle}$, then apply Proposition 46 to obtain a set B such that $B \subseteq^* A^n_0$ for each $n < \omega$ and such that $\{B\} \cup \mathcal{U}_{\langle\rangle}$ has the finite intersection property. In this case, let $\mathcal{U}'_{\langle\rangle}$ denote the filter generated by $\{B\} \cup \mathcal{U}_{\langle\rangle}$. If the sequence $\langle A^n_0 : n < \omega\rangle$ is not contained in $\mathcal{U}_{\langle\rangle}$, then let $\mathcal{U}'_{\langle\rangle} = \mathcal{U}_{\langle\rangle}$. Take α_0 minimal such that both D_{α_0} and $D^c_{\alpha_0}$ are in $(\mathcal{U}'_{\langle\rangle})^+$. Let $\mathcal{U}_{\langle 0\rangle}$ be the filter generated by $\mathcal{U}'_{\langle\rangle} \cup \{D_{\alpha_0}\}$ and let $\mathcal{U}_{\langle 1\rangle}$ be the filter generated by $\mathcal{U}'_{\langle\rangle} \cup \{D^c_{\alpha_0}\}$.

Suppose for $t \in 2^{<\mathfrak{c}}$, the filter $\mathcal{U}_t$ has been constructed and has a filter base of size less than $\mathfrak{c}$. Let β be the length of t. If the sequence $\langle A^n_\beta : n < \omega\rangle$ is contained in $\mathcal{U}_t$, then apply Proposition 46 to obtain a set B such that $B \subseteq^* A^n_\beta$ for each $n < \omega$ and such that $\{B\} \cup \mathcal{U}_t$ has the finite intersection property. In this case, let $\mathcal{U}'_t$ denote the filter generated by $\{B\} \cup \mathcal{U}_t$. If the sequence $\langle A^n_\beta : n < \omega\rangle$ is not contained in $\mathcal{U}_t$, then let $\mathcal{U}'_t = \mathcal{U}_t$. Take α_β minimal such that both D_{α_β} and $D^c_{\alpha_\beta}$ are in $(\mathcal{U}'_t)^+$. Let $\mathcal{U}_{t^\frown 0}$ be the filter generated by $\mathcal{U}'_t \cup \{D_{\alpha_\beta}\}$ and let $\mathcal{U}_{t^\frown 1}$ be the filter generated by $\mathcal{U}'_t \cup \{D^c_{\alpha_\beta}\}$. For $t \in 2^{<\mathfrak{c}}$ such that length of t is some limit ordinal γ, if for all $\beta < \gamma$, $\mathcal{U}_{t\restriction\beta}$ has been constructed, then we let $\mathcal{U}_t = \bigcup_{\beta<\gamma} \mathcal{U}_{t\restriction\beta}$.

For each $x \in 2^{\mathfrak{c}}$, let $\mathcal{U}_x = \bigcup_{\beta<\mathfrak{c}} \mathcal{U}_{x\restriction\beta}$. By similar arguments as for (1), each $\mathcal{U}_x$ is an ultrafilter and for $x \neq y$, $\mathcal{U}_x \neq \mathcal{U}_y$. Moreover, $\mathfrak{d} = \mathfrak{c}$ implies that the cofinality of $\mathfrak{c}$ is uncountable. Thus, any countable collection of elements of $\mathcal{U}_x$ appears in $\mathcal{U}_t$ for some $t \in 2^{<\mathfrak{c}}$ such that $t \sqsubseteq x$ and hence is considered at some stage in the construction of $\mathcal{U}_x$. Thus, $\mathcal{U}_x$ is a p-point. By Theorem 20 and Corollary 23, we obtain $2^{\mathfrak{c}}$ Tukey incomparable p-points. □

Next we take care of the case when $2^{\mathfrak{c}} = \mathfrak{c}^+$. In this case, Corollary 23 does not apply, so we present a new way of constructing $\mathfrak{c}^+$ Tukey incomparable selective ultrafilters (or p-points). To do so we shall need the following notions.

We shall say that a continuous monotone function $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is *presented* by the function $\hat{f} : 2^{<\omega} \to \mathcal{P}(\omega)$ if for for each $m < \omega$ and each $s \in 2^m$, $\hat{f}(s) = \bigcap_{n\geq m} f(\tilde{s} \cup [n, \omega))$, where $\tilde{s}$ denotes $\{i < m : s(i) = 1\}$; and for each $Z \subseteq \omega$, $f(Z) = \bigcup\{\hat{f}(s) : s \sqsubseteq Z\}$, where Z is identified with its characteristic function. Note that $\hat{f}$ has the property that for any $X \subseteq \omega$ and any $l < \omega$, $l \in f(X)$ iff $l \in \hat{f}(X \cap (l+1))$ (where $X \cap (l+1)$ is being considered as an element of 2^{l+1} by identifying $X \cap (l+1)$ with its characteristic function of length $l+1$). In the proof of Theorem 20, it was shown that for any p-point $\mathcal{Z}$, any ultrafilter $\mathcal{U}$, and any monotone cofinal map $f : \mathcal{Z} \to \mathcal{U}$, there is a continuous monotone map $f^* : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ and a cofinal subset $\mathcal{Z} \restriction \tilde{X}$ of $\mathcal{Z}$ such that $f^* \restriction (\mathcal{Z} \restriction \tilde{X})$ equals $f \restriction (\mathcal{Z} \restriction \tilde{X})$. Moreover, the proof of Theorem 20 shows that this f^* is presented by some $\hat{f} : 2^{<\omega} \to \mathcal{P}(\omega)$. Thus, it suffices to consider only continuous monotone maps $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ which are presented by some function $\hat{f} : 2^{<\omega} \to \mathcal{P}(\omega)$.

Lemma 48. *Let $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ be a continuous monotone map presented by a map $\hat{f} : 2^{<\omega} \to \mathcal{P}(\omega)$, let $\mathcal{U}$ be a non-principal ultrafilter, and let $\mathcal{Y}$ be a filter containing the Fréchet filter with a filter base of size less than $\mathfrak{u}$. Then there is a $Y \in \mathcal{Y}^+$ such that for any ultrafilter $\mathcal{Z}$ which extends $\mathcal{Y} \cup \{Y\}$, $f \restriction \mathcal{Z}$ is not a cofinal map from $\mathcal{Z}$ into $\mathcal{U}$.*

Proof. Let f, $\mathcal{U}$, and $\mathcal{Y}$ satisfy the hypotheses. If there is a $Y \in \mathcal{Y}^+$ such that $f(Y) \notin \mathcal{U}$, then we are done. So now suppose that for each $Y \in \mathcal{Y}^+$, $f(Y) \in \mathcal{U}$. If there is a $U \in \mathcal{U}$ such that for each $Y \in \mathcal{Y}^+$, $f(Y) \not\subseteq U$, then for every ultrafilter $\mathcal{Z}$ extending $\mathcal{Y}$, $f''\mathcal{Z}$ is not cofinal in $\mathcal{U}$.

Thus, the remaining case is that $f''\mathcal{Y}^+$ is cofinal in $\mathcal{U}$, which we assume throughout the rest of the proof of the lemma. Let $\hat{f}: 2^{<\omega} \to \mathcal{P}(\omega)$ be given such that f is presented by $\hat{f}$. Note that for each $s \in 2^{<\omega}$, $\hat{f}(s)$ is the set of all k which must be in $f(X)$ for every extension X of s, and $\hat{f}$ has the property that for any $X \subseteq \omega$ and any $l < \omega$, $l \in f(X)$ iff $l \in \hat{f}(X \cap (l+1))$. Recall that for a filter $\mathcal{W}$, the dual ideal is denoted by $\mathcal{W}^*$.

Claim 1. For any ultrafilter $\mathcal{U}$, given any collection $\{C_i : i < \omega\} \subseteq \mathcal{U}^*$ such that each C_i is infinite, there is a $U \in \mathcal{U}$ such that for each $i < \omega$, $C_i \not\subseteq U$.

Proof. Let $\{C_i : i < \omega\}$ be a collection of infinite sets such that each $C_i \in \mathcal{U}^*$. Let $a_0 = \min(C_0)$ and $b_0 = \min(C_0 \setminus \{a_0\})$. Let $I_0 = \{i < \omega : \{a_0, b_0\} \subseteq C_i\}$ and let $I_1 = \{i < \omega : \{a_0, b_0\} \not\subseteq C_i\}$. Let $i_1 = \min(I_1)$. Let $a_1 = \min(C_{i_1} \setminus \{a_0, b_0\})$ and let $b_1 = \min(C_{i_1} \setminus \{a_0, b_0, a_1\})$. Let $I_2 = \{i \in I_1 : \{a_1, b_1\} \not\subseteq C_i\}$. For general m, given I_m, let $i_m = \min(I_m)$, let $a_m = \min(C_{i_m} \setminus (\{a_j : j < m\} \cup \{b_j : j < m\}))$ and let $b_m = \min(C_{i_m} \setminus (\{a_j : j \leq m\} \cup \{b_j : j < m\}))$. Let $I_{m+1} = \{i \in I_m : \{a_m, b_m\} \not\subseteq C_i\}$.

Let $A = \{a_m : m < \omega\}$ and $B = \{b_m : m < \omega\}$. Then A and B are infinite and $A \cap B = \emptyset$. Moreover, for each $j < \omega$, there is an m such that $j < i_{m+1}$, so $\{a_{m'}, b_{m'}\} \subseteq C_j$ for some $m' < m+1$. Hence, $A \cap C_j \neq \emptyset$ and $B \cap C_j \neq \emptyset$. Therefore, for each $j < \omega$, $C_j \not\subseteq A$ and $C_j \not\subseteq B$. Note that one of A and $B \cup (\omega \setminus A)$ must be in $\mathcal{U}$. However, neither A nor $B \cup (\omega \setminus A)$ contains C_j for any $j < \omega$. □

Now we exhaust the possible cases regarding $\hat{f}$.

Case 1. For each $X \in \mathcal{Y}^+$, identifying X with its characteristic function, there is a finite initial segment $s \sqsubseteq X$ such that $\hat{f}(s) \in \mathcal{U}$. Let $\mathcal{S}$ be the collection of $s \in 2^{<\omega}$ such that $\hat{f}(s) \in \mathcal{U}$. Then for each $X \in \mathcal{Y}^+$, $f(X)$ is the union of the $\hat{f}(s)$, where $s \in \mathcal{S}$ and $s \sqsubseteq X$. Since there are only countably many $\hat{f}(s)$, $s \in \mathcal{S}$, they cannot generate the ultrafilter $\mathcal{U}$. Hence, for any ultrafilter extension $\mathcal{Z}$ of $\mathcal{Y}$, $f''\mathcal{Z}$ is not cofinal in $\mathcal{U}$.

Case 2. Not Case 1. Then there is an $X_0 \in \mathcal{Y}^+$ such that for each finite initial segment $s \sqsubseteq X_0$, $\hat{f}(s)$ is not in $\mathcal{U}$.

Subcase 2(a). There is an $X_1 \subseteq X_0$ in $\mathcal{Y}^+$ such that for each $Y \in \mathcal{Y}^+$ with $Y \subseteq X_1$, there is a finite initial segment s of Y such that $\hat{f}(s)$ is infinite. Let $\mathcal{S}$ be the collection of finite initial segments s of members $Y \subseteq X_1$ in $\mathcal{Y}^+$ such that $\hat{f}(s)$ is infinite. Then $\{\hat{f}(s) : s \in \mathcal{S}\}$ satisfies the hypotheses of Claim 1. Thus, there is a $U \in \mathcal{U}$ such that for each $s \in \mathcal{S}$, $\hat{f}(s) \not\subseteq U$. Therefore, for any ultrafilter $\mathcal{Z}$ extending $\mathcal{Y} \cup \{X_1\}$, $f''\mathcal{Z}$ is not cofinal in $\mathcal{U}$, since for any $Z \in \mathcal{Z}$, $f(Z) = \bigcup\{\hat{f}(s) : s \sqsubseteq Z\}$.

Subcase 2(b). For each $X_1 \subseteq X_0$ in $\mathcal{Y}^+$, there is some $X_2 \subseteq X_1$ also in $\mathcal{Y}^+$ such that for each finite initial segment $s \sqsubseteq X_2$, $\hat{f}(s)$ is finite. Fix some such X_2. Then note that for each $s \in 2^{<\omega}$ such that $\tilde{s} \in [X_2]^{<\omega}$, $\hat{f}(s)$ is finite. (Recall that $\tilde{s}$ denotes $\{i \in \mathrm{dom}(s) : s(i) = 1\}$.) Let $\mathcal{S}_2$ denote $\{s \in 2^{<\omega} : \tilde{s} \subseteq X_2\}$.

Claim 2. There is a $Y \in \mathcal{Y}^+$ such that $Y \subseteq X_2$ and $f(Y) \notin \mathcal{U}$.

Proof. Since each $\hat{f}(s)$ is finite for $s \in \mathcal{S}_2$, for each k there is an m such that for each $s \in 2^k \cap \mathcal{S}_2$, $\max(\hat{f}(s)) < m$. Let $j_0 = 0$. Given j_i, choose j_{i+1} to be the least $m > j_i$ such that for each $s \in 2^{j_i} \cap \mathcal{S}_2$, $\max(\hat{f}(s)) < m$. Notice that for each $i < \omega$ and each $s \in 2^{j_i} \cap \mathcal{S}_2$, we have that $\max(\hat{f}(s)) < j_{i+1}$.

Let $\mathcal{W}$ be the filter generated by $\mathcal{Y} \cup \{X_2\}$. Then $\mathcal{W}$ has a base of size less than $\mathfrak{u}$ (since $\mathcal{Y}$ does), so $\mathcal{W}$ is not an ultrafilter. Let $H = \bigcup_{i<\omega}[j_{2i}, j_{2i+1})$. Then H and H^c cannot both be in $\mathcal{W}^*$, since $\mathcal{W}^*$ is a proper ideal. Without loss of generality, assume that $H \notin \mathcal{W}^*$. Then $H \in \mathcal{W}^+$. (If H is in $\mathcal{W}^*$, then use H^c and modify the indexes in the following argument.)

Subclaim. There is an infinite, co-infinite set $K \subseteq \omega$ such that both $\bigcup_{i\in K}[j_{2i}, j_{2i+1})$ and $\bigcup_{i\in K^c}[j_{2i}, j_{2i+1})$ are in $\mathcal{W}^+$.

Proof. For each $i < \omega$, let $\bar{j}_i$ denote the interval $[j_{2i}, j_{2i+1})$. Let $\mathcal{K} = \{K \subseteq \omega : \exists W \in \mathcal{W}\, \forall i < \omega\, (W \cap \bar{j}_i \neq \emptyset \rightarrow i \in K)\}$. Note that $\mathcal{K}$ is a filter: By its definition, $\mathcal{K}$ is closed under supersets and contains the Fréchet filter since $\mathcal{W} \supseteq \mathcal{Y}$ contains the Fréchet filter. Also if K and K' are in $\mathcal{K}$ as witnessed by $W, W' \in \mathcal{W}$, respectively, then $W \cap W' \in \mathcal{W}$, and $W \cap W'$ witnesses that $K \cap K' \in \mathcal{K}$.

Let $\mathcal{C}$ be a base of size less than $\mathfrak{u}$ for the filter $\mathcal{W}$. For each $W \in \mathcal{C}$, define $K_W = \{i \in \omega : W \cap \bar{j}_i \neq \emptyset\}$. Let $\mathcal{B} = \{K_W : W \in \mathcal{C}\}$. Note that $\mathcal{B}$ is a base for the filter $\mathcal{K}$. Also, $|\mathcal{B}| \leq |\mathcal{C}| < \mathfrak{u}$, so $\mathcal{K}$ is not an ultrafilter. Thus, we can fix a $K \in \mathcal{K}^+ \setminus \mathcal{K}$. Then also $K^c \in \mathcal{K}^+ \setminus \mathcal{K}$; so K and K^c are both infinite. Define A to be $\bigcup_{i\in K}[j_{2i}, j_{2i+1})$ and B to be $\bigcup_{i\in K^c}[j_{2i}, j_{2i+1})$. Note that both A and B are subsets of H, $A \cap B = \emptyset$, and $A \cup B = H$.

We claim that both A and B are in $\mathcal{W}^+$. Since $K \in \mathcal{K}^+$, it follows that for each $J \in \mathcal{K}$, $|K \cap J| = \omega$. Since $\mathcal{B}$ generates $\mathcal{K}$, we have that for each $W \in \mathcal{C}$, $|K \cap K_W| = \omega$. Therefore, for each $W \in \mathcal{C}$, $\{i \in K : W \cap \bar{j}_i \neq \emptyset\}$ is infinite. Thus, $A \cap W = (\bigcup_{i\in K} \bar{j}_i) \cap W$ is infinite for each $W \in \mathcal{C}$. Hence, $A \cap W$ is infinite for each $W \in \mathcal{W}$. Thus, $A \in \mathcal{W}^+$. Likewise, since K^c is in $\mathcal{K}^+$, we have that $B \in \mathcal{W}^+$. This finishes the proof of the Subclaim. □

We claim that $f(A) \cap f(B) = \emptyset$. We shall prove more: For any $I \subseteq \omega$, $f(\bigcup_{i\in I}[j_{2i}, j_{2i+1})) \subseteq \bigcup_{i\in I}[j_{2i}, j_{2i+2})$. It suffices to prove this for all finite $I \subseteq \omega$ since for any $I \subseteq \omega$, $f(\bigcup_{i\in I}[j_{2i}, j_{2i+1})) = \bigcup_{k<\omega} \hat{f}(\bigcup_{i\in I\cap k}[j_{2i}, j_{2i+1}))$.

$\hat{f}(\emptyset)$ must be the emptyset, (for if not, then f would not map $\mathcal{Y}^+$ cofinally into $\mathcal{U}$). $\hat{f}([j_0, j_1)) \subseteq [j_0, j_2)$, by definition of j_2. Suppose that $k \geq 1$ and given any finite $I \subseteq k$, $\hat{f}(\bigcup_{i\in I}[j_{2i}, j_{2i+1})) \subseteq \bigcup_{i\in I}[j_{2i}, j_{2i+2})$. Let $I' \subseteq k+1$ be given and let I denote $I' \cap k$. By the induction hypothesis, $\hat{f}(\bigcup_{i\in I}[j_{2i}, j_{2i+1})) \subseteq \bigcup_{i\in I}[j_{2i}, j_{2i+2})$. If $I = I'$, we are done. If $I \neq I'$, then $k \in I'$. Recall the fact that $\hat{f}$ has the property that for any $X \subseteq \omega$ and any $l < \omega$, $l \in f(X)$ iff $l \in \hat{f}(X \cap (l+1))$. Hence, by our choice of the j_i, we have that $\hat{f}(\bigcup_{i\in I'}[j_{2i}, j_{2i+1})) \cap j_{2k} = \hat{f}(\bigcup_{i\in I}[j_{2i}, j_{2i+1}))$. Thus, $\hat{f}(\bigcup_{i\in I'}[j_{2i}, j_{2i+1})) \subseteq \bigcup_{i\in I'}[j_{2i}, j_{2i+2})$.

Thus, $f(A) \cap f(B) = \emptyset$. This implies that at least one of them is not in $\mathcal{U}$. Thus, Claim 2 holds. □

Taking a $Y \in \mathcal{Y}^+$ satisfying Claim 2 contradicts the hypothesis that $f''\mathcal{Y}^+ \subseteq \mathcal{U}$. Thus, the Lemma holds. □

Theorem 49. (1) *Assume cov*$(\mathscr{M}) = \mathfrak{c}$. *Then there are* $\mathfrak{c}^+$ *pairwise Tukey incomparable selective ultrafilters.*

(2) *Assume* $\mathfrak{d} = \mathfrak{u} = \mathfrak{c}$. *Then there are* $\mathfrak{c}^+$ *pairwise Tukey incomparable p-points.*

Proof. Proof of (1). Assume $\text{cov}(\mathscr{M}) = \mathfrak{c}$. To show that there are $\mathfrak{c}^+$ Tukey incomparable selective ultrafilters, we shall show that given $\leq \mathfrak{c}$ selective ultrafilters, there is another selective ultrafilter Tukey incomparable with each of them.

Let $\mathcal{U}_\gamma$, $\gamma < \kappa$, where $\kappa \leq \mathfrak{c}$, be a collection of selective ultrafilters. Fix a listing $\langle D_\alpha : \alpha < \mathfrak{c}\rangle$ of all the infinite subsets of ω. Fix a sequence $\langle \vec{P}_\alpha : \alpha < \mathfrak{c}\rangle$ such that each $\vec{P}_\alpha = \langle P^n_\alpha : n < \omega\rangle$ is a partition of ω and each partition of ω appears in the listing. Fix a listing $\langle f_\beta : \beta < \mathfrak{c}\rangle$ of all continuous monotone maps $f : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ which can be represented by some function $\hat{f} : 2^{<\omega} \rightarrow \mathcal{P}(\omega)$. Finally, fix an onto function $\theta : \mathfrak{c} \rightarrow \{\mathcal{U}_\gamma : \gamma < \kappa\} \times \{f_\beta : \beta < \mathfrak{c}\}$.

We will construct filters $\mathcal{Y}_\alpha$, $\alpha < \mathfrak{c}$, satisfying the following:

(1) For $\alpha < \alpha' < \mathfrak{c}$, $\mathcal{Y}_\alpha \subseteq \mathcal{Y}_{\alpha'}$;
(2) $\mathcal{Y}_\alpha$ has a base of cardinality less than $\mathfrak{c}$;
(3) $\mathcal{Y}_{\alpha+1}$ is selective for $\vec{P}_\alpha$;
(4) Either D_α or D_α^c is in $\mathcal{Y}_{\alpha+1}$;
(5) If $\theta(\alpha)$ is the pair $\langle \mathcal{U}_{\gamma_\alpha}, f_{\beta_\alpha}\rangle$, then for each ultrafilter $\mathcal{Z}$ extending $\mathcal{Y}_{\alpha+1}$, $f_{\beta_\alpha} \restriction \mathcal{U}_{\gamma_\alpha}$ does not map $\mathcal{U}_{\gamma_\alpha}$ cofinally into $\mathcal{Z}$, and $f_{\beta_\alpha} \restriction \mathcal{Z}$ does not map $\mathcal{Z}$ cofinally into $\mathcal{U}_{\gamma_\alpha}$.

We now begin the construction. Let $\mathcal{Y}_0$ be the Fréchet filter. Suppose the filter $\mathcal{Y}_\alpha$ has been constructed. The partition of ω under consideration is $\vec{P}_\alpha = \langle P_\alpha^n : n < \omega\rangle$. If there is an $n < \omega$ such that $P_\alpha^n \in \mathcal{Y}_\alpha$, then let $\mathcal{Y}_{\alpha+1}^{(0)} = \mathcal{Y}_\alpha$. Otherwise, for each $n < \omega$, $\bigcup_{j>n} P_\alpha^j \in \mathcal{Y}_\alpha$. Apply Proposition 45 to find an $X \in [\omega]^\omega$ such that $\{X\} \cup \mathcal{Y}_\alpha$ has the finite intersection property, and such that for each $n < \omega$, $|X \cap P_\alpha^n| \le 1$. Then let $\mathcal{Y}_{\alpha+1}^{(0)}$ be the filter generated by $\{X\} \cup \mathcal{Y}_\alpha$. If $D_\alpha \in (\mathcal{Y}_{\alpha+1}^{(0)})^+$, then let $\mathcal{Y}_{\alpha+1}^{(1)}$ be the filter generated by $\{D_\alpha\} \cup \mathcal{Y}_{\alpha+1}^{(0)}$. Otherwise, let $\mathcal{Y}_{\alpha+1}^{(1)}$ be the filter generated by $\{D_\alpha^c\} \cup \mathcal{Y}_{\alpha+1}^{(0)}$.

Next we consider $\theta(\alpha)$, which is a pair $\langle \mathcal{U}_{\gamma_\alpha}, f_{\beta_\alpha}\rangle$ for some $\gamma_\alpha < \kappa$ and $\beta_\alpha < \mathfrak{c}$. If $f''_{\beta_\alpha}\mathcal{U}_{\gamma_\alpha} \subseteq \mathcal{Y}_{\alpha+1}^{(1)}$, then $f_{\beta_\alpha} \restriction \mathcal{U}_{\gamma_\alpha}$ will not be cofinal into any ultrafilter extending $\mathcal{Y}_{\alpha+1}^{(1)}$. In this case, let $\mathcal{Y}_{\alpha+1}^{(2)} = \mathcal{Y}_{\alpha+1}^{(1)}$. If $f''_{\beta_\alpha}\mathcal{U}_{\gamma_\alpha} \not\subseteq \mathcal{Y}_{\alpha+1}^{(1)}$, then take some $U \in \mathcal{U}_{\gamma_\alpha}$ such that $f_{\beta_\alpha}(U) \notin \mathcal{Y}_{\alpha+1}^{(1)}$ and let $\mathcal{Y}_{\alpha+1}^{(2)}$ be the filter generated by $\mathcal{Y}_{\alpha+1}^{(1)} \cup \{f_{\beta_\alpha}(U)^c\}$. Note that $f_{\beta_\alpha} \restriction \mathcal{U}_{\gamma_\alpha}$ cannot be cofinal into any ultrafilter extending $\mathcal{Y}_{\alpha+1}^{(2)}$. By Lemma 48, there is a $Y \in (\mathcal{Y}_{\alpha+1}^{(2)})^+$ such that for any ultrafilter $\mathcal{Z}$ which extends $\mathcal{Y}_{\alpha+1}^{(2)} \cup \{Y\}$, $f_{\beta_\alpha} \restriction \mathcal{Z}$ is not a cofinal map from $\mathcal{Z}$ into $\mathcal{U}_{\gamma_\alpha}$. Let $\mathcal{Y}_{\alpha+1}$ be the filter generated by $\mathcal{Y}_{\alpha+1}^{(2)} \cup \{Y\}$.

For limit ordinals $\lambda < \mathfrak{c}$, let $\mathcal{Y}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{Y}_\alpha$.

Let $\mathcal{Y} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{Y}_\alpha$. Then $\mathcal{Y}$ is a selective ultrafilter, by (1) - (4). Moreover, $\mathcal{Y}$ is Tukey incomparable with each $\mathcal{U}_\gamma$, $\gamma < \kappa$, by (5).

Since for each collection of selective ultrafilters of cardinality less than or equal to $\mathfrak{c}$ we can build another selective ultrafilter which is Tukey inequivalent to each of them, it follows that there are $\mathfrak{c}^+$ Tukey inequivalent selective ultrafilters.

The proof of (2) of the Theorem follows exactly the same steps as for (1) with only the following modification. Before starting the construction, let $\mathcal{U}_\gamma$, $\gamma < \kappa$, where $\kappa \le \mathfrak{c}$, be a collection of p-points. Fix an enumeration $\langle \vec{A}_\alpha : \alpha < \mathfrak{c}\rangle$, where $\vec{A}_\alpha = \langle A_\alpha^n : n < \omega\rangle$, such that for each countable collection $\vec{B} = \langle B_n : n < \omega\rangle$ of infinite subsets of ω, $\vec{B} = \vec{A}_\alpha$ for cofinally many $\alpha < \mathfrak{c}$.

Let $\mathcal{Y}_0$ be the Fréchet filter. Given the filter $\mathcal{Y}_\alpha$, if the collection $\{A_\alpha^n : n < \omega\}$ is not contained in $\mathcal{Y}_\alpha$, then let $\mathcal{Y}_{\alpha+1}^{(0)} = \mathcal{Y}_\alpha$. If $\{A_\alpha^n : n < \omega\}$ is contained in $\mathcal{Y}_\alpha$, apply Proposition 46 to obtain a set B such that $B \subseteq^* A_\alpha^n$ for each $n < \omega$ and such that $\{B\} \cup \mathcal{Y}_\alpha$ has the finite intersection property. In this case, let $\mathcal{Y}_{\alpha+1}^{(0)}$ denote the filter generated by $\{B\} \cup \mathcal{Y}_\alpha$. The rest of the construction of $\mathcal{Y}_{\alpha+1}$ proceeds exactly as in part (1). Letting $\mathcal{Y} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{Y}_\alpha$, we see that $\mathcal{Y}$ is a p-point which is Tukey inequivalent to every p-point $\mathcal{U}_\gamma$, $\gamma < \kappa$. The Theorem then follows as in part (1). □

Theorem 44 follows from Theorem 47 and Theorem 49.

Remark. The stipulation in (1) in Theorem 44 that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ is optimal, at least for this construction. For by results of Fremlin and Canjar, (see Theorem 4.6.6 of [2]), $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ iff every filter with base of cardinality less than $\mathfrak{c}$ can be extended to a selective ultrafilter. The stipulation in (2) of Theorem 44 that $\mathfrak{u} = \mathfrak{d} = \mathfrak{c}$ is perhaps not optimal, since p-points exist just under the assumption that $\mathfrak{d} = \mathfrak{c}$.

It remains open whether, just assuming $\mathfrak{d} = \mathfrak{c}$, there are 2^κ Tukey incomparable ultrafilters for any κ such that $\mathrm{cf}(\kappa) = \mathrm{cf}(\mathfrak{c})$ and $2^{<\kappa} = \mathfrak{c}$.

One way of making Tukey increasing chains of ultrafilters is by using κ-OK points. We give the following definition straight from [19].

Definition 50 (Kunen [19])**.** Let X be a topological space and κ any cardinal. If $p \in X$ and U_n $(n < \omega)$ are neighborhoods of p, a *κ-refinement system* for $\langle U_n : n < \omega\rangle$ is a κ-sequence of neighborhoods of p, $\langle V_\alpha : \alpha < \kappa\rangle$ such that for all $n \geq 1$,

$$\forall \alpha_1 < \alpha_2 < \cdots < \alpha_n < \kappa \ \ (V_{\alpha_1} \cap \cdots \cap V_{\alpha_n} \subseteq U_n).$$

A point $p \in X$ is *κ-OK* iff whenever U_n $(n < \omega)$ are neighborhoods of p, $\langle U_n : n < \omega\rangle$ has a κ-refinement system.

Translating this into the context of ultrafilters, we let X be the Čech-Stone remainder $\beta\omega\setminus\omega$, the collection of all non-principle ultrafilters on ω. A non-principle ultrafilter $\mathcal{U}$ is κ-OK iff whenever $U_n \in \mathcal{U}$ $(n < \omega)$, there is a κ-sequence $\langle V_\alpha : \alpha < \kappa\rangle$ of elements of $\mathcal{U}$ such that for all $n \geq 1$, for all $\alpha_1 < \cdots < \alpha_n < \kappa$, $V_{\alpha_1} \cap \cdots \cap V_{\alpha_n} \subseteq^* U_n$.

Kunen remarked in [19] that if $\mathcal{U}$ is κ-OK and $\kappa > \mathrm{cof}(\mathcal{U})$, then $\mathcal{U}$ is a p-point. It is easy to see the following.

Proposition 51. *If $\mathcal{U}$ is κ-OK but not a p-point, then $\mathcal{U} \geq_T [\kappa]^{<\omega}$. Hence, if $\mathcal{U}$ is κ-OK but not a p-point, then cof($\mathcal{U}$) $= \kappa$ iff $\mathcal{U} \equiv_T [\kappa]^{<\omega}$.*

Proof. Let $\mathcal{U}$ be κ-OK but not a p-point. Then there are $X_n \in \mathcal{U}$ such that for each $X \in \mathcal{U}$, there is an $n < \omega$ such that $X \not\subseteq^* X_n$. Let $\{C_\alpha : \alpha \in [\kappa]^{<\omega}\} \subseteq \mathcal{U}$ witness that $\mathcal{U}$ is κ-OK for $\langle X_n\rangle_{n<\omega}$. Let $g : [\kappa]^{<\omega} \to \mathcal{U}$ by $g(\alpha) = C_\alpha$ for each $\alpha \in [\kappa]^{<\omega}$. If $\mathcal{X} \subseteq [\kappa]^{<\omega}$ is unbounded, then $\mathcal{X}$ is infinite. Hence, $g''\mathcal{X}$ is infinite, since g is 1-1. Take $\{C_{\alpha_n} : n < \omega\}$ to be any infinite subset of $g''\mathcal{X}$. Suppose $\{C_{\alpha_n} : n < \omega\}$ is $\supseteq^*$ bounded below by $Y \in \mathcal{U}$. Then for each k, $Y \subseteq^* \bigcap_{n\leq k} C_{\alpha_n} \subseteq^* X_k$. But then for each n, $Y \subseteq^* X_n$, contradicting our choice of $\{X_n : n < \omega\}$. Thus, $g : [\kappa]^{<\omega} \to (\mathcal{U}, \supseteq^*)$ is a Tukey map. Therefore, $[\kappa]^{<\omega} \leq_T (\mathcal{U}, \supseteq^*) \leq_T (\mathcal{U}, \supseteq)$.

If $\mathrm{cof}(\mathcal{U}) \neq \kappa$, then $(\mathcal{U}, \supseteq) \not\equiv_T [\kappa]^{<\omega}$; hence, $(\mathcal{U}, \supseteq) >_T [\kappa]^{<\omega}$. If $\mathrm{cof}(\mathcal{U}) = \kappa$, then $\mathcal{U} \leq_T [\kappa]^{<\omega}$. □

It follows that if there are κ-OK non p-points with cofinality κ for each uncountable $\kappa < \mathfrak{c}$, then there is a strictly increasing chain of ultrafilters of length α, where α is such that $\aleph_\alpha = \mathfrak{c}$. We would like to point out that Milovich showed in [22] that and ultrafilter $\mathcal{U}$ is $\mathfrak{c}$-OK and not of top degree iff $\mathcal{U}$ is a p-point.

We now give a general method for building Tukey increasing chains of p-points.

Theorem 52. *Assuming CH, for each p-point D there is a p-point E such that $E >_{RK} D$ and moreover, $E >_T D$.*

Proof. We use the notation from [3]. In [Theorem 6, [3]], Blass proved assuming MA that given a p-point D one can construct a p-point $E >_{RK} D$. Hence, $E \geq_T D$. His construction can be slightly modified to kill all possible cofinal maps from D into E so that we construct a p-point E which is both Rudin-Keisler and Tukey strictly above D.

Let D be a given p-point. Fix a bijective pairing $J : \omega \times \omega \to \omega$ with inverse (π_1, π_2), and identify ω with $\omega \times \omega$ via J. A subset $Y \subseteq \omega \times \omega$ is called *small* iff the function $c_Y(i) := |\{y \in \omega : (i, y) \in Y\}|$ is bounded by some $n < \omega$ for all i in some $X \in D$. Otherwise Y is called *large*. It is useful to note that from [Lemma 1, p152, [3]], it follows that $\omega \times \omega$ is large, the union of any two small sets is small, the complement of a small set is large, and any superset of a large set is large. We give the following characterization of large sets.

Claim 1. Let $Y \subseteq \omega \times \omega$. Y is large iff there is a $W \in D$ such that $c_Y \restriction W$ is bounded below by a non-decreasing, unbounded function on W.

Proof. First note that for any $Y \subseteq \omega \times \omega$, Y is large iff for each $n < \omega$, $\{i < \omega : c_Y(i) \le n\} \notin D$ iff for each $n < \omega$, $\{i < \omega : c_Y(i) > n\} \in D$. Let $Y \subseteq \omega \times \omega$ be large. For each $n < \omega$, define $W_n = \{i < \omega : c_Y(i) > n\}$. Then each $W_n \in D$ and $W_n \supseteq W_{n+1}$. Since D is a p-point, there is a $W \in D$ such that for each $n < \omega$, $W \subseteq^* W_n$. Let k_n be a strictly increasing sequence such that for each $n < \omega$, $W \setminus k_n \subseteq W_n$. Note that for each $i \in W \setminus k_n$, $c_Y(i) > n$. Therefore, for each $n < \omega$, for each $i \in W \cap (k_n, k_{n+1}]$, $c_Y(i) > n$. Hence, c_Y is bounded below on W by the function $g : W \to \omega$, where for each n, for each $i \in W \cap (k_n, k_{n+1}]$, $g(i) = n$.

For the reverse direction, if $Y \subseteq \omega \times \omega$, $W \in D$ and $c_Y \restriction W$ is bounded below by a non-decreasing unbounded function, then for each $n < \omega$, $\{i \in W : c_Y(i) \le n\}$ is finite, hence $\{i < \omega : c_Y(i) \le n\} \notin D$. Therefore, Y is large. □

For the sake of readability, we repeat an argument of Blass [pp 151-152, [3]] in this paragraph. We are going to construct a p-point E on $\omega \times \omega$ such that $\pi_1(E) = D$. To ensure that $E \not\equiv_{RK} D$, it will suffice that π_1 is not one-to-one on any set of E. This means that E must contain the complement of the graph of every function from ω to ω. Hence, E must also contain the complement of every finite union of such graphs. If Y is the graph of a function, then Y is small, for c_Y is bounded by 1 on all of ω. Also, if $A \in D$ and $Y = (\omega \times \omega) - \pi_1^{-1}(A)$, then Y is small, for c_Y is bounded by 0 on A. Therefore, if E is an ultrafilter on $\omega \times \omega$ containing no small set, then $E >_{RK} D$.

We now construct an ultrafilter E in ω_1 stages. Let $\langle f_\alpha : \alpha < \omega_1\rangle$ enumerate all functions from $\omega \times \omega$ into ω, and let $\langle h_\alpha : \alpha < \omega_1\rangle$ enumerate all continuous monotone maps from $\mathcal{P}(\omega)$ into $\mathcal{P}(\omega)$. We build filter bases $\mathcal{Y}_\alpha$, $\alpha < \omega_1$, with the following properties.

(1) Every set in $\mathcal{Y}_\alpha$ is large.
(2) If $\beta < \alpha < \omega_1$, then $\mathcal{Y}_\beta \subseteq \mathcal{Y}_\alpha$.
(3) $\mathcal{Y}_\alpha$ is countable.
(4) f_α is finite-to-one or bounded on some set of $\mathcal{Y}_{\alpha+1}$.
(5) $h_\alpha \restriction D$ is not a cofinal map from D into any ultrafilter extending $\mathcal{Y}_{\alpha+1}$.

Let $\mathcal{Y}_0 = \{\omega \times \omega\}$. If $\alpha < \omega_1$ is a limit ordinal and $\mathcal{Y}_\beta$ has been constructed for all $\beta < \alpha$, then let $\mathcal{Y}_\alpha = \bigcup_{\beta<\alpha} \mathcal{Y}_\alpha$.

If $\mathcal{Y}_\alpha$ is given, do the following. By [Lemma 3, p 153, [3]], there is a set $T \subseteq \omega\times\omega$ on which f_α is finite-to-one or bounded, and such that $T \cap Y$ is large for each $Y \in \mathcal{Y}_\alpha$. Let $\mathcal{Y}'_\alpha$ be the filter base obtained by adjoining T to $\mathcal{Y}_\alpha$ and closing under finite intersections.

Next, consider the continuous monotone map h_α. If h_α"D does not generate an ultrafilter, there is nothing to do; let $\mathcal{Y}_{\alpha+1} = \mathcal{Y}'_\alpha$. Suppose now that h_α"D generates an ultrafilter.

Claim 2. There is a set Z such that $Z \cap Y$ and $Z^c \cap Y$ are large for each $Y \in \mathcal{Y}'_\alpha$.

Proof. By the inductive construction, $\mathcal{Y}'_\alpha$ is countable and every element of $\mathcal{Y}'_\alpha$ is large. Let X_n $(n < \omega)$ be a base for $\mathcal{Y}'_\alpha$ such that each $X_n \supseteq X_{n+1}$. Since each X_n is large, by Claim 1, there is a $W_n \in D$ and a non-decreasing unbounded function $g_n : W_n \to \omega$ such that for each $i \in W_n$, $c_{X_n}(i) \ge g_n(i)$. Without loss of generality, we can assume that each $W_n \supseteq W_{n+1}$. Since D is a p-point, let $W \in D$ satisfy for each $n < \omega$, $W \subseteq^* W_n$.

We shall build disjoint $Z_0, Z_1 \subseteq \omega \times \omega$ and a strictly increasing sequence $\langle k_n : n < \omega\rangle$ as follows. Let k_0 be least such that for each $i \in [k_0, \omega) \cap W_0$, $g_0(i) \ge 2$ and $W \setminus k_0 \subseteq W_0$. In general, choose $k_m > k_{m-1}$ satisfying

(1) for each $j \le m$ and each $i \in [k_m, \omega) \cap W_j$, $g_j(i) \ge 2(m + 1)^2$;

(2) $W \setminus k_m \subseteq W_m$ (and hence for each $j < m$, $W \setminus k_m \subseteq W_j$).

Given $m < \omega$ and $i \in W \cap [k_m, k_{m+1})$, for each $j \leq m$, choose $x_{i,j,l}, y_{i,j,l}$, $l \leq m$, distinct in $\{z \in \omega : (i, z) \in X_j\} \setminus \{x_{i,q,l}, y_{i,q,l} : l \leq m,\ q < j\}$. (This is possible since for each $i \in W \cap [k_m, k_{m+1})$, for each $j \leq m$, $c_{X_j}(i) \geq g_j(i) \geq 2(m+1)^2$.) For each $i \in W$, define m_i to be the integer m for which $i \in [k_m, k_{m+1})$. Define $Z_0 = \{(i, x_{i,j,l}) : i \in W,\ j \leq m_i,\ l \leq m_i\}$; $Z_1 = \{(i, y_{i,j,l}) : i \in W,\ j \leq m_i,\ l \leq m_i\}$. Note that Z_0, Z_1 are large, disjoint, and have large intersection with each X_n. Letting $Z = Z_0$, then both Z and Z^c have the desired properties. □

Take Z as in Claim 2. If $Z \in h_\alpha$"D, let $\mathcal{Y}_{\alpha+1}$ be the filter base obtained by closing $\mathcal{Y}'_\alpha \cup \{Z^c\}$ under finite intersections; and if $Z^c \in h_\alpha$"D then let $\mathcal{Y}_{\alpha+1}$ be the filter base obtained by closing $\mathcal{Y}'_\alpha \cup \{Z\}$ under finite intersections. Then $h_\alpha \restriction D$ cannot be a cofinal map from D into any ultrafilter extending $\mathcal{Y}_{\alpha+1}$.

As in the final argument of [Theorem 6, [3]], let $\mathcal{Y} = \bigcup_{\alpha<\omega_1} \mathcal{Y}_\alpha$, and let $\mathcal{B}$ be the filter of all sets whose complements are small. Every set of $\mathcal{Y}$, being large, has infinite intersection with every set of $\mathcal{B}$, so there is an ultrafilter E extending $\mathcal{Y} \cup \mathcal{B}$. Then $E >_{RK} D$, and E is a p-point since requirement (4) is met for all $\alpha < \omega_1$. Moreover, $E >_T D$, since for every continuous monotone map $h : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, $h \restriction D$ is not a cofinal map from D into E. □

Remark. Dilip Raghavan has independently observed Theorem 52.

Remark. If one is only interested in building an ultrafilter E Tukey strictly above D, then one does not have to use large sets in the previous construction, but one only needs to ensure that E is a p-point and that all continuous monotone maps are prevented from being cofinal maps from D into E. In the above proof, we used large sets to ensure that E also be Rudin-Keisler strictly above D in order to obtain the following Corollary.

Corollary 53. *Assuming CH, there is a Tukey strictly increasing chain of p-points of order type* $\mathfrak{c}$.

Proof. In [Theorem 7, [3]], Blass proved that MA implies that any RK increasing chain of p-points of length ω has an RK upper bound which is a p-point. The p-point E constructed in the above Theorem 52 is also RK strictly above D, so for any $\alpha < \omega_1$, we can construct ω-length chains of p-points $D_{\alpha+n}$, where each $D_{\alpha+n+1} >_T D_{\alpha+n}$ and $D_{\alpha+n+1} >_{RK} D_{\alpha+n}$ $(\alpha < \omega_1)$ and then use [Theorem 7, [3]] to find a p-point RK above each $D_{\alpha+n}$, $n < \omega$, hence also Tukey above them. □

The following questions are to be answered assuming that p-points exist or some assumption that guarantees their existence.

Question 54. Is there a Tukey strictly increasing chain of p-points of length $\mathfrak{c}^+$?

The Tukey increasing chain of p-points constructed in the proof of Theorem 52 is also Rudin-Keisler increasing. This leads to the next question.

Question 55. Given any strictly Tukey increasing sequence of p-points of length ω, is there always a p-point Tukey above all of them?

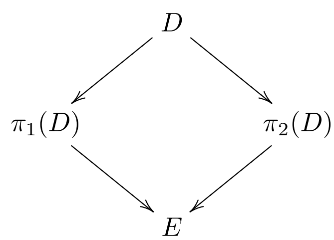
In particular,

Question 56. Given any p-point $\mathcal{V}$, is there a p-point $\mathcal{U}$ such that $\mathcal{U} >_T \mathcal{V}$, but $\mathcal{U}$ and $\mathcal{V}$ are RK-incomparable?

If the answer to Question 56 is no, then the answer to Question 55 is yes.

We now show that, assuming Martin's Axiom, there are incomparable p-points with a common upper bound and a common lower bound which are also p-points.

Theorem 57. *Assume Martin's Axiom. There is a p-point D with two Tukey-incomparable Tukey predecessors $\pi_1(D)$ and $\pi_2(D)$ which are also p-points, which in turn have a common Tukey lower bound E which is also a p-point. (In the following diagram, arrows represent strict Tukey reducibility.)*



Proof. In [Theorem 9, [3]], Blass proved that assuming Martin's Axiom, there is a p-point with two RK-incomparable predecessors. He used the following notions which we shall also use. A subset of $\omega \times \omega$ of the form $P \times Q$, where P and Q are subsets of ω of cardinality $n < \omega$, is called an *n-square*. A subset of $\omega \times \omega$ is called *large* if it includes an n-square for every n, and *small* otherwise. Blass' construction builds a p-point $D \subseteq \omega \times \omega$ consisting of large sets such that $\pi_1(D)$ and $\pi_2(D)$ are RK-incomparable. For $i = 1, 2$, $\pi_i(D) \le_{RK} D$, hence $\pi_i(D)$ are also p-points and are $\le_T D$. The fact that every member of D is large ensures that $\pi_1(D)$ and $\pi_2(D)$ are non-principal.

The following Lemma will be useful for constructing the desired D.

Lemma 58. *Given $\mathcal{Y}$ a filter base on $\omega \times \omega$ of size $< \mathfrak{c}$ and a monotone function $h : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, there is a large set U such that $U \subseteq^* Y$ for each $Y \in \mathcal{Y}$, and for any ultrafilter $D' \supseteq \mathcal{Y} \cup \{U\}$ consisting only of large sets, $h \restriction \pi_1(D')$ is not a cofinal map from $\pi_i(D') \to \pi_j(D')$, for $i \ne j$.*

Proof. By [Lemma 2, Section 6, [3]] (which uses MA), there is a large set X such that $X \subseteq^* Y$ for each $Y \in \mathcal{Y}$. Since X is large, we can choose $L_k \subseteq X$, $k < \omega$, such that L_k is a $(2k)$-square and $\langle \pi_1(L_k) : k < \omega \rangle$, $\langle \pi_2(L_k) : k < \omega \rangle$ form block sequences; that is, for each $k < \omega$ and $i = 1, 2$, each element in $\pi_i(L_k)$ is less than each element in $\pi_i(L_{k+1})$. Let $I = \bigcup_{k<\omega} \pi_1(L_k)$ and $J = \bigcup_{k<\omega} \pi_2(L_k)$.

Case 1. There is an infinite $I' \subseteq I$ such that letting $J' = (\omega \setminus h(I')) \cap J$ and $m_k = \min\{|I' \cap \pi_1(L_k)|, |J' \cap \pi_2(L_k)|\}$, the sequence $\langle m_k : k < \omega \rangle$ is unbounded. Then there is a strictly increasing subsequence $\langle m_{k_n} : n < \omega \rangle$. Let $W = \bigcup_{n<\omega} (I' \cap \pi_1(L_{k_n})) \times (J' \cap \pi_2(L_{k_n}))$. Then $W \subseteq X$ and W is large. Note that if D' is any ultrafilter extending $\mathcal{Y} \cup \{W\}$, then $I' = \pi_1(W)$ is in $\pi_1(D')$ and $h(I')$ is disjoint from $J' = \pi_2(W)$ which is in $\pi_2(D')$. Therefore, $f(I') \notin \pi_2(D')$.

Case 2. Not Case 1. Then for each infinite $I' \subseteq I$, letting $J' = (\omega \setminus h(I')) \cap J$, there is an $m < \omega$ such that $\min\{|I' \cap \pi_1(L_k)|, |J' \cap \pi_2(L_k)|\} \le m$ for each $k < \omega$. Let $W' = \bigcup_{k<\omega} L_k$. Then $W' \subseteq X$ and W' is large.

Claim. For any $I' \subseteq I$ such that $I' = \pi_1(V')$ for some large $V' \subseteq W'$, there is a strictly increasing sequence $\langle k_n : n < \omega \rangle$ and an $m < \omega$ such that for each n, $|h(I') \cap \pi_2(L_{k_n})| \ge 2k_n - m$.

Proof. Let $I' \subseteq I$ be such that $I' = \pi_1(V')$ for some large $V' \subseteq W'$, and let $J' = (\omega \setminus h(I')) \cap J$. Since we are in Case 2, there is an $m < \omega$ satisfying $\min\{|I' \cap \pi_1(L_k)|, |J' \cap \pi_2(L_k)|\} \le m$ for each $k < \omega$. Since V' is large and $V' \subseteq W'$, there is a subsequence $\langle k_n : n < \omega \rangle$ such that $\langle |I' \cap \pi_1(L_{k_n})| : n < \omega \rangle$ is a strictly increasing sequence of numbers greater than m. Then for each $n < \omega$, it must be the case that $|J' \cap \pi_2(L_{k_n})| \le m$. Note that for each n, $(\omega \setminus h(I')) \cap \pi_2(L_{k_n}) = (\omega \setminus h(I')) \cap J \cap \pi_2(L_{k_n}) = J' \cap \pi_2(L_{k_n})$, since $\pi_2(L_{k_n}) = J \cap \pi_2(L_{k_n})$. Thus, for each n, $|(\omega \setminus h(I')) \cap \pi_2(L_{k_n})| = |J' \cap \pi_2(L_{k_n})| \le m$. Since $|\pi_2(L_{k_n})| = 2k_n$, it follows that $|h(I') \cap \pi_2(L_{k_n})| \ge 2k_n - m$. □

Divide each $\pi_2(L_k)$ into two disjoint sets each of size k, labeling one of them M_k. Let $J^* = \bigcup_{k<\omega} M_k$. Let $W = W' \cap (\omega \times J^*)$. Then $W \subseteq X$ and W is large. Let D' be any ultrafilter extending $\mathcal{Y} \cup \{W\}$ consisting only of large sets. Since $W \in D'$, we have that $J^* \in \pi_2(D')$. We claim that for all $I' \in \pi_1(D')$, $h(I') \not\subseteq J^*$.

Let I' be any member of $\pi_1(D')$. Then there is a $V'' \in D'$ such that $V'' \subseteq W$ and $I'' := \pi_1(V'') \subseteq I'$. By the Claim, there is a strictly increasing sequence $\langle k_n : n < \omega\rangle$ and an m such that for each n, $|h(I'') \cap \pi_2(L_{k_n})| \geq 2k_n - m$. However, for each n, $|J^* \cap \pi_2(L_{k_n})| = |M_{k_n}| = k_n$, which is less than $2k_n - m$ for all large enough n. Thus, $h(I'') \not\subseteq J^*$. Since h is monotone, $h(I')$ also cannot be contained in J^*. Thus, $h \restriction \pi_1(D')$ is not a cofinal map from $\pi_1(D')$ into $\pi_2(D')$. This ends Case 2.

Thus, in both Cases 1 and 2, we have found a large W such that $W \subseteq^* Y$ for all $Y \in \mathcal{Y}$ and such that for any ultrafilter D' extending $\mathcal{Y} \cup \{W\}$ consisting only of large sets, $h \restriction \pi_1(D')$ is not a cofinal map from $\pi_1(D')$ into $\pi_2(D')$. Now repeat the entire above argument starting with W in place of X and reversing the roles of π_1 and π_2 to obtain a large $U \subseteq W$ such that for any ultrafilter $D' \supseteq \mathcal{Y} \cup \{U\}$ consisting only of large sets, $h \restriction \pi_2(D')$ is not a cofinal map from $\pi_2(D')$ into $\pi_1(D')$. This finishes the proof of the Lemma. □

Now we construct the desired p-point D on $\omega \times \omega$. Enumerate $\mathcal{P}(\omega \times \omega)$ as A_α, $\alpha < \mathfrak{c}$, and enumerate all continuous monotone maps from $\mathcal{P}(\omega)$ into $\mathcal{P}(\omega)$ as h_α, $\alpha < \mathfrak{c}$. We construct filter bases $\mathcal{Y}_\alpha$, $\alpha < \mathfrak{c}$, which satisfy the following.

(1) $\mathcal{Y}_\alpha$ is a filter base of size less than $\mathfrak{c}$.
(2) Every set in $\mathcal{Y}_\alpha$ is large.
(3) If $\beta < \alpha < \mathfrak{c}$, then $\mathcal{Y}_\beta \subseteq \mathcal{Y}_\alpha$.
(4) Either A_α or $\omega \times \omega \setminus A_\alpha$ is in $\mathcal{Y}_{\alpha+1}$.
(5) There is a $U \in \mathcal{Y}_{\alpha+1}$ such that $U \subseteq^* Y$ for each $Y \in \mathcal{Y}_\alpha$.
(6) For any ultrafilter D' extending $\mathcal{Y}_{\alpha+1}$ consisting only of large sets, $f_\alpha \restriction \pi_1(D')$ is not a cofinal map from $\pi_1(D')$ into $\pi_2(D')$, and $f_\alpha \restriction \pi_2(D')$ is not a cofinal map from $\pi_2(D')$ into $\pi_1(D')$.

Let $\mathcal{Y}_0 = \{\omega \times \omega\}$. If α is a limit ordinal and $\mathcal{Y}_\beta$ has been defined for all $\beta < \alpha$, then let $\mathcal{Y}_\alpha = \bigcup_{\beta<\alpha} \mathcal{Y}_\beta$.

In the case that $\mathcal{Y}_\alpha$ has been constructed, construct $\mathcal{Y}_{\alpha+1}$ as follows. By [Lemma 2, p 162, [3]], there is a large T such that $T \subseteq^* Y$ for each $Y \in \mathcal{Y}_\alpha$. If $A_\alpha \cap T$ is large, then let $\mathcal{Y}'_\alpha = \mathcal{Y}_\alpha \cup \{A_\alpha \cap T\}$. Otherwise, $A_\alpha \cap T$ is small. Since T is large, then $T \setminus A_\alpha$ is large, by [Lemma 1, p 162, [3]]; so let $\mathcal{Y}'_\alpha = \mathcal{Y}_\alpha \cup \{T \setminus A_\alpha\}$.

Next use Lemma 58 for $\mathcal{Y}'_\alpha$ and h_α to obtain a large U_α such that such that $U_\alpha \subseteq^* Y$ for each $Y \in \mathcal{Y}'_\alpha$, and for any ultrafilter $D' \supseteq \mathcal{Y}'_\alpha \cup \{U_\alpha\}$ consisting only of large sets, $h_\alpha \restriction \pi_i(D')$ is not a cofinal map from $\pi_i(D')$ into $\pi_j(D')$, for $i \leq 1$ and $j = 1 - i$. Let $\mathcal{Y}_{\alpha+1} = \mathcal{Y}'_\alpha \cup \{U_\alpha\}$.

Let $\mathcal{D} = \bigcup_{\alpha<\mathfrak{c}} \mathcal{Y}_\alpha$. By (2), $\pi_1(D)$ and $\pi_2(D)$ are non-principal; by (4), D is an ultrafilter; by (5), D is a p-point; and by (6), $\pi_1(D)$ and $\pi_2(D)$ are Tukey-incomparable. Since $\pi_1(D)$ and $\pi_2(D)$ are Rudin-Keisler below D, they are also p-points. Moreover, since the p-point D is Rudin-Keisler above both $\pi_1(D)$ and $\pi_2(D)$, it follows from [Theorem 5, [3]] that there is a p-point which is Rudin-Keisler (hence Tukey) below both $\pi_1(D)$ and $\pi_2(D)$. Thus, assuming MA, the diamond lattice embeds into the Tukey degrees of p-points. □

[Theorem 5, [3]] states that if countably many p-points have an RK upper bound which is a p-point, then they have an RK lower bound (which is necessarily a p-point).

Question 59. If countably many p-points have a Tukey upper bound which is a p-point, do they necessarily have a Tukey lower bound which is a p-point?

Question 60. Does every Tukey strictly decreasing sequence of p-points have a Tukey lower bound which is a p-point?

Remark. Laflamme showed in [20] that in the NCF model of [5], the RK ordering of p-points is upwards directed, and hence also downwards directed. Thus, in the NCF model, the Tukey degrees of p-points are both upwards and downwards directed. (We know by Theorem 16 that the class of basically generated ultrafilters with bases closed under finite intersections is upwards directed.) Recall that the cardinal inequality $\mathfrak{u} < \mathfrak{g}$ implies NCF (see [6]), so it is natural to ask the following.

Question 61. Does $\mathfrak{u} < \mathfrak{g}$ imply there is a minimal Tukey degree in the class of p-points?

6. Block-basic ultrafilters on FIN

In this section we study the Tukey ordering between idempotent ultrafilters $\mathcal{U}$ on the index set FIN and their Rudin-Keisler predecessors $\mathcal{U}_{\min,\max}$, $\mathcal{U}_{\min}$, and $\mathcal{U}_{\max}$. We begin by giving the relevant definitions for this investigation.

The following definitions may all be found in [1]. We let FIN denote the collection of nonempty finite subsets of ω. Note that FIN is countable and can serve as a base set for ultrafilters. Because of the natural structure on FIN, which we shall give shortly, the ultrafilters on FIN may have some extra structure which can be utilized in the study of their Tukey types. The set FIN carries the semigroup operation $\cup$, where for $x, y \in \mathrm{FIN}$ such that $\max(x) < \min(y)$, $x \cup y$ is defined to be $\{i \in \omega : i \in x$ or $i \in y\}$, the usual union. (If $\max(x) \not< \max(y)$, then $x \cup y$ is undefined.) This operation naturally extends to a semigroup operation on the collection $\beta\,\mathrm{FIN}$ of ultrafilters on FIN, that is, the Čech-Stone compactification of FIN, as follows. For $\mathcal{U}$ and $\mathcal{V}$ ultrafilters on FIN, $\mathcal{U} \cup \mathcal{V}$ is defined to be the collection of all $A \subseteq \mathrm{FIN}$ such that $\{x \in \mathrm{FIN} : \{y \in \mathrm{FIN} : x \cup y \in A\} \in \mathcal{U}\} \in \mathcal{V}$. An *idempotent ultrafilter* on the semigroup $(\mathrm{FIN}, \cup)$ is an ultrafilter $\mathcal{U}$ on FIN such that $\mathcal{U} \cup \mathcal{U} = \mathcal{U}$. The existence of idempotent ultrafilters on FIN was established by S. Glazer (see [7]).

At this point, we define some standard maps. The map $\min : \mathrm{FIN} \to \omega$ is given by $\min(x)$ is the least element of x, for any $x \in \mathrm{FIN}$. Likewise, $\max : \mathrm{FIN} \to \omega$ is defined by letting $\max(x)$ be the largest element of x. The map $(\min, \max) : \mathrm{FIN} \to \omega \times \omega$ is defined by $(\min, \max)(x) = (\min(x), \max(x))$. Note that whenever $\mathcal{U}$ is an ultrafilter on FIN, then the following are ultrafilters: $\mathcal{U}_{\min}$ is the ultrafilter on ω generated by the collection of sets $\{\min(x) : x \in U\}$, $U \in \mathcal{U}$. $\mathcal{U}_{\max}$ is the ultrafilter on ω generated by the collection of sets $\{\max(x) : x \in U\}$, $U \in \mathcal{U}$. $\mathcal{U}_{\min,\max}$ is the ultrafilter on $\omega \times \omega$ generated by the collection of sets $\{(\min(x), \max(x)) : x \in U\}$, $U \in \mathcal{U}$. Note that these are all ultrafilters, since they are images of $\mathcal{U}$ under the Rudin-Keisler maps min, max, and (min, max), respectively. Thus, it also follows that $\mathcal{U} \geq_{RK} \mathcal{U}_{\min,\max}$, $\mathcal{U}_{\min,\max} \geq_{RK} \mathcal{U}_{\min}$, and $\mathcal{U}_{\min,\max} \geq_{RK} \mathcal{U}_{\max}$. Thus, the same Tukey reductions between these ultrafilters hold.

In [4], Blass showed that Glazer's proof easily adapts to show the following.

Theorem 62 (Blass, Theorem 2.1, [4])**.** *Let $\mathcal{V}_0$ and $\mathcal{V}_1$ be a pair of nonprincipal ultrafilters on ω. Then there is an idempotent ultrafilter $\mathcal{U}$ on* FIN *such that $\mathcal{U}_{\min} = \mathcal{V}_0$ and $\mathcal{U}_{\max} = \mathcal{V}_1$.*

Corollary 63. *There exist idempotent ultrafilters on* FIN *realizing the maximal Tukey type $\mathcal{U}_{\mathrm{top}}$.*

Proof. Let $\mathcal{V}_0 = \mathcal{V}_1$ be a nonprincipal ultrafilter on ω such that $\mathcal{V}_0 \equiv_T [\mathfrak{c}]^{<\omega}$. Then by Theorem 62, $\mathcal{U}_{\min} = \mathcal{U}_{\max} = \mathcal{V}_0$. Since $\mathcal{U} \geq_{RK} \mathcal{U}_{\min}$, we have that $\mathcal{U} \geq_T \mathcal{V}_0$, which implies that $\mathcal{U}$ has the top Tukey type. □

Thus, one is naturally led to consider the conditions on idempotent ultrafilters $\mathcal{U}$ on FIN that would prevent $\mathcal{U}$ from having the maximal Tukey type.

Definition 64. A *block-sequence* of FIN is an infinite sequence $X = (x_n)_{n<\omega}$ of elements of FIN such that for each $n < \omega$, $\max(x_n) < \min(x_{n+1})$. For a block-sequence X, we let $[X]$ denote $\{x_{n_1} \cup \cdots \cup x_{n_k} : k < \omega \text{ and } n_1 < \cdots < n_k\}$, the set of finite unions of elements of X. For any $m < \omega$, let X/m denote $(x_n)_{n\geq k}$ where k is least such that $\min(x_k) \geq m$.

The collection of block-sequences carry the following partial ordering $\leq$. For two infinite block-sequences $X = (x_n)_{n<\omega}$ and $Y = (y_n)_{n<\omega}$, define $Y \leq X$ iff each member of Y is a finite union of elements of X; i.e. $y_n \in [X]$ for each n. We write $Y \leq^* X$ to mean that $Y/m \leq X$ for some $m < \omega$. That is, $Y \leq^* X$ iff there is some k such that for all $n \geq k$, $y_n \in [X]$.

An idempotent ultrafilter $\mathcal{U}$ on FIN is called *block-generated* if it is generated by sets of the form $[X]$ where X is an infinite block-sequence. (Block-generated ultrafilters are called *ordered-union ultrafilters* in [4].)

We now state some relevant information about block-generated ultrafilters, much of which was proved by Blass in [4].

Fact 65. *Let $\mathcal{U}$ be any nonprincipal block-generated ultrafilter on* FIN.

1. *(Proposition 3.3* [4]*) $\mathcal{U}$ is idempotent.*
2. *(Corollary 3.6* [4]*) $\mathcal{U}$ is not a p-point.*
3. *$\mathcal{U}$ is not a q-point.*
4. *(Corollary 3.7* [4]*) $\mathcal{U}_{\min,\max}$ is isomorphic (i.e. Rudin-Keisler equivalent) to $\mathcal{U}_{\min} \cdot \mathcal{U}_{\max}$.*
5. *(Proposition 3.9* [4]*) $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ are q-points.*
6. *$\mathcal{U}_{\min,\max}$ is neither a p-point nor a q-point.*
7. *If $\mathcal{U}_{\min}$ is selective, then $\mathcal{U}_{\min,\max}$ is rapid.*

Proof. (3) We provide a proof of (3) since it does not seem to yet be in the literature, though most likely it has been noticed before. Let $\mathcal{U}$ be a nonprincipal block-generated ultrafilter on FIN and let $P_n = \{x \in \text{FIN} : \max(x) = n\}$. Then $(P_n)_{n<\omega}$ forms a partition of FIN into finite sets. Let U be any element of $\mathcal{U}$. Since $\mathcal{U}$ is block-generated, there is some block sequence X such that $[X] \in \mathcal{U}$ and $[X] \subseteq U$. Let y be any member of X except the first member of X, and let $n = \max(y)$. Then there is an $x \in X$ such that $\max(x) < \min(y)$. Thus, both $x \cup y$ and y are in $[X] \cap P_n$. Hence, for each $U \in \mathcal{U}$, there is some n such that $|U \cap P_n| \geq 2$. Therefore, $\mathcal{U}$ is not a q-point.

(6) Let $M_n = \{x_{\min,\max} : \min(x) = n\}$. Then $\{M_n : n < \omega\}$ is a partition of ω. If X is any block sequence, then $|[X]_{\min,\max} \cap M_n| = \omega$ for infinitely many n. So $\mathcal{U}_{\min,\max}$ is not a p-point. Let $P_n = \{\iota(\{k, n\}) : k < n\}$, where ι is some fixed pairing function. Then for each $n \geq 1$, P_n is finite, and $\{P_n : n \geq 1\}$ is a partition of ω. If X is a block sequence, then $|[X]_{\min,\max} \cap P_n| > 1$ for infinitely many n. Hence, $\mathcal{U}_{\min,\max}$ is not a q-point.

(7) Given a strictly increasing function $g : \omega \to \omega$, without loss of generality assuming the coding function $\iota : [\omega]^2 \to \omega$ has the property that $\iota(\{m, n\}) \geq n$ for each $m < n$, let $k_l = 2^{l+1}$ for all $l < \omega$. Since $\mathcal{U}_{\min}$ is selective, there is an infinite block-sequence X such that $[X] \in \mathcal{U}$, $|X_{\min} \cap [0, g(k_2)]| = 0$, and for each $l \geq 2$, $|X_{\min} \cap (g(k_l), g(k_{l+1})]| \leq 1$. Then $|[X]_{\min,\max} \cap g(n)| < n$ for each $n < \omega$. □

By (5), the existence of block-generated ultrafilters on FIN cannot be proved on the basis of the usual ZFC axioms of set theory, though using Hindman's Theorem one can easily establish the existence of such ultrafilters using CH or MA.

As noted above, no nontrivial idempotent ultrafilter on FIN is basic, since such an ultrafilter is never a p-point, so we are naturally led to the following relaxation of this notion.

Definition 66. For infinite block sequences $X_n = (x^n_k)_{k<\omega}$ and $X = (x_k)_{k<\omega}$, the sequence $(X_n)_{n<\omega}$ *converges* to X (written $X_n \to X$ as $n \to \infty$) if for each $l < \omega$

there is an $m < \omega$ such that for all $n \geq m$ and all $k \leq l$, $x_k^n = x_k$. A block-generated ultrafilter $\mathcal{U}$ is *block-basic* if whenever we are given a sequence $(X_n)_{n<\omega}$ of infinite block sequences of elements of FIN such that each $[X_n] \in \mathcal{U}$ and $(X_n)_{n<\omega}$ converges to some infinite block sequence X such that $[X] \in \mathcal{U}$, then there is an infinite subsequence $(X_{n_k})_{k<\omega}$ such that $\bigcap_{k<\omega}[X_{n_k}] \in \mathcal{U}$.

Definition 67. Let $\mathrm{FIN}^{[n]}$ denote the collection of all block sequences of elements of FIN of length n. A block-generated ultrafilter $\mathcal{U}$ on FIN has the *2-dimensional Ramsey Property* if for each finite coloring of $\mathrm{FIN}^{[2]}$, there is an infinite block sequence X such that $[X] \in \mathcal{U}$ and $[X]^{[2]}$ is monochromatic. A block-generated ultrafilter $\mathcal{U}$ on FIN has the *Ramsey Property* if for each $n < \omega$ and each finite coloring of $\mathrm{FIN}^{[n]}$, there is an infinite block sequence X such that $[X] \in \mathcal{U}$ and $[X]^{[n]}$ is monochromatic. Let $\mathrm{FIN}^{[\infty]}$ denote the collection of all infinite block sequences of elements of FIN. A block-generated ultrafilter $\mathcal{U}$ on FIN has the *∞-dimensional Ramsey Property* if for every analytic subset $\mathcal{A}$ of $\mathrm{FIN}^{[\infty]}$ there is an infinite block sequence X such that $[X] \in \mathcal{U}$ and $[X]^{[\infty]}$ is either included in or disjoint from $\mathcal{A}$. (For more information about ∞-dimensional Ramsey Theory, see [32].)

The following theorem shows how the notion of block-basic ultrafilters fits with several equivalences shown by Blass in [4].

Theorem 68. *The following are equivalent for a block-generated ultrafilter $\mathcal{U}$ on* FIN*.*

1. *$\mathcal{U}$ is block-basic.*
2. *For every sequence (X_n) of infinite block sequences of* FIN *such that $[X_n] \in \mathcal{U}$ and $X_{n+1} \leq^* X_n$ for each n, there is an infinite block sequence X such that $[X] \in \mathcal{U}$ and $X \leq^* X_n$ for each n.*
3. *$\mathcal{U}$ has the 2-dimensional Ramsey property.*
4. *$\mathcal{U}$ has the Ramsey property.*
5. *$\mathcal{U}$ has the ∞-dimensional Ramsey property.*

Remark. (2) is called a *stable ordered-union ultrafilter* in [4].

Proof. The equivalence of (2), (3), (4) and (5) were established in Theorem 4.2 of [4].

(1) implies (2). Suppose $\mathcal{U}$ is block-basic. Let $(X_n)_{n<\omega}$ be a sequence of block-sequences of FIN such that $[X_n] \in \mathcal{U}$ and $X_{n+1} \leq^* X_n$ for each n. Let $(m_n)_{n<\omega}$ be a strictly increasing sequence such that $X_0 \geq X_1/m_1 \geq X_2/m_2 \geq \dots$. Let $Y_n = (\{l\})_{l \leq m_n}^\frown (X_n/m_n)$. Then each $Y_n =^* X_n$ and $Y_n \to (\{l\})_{l<\omega}$. By (1) there is a subsequence $(n_k)_{k<\omega}$ such that $\bigcap_{k<\omega}[Y_{n_k}] \in \mathcal{U}$. Since $\mathcal{U}$ is block-generated, there is a Z such that $[Z] \in \mathcal{U}$ and $[Z] \subseteq \bigcap_{k<\omega}[Y_{n_k}]$. Then for each $n < \omega$, taking k such that $n_k > n$, we have that $X_n =^* Y_n \geq^* Y_{n_k} \geq Z$. Thus, (2) holds.

Now suppose that (2) holds. Since $\mathcal{U}$ is block-generated, (2) is equivalent to the statement (2)$'$: For every sequence $(X_n)_{n<\omega}$ of infinite block sequences of FIN such that each $[X_n] \in \mathcal{U}$, there is an infinite block sequence X such that $[X] \in \mathcal{U}$ and $X \leq^* X_n$ for each n. Let $(X_n)_{n<\omega}$ be a sequence of block sequences such that each $[X_n] \in \mathcal{U}$ and $(X_n)_{n<\omega} \to X$. By (2)$'$, there is a $Z \leq X_0$ such that $[Z] \in \mathcal{U}$ and for each $n < \omega$, $Z \leq^* X_n$. Thus, there is a strictly increasing sequence $(m_k)_{k<\omega}$ such that each $m_k = \min(z)$ for some $z \in Z$ and

(a) $n \geq m_{k+1}$ implies $X_n \cap m_k = X \cap m_k$;
(b) $n \leq m_k$ implies $X_n/m_{k+1} \geq Z$.

Let $Z_0 = \{z \in Z : \exists k(m_{4k} \leq \min(z) < m_{4k+2})\}$. If $[Z_0] \in \mathcal{U}$, then take some $Y \leq Z_0, X$ such that $[Y] \in \mathcal{U}$. For each $k < \omega$, $X_{4k+3} \cap m_{4k+2} = X \cap m_{4k+2} \geq Y \cap m_{4k+2}$. For each $y \in Y$, $y \cap [m_{4k+2}, m_{4k+4}) = \emptyset$. $X_{4k+3}/m_{4k+4} \geq Z \geq Y$. Therefore, $\bigcap_{k<\omega}[X_{4k+3}] \supseteq [Y]$. If $[Z_0] \notin \mathcal{U}$, then since $\mathcal{U}$ is block-generated, there is a Z_1 such that $[Z_1] \in \mathcal{U}$ and $[Z_1] \subseteq [Z] \setminus [Z_0]$. Since $Z_1 \le Z$ and $[Z_1] \cap [Z_0] = \emptyset$, for each $z \in Z$, if $\min(z) \in [m_{4k}, m_{4k+2})$ then $z \notin Z_1$. Therefore, $Z_1 \cap Z_0 = \emptyset$. Hence, for each $z \in Z_1$, $\min(z) \in [m_{4k+2}, m_{4k+4})$. Letting $Y \le Z_1, X$ such that $[Y] \in \mathcal{U}$, $\bigcap_{k<\omega}[X_{4k+1}] \supseteq [Y]$. Hence, (1) holds. □

Remark. Blass showed in [4], that for every stable ordered-union ultrafilter $\mathcal{U}$ on FIN, both $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ are non-isomorphic selective ultrafilters. Thus, we have the following corollary.

Corollary 69. *If $\mathcal{U}$ is a block-basic ultrafilter on* FIN*, then $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ are Rudin-Keisler incomparable selective ultrafilters on ω.*

Remark. It follows by a theorem of Todorcevic in [24], that for any block-basic ultrafilter $\mathcal{U}$ on FIN, $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ are Tukey-incomparable.

Applying [Theorem 2.4, [4]] of Blass, we get some sort of converse to the previous corollary.

Corollary 70. *Assuming CH, for every pair $\mathcal{V}_0$ and $\mathcal{V}_1$ of non-isomorphic selective ultrafilters on ω, there is a block-basic ultrafilter $\mathcal{U}$ on* FIN *such that $\mathcal{U}_{\min} = \mathcal{V}_0$ and $\mathcal{U}_{\max} = \mathcal{V}_1$.*

Our interest in block-basic ultrafilters on FIN is based on the following fact whose proof is analogous to that of Theorem 20.

Theorem 71. *Suppose $\mathcal{U}$ is a block-basic ultrafilter on* FIN *and that $\mathcal{U} \ge_T \mathcal{V}$ for some ultrafilter $\mathcal{V}$ on any countable index set I. Then there is a monotone continuous map $f : \mathcal{P}(\mathrm{FIN}) \to \mathcal{P}(I)$ such that $f''\mathcal{U}$ is a cofinal subset of $\mathcal{V}$.*

Though the proof the next theorem follows the general outline of that of Theorem 20, we include the proof here since it does use some extra arguments.

Theorem 72. *Suppose $\mathcal{U}$ is a block-basic ultrafilter on* FIN *and $\mathcal{V}$ is any ultrafilter on a countable index set I. If $\mathcal{U}_{\min,\max} \ge_T \mathcal{V}$, then there are an infinite block sequence $\tilde{X}$ such that $[\tilde{X}] \in \mathcal{U}$ and a monotone continuous function f from $\{[X]_{\min,\max} : X \le \tilde{X}\}$ into $\mathcal{P}(I)$ whose restriction to $\{[X]_{\min,\max} : X \le \tilde{X},\ [X] \in \mathcal{U}\}$ has cofinal range in $\mathcal{V}$.*

Proof. Let $\mathcal{B}$ be the collection of block sequences X such that $[X] \in \mathcal{U}$. Then $\{[X] : X \in \mathcal{B}\}$ is a base for $\mathcal{U}$. Let $\mathcal{C} = \{[X]_{\min,\max} : X \in \mathcal{B}\}$. Then $\mathcal{C}$ is a base for $\mathcal{U}_{\min,\max}$. For the sake of notation, let $\mathcal{W}$ denote $\mathcal{U}_{\min,\max}$. Let $\mathcal{V}$ be any ultrafilter on some countable base set I such that $\mathcal{W} \ge_T \mathcal{V}$ and let $f : \mathcal{W} \to \mathcal{V}$ be a monotone cofinal map witnessing that $\mathcal{W} \ge_T \mathcal{V}$. Then $f \restriction \mathcal{C}$ is also a monotone cofinal map from $\mathcal{C}$ into $\mathcal{V}$.

In a similar manner as in the proof of Theorem 20, we construct an $\tilde{X} \in \mathcal{B}$ such that the map f is continuous on $\{[W]_{\min,\max} : W \in \mathcal{B},\ W \le \tilde{X}\}$. Let $\langle i_n : n < \omega\rangle$ be an enumeration of I. Let $X_0 = (\{0\}, \{1\}, \{2\}, \dots)$. Given X_n, take $X_{n+1} \le X_n$ such that, letting $(x_i^{n+1})_{i<\omega}$ denote X_{n+1},

1. $\min(x_0^{n+1}) \ge n+1$;
2. For each finite block sequence $s \subseteq \mathcal{P}(n+1)$, for each $k \le n$, if there is a $Z \in \mathcal{B}$ such that $\min(Z) \ge n+1$ and $i_k \notin f([s \cup Z]_{\min,\max})$, then $i_k \notin f([s \cup X_n]_{\min,\max})$.

Since $\mathcal{U}$ is block-basic, there is a $Y \in \mathcal{B}$ such that for each $n < \omega$, $Y \le^* X_n$. Let $l_0 = 0$ and for each $n < \omega$, let $l_{n+1} > l_n$ satisfy $l_{n+1} = \min(y)$ for some $y \in Y$ and $Y/l_{n+1} \le X_{l_n}$.

Color $[Y]^{[2]}$ as follows: Let $h((y_0, y_1)) = 0$ if there is an $n < \omega$ such that $\max(y_0) < l_n$ and $l_{n+2} \le \min(y_1)$; 1 otherwise. Since $\mathcal{U}$ has the Ramsey property for pairs, there is a block-sequence $\tilde{X} \le Y$ such that h is constant on $[\tilde{X}]^{[2]}$.

h cannot be constantly 1 on $[W]^{[2]}$ for any block-sequence W, since for any block-sequence (z_k), there will be some n and some $k < k'$ such that $\max(z_k) < l_n$ and $l_{n+2} \le \min(z_{k'})$; and such a pair will have color 0. Thus, h is constantly 0 on $[\tilde{X}]^{[2]}$.

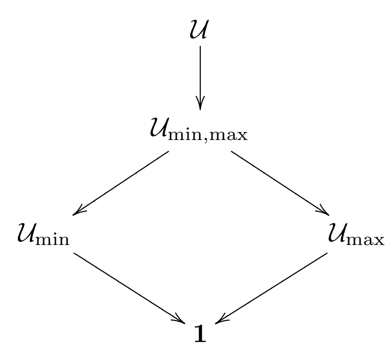
Let (l_{n_j}) be a subsequence of (l_n) such that for each x in $\tilde{X}$, either $\max(x) < l_{n_{2j+1}}$ or $l_{n_{2j+2}} \le \min(x)$. Suppose $W = (w_0, w_1, w_2, \dots) \le \tilde{X}$ and is in $\mathcal{B}$. Let $C = [W]_{\min,\max}$ and let $i \in I$. Let k be such that $i = i_k$. Take j large enough that $k < l_{n_{2j+1}}$ and there is an m such that $\max(w_m) < l_{n_{2j+1}}$ and $\min(w_{m+1}) \ge l_{n_{2j+2}}$. Note that $W/l_{n_{2j+1}} \le \tilde{X}/l_{n_{2j+1}} \subseteq Y/l_{n_{2j+1}} = Y/l_{n_{2j+2}} \le X_{l_{n_{2j+1}}}$. Thus, $i \notin f(C)$ iff $i \notin f([t \cup (W/l_{n_{2j+1}})]_{\min,\max})$, where $t = (w_0, \dots, w_m)$, iff $i \notin f([t \cup X_{l_{n_{2j+1}}}]_{\min,\max})$ iff $i \notin f([t \cup (\tilde{X}/l_{n_{2j+1}})]_{\min,\max})$. Thus, f is continuous on $\{[W]_{\min,\max} : W \in \mathcal{B},\ W \le \tilde{X}\}$.

In the following natural way, $f \restriction \{[W]_{\min,\max} : W \in \mathcal{B},\ W \le \tilde{X}\}$ can be extended to a continuous monotone map from $\{[X]_{\min,\max} : X \le \tilde{X}\}$ into $\mathcal{P}(I)$. For any $X \le \tilde{X}$, define $f'([X]_{\min,\max})$ to be $\bigcap\{f([W]_{\min,\max}) : W \in \mathcal{B},\ W \le \tilde{X},$ and $X \le W\}$. It follows from the definition of f' and the fact that f is monotone on $\{[W]_{\min,\max} : W \in \mathcal{B},\ W \le \tilde{X}\}$ that f' is monotone on $\{[X]_{\min,\max} : X \le \tilde{X}\}$. Note also that when restricted to $\{[W]_{\min,\max} : W \in \mathcal{B},\ W \le \tilde{X}\}$ f' is the same as f. Finally, f' is continuous, since for any $X \le \tilde{X}$ and any $k < \omega$, $k \in f'([X]_{\min,\max})$ iff for all $W \in \mathcal{B},\ W \le \tilde{X}$, and $X \le W$, $k \in f([W]_{\min,\max})$, and each of these is determined by the initial segment of W lying strictly below some particular $l_{n_{2_j+2}}$ depending only on k. So a finite amount of information which depends only on k and X determines whether or not k is in $f'([X]_{\min,\max})$. Hence, f' is continuous. □

Recall the following theorem of Hindman from [12], which is useful for constructing block-basic ultrafilters.

Theorem 73 (Hindman's Theorem)**.** *For every finite coloring of* FIN*, there is an infinite block sequence* $X = (x_n)$ *of members of* FIN *such that the set* $[X]$ *of all finite unions of members of* X *is monochromatic.*

Theorem 74. *Assuming CH, there is a block-basic ultrafilter* $\mathcal{U}$ *on* FIN *such that* $\mathcal{U}_{\min,\max} <_T \mathcal{U}$ *and* $\mathcal{U}_{\min}$ *and* $\mathcal{U}_{\max}$ *are Tukey incomparable. (In the following diagram, arrows represent strict Tukey reducibility.)*



Proof. Recall that for every block-generated ultrafilter $\mathcal{U}$ on FIN, $\mathcal{U}_{\min,\max} \equiv_{RK} \mathcal{U}_{\min} \cdot \mathcal{U}_{\max}$, and by Fact 30 and Corollary 34, $\mathcal{U}_{\min} \cdot \mathcal{U}_{\max} \equiv_T \mathcal{U}_{\min} \times \mathcal{U}_{\max}$. Recall that $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ are Tukey incomparable, since they are non-isomorphic selective ultrafilters. Thus, it suffices to construct a block-basic ultrafilter $\mathcal{U}$ on FIN such that $\mathcal{U}_{\min,\max} <_T \mathcal{U}$. Assuming CH, one can construct a block-basic ultrafilter on FIN in the standard way (see [4]).

Fix a well-ordering $\langle A_\beta : \beta < \omega_1\rangle$ of $\mathcal{P}(\mathrm{FIN})$. By Theorem 72, we can enumerate as $\langle (f_\beta, \tilde{X}_\beta) : \beta < \omega_1\rangle$, all pairs $(f, \tilde{X})$ such that $\tilde{X} \in \mathrm{FIN}^{[\infty]}$ and $f : \{[Z]_{\min,\max} :$

$Z \le \tilde{X}\} \to \mathcal{P}(\mathrm{FIN})$ is a monotone continuous function. We build a sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ of elements of $\mathrm{FIN}^{[\infty]}$ such that for each $\alpha < \omega_1$,

(i) For all $\beta < \alpha$, $S_\alpha \le^* S_\beta$;
(ii) Either $[S_\alpha] \subseteq A_\alpha$ or else $[S_\alpha] \cap A_\alpha = \emptyset$;
(iii) One of the following hold:
  (a) $[S_\alpha] \cap [\tilde{X}_\alpha] = \emptyset$; or
  (b) for each $W' \le S_\alpha$, $f_\alpha([W']_{\min,\max}) \not\subseteq [S_\alpha]$; or
  (c) $f_\alpha([S_\alpha]_{\min,\max}) \cap [S_\alpha] = \emptyset$.

Let S_0 be any block sequence such that either $[S_0] \subseteq A_0$ or else $[S_0] \cap A_0 = \emptyset$. Such an S_0 exists by Hindman's Theorem. At stage α in the construction, let Y be a block sequence such that

(i) for all $\beta < \alpha$, $Y \le^* S_\beta$, and
(ii) either $[Y] \subseteq A_\alpha$ or else $[Y] \cap A_\alpha = \emptyset$.

(The standard argument using Hindman's Theorem to find such a Y can be found on p. 93 of [4].)

Now we show there is an $S_\alpha \le Y$ satisfying (iii). If there is no block sequence $Z \le Y, \tilde{X}_\alpha$, then the domain of f_α is not contained in $\mathcal{U}_{\min,\max}$ for any block-generated ultrafilter $\mathcal{U}$ extending $\{[S_\beta] : \beta < \alpha\}$. In this case, use Hindman's Theorem to find an $S_\alpha \le Y$ such that $[S_\alpha] \cap [\tilde{X}_\alpha] = \emptyset$.

Now suppose there is a $Z \le Y, \tilde{X}_\alpha$. If there is a $W \le Z$ such that for each $W' \le W$, $f_\alpha([W']_{\min,\max}) \not\subseteq [W]$, then let $S_\alpha = W$. This ensures that f_α cannot be cofinal into any block-generated ultrafilter extending the filter generated by $\{[S_\beta] : \beta \le \alpha\}$, since f_α is monotone.

Otherwise, for each $W \le Z$, there is a $W' \le W$ such that $f_\alpha([W']_{\min,\max}) \subseteq [W]$. Let (z_i) denote Z. Fix $W = (w_i)$ to be the block sequence where each $w_i = z_{3i} \cup z_{3i+1} \cup z_{3i+2}$. Thus, $W \le Z$. Fix some $W' \le W$ such that $f_\alpha([W']_{\min,\max}) \subseteq [W]$. $W' \le W$ means that $W' = (w'_j)$, where each $w'_j = \bigcup_{i \in I_j} w_i$, where each I_j is some finite set. Let $m_j = \min(I_j)$ and $k_j = \max(I_j)$. Let $S_\alpha = (s_j)$, where each $s_j = z_{3m_j} \cup z_{3k_j+2}$. Then $\min(s_j) = \min(w'_j)$ and $\max(s_j) = \max(w'_j)$ for all $j < \omega$; so $[W']_{\min,\max} = [S_\alpha]_{\min,\max}$. Note that $[W] \cap [S_\alpha] = \emptyset$, and $S_\alpha \le Z$. Note that for any ultrafilter $\mathcal{U}$ extending $\{[S_\beta] : \beta \le \alpha\}$, $[S_\alpha]_{\min,\max} \in \mathcal{U}_{\min,\max}$. Hence, $f_\alpha([S_\alpha]_{\min,\max}) = f_\alpha([W']_{\min,\max}) \subseteq [W]$ which is disjoint from $[S_\alpha]$. Thus, the range of f_α will not be contained in $\mathcal{U}$. By this and the previous two paragraphs, we have satisfied (iii).

Let $\mathcal{U}$ be the filter generated by $\{[S_\alpha] : \alpha < \omega_1\}$. Condition (ii) ensures that $\mathcal{U}$ is an ultrafilter which is block-generated. Condition (iii) ensures that $\mathcal{U}_{\min,\max} \not\ge_T \mathcal{U}$, and thus $\mathcal{U} >_T \mathcal{U}_{\min,\max}$. □

Question 75. If $\mathcal{U}$ is any block-basic ultrafilter, does it follow that $\mathcal{U} >_T \mathcal{U}_{\min,\max}$?

Remark. Note that the proof of Theorem 74 shows that the generic filter for the forcing notion $(\mathrm{FIN}^{[\infty]}, \le^*)$ adjoins a block-basic ultrafilter $\mathcal{U}$ on FIN with the properties stated in Theorem 68. On the other hand, an argument analogous with the case of selective ultrafilters on ω (see the theorem of Todorcevic appearing in [8; 4.9]) shows that if there is a supercompact cardinal, then every block-basic ultrafilter $\mathcal{U}$ on FIN is generic over $L(\mathbb{R})$ for the forcing notion $(\mathrm{FIN}^{[\infty]}, \le^*)$. Thus, the conclusion of Theorem 4.9 in [9] is true for any block-basic ultrafilter $\mathcal{U}$ on FIN assuming the existence of a supercompact cardinal. This leads us also to the following related problem.

Problem 76. Assume the existence of a supercompact cardinal. Let $\mathcal{U}$ be an arbitrary block-basic ultrafilter on FIN. Show that the inner model $L(\mathbb{R})[\mathcal{U}]$ has exactly five Tukey types of ultrafilters on a countable index set.

This problem is based on the $\mathcal{U}$-version of Taylor's canonical Ramsey Theorem for FIN stating that for each map $f : \mathrm{FIN} \to \omega$, there is an $[X] \in \mathcal{U}$ such that $f \restriction [X]$ is equivalent to one of the five mappings: constant, identity, min, max, (min,max) (see [1], [29]). If the answer to this problem is positive, then one can look at ultrafilters $\mathcal{U}$ on the index set FIN_k ($k = 1, 2, 3, \dots$) with analogous Ramsey-theoretic properties whose corresponding inner models $L(\mathbb{R})[\mathcal{U}]$ have different finite numbers of Tukey types. This will of course be based on Gower's Theorem for FIN_k and Lopez-Abad's canonical Ramsey Theorem for FIN_k (see [1], [21], [11]). For example, for a block-basic ultrafilter $\mathcal{U}$ on FIN_2, one could expect exactly 43 Tukey types of ultrafilters in $L(\mathbb{R})[\mathcal{U}]$.

The following is a subproblem of Problem 76.

Question 77. Is it true that for each block-basic $\mathcal{U}$, there are no Tukey types (a) strictly between $\mathcal{U}$ and $\mathcal{U}_{\min,\max}$, (b) strictly between $\mathcal{U}_{\min,\max}$ and $\mathcal{U}_{\min}$, and (c) strictly between $\mathcal{U}_{\min,\max}$ and $\mathcal{U}_{\max}$?

Question 78. Are there block-basic ultrafilters $\mathcal{U}, \mathcal{V}$ on FIN which are Tukey equivalent but RK incomparable?

7. A characterization of ultrafilters which are not of Tukey top degree

In this section we investigate Isbell's question of whether ZFC implies that there is always an ultrafilter which does not have top Tukey degree. It will be useful here to consider the directed partial ordering $\supseteq^*$ on ultrafilters as well as the one we have been considering all along, namely $\supseteq$. We note that always $(\mathcal{U}, \supseteq) \leq_T (\mathcal{U}, \supseteq^*)$; for any subset $\mathcal{X} \subseteq \mathcal{U}$ which is unbounded in $(\mathcal{U}, \supseteq^*)$ is also unbounded in $(\mathcal{U}, \supseteq)$, so the identity map $id_{\mathcal{U}} : (\mathcal{U}, \supseteq) \to (\mathcal{U}, \supseteq^*)$ is a Tukey map. Hence, if $(\mathcal{U}, \supseteq^*) <_T [\mathfrak{c}]^\omega$, then also $(\mathcal{U}, \supseteq) <_T [\mathfrak{c}]^\omega$. Milovich showed in [22] that for any ultrafilter $\mathcal{U}$, there is an ultrafilter $\mathcal{W}$ such that $(\mathcal{W}, \supseteq^*) \leq_T (\mathcal{U}, \supseteq)$. Thus, there is an ultrafilter $\mathcal{U}$ such that $(\mathcal{U}, \supseteq) <_T ([\mathfrak{c}]^{<\omega}, \subseteq)$ if and only if there is an ultrafilter $\mathcal{W}$ such that $(\mathcal{W}, \supseteq^*) <_T ([\mathfrak{c}]^{<\omega}, \subseteq)$.

CH implies the existence of p-points, which solves Isbell's problem, since p-points have Tukey type strictly below the top, by Corollary 19. Thus, we now assume $\neg$CH throughout this section. Assuming $\neg$CH, the following combinatorial principle holds.

Definition 79 (Todorcevic)**.** $\diamondsuit_{[\mathfrak{c}]^\omega}$ is the statement: There exist sets $S_A \subseteq A$, $A \in [\mathfrak{c}]^\omega$, such that for each $X \subseteq \mathfrak{c}$, $\{A \in [\mathfrak{c}]^\omega : X \cap A = S_A\}$ is stationary in $[\mathfrak{c}]^\omega$.

This implies the next combinatorial principle in the same way that the standard $\diamondsuit$ implies $\diamondsuit^-$.

Definition 80 (Todorcevic)**.** $\diamondsuit^-_{[[\omega]^\omega]^\omega}$ is the statement: There exist ordered pairs $(\mathcal{U}_A, \mathcal{X}_A)$, where $A \in [[\omega]^\omega]^\omega$ and $\mathcal{X}_A \subseteq \mathcal{U}_A \subseteq A$, such that for each pair $(\mathcal{U}, \mathcal{X})$ with $\mathcal{X} \subseteq \mathcal{U}$ and $\mathcal{X}, \mathcal{U} \in [[\omega]^\omega]^\mathfrak{c}$, $\{A \in [[\omega]^\omega]^\omega : \mathcal{U}_A = \mathcal{U} \cap A,\ \mathcal{X}_A = \mathcal{X} \cap A\}$ is stationary in $[[\omega]^\omega]^\omega$.

We now proceed to define some dense subsets of $[\omega]^\omega$, denoted D_A and D'_A which can be used to give conditions under which an ultrafilter on ω is a p-point, and other conditions under which it has Tukey type less than the top type.

For the rest of this section, fix a $\diamondsuit^-_{[[\omega]^\omega]^\omega}$ sequence $(\mathcal{U}_A, \mathcal{X}_A)$, where $A \in [[\omega]^\omega]^\omega$.

Definition 81. Let $P_A = \{W \in [\omega]^\omega : \exists X \in \mathcal{U}_A (W \cap X = \emptyset)\}$, and let $Q_A = \{W \in [\omega]^\omega : \exists (B_n)_{n<\omega} \subseteq \mathcal{X}_A (\forall n < \omega,\ W \subseteq^* B_n)\}$. Let $D_A = P_A \cup Q_A$.

Fact 82. *For each $A \in [[\omega]^\omega]^\omega$, D_A is dense open in the partial ordering $([\omega]^\omega, \subseteq)$.*

Proof. Let $Y \in [\omega]^\omega$. Suppose there are $U, V \in \mathcal{U}_A$ such that $|U \cap V| < \omega$. Then either $|Y \setminus U| = \omega$ or $|Y \setminus V| = \omega$. Thus, there is a $W \in [Y]^\omega$ such that for some $X \in \mathcal{U}_A$, $W \cap X = \emptyset$. Hence $W \in P_A$, and moreover, any $W' \in [W]^\omega$ is also in P_A. Suppose that for any $U, V \in \mathcal{U}$, U and V have infinite intersection. If $Y \notin \mathcal{U}_A^+$, then there is an $X \in \mathcal{U}_A$ such that $|Y \cap X| < \omega$. So $W = Y \setminus X \in P_A$, and any $W' \in [W]^\omega$ is also in P_A. Otherwise, $Y \in \mathcal{U}_A^+$. Then there is a $W \in [Y]^\omega$ such that for each $B \in \mathcal{U}_A$, $W \subseteq^* B$, since $|\mathcal{U}_A| \le \omega$. Thus, $W \in Q_A$. Moreover, any $W' \in [W]^\omega$ is also in Q_A. Therefore, D_A is dense open in $[\omega]^\omega$. □

Fact 83. *For any ultrafilter* $\mathcal{U}$, $\{A \in [[\omega]^\omega]^\omega : \mathcal{U} \cap D_A \neq \emptyset\}$ *is stationary.*

Proof. Let $\mathcal{U}$ be an ultrafilter and suppose that $\{A \in [[\omega]^\omega]^\omega : \mathcal{U} \cap D_A \neq \emptyset\}$ is not stationary. Then $\{A \in [[\omega]^\omega]^\omega : \mathcal{U} \cap D_A = \emptyset\}$ contains a club set, call it $\mathcal{C}$. Let $\mathcal{X} = \bigcup\{\mathcal{X}_A : A \in \mathcal{C}\}$. Let $X \in \mathcal{U}$. There are club many $A \in [[\omega]^\omega]^\omega$ with $X \in A$. Thus, there are club many A with $(A, \mathcal{U} \cap A) \prec ([\omega]^\omega, \mathcal{U})$. By $\diamondsuit^-_{[[\omega]^\omega]^\omega}$, there is an $A \in [\omega]^\omega$ with $X \in A$ such that $\mathcal{U} \cap A = \mathcal{U}_A$ and $\mathcal{U} \cap A = \mathcal{X}_A$. Therefore, $\mathcal{U} \subseteq \mathcal{X}$.

We claim that for each $\mathcal{Y} \in [\mathcal{U}]^\omega$, there is no $X \in \mathcal{U}$ such that $X \subseteq^* Y$ for each $Y \in \mathcal{Y}$. Take an $A \in \mathcal{C}$ with $\mathcal{Y} \subseteq A$ such that $\mathcal{U} \cap D_A = \emptyset$, $\mathcal{U}_A = \mathcal{U} \cap A$, and $\mathcal{X}_A = \mathcal{U} \cap A$. Then $\mathcal{Y} \subseteq \mathcal{U} \cap A = \mathcal{X}_A$ and for each infinite subset $\mathcal{Z}$ of $\mathcal{Y}$ in $\mathcal{U}$, there is no $U \in \mathcal{U}$ such that $U \subseteq^* Z$ for all $Z \in \mathcal{Z}$, since $Q_A \cap \mathcal{U} = \emptyset$. Contradiction, since $\mathcal{U}$ contains the Fréchet filter. □

Fact 84. *If* $\mathcal{U} \cap D_A \neq \emptyset$ *for club many* $A \in [[\omega]^\omega]^\omega$, *then* $\mathcal{U} <_T \mathcal{U}_{\mathrm{top}}$.

Proof. Let $\mathcal{X} \in [\mathcal{U}]^{\mathfrak{c}}$. $\{A \in [[\omega]^\omega]^\omega : (A, \mathcal{U} \cap A, \mathcal{X} \cap A) \prec ([\omega]^\omega, \mathcal{U}, \mathcal{X})\}$ is club in $[[\omega]^\omega]^\omega$. $\{A \in [[\omega]^\omega]^\omega : A \cap \mathcal{U} = \mathcal{U}_A,\ A \cap \mathcal{X} = \mathcal{X}_A\}$ is stationary. If $\mathcal{U} \cap D_A \neq \emptyset$ for club many A, then there are stationary many A such that $\mathcal{U} \cap A = \mathcal{U}_A$, $\mathcal{X} \cap A = \mathcal{X}_A$, and either there is a $U \in \mathcal{U}_A$ and a $W \in \mathcal{U}$ such that $U \cap W = \emptyset$, which is impossible, or else there is a $W \in \mathcal{U}$ and $(B_n)_{n<\omega} \subseteq \mathcal{X}_A = A \cap \mathcal{X}$ such that for each $n < \omega$, $W \subseteq^* B_n$. Therefore, $\mathcal{U}$ is not of Tukey top degree. □

Fact 85. *If* $\mathcal{U}$ *is a p-point, then* $\mathcal{U} \cap D_A \neq \emptyset$ *for all* $A \in [[\omega]^\omega]^\omega$.

Proof. Let $A \in [[\omega]^\omega]^\omega$ be given. If $\mathcal{U}_A \not\subseteq \mathcal{U}$, then taking an $X \in \mathcal{U}_A \setminus \mathcal{U}$, we have $\omega \setminus X \in \mathcal{U} \cap P_A$. If $\mathcal{U}_A \subseteq \mathcal{U}$, then since $\mathcal{X}_A$ is countable, there is a $W \in \mathcal{U}$ which almost contains every member of $\mathcal{X}_A$. Hence, $W \in \mathcal{U} \cap Q_A$. □

Let $P'_A = \{W \in [\omega]^\omega : \forall X \in \mathcal{X}_A (W \subseteq^* X)\}$. Let $D'_A = P'_A \cup Q_A$. By the same proof as for D_A, we see that D'_A is dense open in $[\omega]^\omega$.

Fact 86. *If* $\mathcal{U} \cap D'_A \neq \emptyset$ *for club many* A, *then* $\mathcal{U}$ *is a p-point.*

Proof. Suppose that $\mathcal{C}$ is club in $[[\omega]^\omega]^\omega$ and for each $A \in \mathcal{C}$, $\mathcal{U} \cap D'_A \neq \emptyset$. Let $\mathcal{Y} \in [\mathcal{U}]^\omega$. Take A such that $\mathcal{Y} \subseteq A$, $(A, \mathcal{U} \cap A, \mathcal{U} \cap A) \prec ([\omega]^\omega, \mathcal{U}, \mathcal{U})$, $\mathcal{U}_A = \mathcal{U} \cap A$, $\mathcal{X}_A = \mathcal{X} \cap A$, and $\mathcal{U} \cap D'_A \neq \emptyset$. Then $\mathcal{Y} \subseteq \mathcal{U} \cap A = \mathcal{X}_A$. So since there is a $W \in \mathcal{U}$ such that for each $X \in \mathcal{X}_A$, $W \subseteq^* X$, there is a $U \in \mathcal{U}$ such that $U \subseteq^* Y$ for each $Y \in \mathcal{Y}$. □

Remark. Assuming ¬CH and that there are no p-points (the remaining open case for Isbell's Problem), if Isbell's Problem is solved in the affirmative with an ultrafilter $\mathcal{U} <_T [\mathfrak{c}]^{<\omega}$, then $\mathcal{U}$ *must* have the following properties.

(1) $\mathcal{U} \cap D_A \neq \emptyset$ for club many $A \in [[\omega]^\omega]^\omega$.
(2) The collection of $A \in [[\omega]^\omega]^\omega$ such that $\mathcal{U} \cap D'_A \neq \emptyset$ does not contain a club set.

To solve Isbell's Problem under the assumptions ¬CH and there are no p-points, it suffices to find an ultrafilter $\mathcal{U}$ such that (1) holds and

(3) There is some $A \in [[\omega^\omega]]^\omega$ such that $\mathcal{U} \cap D_A = \emptyset$.

Question 87. Assume ¬CH and there are no p-points. Can we use these dense sets, or similar ones, to obtain

(1) an ultrafilter which is not Tukey top?
(2) an ultrafilter which is not Tukey top but also is not basically generated?

8. Concluding remarks and problems

Recall that the properties of p-point and rapid are preserved under Rudin-Keisler reducibility.

Question 88. Which properties of ultrafilters are preserved under Tukey reducibility?

By Theorem 35, if a p-point $\mathcal{U} \geq_T \omega^\omega$, then $\mathcal{U} \equiv_T \mathcal{U} \cdot \mathcal{U}$, which is not a p-point, so the property of being a p-point is not preserved by Tukey reducibility. However, we may ask the following.

Question 89. If $\mathcal{U}$ is a p-point and $\mathcal{U} \geq_T \mathcal{V}$, then is there a p-point $\mathcal{W}$ such that $\mathcal{W} \equiv_T \mathcal{V}$?

Question 90. Which lattices can be embedded into the Tukey types of p-points? In particular, are there two Tukey incomparable p-points which have no p-point as a common Tukey upper bound?

Question 91. Are there two Tukey non-comparable ultrafilters whose least upper bound is the top Tukey type?

Question 92. Does every Tukey minimal type contain a selective ultrafilter?

Question 93. What is the structure of the Rudin-Keisler types within the top Tukey type?

References

1. Spiros A. Argyros and Stevo Todorcevic, *Ramsey Methods in Analysis*, Birkhäuser, 2005.
2. S. Tomek Bartoszyński and Haim Judah, *Set Theory on the Structure of the Real Line*, A. K. Peters, Ltd., 1995.
3. Andreas Blass, *The Rudin-Keisler ordering of P-Points*, Transactions of the American Mathematical Society **179** (1973), 145–166.
4. ______, *Ultrafilters related to Hindman's finite-unions theorem and its extensions*, Contemporary Mathematics **65** (1987), 89–124.
5. ______, *Near coherence of filters III: A simplified consistency proof*, Notre Dame Journal of Formal Logic **30** (1989), no. 4, 530–538.
6. Andreas Blass and Claude Laflamme, *Consistency results about filters and the number of inequivalent growth types*, Journal of Symbolic Logic **54** (1989), no. 1, 50–56.
7. W. Wistar Comfort, *The Theory of Ultrafilters*, Springer-Verlag, 1974.
8. Mahlon M. Day, *Oriented systems*, Duke Mathematical Journal **11** (1944), 201–229.
9. Ilijas Farah, *Semiselective coideals*, Mathematika **45** (1998), no. 1, 79–103.
10. David Fremlin, *The partially ordered sets of measure theory and Tukey's ordering*, Note di Matematica **XI** (1991), 177–214.
11. W. Timothy Gowers, *An infinite Ramsey Theorem and some Banach-space dichotomies*, Annals of Mathematics **156** (2002), 797–833.
12. Neil Hindman, *Finite sums from sequences within cells of a partition of n*, Journal of Combinatorial Theory. Series A **17** (1974), 1–11.
13. John Isbell, *The category of cofinal types. II*, Transactions of the American Mathematical Society **116** (1965), 394–416.
14. S.A. Jalali-Naini, *The Monotone Subsets of Cantor Space, Filters and Descriptive Set Theory*, Ph.D. thesis, Oxford, 1976.
15. Istvan Juhász, *Remarks on a theorem of B. Pospíšil*, General Topology and its Relations to Modern Analysis and Algebra, Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1967, pp. 205–206.
16. ______, *Remarks on a theorem of B. Pospíšil. (Russian)*, Commentationes Mathematicae Universitatis Carolinae **8** (1967), 231–247.
17. Istvan Juhasz, *Cardinal functions in topology*, Math, Centre tracts **34** (1975).

18. Jussi Ketonen, *On the existence of P-points in the Stone-Cech compactification of integers*, Fundamenta Mathematicae **92** (1976), no. 2, 91–94.
19. Kenneth Kunen, *Weak P-points in N^**, Colloquia Mathematica Societatis János Bolyai, 23. Topology, Budapest (1978), 741–749.
20. Claude Laflamme, *Upward directedness of the Rudin-Keiser ordering of p-points*, The Journal of Symbolic Logic **55** (1990), no. 2, 449–456.
21. Jordi Lopez-Abad, *Canonical equivalence relations on nets of* PS_{c_0}, Discrete Mathematics **307** (2007), no. 23, 2943–2978.
22. David Milovich, *Tukey classes of ultrafilters on ω*, Topology Proceedings **32** (2008), 351–362.
23. Bedřich Pospíšil, *On bicompact spaces*, Publ. Fac. Sci. Univ. Masaryk **270** (1939).
24. Dilip Raghavan and Stevo Todorcevic, *Cofinal types of ultrafilters*, (2010), Preprint.
25. Jürgen Schmidt, *Konfinalität*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **1** (1955), 271–303.
26. Saharon Shelah and M.E. Rudin, *Unordered types of ultrafilters*, Topology Proceedings **3** (1978), 199–204.
27. Slawomir Solecki and Stevo Todorcevic, *Cofinal types of topological directed orders*, Annales de L'Institut Fourier **54** (2004), no. 6, 1877–1911.
28. Michel Talagrand.
29. Alan D. Taylor, *A canonical Partition Relation for finite Subsets of ω*, Journal of Combinatorial Theory (A) **21** (1970), 137–146.
30. Stevo Todorcevic, *Directed sets and cofinal types*, Transactions of the American Mathematical Society **290** (1985), no. 2, 711–723.
31. ______, *A classification of transitive relations on ω_1*, Proceedings of the London Mathematical Society **(3) 73** (1996), no. 3, 501–533.
32. ______, *Introduction to Ramsey Spaces*, Princeton University Press, 2010.
33. John W. Tukey, *Convergence and uniformity in topology*, Princeton University Press, 1940.

Department of Mathematics, University of Denver, 2360 S Gaylord St, Denver, CO 80208, USA

E-mail address: `natasha.dobrinen@du.edu`

Department of Mathematics, University of Toronto, 40 St George St, Toronto, ON, CANADA M5S 2E4

E-mail address: `stevo@utoronto.edu`