QUANTUM REALITY FILTERS

Stan Gudder
Department of Mathematics
University of Denver
Denver, Colorado 80208
sgudder@math.du.edu

Abstract

An anhomomorphic logic $\mathcal{A}^*$ is the set of all possible realities for a quantum system. Our main goal is to find the "actual reality" $\phi_a \in \mathcal{A}^*$ for the system. Reality filters are employed to eliminate unwanted potential realities until only ϕ_a remains. In this paper, we consider three reality filters that are constructed by means of quantum integrals. A quantum measure μ can generate or actualize a $\phi \in \mathcal{A}^*$ if $\mu(A)$ is a quantum integral with respect to ϕ for a density function f over events A. In this sense, μ is an "average" of the truth values of ϕ with weights given by f. We mainly discuss relations between these filters and their existence and uniqueness properties. For example, we show that a quadratic reality generated by a quantum measure is unique. In this case we obtain the unique actual quadratic reality.

"Reality is merely an illusion, albeit a very persistent one."

—Albert Einstein

1 Introduction

In the past, one of the main goals of physics has been to describe physical reality. Recently however, physicists have embarked on the even more ambitious program of actually *finding* physical reality. Specifically, their quest is to find the universal truth function ϕ_a. If A is any proposition concerning the physical universe, then $\phi_a(A)$ is 0 or 1 depending on whether A is false or true. Assuming that the set of propositions is a Boolean algebra $\mathcal{A}$, we can think of $\mathcal{A}$ as an algebra of subsets of a universe Ω of outcomes. The outcomes are frequently interpreted as paths (or trajectories or histories) of a physical world. The actual physical universe Ω_1 is vast, complicated and probably infinite. To make the situation more manageable, we shall only consider a toy universe Ω with a finite number of elements. This will still enable us to investigate structures that may be applicable to Ω_1.

The theory that we present originated with the work of R. Sorkin [12, 13] who was motivated by the histories approach to quantum mechanics and quantum gravity and cosmology [9, 10, 14]. Sorkin called this quantum measure theory and anhomomorphic logic. After Sorkin's pioneering work, many investigators have developed various aspects of the theory [1, 2, 3, 4, 5, 6, 7, 8, 11, 15, 16]. For a classical universe, a truth function would be a homomorphism ϕ from the algebra $\mathcal{A}$ of propositions or events to the two-element Boolean algebra $\mathbb{Z}_2 = \{0, 1\}$. However, because of quantum interference, a truth function describing a quantum reality need not be a homomorphism. We call the set

$$\mathcal{A}^* = \{\phi\colon \mathcal{A} \to \mathbb{Z}_2 \colon \phi(\emptyset) = 0\}$$

the *full anhomomorphic logic*. The elements of $\mathcal{A}^*$ are interpreted as potential realities for a physical system and are called *coevents*. Our task is to find the actual reality $\phi_a \in \mathcal{A}^*$ which describes what actually happens.

Even if the cardinality $n = |\Omega|$ of Ω is small, the cardinality $|\mathcal{A}^*| = 2^{(2^n - 1)}$ of $\mathcal{A}^*$ can be immense. Hence, it is important to establish reality filters that filter out unwanted potential realities until we are left with the actual reality ϕ_a. Mathematically, reality filters are requirements that can be employed to distinguish ϕ_a from other possible coevents. One of the main reality filters that has been used is called preclusivity [2, 3, 12, 13, 15]. Nature has provided us with a quantum measure μ which is related to the state of the system. The measure μ is where the physics is contained and information about μ is obtained by observing the physical universe. Specifically, μ is a nonnegative set function on $\mathcal{A}$ that is more general than an ordinary measure. For $A \in \mathcal{A}^*$, $\mu(A)$ is interpreted as the propensity that the event A occurs. If $\mu(A) = 0$, then A does not occur and we say that A is *precluded*. We say that $\phi \in \mathcal{A}^*$ is *preclusive* if $\phi(A) = 0$ for all precluded $A \in \mathcal{A}$. It is generally agreed that ϕ_a should be preclusive.

Although preclusivity is an important reality filter, it is too weak to determine ϕ_a uniquely. Other reality filters that have been used involve the algebraic properties of coevents and are called unital, additive, multiplicative and quadratic properties [2, 7, 8, 12, 13, 15]. Unfortunately, there does not seem to be agreement on which, if any, of these properties is appropriate. In this paper we shall consider three other reality filters, two of which were proposed in [8]. These filters involve coevents that determine the quantum measure using an averaging process called a quantum integral. These filters are called 1-generated, 2-generated and actualized.

The present paper continues our study of 1-generated and 2-generated coevents and introduces the concept of an actualized coevent. One of our results shows that if ϕ and ψ are 1-generated by the same quantum measure, then $\phi = \psi$. Another result shows that if ϕ and ψ are quadratic coevents that are 2-generated by the same quantum measure, then $\phi = \psi$. It follows that the 1- and 2-generated filters uniquely determine a quadratic reality. We also demonstrate that this result does not hold for actualized coevents. Still another result shows that 1-generated coevents are 2-generated and that unital 1-generated coevents are actualized. We give examples of coevents that are 2-generated and actualized but are not 1-generated. We also give an example of a coevent that is actualized but we conjecture is not 2-generated.

Quantum measures that 1- or 2-generate a coevent appear to belong to a rather restricted class. The main reason for introducing the actualized filter is that it appears to be more general so it admits a larger set of actualizing quantum measures. We present evidence of this fact by considering a sample space Ω with $|\Omega| = 2$. Various open problems are presented. These problems mainly concern quantum measures and coevents that correspond to our three reality filters. We begin an approach to one of these problems by characterizing 1-generated coevents.

2 Anhomomorphic Logic and Quantum Integrals

Let Ω be a finite nonempty set with cardinality $|\Omega| = n$. We call Ω a *sample space*. We think of the elements of Ω as outcomes or trajectories of an experiment or physical system and the collection of subsets $\mathcal{A} = 2^{\Omega}$ of Ω as the possible events. We can also view the elements of $\mathcal{A}$ as propositions concerning the system. Contact with reality is given by a truth function $\phi\colon \mathcal{A} \to \mathbb{Z}_2$ where $\mathbb{Z}_2 = \{0, 1\}$ is the two-element Boolean algebra with the usual multiplication and addition given by $0 \oplus 0 = 1 \oplus 1 = 0$ and $0 \oplus 1 = 1 \oplus 0 = 1$.

For $\omega \in \Omega$ we define *evaluation map* $\omega^* \colon \mathcal{A} \to \mathbb{Z}_2$ given by

$$\omega^*(A) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

For classical systems, it is assumed that a truth function ϕ is a homomorphism; that is, ϕ satisfies

(H1) $\phi(\Omega) = 1$ (unital)

(H2) $\phi(A \mathbin{\dot\cup} B) = \phi(A) \oplus \phi(B)$ whenever $A \cap B = \emptyset$ (additivity)

(H3) $\phi(A \cap B) = \phi(A)\phi(B)$ (multiplicativity)

In (H2) $A \mathbin{\dot\cup} B$ denotes $A \cup B$ whenever $A \cap B = \emptyset$. It is well-known that ϕ is a homomorphism if and only if $\phi = \omega^*$ for some $\omega \in \Omega$. Thus, there are n truth functions for classical systems.

As discussed in [2, 12, 13, 15], for a quantum system a truth function need not be a homomorphism. Various conditions for quantum truth functions have been proposed. In [12, 13] it is assumed that quantum truth functions satisfy (H2) and these are called *additive* truth functions while in [2, 15] it is assumed that quantum truth functions satisfy (H3) and these are called *multiplicative* truth functions. In [7] it is argued that quantum truth functions need not satisfy (H1), (H2) or (H3) but should be *quadratic* or *grade*-2 *additive* [2, 13] in the sense that

(Q4) $\phi(A \mathbin{\dot\cup} B \mathbin{\dot\cup} C) = \phi(A \mathbin{\dot\cup} B) \oplus \phi(A \mathbin{\dot\cup} C) \oplus \phi(B \mathbin{\dot\cup} C) \oplus \phi(A) \oplus \phi(B) \oplus \phi(C)$

If $\phi, \psi \colon \mathcal{A} \to \mathbb{Z}_2$ we define $\phi\psi \colon \mathcal{A} \to \mathbb{Z}_2$ by $(\phi\psi)(A) = \phi(A)\psi(A)$ and $\phi \oplus \psi \colon \mathcal{A} \to \mathbb{Z}_2$ by $(\phi \oplus \psi)(A) = \phi(A) \oplus \psi(A)$ for all $A \in \mathcal{A}$. We define the 0 and 1 truth functions by $0(A) = 0$ for all $A \in \mathcal{A}$ and $1(A) = 1$ if and only if $A \neq \emptyset$. It can be shown [2, 7, 12] that ϕ is additive if and only if ϕ is a degree-1 polynomial

$$\phi = \omega_1^* \oplus \cdots \oplus \omega_m^*$$

and that ϕ is multiplicative if and only if ϕ is a monomial

$$\phi = \omega_1^* \omega_2^* \cdots \omega_m^*$$

Moreover, one can show [2, 7] that ϕ is quadratic if and only if ϕ is a degree-1 polynomial or ϕ is a degree-2 polynomial of the form

$$\phi = \omega_1^* \oplus \cdots \oplus \omega_m^* \oplus \omega_i^* \omega_j^* \oplus \cdots \oplus \omega_r^* \omega_s^*$$

We call $\mathcal{A}^* = \{\phi\colon \mathcal{A} \to \mathbb{Z}_2 \colon \phi(\emptyset) = 0\}$ the full *anhomomorphic logic* and the elements of $\mathcal{A}^*$ are called *coevents*. There are $2^{(2^n-1)}$ coevents of which n are classical, $2^n - 1$ are additive, $2^n - 1$ are multiplicative and $2^{n(n+1)/2}$ are quadratic. It can be shown that any coevent can be written as a polynomial in the evaluation maps and that such an evaluation map representation is unique up to the order of its terms [2, 7].

Applying (Q4) one can prove by induction that $\phi \in \mathcal{A}^*$ is quadratic if and only if

$$\phi\left(A_1, \dot\cup \cdots \dot\cup A_m\right) = \bigoplus_{i<j=1}^{m} \phi(A_i \dot\cup A_j) \oplus \tfrac{1}{2}\left[1 - (-1)^m\right] \bigoplus_{i=1}^{m} \phi(A_i) \tag{2.1}$$

for all $m \geq 3$. It follows from (2.1) that a quadratic coevent is determined by its values on singleton and doubleton sets in $\mathcal{A}$. Moreover, given any assignment of zeros and ones to the singleton and doubleton sets in $\mathcal{A}$, there exists a unique quadratic coevent that has these values.

Following [6], if $f\colon \Omega \to \mathbb{R}$ and $\phi \in \mathcal{A}^*$, we define the *q-integral*

$$\int f d\phi = \int_0^\infty \phi\left(\{\omega\colon f(\omega) > \lambda\}\right) d\lambda - \int_0^\infty \phi\left(\{\omega\colon f(\omega) < -\lambda\}\right) d\lambda$$

where $d\lambda$ denotes Lebesgue measure on $\mathbb{R}$. If $f \geq 0$, then

$$\int f d\phi = \int_0^\infty \left(\{\omega\colon f(\omega) > \lambda\}\right) d\lambda \tag{2.2}$$

and we shall only integrate nonnegative functions here. Denoting the characteristic function of a set A by χ_A, any $f\colon \Omega \to \mathbb{R}^+$ has the canonical representation $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$ where $0 < \alpha_1 < \cdots < \alpha_n$ and $A_i \cap A_j = \emptyset$, $i \neq j$. It follows from (2.2) that

$$\begin{aligned}\int f d\phi &= \alpha_1 \phi\left(\dot\bigcup_{i=1}^n A_i\right) + (\alpha_2 - \alpha_1)\phi\left(\dot\bigcup_{i=2}^n A_i\right) + \cdots + (\alpha_n - \alpha_{n-1})\phi(A_n) \\ &= \sum_{j=1}^n \alpha_j \left[\phi\left(\dot\bigcup_{i=j}^n A_i\right) - \phi\left(\dot\bigcup_{i=j+1}^n A_i\right)\right] \end{aligned} \tag{2.3}$$

It is clear from (2.2) or (2.3) that $\int f d\phi \geq 0$ if $f \geq 0$. Also, it is easy to check that $\int \alpha f d\phi = \alpha \int f d\phi$ for all $\alpha \in \mathbb{R}$. However, the q-integral is not linear because

$$\int (f+g) d\phi \neq \int f d\phi + \int g d\phi$$

in general. Moreover, in general we have

$$\int f d(\phi \oplus \psi) \neq \int f d\phi + \int f d\psi$$

As usual in integration theory, for $A \in \mathcal{A}$ and $\phi \in \mathcal{A}^*$ we define

$$\int_A f d\phi = \int f \chi_A d\phi$$

In general,

$$\int_{A \cup B} f d\phi \neq \int_A f d\phi + \int_B f d\phi$$

The q-integral is not even grade-2 additive because, in general

$$\int_{A \cup B \cup C} f d\phi \neq \int_{A \cup B} f d\phi + \int_{A \cup C} f d\phi + \int_{B \cup C} f d\phi - \int_A f d\phi - \int_B f d\phi - \int_C f d\phi$$

3 Reality Filters

This section introduces the three quantum reality filters discussed in the introduction. A *q-measure* is a set function $\mu \colon \mathcal{A} \to \mathbb{R}^+$ that satisfies the *grade*-2 *additivity condition*

$$\mu(A \cup B \cup C) = \mu(A \cup B) + \mu(A \cup C) + \mu(B \cup C) - \mu(A) - \mu(B) - \mu(C) \quad (3.1)$$

Condition (3.1) is more general than the usual (*grade*-1) *additivity* $\mu(A \cup B) = \mu(A) + \mu(B)$ for measures. A q-measure μ is *regular* if it satisfies

(R1) $\mu(A) = 0$ implies $\mu(A \cup B) = \mu(B)$.

(R2) $\mu(A \cup B) = 0$ implies $\mu(A) = \mu(B)$.

It is frequently assumed that a q-measure is regular but, for generality, we shall not make that assumption here. An example of a regular q-measure is a map $\mu(A) = |\nu(A)|^2$ where $\nu \colon \mathcal{A} \to \mathbb{C}$ is a complex measure. Of course, complex measures arise in quantum mechanics as amplitude measures. A more general example of a regular q-measure is a decoherence functional that is employed in the histories approach to quantum mechanics [9, 10, 13]. A q-measure μ is determined by its values on singleton and doubleton sets because it follows from (3.1) and induction that

$$\mu\left(\{\omega_1,\ldots,\omega_m\}\right)=\sum_{i<j=1}^{m}\mu\left(\{\omega_i,\omega_j\}\right)-(m-2)\sum_{i=1}^{m}\mu(\omega_i) \tag{3.2}$$

for all $m\geq 3$, where $\mu(\omega_i)$ is shorthand for $\mu\left(\{\omega_i\}\right)$.

We assume that nature provides us with a fixed q-measure $\mu\colon\mathcal{A}\to\mathbb{R}^+$ and that $\mu(A)$ can be interpreted as the propensity that A occurs. If $\mu(A)=0$, then A does not occur and A is μ*-precluded*. We say that $\phi\in\mathcal{A}^*$ is μ*-preclusive* if $\phi(A)=0$ whenever $\mu(A)=0$. The q-measure μ 1-*generates* $\phi\in\mathcal{A}^*$ if there exists a strictly positive function $f\colon\Omega\to\mathbb{R}$ such that $\mu(A)=\int_A fd\phi$ for all $A\in\mathcal{A}$. We call f a ϕ*-density* for μ. Thus, μ is an "average" over the truth values of ϕ weighted by the density f.

Unfortunately, there are many q-measures that do not 1-generate any coevent. One reason for this is that when $|\Omega|=n$, then $f\colon\Omega\to\mathbb{R}$ gives at most n pieces of information, while a q-measure is determined by its values on singleton and doubleton sets so $n(n+1)/2$ pieces of information may be needed. We therefore introduce a more complicated (and as we shall show, more general) definition. A function $f\colon\Omega\times\Omega\to\mathbb{R}$ is *symmetric* if $f(\omega,\omega')=f(\omega',\omega)$ for all $\omega,\omega'\in\Omega$. Notice that a symmetric function on $\Omega\times\Omega$ has $n(n+1)/2$ possible values. A q-measure μ on $\mathcal{A}$ 2-*generates* $\phi\in\mathcal{A}^*$ if there exists a strictly positive symmetric function $f\colon\Omega\times\Omega\to\mathbb{R}$ such that

$$\mu(A)=\int_A\left[\int_A f(\omega,\omega')d\phi(\omega)\right]d\phi(\omega')$$

for every $A\in\mathcal{A}$. Again, f is a ϕ*-density* for μ. It can be shown that if ϕ is 1- or 2-generated by μ, then ϕ is μ-preclusive [8].

There are still a considerable number of coevents that are not 2-generated by any q-measures. For example, we conjecture that $\omega_1^*\oplus\omega_2^*\oplus\omega_3^*$ is not 2-generated by a q-measure. For this reason we introduce what we believe (but have not yet proved) is a more general condition. A q-measure μ on $\mathcal{A}$ *actualizes* $\phi\in\mathcal{A}^*$ if there exists a strictly positive symmetric function $f\colon\Omega\times\Omega\to\mathbb{R}$ such that

$$\mu(A)=\int\left[\int_A f(\omega,\omega')d\phi(\omega)\right]d\phi(\omega')$$

for all $A \in \mathcal{A}$. Although we have not been able to show that actualization is more general than 1-generation, the next result shows that this is usually the case.

Theorem 3.1. *If μ 1-generates ϕ and $\mu(\Omega) \neq 0$, then μ actualizes ϕ.*

Proof. Since μ 1-generates ϕ, there exists a density f such that $\mu(A) = \int_A f d\phi$ for all $A \in \mathcal{A}$. Define the strictly positive symmetric function $g\colon \Omega \times \Omega \to \mathbb{R}$ by $g(\omega, \omega') = f(\omega)f(\omega')/\mu(\Omega)$. We then have that

$$\int \left[\int_A g(\omega,\omega')d\phi(\omega)\right] d\phi(\omega') = \frac{1}{\mu(\Omega)} \int \left[\int_A f(\omega)f(\omega')d\phi(\omega)\right] d\phi(\omega')$$
$$= \frac{1}{\mu(\Omega)} \int f(\omega')\mu(A)d\phi(\omega') = \mu(A)$$

for all $A \in \mathcal{A}$. Hence, μ actualizes ϕ with density g. □

4 Actualization

The 1- and 2-generation filters have already been considered in [8]. Since we have just introduced the actualization filter in this paper, we shall now discuss it in more detail. Let $\Omega_2 = \{\omega_1, \omega_2\}$, $\mathcal{A}_2 = 2^{\Omega_2}$ and let $\mathcal{A}_2^*$ be the corresponding full anhomomorphic logic. We shall show that every coevent in $\mathcal{A}_2^*$ is actualized and shall characterize the actualizing q-measures. Now $\mathcal{A}_2^*$ has the eight elements 0, ω_1^*, ω_2^*, $\omega_1^* \oplus \omega_2^*$, $\omega_1^*\omega_2^*$, $\omega_1^* \oplus \omega_1^*\omega_2^*$, $\omega_2^* \oplus \omega_1^*\omega_2^*$ and $1 = \omega_1^* \oplus \omega_2^* \oplus \omega_1^*\omega_2^*$. It is clear that 0 is actualized by the zero q-measure. For $a > 0$ we define the *Dirac measure* $a\delta_{\omega_1}$ by

$$a\delta_{\omega_1}(A) = \begin{cases} a & \text{if } \omega_1 \in A \\ 0 & \text{if } \omega_1 \notin A \end{cases}$$

Since

$$a\delta_{\omega_1}(A) = \int \left[\int_A a d\omega_1^*(\omega)\right] d\omega_1^*(\omega')$$

it follows that $a\delta_{\omega_1}$ are the only q-measures that actualize ω_1^* and a similar result holds for ω_2^*.

Lemma 4.1. *A q-measure μ on $\mathcal{A}_2$ actualizes $\omega_1^* \oplus \omega_2^*$ if and only if*

$$\mu(\Omega_2) = \max\left(\mu(\omega_1), \mu(\omega_2)\right) - \min\left(\mu(\omega_1), \mu(\omega_2)\right) \tag{4.1}$$

Proof. Let $\phi = \omega_1^* \oplus \omega_2^*$ and let $f\colon \Omega_2 \times \Omega_2 \to \mathbb{R}$ be a strictly positive symmetric function satisfying

$$f(\omega_1, \omega_2) \leq f(\omega_1, \omega_1) \leq f(\omega_2, \omega_2) \tag{4.2}$$

For $A \in \mathcal{A}_2$ define $g_A\colon \Omega_2 \to \mathbb{R}^+$ by

$$g_A(\omega') = \int_A f(\omega, \omega') d\phi(\omega) \tag{4.3}$$

We then have

$$\begin{aligned} g_{\{\omega_1\}}(\omega_1) &= \int_{\{\omega_1\}} f(\omega, \omega_1) d\phi(\omega) = f(\omega_1, \omega_1) \\ g_{\{\omega_1\}}(\omega_2) &= \int_{\{\omega_1\}} f(\omega, \omega_2) d\phi(\omega) = f(\omega_1, \omega_2) \end{aligned}$$

Hence, if μ actualizes ϕ with ϕ-density f then

$$\mu(\omega_1) = \int g_{\{\omega_1\}}(\omega') d\phi(\omega') = f(\omega_1, \omega_1) - f(\omega_1, \omega_2)$$

and similarly, $\mu(\omega_2) = f(\omega_2, \omega_2) - f(\omega_1, \omega_2)$. We also have that

$$\begin{aligned} g_{\Omega_2}(\omega_1) &= \int f(\omega, \omega_1) d\phi(\omega) = f(\omega_1, \omega_1) - f(\omega_1, \omega_2) \\ g_{\Omega_2}(\omega_2) &= \int f(\omega, \omega_2) d\phi(\omega) = f(\omega_2, \omega_2) - f(\omega_1, \omega_2) \end{aligned}$$

Hence,

$$\begin{aligned} \mu(\Omega_2) &= \int g_{\Omega_2}(\omega') d\phi(\omega') = f(\omega_2, \omega_2) - f(\omega_1, \omega_1) = \mu(\omega_2) - \mu(\omega_1) \\ &= \max\left(\mu(\omega_1), \mu(\omega_2)\right) - \min\left(\mu(\omega_1), \mu(\omega_2)\right) \end{aligned}$$

The "only if" statement of the theorem holds because we obtain similar results for all orderings of $f(\omega_1, \omega_1)$, $f(\omega_2, \omega_2)$, $f(\omega_1, \omega_2)$.

Conversely, suppose that (4.1) holds. We can assume without loss of generality that $\mu(\omega_1) \leq \mu(\omega_2)$ so that $\mu(\Omega_2) = \mu(\omega_2) - \mu(\omega_1)$. Let $f\colon \Omega_2 \times \Omega_2 \to \mathbb{R}$ be the strictly positive symmetric function given by $f(\omega_1, \omega_2) =$

$f(\omega_2, \omega_1) = 1$, $f(\omega_i, \omega_i) = \mu(\omega_i) + 1$, $i = 1, 2$. Then (4.2) holds so by our previous work we have for $i = 1, 2$ that

$$\int \left[\int_{\{\omega_i\}} f(\omega, \omega') d\phi(\omega) \right] d\phi(\omega') = f(\omega_i, \omega_i) - f(\omega_1, \omega_2) = \mu(\omega_i)$$

and

$$\begin{aligned} \int \left[\int f(\omega, \omega') d\phi(\omega) \right] d\phi(\omega') &= f(\omega_2, \omega_2) - f(\omega_1, \omega_1) \\ &= \mu(\omega_2) - \mu(\omega_1) = \mu(\Omega_2) \end{aligned}$$

Hence, μ actualizes ϕ with ϕ-density f. □

Notice if we let $\mu(\omega_2) = 0$ and $\mu(\omega_1) = \mu(\Omega_2) = 1$ in Lemma 4.1, μ becomes the Dirac measure δ_{ω_1}. This shows that δ_{ω_1} actualizes both ω_1^* and $\omega_1^* \oplus \omega_2^*$ so actualizing q-measures need not be unique. Also, note that $\omega_1^* \oplus \omega_2^*$ is not δ_{ω_1}-preclusive.

Lemma 4.2. *A q-measure μ on $\mathcal{A}_2$ actualizes $\omega_1^* \omega_2^*$ if and only if $\mu(\omega_1) = \mu(\omega_2) = 0$ and $\mu(\Omega_2) > 0$*

Proof. Let $\phi = \omega_1^* \omega_2^*$ and let $f \colon \Omega_2 \times \Omega_2 \to \mathbb{R}$ be a strictly positive symmetric function satisfying

$$f(\omega_1, \omega_2) \leq f(\omega_1, \omega_1) \leq f(\omega_2, \omega_2)$$

Employing the notation of (4.3) we have that

$$g_{\{\omega_1\}}(\omega_1) = \int_{\{\omega_1\}} f(\omega, \omega_1) d\phi(\omega) = 0$$

and similarly, $g_{\{\omega_1\}}(\omega_2) = 0$. Hence, if μ actualizes ϕ with ϕ-density f then

$$\mu(\omega_1) = \int g_{\{\omega_1\}}(\omega') d\phi(\omega') = 0$$

and similarly, $\mu(\omega_2) = 0$. We also have that

$$\begin{aligned} g_{\Omega_2}(\omega_1) &= \int f(\omega, \omega_1) d\phi(\omega) = f(\omega_1, \omega_2) \\ g_{\Omega_2}(\omega_2) &= \int f(\omega, \omega_2) d\phi(\omega) = f(\omega_1, \omega_2) \end{aligned}$$

Hence,

$$\mu(\Omega_2) = \int g_{\Omega_2}(\omega')d\phi(\omega') = f(\omega_1, \omega_2)$$

The "only if" statement of the theorem holds because we obtain similar results for all orderings of $f(\omega_1, \omega_1)$, $f(\omega_2, \omega_2)$, $f(\omega_1, \omega_2)$.

Conversely, suppose that $\mu(\omega_1) = \mu(\omega_2) = 0$ and $\mu(\Omega_2) > 0$. Let $f\colon \Omega_2 \times \Omega_2 \to \mathbb{R}$ be the strictly positive symmetric function given by $f(\omega_i, \omega_j) = \mu(\Omega_2)$, $i = 1, 2$. By our previous work μ actualizes ϕ with ϕ-density f. □

Notice that the q-measure in Lemma 4.2 is not regular.

Lemma 4.3. *A q-measure μ on $\mathcal{A}_2$ actualizes $1 = \omega_1^* \oplus \omega_2^* \oplus \omega_1^*\omega_2^*$ if and only if $\mu(\omega_1), \mu(\omega_2) \neq 0$ and $\mu(\Omega_2) = \max(\mu(\omega_1), \mu(\omega_2))$.*

Proof. Let $\phi = 1$ and let $f\colon \Omega_2 \times \Omega_2 \to \mathbb{R}$ be a strictly positive symmetric function satisfying

$$f(\omega_1, \omega_2) \leq f(\omega_1, \omega_1) \leq f(\omega_2, \omega_2)$$

Employing the notation of (4.3) we have that

$$g_{\{\omega_1\}}(\omega_1) = \int_{\{\omega_1\}} f(\omega, \omega_1)d\phi(\omega) = f(\omega_1, \omega_1)$$

$$g_{\{\omega_1\}}(\omega_2) = \int_{\{\omega_1\}} f(\omega, \omega_2)d\phi(\omega) = f(\omega_1, \omega_2)$$

Hence, if μ actualizes ϕ with ϕ-density f then

$$\mu(\omega_1) = \int g_{\{\omega_1\}}(\omega')d\phi(\omega') = f(\omega_1, \omega_1)$$

and similarly, $\mu(\omega_2) = f(\omega_2, \omega_2)$. We also have that

$$g_{\Omega_2}(\omega_1) = \int f(\omega, \omega_1)d\phi(\omega) = f(\omega_1, \omega_1)$$

$$g_{\Omega_2}(\omega_2) = \int f(\omega, \omega_2)d\phi(\omega) = f(\omega_2, \omega_2)$$

Hence,

$$\begin{aligned}\mu(\Omega_2) &= \int g_{\Omega_2}(\omega')d\phi(\omega') = f(\omega_2, \omega_2) = \mu(\omega_2)\\ &= \max(\mu(\omega_1), \mu(\omega_2))\end{aligned}$$

The "only if" statement of the theorem holds because we obtain similar results for all orderings of $f(\omega_1,\omega_1)$, $f(\omega_2,\omega_2)$, $f(\omega_1,\omega_2)$.

Conversely, suppose that $\mu(\omega_1),\mu(\omega_2)\neq 0$ and $\mu(\Omega_2)=\max\left(\mu(\omega_1),\mu(\omega_2)\right)$. We can assume without loss of generality that $\mu(\omega_1)\leq\mu(\omega_2)$ so that $\mu(\Omega_2)=\mu(\omega_2)$. Define the strictly positive symmetric function $f\colon\Omega_2\times\Omega_2\to\mathbb{R}$ by $f(\omega_2,\omega_2)=\mu(\omega_2)$ and

$$f(\omega_1,\omega_2)=f(\omega_2,\omega_1)=f(\omega_1,\omega_1)=\mu(\omega_1)$$

By our previous work, μ actualizes ϕ with ϕ-density f. □

Lemma 4.4. *A q-measure μ on $\mathcal{A}_2$ actualizes $\omega_1^*\oplus\omega_1^*\omega_2^*$ if and only if $\mu(\omega_2)=0$.*

Proof. Let $\phi=\omega_1^*\oplus\omega_1^*\omega_2^*$. If μ actualizes ϕ then $\mu(\omega_2)=0$ because $\phi(\omega_2)=0$. Conversely, suppose $\mu(\omega_2)=0$. Let $f\colon\Omega_2\times\Omega_2\to\mathbb{R}$ be the strictly positive symmetric function given by $f(\omega_2,\omega_2)=1$,

$$\begin{aligned}f(\omega_1,\omega_2)&=f(\omega_2,\omega_1)=1+\mu(\Omega_2)+\mu(\omega_1)\\ f(\omega_1,\omega_1)&=1+\mu(\Omega_2)+2\mu(\omega_1)\end{aligned}$$

Since $g_{\{\omega_1\}}(\omega_1)=f(\omega_1,\omega_1)$, $g_{\{\omega_1\}}(\omega_2)=f(\omega_1,\omega_2)$ we have that

$$\int g_{\{\omega_2\}}(\omega')d\phi(\omega')=f(\omega_1,\omega_1)-f(\omega_1,\omega_2)=\mu(\omega_1)$$

Of course, $g_{\{\omega_2\}}=0$ so that

$$\int g_{\{\omega_2\}}(\omega')d\phi(\omega')=0=\mu(\omega_2)$$

We also have

$$\begin{aligned}g_{\Omega_2}(\omega_2)&=\int f(\omega,\omega_1)d\phi(\omega)=f(\omega_1,\omega_1)-f(\omega_1,\omega_2)=\mu(\omega_1)\\ g_{\Omega_2}(\omega_2)&=\int f(\omega,\omega_2)d\phi(\omega)=f(\omega_1,\omega_2)-f(\omega_2,\omega_2)=\mu(\Omega_2)+\mu(\omega_1)\end{aligned}$$

Hence,

$$\int g_{\Omega_2}(\omega')d\phi(\omega')=g_{\Omega_2}(\omega_2)-g_{\Omega_2}(\omega_1)=\mu(\Omega_2)$$

We conclude that μ actualizes ϕ with ϕ-density f. □

These lemmas show that the actualization filter need not produce a unique coevent. That is, a q-measure may actualize more than one coevent. For example let f be the density function given by $f(\omega_1,\omega_1)=2$ and

$$f(\omega_1,\omega_2)=f(\omega_2,\omega_1)=f(\omega_2,\omega_2)=1$$

Then the Dirac measure δ_{ω_1} actualizes both $\omega_1^*\oplus\omega_2^*$ and $\omega_1^*\oplus\omega_1^*\omega_2^*$ with density f. However, $\omega_1^*\oplus\omega_2^*$ is not δ_{ω_1}-preclusive so a preclusivity filter would eliminate $\omega_1^*\oplus\omega_2^*$. Unfortunately, δ_{ω_1} also actualizes ω_1^* with density $g(\omega_i,\omega_j)=1$, $i,j=1,2$. Of course, all these coevents are quadratic. By contrast, we shall show in Section 5 that if a quadratic coevent is 1- or 2-generated by a q-measure, then it is unique.

Let $\Omega_3=\{\omega_1,\omega_2,\omega_3\}$, $\mathcal{A}_3=2^{\Omega_3}$ and let $\mathcal{A}_3^*$ be the corresponding full anhomomorphic logic. Since $|\mathcal{A}_3^*|=2^7=128$ we cannot discuss them all so we consider a few examples.

Example 1. We have shown in [8] that $\phi=\omega_1^*\oplus\omega_2^*\oplus\omega_3^*$ is not 1-generated and we conjecture that ϕ is also not 2-generated. We now show that ϕ is actualized by the q-measure μ on $\mathcal{A}_3$ given by $\mu(\emptyset)=0$, $\mu(\omega_1)=5$, $\mu(\omega_2)=\mu(\{\omega_2,\omega_3\})=3$, $\mu(\omega_3)=\mu(\{\omega_1,\omega_2\})=6$, $\mu(\{\omega_1,\omega_3\})=9$ and $\mu(\Omega_3)=4$. To show that μ is indeed a q-measure we have that

$$\sum_{i<j=1}^{3}\mu\left(\{\omega_i,\omega_j\}\right)-\sum_{i=1}^{3}\mu(\omega_i)=6+9+3-5-3-6=4=\mu(\Omega_3)$$

Define the density function f by $f(\omega_1,\omega_2)=f(\omega_1,\omega_3)=1$, $f(\omega_2,\omega_3)=f(\omega_1,\omega_1)=5$, $f(\omega_2,\omega_2)=7$, $f(\omega_3,\omega_3)=10$. For all $A\in\mathcal{A}_3$ let

$$\mu'(A)=\int\left[\int_A f(\omega,\omega')d\phi(\omega)\right]d\phi(\omega') \tag{4.4}$$

Employing the notation (4.3) we obtain

$$\begin{aligned}&g_{\{\omega_1\}}(\omega_1)=f(\omega_1,\omega_1)=5,\quad g_{\{\omega_1\}}(\omega_2)=f(\omega_1,\omega_2)=1,\\&g_{\{\omega_1\}}(\omega_3)=f(\omega_1,\omega_3)=1\end{aligned}$$

It follows that

$$\mu'(\omega_1)=f(\omega_1,\omega_2)+f(\omega_1,\omega_1)-f(\omega_1,\omega_3)=5$$

In a similar way we have that

$$\mu'(\omega_2) = f(\omega_1,\omega_2) + f(\omega_2,\omega_2) - f(\omega_2,\omega_3) = 3$$
$$\mu'(\omega_3) = f(\omega_1,\omega_3) + f(\omega_3,\omega_3) - f(\omega_2,\omega_3) = 6$$

We also have that

$$g_{\{\omega_1,\omega_2\}}(\omega_1) = f(\omega_1,\omega_1) - f(\omega_1,\omega_2) = 4$$
$$g_{\{\omega_1,\omega_2\}}(\omega_2) = f(\omega_2,\omega_2) - f(\omega_1,\omega_2) = 6$$
$$g_{\{\omega_1,\omega_2\}}(\omega_3) = f(\omega_2,\omega_3) - f(\omega_1,\omega_3) = 4$$

and hence, $\mu'\left(\{\omega_1,\omega_2\}\right) = 6$. Continuing the computations gives

$$g_{\{\omega_1,\omega_3\}}(\omega_1) = f(\omega_1,\omega_1) - f(\omega_1,\omega_3) = 4$$
$$g_{\{\omega_1,\omega_3\}}(\omega_2) = f(\omega_2,\omega_3) - f(\omega_1,\omega_2) = 4$$
$$g_{\{\omega_1,\omega_3\}}(\omega_3) = f(\omega_3,\omega_3) - f(\omega_1,\omega_3) = 9$$

and hence, $\mu'\left(\{\omega_1,\omega_3\}\right) = 9$. We also have that

$$g_{\{\omega_2,\omega_3\}}(\omega_1) = f(\omega_1,\omega_3) - f(\omega_1,\omega_2) = 0$$
$$g_{\{\omega_2,\omega_3\}}(\omega_2) = f(\omega_2,\omega_2) - f(\omega_2,\omega_3) = 2$$
$$g_{\{\omega_2,\omega_3\}}(\omega_3) = f(\omega_3,\omega_3) - f(\omega_2,\omega_3) = 5$$

and hence, $\mu'\left(\{\omega_2,\omega_3\}\right) = 3$. Finally,

$$g_{\Omega_3}(\omega_1) = f(\omega_1,\omega_2) + f(\omega_1,\omega_1) - f(\omega_1,\omega_3) = 5$$
$$g_{\Omega_3}(\omega_2) = f(\omega_1,\omega_2) + f(\omega_2,\omega_2) - f(\omega_2,\omega_3) = 3$$
$$g_{\Omega_3}(\omega_3) = f(\omega_1,\omega_2) + f(\omega_3,\omega_3) - f(\omega_2,\omega_3) = 6$$

and hence, $\mu'(\Omega_3) = 4$. Since $\mu(A) = \mu'(A)$ for all $A \in \mathcal{A}_3$ we conclude that μ actualizes ϕ. $\square$

Example 2. This example shows that δ_{ω_1} actualizes the coevent $\phi = \omega_1^* \oplus \omega_1^*\omega_2^*\omega_3^*$ with density given by $f(\omega_i,\omega_j) = 1$ for $i \neq j = 1,2,3$, $f(\omega_i,\omega_i) = 2$, $i = 1,2,3$. As before, define μ' by (**??**). As in previous calculations, we have $g_{\{\omega_1\}}(\omega_i) = f(\omega_1,\omega_i)$, $i = 1,2,3$ and hence,

$$\mu'(\omega_1) = f(\omega_1,\omega_1) - f(\omega_1,\omega_3) = 1$$

It is clear that

$$g_{\{\omega_2\}}(\omega_i) = g_{\{\omega_3\}}(\omega_i) = 0$$

for $i = 1, 2, 3$ and hence, $\mu'(\omega_2) = \mu'(\omega_3) = 0$. We also have that

$$g_{\{\omega_1,\omega_2\}}(\omega_i) = g_{\{\omega_1,\omega_3\}}(\omega_i) = f(\omega_1, \omega_i)$$

for $i = 1, 2, 3$ and hence,

$$\mu'\left(\{\omega_1, \omega_2\}\right) = \mu'\left(\{\omega_1\omega_3\}\right) = f(\omega_1, \omega_1) - f(\omega_1, \omega_2) = 1$$

Moreover, $g_{\{\omega_2,\omega_3\}}(\omega_i) = 0$ for $i = 1, 2, 3$ so that $\mu'\left(\{\omega_2, \omega_3\}\right) = 0$. Finally, $g_{\Omega_3}(\omega_1) = f(\omega_1, \omega_1) - f(\omega_1, \omega_2) = 1$, $g_{\Omega_3}(\omega_2) = g_{\Omega_3}(\omega_3) = 0$ so that $\mu'(\Omega_3) = 1$. Since $\delta_{\omega_1}(A) = \mu'(A)$ for all $A \in \mathcal{A}_3$ we conclude that δ_{ω_1} actualizes ϕ. Of course, δ_{ω_1} also actualizes ω_1^* so we again have nonuniqueness. □

Example 3. Calculations similar to those in the previous two examples show that $\phi = \omega_1^* \oplus \omega_2^* \oplus \omega_3^* \oplus \omega_1^*\omega_2^*$ is actualized by the q-measure μ given by $\mu(\emptyset) = 0$, $\mu(\omega_1) = \mu(\Omega_3) = 1$, $\mu(\omega_2) = \mu\left(\{\omega_1, \omega_2\}\right) = 4$, $\mu(\omega_3) = \mu\left(\{\omega_1, \omega_3\}\right) = 3$ and $\mu\left(\{\omega_2, \omega_3\}\right) = 2$. The corresponding density is given by

$$\begin{aligned} &f(\omega_1, \omega_1) = f(\omega_2, \omega_2) = 2 \\ &f(\omega_3, \omega_3) = f(\omega_1, \omega_2) = 4 \\ &f(\omega_1, \omega_3) = 5,\ f(\omega_2, \omega_3) = 8 \end{aligned}$$

It can be shown that ϕ is also actualized by the q-measure ν given by $\nu(\emptyset) = 0$,

$$\begin{aligned} &\nu(\omega_1) = \nu\left(\{\omega_2, \omega_3\}\right) = \nu(\Omega_3) = 0 \\ &\nu(\omega_2) = \nu(\omega_3) = \nu\left(\{\omega_1, \omega_2\}\right) = \nu\left(\{\omega_1, \omega_3\}\right) = 1 \end{aligned}$$

The corresponding density is given by $f(\omega_i, \omega_j) = 1$ for $i, j = 1, 2, 3$, $(i, j) \neq (2, 3)$ or $(3, 2)$ and $f(\omega_2, \omega_3) = 2$. In the second case, ϕ is not ν-preclusive.□

5 Generation

This section discusses existence and uniqueness properties of 1- and 2-generated coevents. We first consider existence. It is clear that any coevent in $\mathcal{A}_2^*$ is 1- and 2-generated (and actualized). We now discuss coevents in $\mathcal{A}^* = \mathcal{A}_n^*$ that are 1-generated.

A q-measure on $\mathcal{A}$ whose only values are 0 or 1 is called a *pure q-measure*. A pure q-measure can also be thought of as a coevent in $\mathcal{A}^*$ and such coevents are called *pure coevents*. Thus, a pure coevent is an element of $\mathcal{A}^*$ that is also a q-measure. Although this appears to be rather specialized, there are quite a few pure coevents and most q-measures can be written as convex combinations of pure q-measures.

It is clear that any $\phi \in \mathcal{A}_2^*$ is a pure coevent. It can be shown that of the 128 coevents in $\mathcal{A}_3^*$, 34 are pure [7].

Example 4. Examples of pure coevents in $\mathcal{A}_3^*$ are ω_1^*, $\omega_1^* \oplus \omega_2^*$, $\omega_1^* \oplus \omega_1^*\omega_2^*$, $\omega_1^*\omega_2^*$, $\omega_1^* \oplus \omega_2^* \oplus \omega_1^*\omega_2^*$, $\omega_1^* \oplus \omega_1^*\omega_2^* \oplus \omega_2^*\omega_3^*$, $\omega_1^* \oplus \omega_2^* \oplus \omega_1^*\omega_2^* \oplus \omega_1\omega_3^*$, $\omega_1^* \oplus \omega_1^*\omega_2^* \oplus \omega_1^*\omega_3^* \oplus \omega_2^*\omega_3^*$, $\omega_1^* \oplus \omega_2^* \oplus \omega_3^* \oplus \omega_1^*\omega_2^* \oplus \omega_1^*\omega_3^* \oplus \omega_2^*\omega_3^*$ and the rest are obtained by symmetry. An example of a $\phi \in \mathcal{A}_3^*$ that is not pure is $\phi = \omega_1^* \oplus \omega_2^* \oplus \omega_3^*$. Indeed, $\phi(\Omega_3) = 1$ and

$$\phi\left(\{\omega_1,\omega_2\}\right) + \phi\left(\{\omega_1,\omega_3\}\right) + \phi\left(\{\omega_2,\omega_3\}\right) - \phi(\omega_1) - \phi(\omega_2) - \phi(\omega_3) = -3$$

Another example of a nonpure element of $\mathcal{A}_3^*$ is $\psi = \omega_1^* \oplus \omega_2^*\omega_3^*$. Indeed, $\psi(\Omega_3) = 0$ and

$$\psi\left(\{\omega_1,\omega_2\}\right) + \psi\left(\{\omega_1,\omega_3\}\right) + \psi\left(\{\omega_2,\omega_3\}\right) - \psi(\omega_1) - \psi(\omega_2) - \psi(\omega_3) = 2 \quad \square$$

Lemma 5.1. *If $\phi \in \mathcal{A}^*$ is pure, then ϕ is quadratic.*

Proof. We must show that if ϕ satisfies

$$\phi(A \cup B \cup C) = \phi(A \cup B) + \phi(A \cup C) + \phi(B \cup C) - \phi(A) - \phi(B) - \phi(C) \quad (5.1)$$

then ϕ satisfies

$$\phi(A \cup B \cup C) = \phi(A \cup B) \oplus \phi(A \cup C) \oplus \phi(B \cup C) \oplus \phi(A) \oplus \phi(B) \oplus \phi(C) \quad (5.2)$$

Suppose the left hand side of (5.2) is 1. Then there are an odd number of 1s on the right hand side of (5.1). Hence, the right hand side of (5.2) is 1. Suppose the left hand side of (5.2) is 0. Then there are an even number of 1s on the right hand side of (5.1). Hence, the right hand side of (5.2) is 0. We conclude that (5.2) holds so ϕ is quadratic. $\square$

The converse Lemma 5.1 does not hold. For instance, in Example 4 we showed that the quadratic coevent $\omega_1^* \oplus \omega_2^* \oplus \omega_3^*$ is not pure.

Theorem 5.2. *A coevent $\phi \in \mathcal{A}^*$ is 1-generated if and only if ϕ is pure.*

Proof. It is clear that if ϕ is pure, then ϕ 1-generates itself. Conversely, suppose $\phi \in \mathcal{A}_n^*$ is 1-generated by the q-measure μ with ϕ-density f. We can reorder the $\omega_i \in \Omega_n$ if necessary and assume that $f(\omega_i) = a_i$, $i = 1, \ldots, n$, where $0 < a_1 \leq a_2 \leq \cdots \leq a_n$. Since $\mu(A) = \int_A f d\phi$ for every $A \in \mathcal{A}_n$ we have $\mu(\omega_i) = a_i\phi(\omega_i)$, $i = 1, 2, \ldots, n$ and for $i < j = 1, 2, \ldots, n$ that

$$\mu\left(\{\omega_i, \omega_j\}\right) = \int_{\{\omega_i,\omega_j\}} f d\phi = a_i\phi\left(\{\omega_i, \omega_j\}\right) + (a_j - a_i)\phi(\omega_j) \tag{5.3}$$

Letting $i < j < k$ with $i, j, k = 1, \ldots, n$, and $A = \{\omega_i, \omega_j, \omega_k\}$ we have

$$\mu(A) = \int_A f d\phi = a_i\phi(A) + (a_j - a_i)\phi\left(\{\omega_j, \omega_k\}\right) + (a_k - a_j)\phi(\omega_k) \tag{5.4}$$

Applying (5.3) and grade-2 additivity of μ gives

$$\begin{aligned}\mu(A) = {} & a_i\phi\left(\{\omega_i, \omega_j\}\right) + (a_j - a_i)\phi(\omega_j) + a_i\phi\left(\{\omega_i, \omega_k\}\right) + (a_k - a_i)\phi(\omega_k) \\ & + a_j\phi\left(\{\omega_j, \omega_k\}\right) + (a_k - a_j)\phi(\omega_k) - a_i\phi(\omega_i) - a_j\phi(\omega_j) - a_k\phi(\omega_k)\end{aligned} \tag{5.5}$$

Equating (5.4) and (5.5) we see that all the terms cancel except those with a factor of a_i. Canceling the a_i gives

$$\phi(A) = \phi\left(\{\omega_i, \omega_j\}\right) + \phi\left(\{\omega_i, \omega_k\}\right) + \phi\left(\{\omega_j, \omega_k\}\right) - \phi(\omega_i) - \phi(\omega_j) - \phi(\omega_k) \tag{5.6}$$

We can now proceed by induction to show that ϕ satisfies (3.2) and thus is a q-measure. Instead of carrying out the general induction step which is quite cumbersome, we shall verify (3.2) for $B = \{\omega_i, \omega_j, \omega_k, \omega_l\}$ where $i, j, k, l \in \{1, 2, \ldots n\}$ with $i < j < k < l$. As in (5.3) we have by (5.6) that

$$\begin{aligned}\mu(B) &= \int_B f d\phi = a_i\phi(B) + (a_j - a_i)\phi\left(\{\omega_j, \omega_k, \omega_l\}\right) \\ &\quad + (a_k - a_j)\phi\left(\{\omega_k, \omega_l\}\right) + (a_l - a_k)\phi(\omega_l) \\ &= a_i\phi(B) + (a_j - a_i)\left[\phi\left(\{\omega_j, \omega_k\}\right) + \phi\left(\{\omega_j, \omega_l\}\right) + \phi\left(\{\omega_k, \omega_l\}\right)\right. \\ &\quad \left.-\phi(\omega_j) - \phi(\omega_k) - \phi(\omega_l)\right] + (a_k - a_j)\phi\left(\{\omega_k, \omega_l\}\right) + (a_l - a_k)\phi(\omega_l)\end{aligned} \tag{5.7}$$

Again, applying (5.3) and grade-2 additivity of μ gives

$$\begin{aligned}\mu(B) = {} & a_i\phi\left(\{\omega_i,\omega_j\}\right) + (a_j - a_i)\phi(\omega_j) + a_i\phi\left(\{\omega_i,\omega_k\}\right) + (a_k - a_i)\phi(\omega_k)\\ & + a_i\phi\left(\{\omega_i,\omega_l\}\right) + (a_l - a_i)\phi(\omega_l) + a_j\phi\left(\{\omega_j,\omega_k\}\right) + (a_k - a_j)\phi(\omega_k)\\ & + a_j\phi\left(\{\omega_j,\omega_l\}\right) + (a_l - a_j)\phi(\omega_l) + a_k\phi\left(\{\omega_k,\omega_l\}\right) + (a_l - a_k)\phi(\omega_l)\\ & - 2\left[a_i\phi(\omega_i) + a_j\phi(\omega_j) + a_k\phi(\omega_k) + a_l\phi(\omega_l)\right] \qquad (5.8)\end{aligned}$$

Equating (5.7) and (5.8) we see that all the terms cancel except those with a factor a_i. Canceling the a_i shows that (3.2) holds for ϕ with $m = 4$. □

It follows from Theorem 5.2 that if ϕ is 1-generated then ϕ is 2-generated. Indeed if ϕ is 1-generated then ϕ is 2-generated by itself because

$$\int_A\left[\int_A d\phi(\omega)\right]d\phi(\omega') = \int_A \phi(A)d\phi = \phi(A)^2 = \phi(A)$$

Similar to Theorem 3.1 it also follows from Theorem 5.2 that if ϕ is unital and 1-generated then ϕ is actualized. Indeed, we then have

$$\int\left[\int_A d\phi(\omega)\right]d\phi(\omega') = \int \phi(A)d\phi = \phi(A)\phi(\Omega) = \phi(A)$$

Using these filters we see that the actual reality corresponding to a pure q-measure is itself. The next example shows that 2-generation is strictly more general than 1-generation.

Example 5. It is easy to check that $\phi = \omega_1^*\oplus\omega_2^*\oplus\omega_3^*\oplus\omega_1^*\omega_2^*$ is not pure so ϕ is not 1-generated. Let μ be the q-measure on $\mathcal{A}_3$ defined by $\mu(\emptyset) = \mu(\Omega_3) = 0$, $\mu(\omega_3) = \mu\left(\{\omega_1,\omega_2\}\right) = 2$,

$$\mu(\omega_1) = \mu(\omega_2) = \mu\left(\{\omega_1,\omega_3\}\right) = \mu\left(\{\omega_2,\omega_3\}\right) = 1$$

It can be shown that μ 2-generates ϕ with ϕ-density f given by $f(\omega_1,\omega_1) = f(\omega_2,\omega_2) = 1$ and

$$f(\omega_3,\omega_3) = f(\omega_1,\omega_2) = f(\omega_1,\omega_3) = f(\omega_2,\omega_3) = 2$$ □

The next two results concern the uniqueness of 1- and 2-generated coevents.

Theorem 5.3. *If μ 1-generates ϕ and ψ then $\phi = \psi$.*

Proof. We have that

$$\mu(A) = \int_A f d\phi = \int_A g d\psi$$

for all $A \in \mathcal{A}$ where f is a ϕ-density and g is a ψ-density for μ. Since

$$\int_{\{\omega\}} f d\phi = f(\omega)\phi(\omega)$$

we conclude that $f(\omega)\phi(\omega) = g(\omega)\psi(\omega)$ for every $\omega \in \Omega$. Hence, $\phi(\omega) = 1$ if and only if $\psi(\omega) = 1$ and in this case

$$f(\omega) = g(\omega) = \mu(\omega)$$

Thus, ϕ and ψ agree on singleton sets and $\mu(\omega) = 0$ if and only if $\phi(\omega) = 0$. Suppose that $0 < \mu(\omega_1) \leq \mu(\omega_2)$. Then $f(\omega_1) \leq f(\omega_2)$ and

$$\phi(\omega_1) = \phi(\omega_2) = \psi(\omega_1) = \psi(\omega_2) = 1$$

Similarly, $g(\omega_1) \leq g(\omega_2)$. Since

$$\int_{\{\omega_1, \omega_2\}} f d\phi = \int_{\{\omega_1, \omega_2\}} g d\psi$$

we have that

$$\begin{aligned} & f(\omega_1)\phi\left(\{\omega_1, \omega_2\}\right) + \left[f(\omega_2) - f(\omega_1)\right]\phi(\omega_2) \\ & \quad = g(\omega_1)\psi\left(\{\omega_1, \omega_2\}\right) + \left[g(\omega_2) - g(\omega_1)\right]\psi(\omega_2) \end{aligned} \tag{5.9}$$

Hence, $\phi\left(\{\omega_1, \omega_2\}\right) = \psi\left(\{\omega_1, \omega_2\}\right)$. Next suppose that $\mu(\omega_1) = 0$ and $\mu(\omega_2) \neq 0$. Then $\phi(\omega_1) = \psi(\omega_1) = 0$ and $\phi(\omega_2) = \psi(\omega_2) = 1$. Again (5.9) holds. Hence, $\phi\left(\{\omega_1, \omega_2\}\right) = 0$ if and only if $\psi\left(\{\omega_1, \omega_2\}\right) = 0$. The case $\mu(\omega_1) = \mu(\omega_2) = 0$ is similar. We conclude that ϕ and ψ agree on doubleton sets. Since ϕ and ψ are quadratic by Lemma 5.1, and quadratic coevents are determined by their values on singleton and doubleton sets, ϕ and ψ coincide. □

Theorem 5.4. *If μ 2-generates ϕ and ψ and both ϕ and ψ are quadratic, then $\phi = \psi$.*

Proof. We have that

$$\mu(A) = \int_A \left[\int_A f(\omega,\omega')d\phi(\omega)\right] d\phi(\omega')$$
$$= \int_A \left[\int_A g(\omega,\omega')d\psi(\omega)\right] d\psi(\omega') \tag{5.10}$$

for all $A \in \mathcal{A}$. Letting $A = \{\omega\}$ in (5.10) we conclude that

$$f(\omega,\omega)\phi(\omega) = g(\omega,\omega)\psi(\omega)$$

for all $\omega \in \Omega$. Hence, $\phi(\omega) = 1$ if and only if $\psi(\omega) = 1$ and in this case $f(\omega,\omega) = g(\omega,\omega) = \mu(\omega)$. We conclude that ϕ and ψ agree on singleton sets and $\mu(\omega) = 0$ if and only if $\psi(\omega) = 0$. Suppose $0 < \mu(\omega_1) \le \mu(\omega_2)$. Then $f(\omega_1,\omega_1) \le f(\omega_2,\omega_2)$ and

$$\phi(\omega_1) = \phi(\omega_2) = \psi(\omega_1) = \psi(\omega_2) = 1$$

Assume that $f(\omega_2,\omega_2) \le f(\omega_1,\omega_2)$.
Case 1. $g(\omega_1,\omega_1) \le g(\omega_2,\omega_2) \le g(\omega_1,\omega_2)$
Letting

$$h_f(\omega') = \int_{\{\omega_1,\omega_2\}} f(\omega,\omega')d\phi(\omega), \quad h_g(\omega') = \int_{\{\omega_1,\omega_2\}} g(\omega,\omega')d\phi(\omega)$$

we have that

$$h_f(\omega_1) = \mu(\omega_1)\phi\left(\{\omega_1,\omega_2\}\right) + f(\omega_1,\omega_2) - \mu(\omega_1)$$
$$h_f(\omega_2) = \mu(\omega_2)\phi\left(\{\omega_1,\omega_2\}\right) + f(\omega_1,\omega_2) - \mu(\omega_2)$$

Hence,

$$\begin{aligned}\mu\left(\{\omega_1,\omega_2\}\right) &= \int_{\{\omega_1,\omega_2\}} h_f(\omega')d\phi(\omega') \\ &= \left[\mu(\omega_2)\phi\left(\{\omega_1,\omega_2\}\right) + f(\omega_1,\omega_2) - \mu(\omega_2)\right]\phi\left(\{\omega_1,\omega_2\}\right) \\ &\quad + \left[\mu(\omega_1) - \mu(\omega_2)\right]\phi\left(\{\omega_1,\omega_2\}\right) + \mu(\omega_2) - \mu(\omega_1)\end{aligned} \tag{5.11}$$

If $\phi\left(\{\omega_1,\omega_2\}\right) = 0$, then $\mu\left(\{\omega_1,\omega_2\}\right) = \mu(\omega_2) - \mu(\omega_1)$. Since (5.11) also applies for ψ, we conclude that $\psi\left(\{\omega_1,\omega_2\}\right) = 0$. If $\phi\left(\{\omega_1,\omega_2\}\right) = 1$, then $\mu\left(\{\omega_1,\omega_2\}\right) = f(\omega_1,\omega_2)$. Again, (5.11) also applies for ψ so $\psi\left(\{\omega_1,\omega_2\}\right) = 1$.

Case 2. $g(\omega_1, \omega_1) \leq g(\omega_1, \omega_2) \leq g(\omega_2, \omega_2)$
We now have that

$$h_g(\omega_1) = \mu(\omega_1)\psi\left(\{\omega_1, \omega_2\}\right) + g(\omega_1, \omega_2) - \mu(\omega_1)$$
$$h_g(\omega_2) = g(\omega_1, \omega_2)\psi\left(\{\omega_1, \omega_2\}\right) + \mu(\omega_2) - g(\omega_1, \omega_2)$$

If $\psi\left(\{\omega_1, \omega_2\}\right) = 1$, then

$$\mu\left(\{\omega_1, \omega_2\}\right) = \int_{\{\omega_1, \omega_2\}} h_g(\omega')d\psi(\omega') = \mu(\omega_2)$$

and hence $f(\omega_1, \omega_2) = \mu(\omega_2)$ so that $\phi\left(\{\omega_1, \omega_2\}\right) = 1$. If $\psi\left(\{\omega_1, \omega_2\}\right) = 0$, then

$$\mu\left(\{\omega_1, \omega_2\}\right) = \int_{\{\omega_1, \omega_2\}} h_g(\omega')d\psi(\omega') = 2g(\omega_1, \omega_2) - \mu(\omega_1) - \mu(\omega_2)$$

or $\mu\left(\{\omega_1, \omega_2\}\right) = \mu(\omega_2) + \mu(\omega_1) - 2g(\omega_1, \omega_2)$ whichever is nonnegative. If $\phi\left(\{\omega_1, \omega_2\}\right) = 1$, then

$$2g(\omega_1, \omega_2) - \mu(\omega_1) - \mu(\omega_2) = f(\omega_1, \omega_2) \geq \mu(\omega_2)$$

so that $2g(\omega_1, \omega_2) \geq 2\mu(\omega_2) + \mu(\omega_1)$ which is a contradiction. We could also have

$$\mu(\omega_2) + \mu(\omega_1) - 2g(\omega_1, \omega_2) = f(\omega_1, \omega_2) \geq \mu(\omega_2)$$

so that $2g(\omega_1, \omega_2) \leq \mu(\omega_1)$ which is a contradiction. We conclude that $\psi\left(\{\omega_1, \omega_2\}\right) = 0$ if and only if $\phi\left(\{\omega_1, \omega_2\}\right) = 0$.

Case 3. $g(\omega_1, \omega_2) \leq g(\omega_1, \omega_1) \leq g(\omega_2, \omega_2)$
We now have that

$$h_g(\omega_1) = g(\omega_1, \omega_2)\psi\left(\{\omega_1, \omega_2\}\right) + \mu(\omega_1) - g(\omega_1, \omega_2)$$
$$h_g(\omega_2) = g(\omega_1, \omega_2)\psi\left(\{\omega_1, \omega_2\}\right) + \mu(\omega_2) - g(\omega_1, \omega_2)$$

Hence,

$$\begin{aligned}\mu\left(\{\omega_1, \omega_2\}\right) &= \int_{\{\omega_1, \omega_2\}} h_g(\omega')d\psi(\omega') \\ &= \left[g(\omega_1, \omega_2)\psi\left(\{\omega_1, \omega_2\}\right) + \mu(\omega_1) - g(\omega_1, \omega_2)\right]\psi\left(\{\omega_1, \omega_2\}\right) \\ &\quad + \mu(\omega_2) - \mu(\omega_1)\end{aligned}$$

If $\psi(\{\omega_1,\omega_2\}) = 0$, then $\mu(\{\omega_1,\omega_2\}) = \mu(\omega_2) - \mu(\omega_1)$ so that $\phi(\{\omega_1,\omega_2\}) = 0$. If $\psi(\{\omega_1,\omega_2\}) = 1$, then

$$\mu(\{\omega_1,\omega_2\}) = \mu(\omega_2) > \mu(\omega_2) - \mu(\omega_1)$$

This implies that $\phi(\{\omega_1,\omega_2\}) = 1$. There are other cases, including the case where $f(\omega_1,\omega_2) \le f(\omega_2,\omega_2)$, but the results are similar. It follows that in all possible cases where $\mu(\omega_1), \mu(\omega_2) \ne 0$ we have $\phi(\{\omega_1,\omega_2\}) = \psi(\{\omega_1,\omega_2\})$.

We now consider the situation in which $\mu(\omega_1) = \mu(\omega_2) = 0$. Then

$$\phi(\omega_1) = \phi(\omega_2) = \psi(\omega_1) = \psi(\omega_2) = 0$$

Assume that

$$f(\omega_1,\omega_1) \le f(\omega_2,\omega_2) \le f(\omega_1,\omega_2) \tag{5.12}$$

and

$$g(\omega_1,\omega_1) \le g(\omega_2,\omega_2) \le g(\omega_1,\omega_2)$$

We then have that

$$h_f(\omega_1) = f(\omega_1,\omega_1)\phi(\{\omega_1,\omega_2\}), \quad h_f(\omega_2) = f(\omega_2,\omega_2)\phi(\{\omega_1,\omega_2\})$$

Hence,

$$\mu(\{\omega_1,\omega_2\}) = f(\omega_1,\omega_1)\phi(\{\omega_1,\omega_2\}) = g(\omega_1,\omega_1)\psi(\{\omega_1,\omega_2\})$$

and $\phi(\{\omega_1,\omega_2\}) = \psi(\{\omega_1,\omega_2\})$. All the other cases in this situation are similar.

The last situation that needs to be considered is $\mu(\omega_1) = 0$, $\mu(\omega_2) > 0$. Then $\phi(\omega_1) = \psi(\omega_1) = 0$, $\phi(\omega_2) = \psi(\omega_2) = 1$ and

$$f(\omega_2,\omega_2) = g(\omega_2,\omega_2) = \mu(\omega_2)$$

Assuming that (5.12) holds, we treat the three cases considered before.

Case 1. $g(\omega_1,\omega_1) \le g(\omega_2,\omega_2) \le g(\omega_1,\omega_2)$
We now have that

$$\begin{aligned} h_f(\omega_1) &= f(\omega_1,\omega_1)\phi(\{\omega_1,\omega_2\}) + f(\omega_1,\omega_2) - f(\omega_1,\omega_1) \\ h_f(\omega_2) &= \mu(\omega_2)\phi(\{\omega_1,\omega_2\}) \end{aligned}$$

Since we obtain similar results for h_g, we conclude that

$$\mu(\{\omega_1,\omega_2\}) = \mu(\omega_2)\phi(\{\omega_1,\omega_2\}) = \mu(\omega_2)\psi(\{\omega_1,\omega_2\})$$

Hence, $\phi(\{\omega_1,\omega_2\}) = \psi(\{\omega_1,\omega_2\})$.

Case 2. $g(\omega_1,\omega_1) \le g(\omega_1,\omega_2) \le g(\omega_2,\omega_2)$
As in Case 1 we have that $\mu(\{\omega_1,\omega_2\}) = \mu(\omega_2)\phi(\{\omega_1,\omega_2\})$. Also,

$$\begin{aligned} h_g(\omega_1) &= g(\omega_1,\omega_1)\psi(\{\omega_1,\omega_2\}) + g(\omega_1,\omega_2) - g(\omega_1,\omega_1) \\ h_g(\omega_2) &= g(\omega_1,\omega_2)\psi(\{\omega_1,\omega_2\}) + \mu(\omega_2) - g(\omega_1,\omega_2) \end{aligned}$$

If $\psi(\{\omega_1,\omega_2\}) = 1$, then

$$\mu(\{\omega_1,\omega_2\}) = \mu(\omega_2) = \mu(\omega_2)\phi(\{\omega_1,\omega_2\})$$

so that $\phi(\{\omega_1,\omega_2\}) = 1$. If $\psi(\{\omega_1,\omega_2\}) = 0$, then

$$\mu(\{\omega_1,\omega_2\}) = \mu(\omega_2) + g(\omega_1,\omega_1) - 2g(\omega_1,\omega_2) = \mu(\omega_2)\phi(\{\omega_1,\omega_2\})$$

If $\phi(\{\omega_1,\omega_2\}) = 1$ we obtain the contradiction, $g(\omega_1,\omega_1) = 2g(\omega_1,\omega_2)$ so $\phi(\{\omega_1,\omega_2\}) = 0$. Alternatively, we could have $\mu(\{\omega_1,\omega_2\}) = 0$ so again, $\phi(\{\omega_1,\omega_2\}) = 0$.

Case 3. $g(\omega_1,\omega_2) \le g(\omega_1,\omega_1) \le g(\omega_2,\omega_2)$
We now have that

$$\begin{aligned} \mu(\{\omega_1,\omega_2\}) &= g(\omega_1,\omega_2)\psi(\{\omega_1,\omega_2\}) + \mu(\omega_2) - g(\omega_1,\omega_2) \\ &= \mu(\omega_2)\phi(\{\omega_1,\omega_2\}) \end{aligned}$$

Just as in Case 2 we conclude that $\psi(\{\omega_1,\omega_2\}) = \phi(\{\omega_1,\omega_2\})$. The other cases are similar to the three cases considered.

We have shown that ϕ and ψ coincide for all singleton and doubleton sets. Since ϕ and ψ are quadratic, it follows that $\phi = \psi$. □

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