Lifting Module Maps Between Different Noncommutative Domain Algebras

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March 17, 2010

Abstract

In this paper we use a renorming technique to lift module maps between $*$-invariant submodules of domain algebras recently introduced by Popescu.

1 Introduction

In [8], Popescu studied noncommutative domains $\mathcal{D}_f(\mathcal{H}) \subset B(\mathcal{H})^n$ generated by positive regular free holomorphic functions f. He proved that each such domain has a universal model $(W_1, W_2, \ldots, W_n)$ of weighted orthogonal shifts acting on the full Fock space with n generators. In this paper, we will look at a unitarily equivalent version of the weighted Fock spaces used by Popescu, which we will denote $\mathcal{F}^2(f)$. We will use a Hilbert module language similar to that of Douglas and Paulsen [5] and Muhly and Solel [6].

As stated in [8], a commutant lifting theorem for $\mathcal{F}^2(f)$ follows directly from [1], which was adapted from a paper of Clancy and McCullough [4]. Assume $\mathcal{M}$ and $\mathcal{N}$ are both $*$-submodules of $\mathcal{F}^2(f)$ and $X : \mathcal{M} \to \mathcal{N}$ is a non-zero module map. This commutant lifting theorem states that there exists a module map $\widehat{X} : \mathcal{F}^2(f) \to \mathcal{F}^2(f)$ with $\|X\| = \left\|\widehat{X}\right\|$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}^2(f) \arrow[r, two heads, "P_{\mathcal{M}}"] \arrow[d, "\exists"', "\widehat{X}"] & \mathcal{M} \arrow[r] \arrow[d, "X"] & 0 \\
\mathcal{F}^2(f) \arrow[r, two heads, "P_{\mathcal{N}}"] & \mathcal{N} \arrow[r] & 0
\end{tikzcd}$$

Notice here that both $\mathcal{N}$ and $\mathcal{M}$ are $*$-submodules of the same Hilbert module $\mathcal{F}^2(f)$. In this paper we ask a more general question. What if $\mathcal{M}$ and $\mathcal{N}$ are instead $*$-submodules of $\mathcal{F}^2(f)$ and $\mathcal{F}^2(g)$ respectively? This gives the following diagram:

$$\begin{tikzcd}
\mathcal{F}^2(f) \arrow[r, two heads, "P_{\mathcal{M}}"] & \mathcal{M} \arrow[r] \arrow[d, "X"] & 0 \\
\mathcal{F}^2(g) \arrow[r, two heads, "P_{\mathcal{N}}"] & \mathcal{N} \arrow[r] & 0
\end{tikzcd}$$

In this case, there is certainly no reason why we should be able to lift X to a module map between $\mathcal{F}^2(f)$ and $\mathcal{F}^2(g)$ in general. Indeed, there are examples of spaces $\mathcal{F}^2(f)$ and $\mathcal{F}^2(g)$ such that there does not exist a non-zero module map $\widehat{X} : \mathcal{F}^2(f) \to \mathcal{F}^2(g)$ of any kind. But if non-zero module maps do exist between $\mathcal{F}^2(f)$ and $\mathcal{F}^2(g)$, under what cases can a lifting theorem be proved?

Assuming that the "formal identity map" $\varepsilon : \mathcal{F}^2(f) \to \mathcal{F}^2(g)$ is a contraction, the commutant lifting theorems of [1] and [4] can be adjusted with some care to prove that there exists an $\widehat{X}$ with $\|X\| = \left\|\widehat{X}\right\|$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}^2(f) \arrow[r, two heads, "P_{\mathcal{M}}"] \arrow[d, "\widehat{X}", "\exists"'] & \mathcal{M} \arrow[r] \arrow[d, "X"] & 0 \\
\mathcal{F}^2(g) \arrow[r, two heads, "P_{\mathcal{N}}"] & \mathcal{N} \arrow[r] & 0
\end{tikzcd}$$

We follow the standard argument of lifting element by element, then iterating using Parrott's lemma [7]. However, this technique breaks down if the "formal identity map" $\varepsilon : \mathcal{F}^2(f) \to \mathcal{F}^2(g)$ is not a contraction.

In the case where $\mathcal{F}^2(g)$ is the full Fock space and $\varepsilon : \mathcal{F}^2(f) \to \mathcal{F}^2(g)$ is bounded, it immediately follows that ε is a contraction, and thus any module map $X : \mathcal{M} \to \mathcal{N}$ can be lifted to an $\widehat{X} : \mathcal{F}^2(f) \to \mathcal{F}^2(g)$ with $\|X\| = \left\|\widehat{X}\right\|$. This suggests that $X : \mathcal{M} \to \mathcal{N}$ can be lifted (with some penalty in the norm) in general, as long as there exists a non-zero module map between $\mathcal{F}^2(f)$ and $\mathcal{F}^2(g)$.

To prove that this is indeed the case, we apply a renorming technique to $\mathcal{F}^2(f)$ to create the situation where we can apply our lifting theorem. It turns out that if the "formal identity map" $\varepsilon : \mathcal{F}^2(f) \to \mathcal{F}^2(g)$ has norm $\|\varepsilon\| = C$, then there exists a module map $\widehat{X} : \mathcal{F}^2(f) \to \mathcal{F}^2(g)$ with $\left\|\widehat{X}\right\| \leq C\,\|X\|$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}^2(f) \arrow[r, two heads, "P_{\mathcal{M}}"] \arrow[d, "\widehat{X}", "\exists"'] & \mathcal{M} \arrow[r] \arrow[d, "X"] & 0 \\
\mathcal{F}^2(g) \arrow[r, two heads, "P_{\mathcal{N}}"] & \mathcal{N} \arrow[r] & 0
\end{tikzcd}$$

In section 2 we will give a brief overview overview of the Hilbert module language. Section 3 will provide a unitarily equivalent description of the Fock spaces given in [1] and [8] which will allow some aspects of weighted Fock spaces to be addressed with greater transparency. Finally, a lifting theorem for a specific class of weighted Fock spaces will be given in section 4. For convenience, we will assume that our functions f and g are finite.

2 Hilbert Modules

Let $\mathcal{H}$ be a Hilbert space, and $L_i : \mathcal{H} \to \mathcal{H}$ be bounded linear operators for $1 \leq i \leq n$. Then $(\mathcal{H}; L_1, L_2, \ldots, L_n)$ is called a Hilbert module over the free algebra generated by n-noncommutative variables. In this paper, we will primarily consider Hilbert modules of the form $(\mathcal{F}^2(f) \otimes \mathcal{H}; L_1 \otimes I_{\mathcal{H}}, L_2 \otimes I_{\mathcal{H}}, \ldots, L_n \otimes I_{\mathcal{H}})$, where $\mathcal{F}^2(f)$ is a weighted Fock space, defined in the next section, and $\mathcal{H}$ is an arbitrary Hilbert space. If $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ is invariant under $L_i \otimes I_{\mathcal{H}}$ for $1 \leq i \leq n$, we say that $(\mathcal{N}; L_1 \otimes I_{\mathcal{H}}, L_2 \otimes I_{\mathcal{H}}, \ldots, L_n \otimes I_{\mathcal{H}})$ is a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$. Similarly, if $\mathcal{M} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ is invariant under $(L_i \otimes I_{\mathcal{H}})^*$ for $1 \leq i \leq n$ and $V_i = P_{\mathcal{M}}(L_i \otimes I_{\mathcal{H}})|_{\mathcal{M}}$ for $1 \leq i \leq n$, we say that $(\mathcal{M}; V_1, V_2, \ldots, V_n)$ is a $*$-submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$.

Let $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ and an algebra $A \subset B(\mathcal{F}^2(f) \otimes \mathcal{H})$ be given. Then $\mathcal{N}$ is said to be semi-invariant under A if for every $a, b \in A$ it follows that $P_N a P_N b P_N = P_N ab P_N$. In [9] Sarason proved that $\mathcal{N}$ is semi-invariant if and only if there exist two submodules $\mathcal{N}_1$ and $\mathcal{N}_2$ with $\mathcal{N}_1 \oplus \mathcal{N} = \mathcal{N}_2$. Now if $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ is semi-invariant under $L_i \otimes I_{\mathcal{H}}$ for $i \leq n$ and $W_i = P_N(L_i \otimes I_{\mathcal{H}})|_{\mathcal{N}}$ for $i \leq n$, then $(\mathcal{N}; W_1, W_2, \ldots, W_n)$ is called a subquotient of $\mathcal{F}^2(f) \otimes \mathcal{H}$. It is easy to see that every submodule and every $*$-submodule is a subquotient, but not every subquotient is a submodule or a $*$-submodule.

Let $X : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator between the Hilbert modules $(\mathcal{H}; L_1, L_2, \ldots, L_n)$ and $(\mathcal{K}; V_1, V_2, \ldots, V_n)$. If $X(L_i h) = V_i(Xh)$ for every $h \in \mathcal{H}$ and $1 \leq i \leq n$, we say that X is a module map. For our purposes, it is important to determine when the orthogonal projection and the inclusion map are module maps. It is well known and easy to verify that $P_{\mathcal{N}} : \mathcal{F}^2(f) \otimes \mathcal{H} \to \mathcal{N}$ is a module map if and only if $\mathcal{N} \subset \mathcal{F}^2(f) \otimes \mathcal{H}$ is a $*$-submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$. Similarly, the inclusion map $\iota : \mathcal{N} \to \mathcal{F}^2(f) \otimes \mathcal{H}$ is a module map if and only if $\mathcal{N}$ is a submodule of $\mathcal{F}^2(f) \otimes \mathcal{H}$.

3 Non-commutative domains as module maps - Weighted Fock Spaces

In this section we give a unitarily equivalent description of the non-commutative domain algebras of Popescu. One representation of the full Fock space is given using the free semigroup on n generators. Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators, $g_1, g_2, \ldots, g_n$, together with the identity g_0. Define the length of an element $\alpha \in \mathbb{F}_n^+$ as $|\alpha| := 0$ if $\alpha = g_0$ and $|\alpha| := k$ if $\alpha = g_{i_1} g_{i_2} \cdots g_{i_k}$, where $1 \leq i_1, i_2, \cdots, i_k \leq n$. For each $\alpha \in \mathbb{F}_n^+$, define

$$e_\alpha := \begin{cases} e_{g_{i_1}} \otimes e_{g_{i_2}} \otimes \ldots \otimes e_{g_{i_k}} & if \quad \alpha = g_{i_1} g_{i_2} \cdots g_{i_k} \\ 1 & if \quad \alpha = g_0 \end{cases}$$

It is clear that the set $\{e_\alpha : \alpha \in \mathbb{F}_n^+\}$ is an orthonormal basis for the full Fock space $l_2(\mathbb{F}_n^+)$.

For each $i = 1, 2, \ldots, n$, the left creation operators $L_i : l_2(\mathbb{F}_n^+) \to l_2(\mathbb{F}_n^+)$ are defined by $L_i e_\alpha = e_{g_i \alpha}$ for every $\alpha \in \mathbb{F}_n^+$. Similarly, for each $i = 1, 2, \ldots, n$, the right creation operators $R_i : l_2(\mathbb{F}_n^+) \to l_2(\mathbb{F}_n^+)$ are defined by $R_i e_\alpha = e_{\alpha g_i}$ for every $\alpha \in \mathbb{F}_n^+$. The left and right creation operators are isometries with orthogonal ranges. Furthermore, they are row contractions: $\sum\limits_{i=1}^{n} L_i L_i^* \leq I$, $\sum\limits_{i=1}^{n} R_i R_i^* \leq I$.

For a Hilbert space $\mathcal{H}$, let $B(\mathcal{H})$ be the algebra of all bounded linear operators over $\mathcal{H}$. If $(T_1, T_2, \ldots, T_n) \in B(\mathcal{H})^n$ and $\alpha = g_{i_1} g_{i_2} \ldots g_{i_k} \in \mathbb{F}_n^+$, define T_α to be the product $T_{i_1} T_{i_2} \ldots T_{i_k}$.

Definition 1 *(Popescu, [8]) Look at the formal power series over n free variables* $(X_1, X_2, \ldots, X_n)$ *given by* $f = \sum\limits_{\alpha \in \mathbb{F}_n^+} a_\alpha^f X_\alpha$ *for scalar* a_α^f. *Then* f *is a positive regular free holomorphic function on* $B(\mathcal{H})^n$ *if the following properties on* a_α^f *are satisfied:*

$$\begin{aligned} a_{g_0}^f &= 0 \\ a_{g_i}^f &> 0 \qquad 1 \leq i \leq n \\ a_\alpha^f &\geq 0 \qquad \alpha \in \mathbb{F}_n^+, \ |\alpha| > 1 \\ \limsup_{k \longrightarrow \infty} \left(\sum_{|\alpha|=k} \left| a_\alpha^f \right|^2 \right)^{\frac{1}{2k}} &< \infty \end{aligned}$$

In [8], Popescu proved that the a_α's given above induce a family of positive real numbers $(b_\alpha)_{\alpha \in \mathbb{F}_n^+}$ with the following properties:

$$\begin{aligned} b_{g_0} &= 1 \\ b_\alpha &> 0 \text{ for all } \alpha \in \mathbb{F}_n^+ \\ b_\gamma &= \sum_{\substack{\beta\alpha = \gamma \\ |\beta| \geq 1}} a_\beta b_\alpha \text{ if } |\gamma| \geq 1 \\ b_\gamma &= \sum_{\substack{\alpha\beta = \gamma \\ |\beta| \geq 1}} a_\beta b_\alpha \text{ if } |\gamma| \geq 1 \\ b_\alpha b_\beta &\leq b_{\alpha\beta} \text{ for any } \alpha, \beta \in \mathbb{F}_n^+ \end{aligned}$$

Theorem 2 *(G. Popescu, [8]) Let* $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^f X_\alpha$ *be a positive regular free holomorphic function. If we define* $W_i^f : \mathcal{F}^2(f) \to \mathcal{F}^2(f)$ *by* $W_i^f(e_\alpha^f) = \sqrt{\frac{b_\alpha}{b_{g_i\alpha}}} e_{g_i\alpha}^f$, *then:*

$$\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^f W_\alpha^f W_\alpha^{f*} \leq I$$

Furthermore, if $(T_1, T_2, \ldots, T_n)$ *is an* n*-tuple of operators acting on a Hilbert space* $\mathcal{H}$*, then* $\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha^f T_\alpha T_\alpha^* \leq I$ *if and only if there exists a unique unital completely contractive morphism* Φ *from the algebra generated by* $\left\{W_1^f, W_2^f, \ldots, W_n^f\right\}$ *into* $B(\mathcal{H})$ *such that* $W_i^f \mapsto T_i$ *for* $1 \leq i \leq n$.

If we consider the Fock space $l_2(\mathbb{F}_n^+)$ together with the weighted shifts $W_1^f, W_2^f, \ldots, W_n^f$ defined in the previous theorem, we get the Hilbert module $(l_2(\mathbb{F}_n^+); W_1^f, W_2^f, \ldots, W_n^f)$.

It is more useful for us to work with a unitarily equivalent version of the weighted Fock space, which we will call $\mathcal{F}^2(f)$. For a positive regular free holomorphic function f, set $\mathcal{F}^2(f)$ to be the Hilbert space with a complete orthogonal basis $\left\{\delta_\alpha^f : \alpha \in \mathbb{F}_n^+\right\}$, where the norm is given by $\left\|\delta_\alpha^f\right\| = \frac{1}{\sqrt{b_\alpha^f}}$. For each $i \leq n$, define the left creation operators $L_i : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(f)$ by $L_i\delta_\alpha = \delta_{g_i\alpha}$. Similarly, for each $i \leq n$, define the right creation operators $R_i : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(f)$ by $R_i\delta_\alpha = \delta_{\alpha g_i}$. We obtain this unitarily equivalent version via the unitary operator $U : l_2(\mathbb{F}_n^+) \longrightarrow \mathcal{F}^2(f)$ defined as $Ue_\alpha = \sqrt{b_\alpha}\delta_\alpha$. Furthermore, $U^*\delta_\alpha = \frac{1}{\sqrt{b_\alpha}} e_\alpha$. Verifying that this unitary satisfies the conditions is easy.

The correspondence with Popescu's version is as follows, with Popescu's version on the left and our version on the right:

$$\begin{array}{rcl}
l_2(\mathbb{F}_n^+) & \Longleftrightarrow & \mathcal{F}^2(f) \\
e_\alpha & \Longleftrightarrow & \delta_\alpha \\
a_\alpha & \Longleftrightarrow & a_\alpha \\
\|e_\alpha\| = 1 & \Longleftrightarrow & \|\delta_\alpha\| = \sqrt{\frac{1}{b_\alpha}} \\
W_\beta e_\alpha = \sqrt{\frac{b_\alpha}{b_{\beta\alpha}}} e_{\beta\alpha} & \Longleftrightarrow & L_\beta\delta_\alpha = \delta_{\beta\alpha} \\
\|W_\beta\| = \sqrt{\frac{1}{b_\beta}} & \Longleftrightarrow & \|L_\beta\| = \sqrt{\frac{1}{b_\beta}} \\
\Lambda_\beta e_\alpha = \sqrt{\frac{b_\alpha}{b_{\beta\alpha}}} e_{\alpha\beta} & \Longleftrightarrow & R_\beta\delta_\alpha = \delta_{\alpha\beta} \\
\|\Lambda_\beta\| = \sqrt{\frac{1}{b_\beta}} & \Longleftrightarrow & \|R_\beta\| = \sqrt{\frac{1}{b_\beta}}
\end{array}$$

If we consider the weighted Fock space $\mathcal{F}^2(f)$ together with the weighted shifts $L_1^f, L_2^f, \ldots, L_n^f$, we get the Hilbert module $(\mathcal{F}^2(f); L_1^f, L_2^f, \ldots, L_n^f)$. If the underlying positive regular free holomorphic function is understood, we will refer to L_α^f and δ_α^f simply as L_α and δ_α, respectively.

It turns out that not only is U a unitary operator, but it is also a module map, as shown in the following lemma whose proof we omit.

Lemma 3 *Let* $(l_2(\mathbb{F}_n^+); W_1^f, W_2^f, \ldots, W_n^f)$ *and* $\left(\mathcal{F}^2(f); L_1, L_2, \ldots, L_n\right)$ *be given, and let* $Ue_\alpha = \sqrt{b_\alpha}\delta_\alpha$ *as above. Then* $U : l_2(\mathbb{F}_n^+) \longrightarrow \mathcal{F}^2(f)$ *is an isometric module map.*

Notice that the b_α's end up being identical between spaces, as they are solely determined by the a_α's. Let $\alpha = g_{i_1}g_{i_2}\ldots g_{i_k}$ be given. Denote by $\widetilde{\alpha}$ the reverse of α, $\widetilde{\alpha} = g_{i_k}\ldots g_{i_2}g_{i_1}$. It is clear to see that if $f(X_1, X_2, \ldots, X_n) = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha X_\alpha$ is a positive regular free holomorphic function on $B(\mathcal{H})^n$, then so is $\widetilde{f}(X_1, X_2, \ldots, X_n) = \sum_{\alpha\in\mathbb{F}_n^+} a_{\widetilde{\alpha}} X_\alpha$. We will now state a result we will need for this work:

Proposition 4 *(G. Popescu, [8]) Let* $(L_1, L_2, \ldots, L_n)$, $(R_1, R_2, \ldots, R_n)$, *and* $f = \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$ *be given. Then:*

(i) $\sum_{|\beta|\geq 1} a_\beta L_\beta L_\beta^* \leq I$

(ii) $\sum_{|\beta|\geq 1} a_{\widetilde{\beta}} R_\beta R_\beta^* \leq I$

3.1 Module maps between $\mathcal{F}^2(f)$ and $\mathcal{F}^2(g)$

Let $(\mathcal{F}^2(f); L_1, L_2, \ldots, L_n)$ and $(\mathcal{F}^2(g); L_1, L_2, \ldots, L_n)$ be two Hilbert modules. Define the "formal identity map" $\varepsilon(f, g) : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$ to be the linear map $\varepsilon(\delta_\alpha^f) = \delta_\alpha^g$. When f and g are clear, we will just refer to $\varepsilon(f, g)$ as ε. By this definition, ε, if bounded, acts like the "formal identity map." However, it is important to note that ε does not have to be isometric, needs not be bounded, and isn't necessarily well-defined. Now an element $x \in \mathcal{F}^2(f)$ looks like $x = \sum_{\alpha\in\mathbb{F}_n^+} x_\alpha\delta_\alpha$, so we can think of an element in $\mathcal{F}^2(f)$ as the sequence $x = \{x_\alpha : \alpha \in \mathbb{F}_n^+\}$. This allows us to define $\mathcal{F}^2(f) \subset \mathcal{F}^2(g)$ as sequence inclusion.

The main result of this section is

Theorem 5 *Let* $(\mathcal{F}^2(f); L_1, L_2, \ldots, L_n)$ *and* $(\mathcal{F}^2(g); L_1, L_2, \ldots, L_n)$ *be two Hilbert modules. Then the following are equivalent:*

(i) $\mathcal{F}^2(f) \subset \mathcal{F}^2(g)$.

(ii) $\varepsilon : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$ *is well defined.*

(iii) $\varepsilon : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$ *is bounded.*

(iv) $\varepsilon : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$ *is a module map.*

(v) $\sup_{\alpha \in \mathbb{F}_n^+} \dfrac{b_\alpha^f}{b_\alpha^g} < \infty$

Furthermore, if f has finitely many terms, then:

(vi) There exists a non-zero module map $X : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$.

We start with a Lemma.

Lemma 6 *Let* $f = \sum_{|\alpha| \geq 1} a_\alpha X_\alpha$, *where f has only finitely many terms. Then there exists $C > 0$ such that for all $\alpha \in \mathbb{F}_n^+$ and $i \leq n$,*

$$\frac{b_\alpha}{C_i} \geq b_{g_i \alpha} \geq b_{g_i} b_\alpha \qquad\qquad \frac{b_\alpha}{C_i} \geq b_{\alpha g_i} \geq b_\alpha b_{g_i}$$

Proof. We will only prove the first inequality. The other one has a similar proof. The right inequality was proven by Popescu in [8]. Thus, it remains to prove the left inequality. Now

$$b_{g_i \alpha} = \sum_{\substack{\beta\gamma = g_i \alpha \\ |\beta| \geq 1}} a_\beta b_\gamma$$

Since f is finite, there exists a $k \in \mathbb{N}$ such that for all α with $|\alpha| > k$, $a_\alpha = 0$. Thus:

$$b_{g_i \alpha} = \sum_{\substack{\beta\gamma = g_i \alpha \\ 1 \leq |\beta| \leq k}} a_\beta b_\gamma \leq \max_{\substack{\beta\gamma = g_i \alpha \\ 1 \leq |\beta| \leq k}} \{a_\beta\} \sum_{\substack{\beta\gamma = g_i \alpha \\ 1 \leq |\beta| \leq k}} b_\gamma$$

Let $A = \max_{1 \leq |\beta| \leq k} \{a_\beta\}$. Then:

$$\begin{aligned} b_{g_i \alpha} &\leq A \sum_{\substack{\sigma\gamma = \alpha \\ 0 \leq |\sigma| \leq k-1}} b_\gamma = A \sum_{\substack{\sigma\gamma = \alpha \\ 0 \leq |\sigma| \leq k-1}} \frac{b_\sigma b_\gamma}{b_\sigma} \leq A \sum_{\substack{\sigma\gamma = \alpha \\ 0 \leq |\sigma| \leq k-1}} \frac{b_{\sigma\gamma}}{b_\sigma} \\ &= A \sum_{\substack{\sigma\gamma = \alpha \\ 0 \leq |\sigma| \leq k-1}} \frac{b_\alpha}{b_\sigma} = A b_\alpha \sum_{\substack{\sigma\gamma = \alpha \\ 0 \leq |\sigma| \leq k-1}} \frac{1}{b_\sigma} \\ &\leq C b_\alpha \end{aligned}$$

where $C = A \sum_{|\sigma| \leq k-1} \frac{1}{b_\sigma}$. Thus, $\frac{b_\alpha}{C} \geq b_{g_i \alpha}$, as desired. ∎

It is important to note that this result does not extend to arbitrary f's, as shown in the following example:

Example 7 *Define* $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha X_\alpha$ *by:*

$$a_\alpha = \begin{cases} 1 & if \quad |\alpha| = 1 \\ 2^k & if \quad \alpha = 12^k 1 \\ 0 & \quad else \end{cases}$$

Let $\alpha = 2^k 1$ *and* $i = 1$*. Let's calculate* $\frac{b_\alpha}{b_{g_i \alpha}}$*:*

$$\frac{b_\alpha}{b_{g_i \alpha}} = \frac{b_{2^k 1}}{b_{12^k 1}} = \frac{\sum_{\substack{\beta\gamma = 2^k 1 \\ |\beta| \geq 1}} a_\beta b_\gamma}{\sum_{\substack{\beta\gamma = 12^k 1 \\ |\beta| \geq 1}} a_\beta b_\gamma} = \frac{(a_2)^k a_1}{a_1 (a_2)^k a_1 + a_{12^k 1}} = \frac{1}{1 + 2^k}$$

This converges to 0 *as* $k \to \infty$*, and therefore, no* $C > 0$ *exists such that* $\frac{b_\alpha}{b_{g_i \alpha}} \geq C$ *and therefore no* C *exists such that* $\frac{b_\alpha}{C} \geq b_{g_i \alpha}$*.*

We are now ready to complete the proof of Theorem 5.

Proof. The following implications are clear: (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i). We will now complete the loop by showing that (i) $\Longrightarrow$ (iv):

Assume that $x_m \longrightarrow x$ in $\mathcal{F}^2(f)$ and that $\varepsilon(x_m) \longrightarrow y$ in $\mathcal{F}^2(g)$. Now we know that $x_m = \sum_{\beta \in \mathbb{F}_n^+} c_{m,\beta} \delta_\beta$. A quick calculation reveals that $x = \sum_{\beta \in \mathbb{F}_n^+} c_\beta \delta_\beta$, where $c_\beta = \lim_{m \to \infty} c_{m,\beta}$ for every $\beta \in \mathbb{F}_n^+$:

$$\left\langle \lim_{m \to \infty} x_m, \delta_\alpha \right\rangle_f = \lim_{m \to \infty} \left\langle \sum_{\beta \in \mathbb{F}_n^+} c_{m,\beta} \delta_\beta, \delta_\alpha \right\rangle_f = \lim_{m \to \infty} c_{m,\alpha} \langle \delta_\alpha, \delta_\alpha \rangle_f$$

On the other hand:

$$\langle x, \delta_\alpha \rangle_f = \left\langle \sum_{\beta \in \mathbb{F}_n^+} c_\beta \delta_\beta, \delta_\alpha \right\rangle_f = c_\alpha \langle \delta_\alpha, \delta_\alpha \rangle_f$$

Notice that the above calculations will look exactly the same in $\mathcal{F}^2(g)$. Since $\varepsilon(\delta_\alpha^f) = \delta_\alpha^g$, it is clear to see that $\varepsilon(x) = y$. Thus, by the closed graph theorem, ε is bounded. Now it is trivial to show that ε is a module map:

$$\varepsilon L_\beta^f \delta_\alpha^f = \varepsilon \delta_{\beta\alpha}^f = \delta_{\beta\alpha}^g = L_\beta^g \delta_\alpha^g = L_\beta^g \varepsilon \delta_\alpha^f$$

As desired. Thus, (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv).

Now we will show that (v) $\Longleftrightarrow$ (iii): Notice the following:

$$\begin{aligned} \|\varepsilon\| &= \sup_{\alpha \in \mathbb{F}_n^+} \left\| \varepsilon \sqrt{b_\alpha^f} \delta_\alpha \right\|_g = \sup_{\alpha \in \mathbb{F}_n^+} \left\| \sqrt{b_\alpha^f} \delta_\alpha \right\|_g \\ &= \sup_{\alpha \in \mathbb{F}_n^+} \sqrt{b_\alpha^f} \|\delta_\alpha\|_g = \sup_{\alpha \in \mathbb{F}_n^+} \frac{\sqrt{b_\alpha^f}}{\sqrt{b_\alpha^g}} \end{aligned}$$

Thus, ε is bounded if and only if $\sup_{\alpha\in\mathbb{F}_n^+} \frac{b_\alpha^f}{b_\alpha^g} < \infty$, as desired. Thus, (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v).

Now assume that f has finitely many terms. Since (iv) $\Longrightarrow$ (vi) always holds, we need only prove (vi) $\Longrightarrow$ (v): Assume $X : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$ is a non-zero module map. Then X is completely determined by what it does to δ_0: $X\delta_0 = \sum_{\beta\in\mathbb{F}_n^+} c_\beta\delta_\beta$. Then in general, we have:

$$X\delta_\alpha = XL_\alpha\delta_0 = L_\alpha X\delta_0 = L_\alpha \sum_{\beta\in\mathbb{F}_n^+} c_\beta\delta_\beta = \sum_{\beta\in\mathbb{F}_n^+} c_\beta L_\alpha\delta_\beta = \sum_{\beta\in\mathbb{F}_n^+} c_\beta\delta_{\alpha\beta}$$

Let β_0 be such that $\beta_0 \neq 0$, but for all other $\beta \in \mathbb{F}_n^+$ such that $|\beta| < |\beta_0|$ it follows that $\beta = 0$. Then:

$$X\delta_\alpha = c_{\beta_0}\delta_{\alpha\beta_0} + \sum_{\substack{|\beta|\geq|\beta_0| \\ \beta\neq\beta_0}} c_\beta\delta_{\alpha\beta}$$

Now $\delta_{\alpha\beta_0}$ is orthogonal to $\delta_{\alpha\beta}$ for every $\beta \neq \beta_0$. Thus:

$$\|X\delta_\alpha\|_g = \left\| c_{\beta_0}\delta_{\alpha\beta_0} + \sum_{\substack{|\beta|\geq|\beta_0| \\ \beta\neq\beta_0}} c_\beta\delta_{\alpha\beta} \right\|_g \geq \|c_{\beta_0}\delta_{\alpha\beta_0}\|_g = |c_{\beta_0}| \, \|\delta_{\alpha\beta_0}\|_g = \frac{|c_{\beta_0}|}{\sqrt{b_{\alpha\beta_0}^g}}$$

By applying lemma 6 iteratively, we can peel off the β_0 on the bottom of the fraction. This gives us:

$$\|X\delta_\alpha\|_g \geq \frac{|c_{\beta_0}|\,K}{\sqrt{b_\alpha^g}}$$

Therefore, it follows that $\frac{1}{\sqrt{b_\alpha^g}} \leq C\,\|X\delta_\alpha\|_g \leq C\,\|X\|\,\|\delta_\alpha\|_f = C\,\|X\|\,\frac{1}{\sqrt{b_\alpha^f}}$. Thus,

$$\frac{\sqrt{b_\alpha^f}}{\sqrt{b_\alpha^g}} \leq C\,\|X\|$$

Since α was arbitrary, it follows that

$$\sup_{\alpha\in\mathbb{F}_n^+} \frac{b_\alpha^f}{b_\alpha^g} < \infty$$

as desired. ■

This leads immediately to the following theorem, which gives a simple method to calculate the norm of the ε map:

Corollary 8 *Let $(\mathcal{F}^2(f); L_1, L_2, \ldots, L_n)$ and $(\mathcal{F}^2(g); L_1, L_2, \ldots, L_n)$ be two Hilbert modules. Assume that f has finitely many terms. Then if $\varepsilon : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$ is a module map, its norm is given by:*

$$\|\varepsilon\| = \sup_{\alpha\in\mathbb{F}_n^+} \frac{\sqrt{b_\alpha^f}}{\sqrt{b_\alpha^g}}$$

3.2 Examples

In order to get some intuition about the b_α's, we will look at some examples. These examples will show some nice combinatorial connections between the a's and the b's for certain classes of functions f:

Example 9 *Let* $f = \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$*, where* $a_\alpha \in \{0,1\}$ *for all* $\alpha \in \mathbb{F}_n^+$*. Then* $b_\alpha = \left|\left\{\alpha_1\alpha_2\cdots\alpha_i = \alpha : a_{\alpha_j} = 1, 1 \leq j \leq i\right\}\right|$*. In other words,* b_α *counts the number of ways you can decompose* α *into subwords* $\alpha = \alpha_1\alpha_2\cdots\alpha_i$ *such that* $a_{\alpha_1} = a_{\alpha_2} = \cdots = a_{\alpha_i} = 1$*.*

Proof. Let $f = \sum_{|\alpha|\geq 1} a_\alpha X_\alpha$, with $a_\alpha \in \{0,1\}$ for all $\alpha \in \mathbb{F}_n^+$ be given. The proof will be by strong induction.

Base: Let $\alpha = g_i$. Now since $a_{g_i} > 0$ for all $i \leq n$ and $a_\alpha \in \{0,1\}$ for all $\alpha \in \mathbb{F}_n^+$, it follows that $a_{g_i} = 1$. Also recall that $b_0 = 1$. Now

$$b_{g_i} = \sum_{\substack{\beta\gamma = g_i \\ |\beta|\geq 1}} a_\beta b_\gamma = a_{g_i} b_0 = 1$$

But there is clearly only one way to decompose $\alpha = g_i$, so we're done.

Induction: Assume the claim is true for all α with $|\alpha| \leq m$. Let α be given such that $|\alpha| = m + 1$. Then

$$b_\alpha = \sum_{\substack{\beta\gamma = \alpha \\ |\beta|\geq 1}} a_\beta b_\gamma$$

By the induction hypothesis, b_γ counts the number of ways γ can be decomposed into subwords $\gamma = \gamma_{1\gamma 2}\cdots\gamma_i$ such that $a_{\gamma_1} = a_{\gamma_2} = \cdots = a_{\gamma_i} = 1$. Now if $a_\beta = 1$, then any of the decompositions of γ will lead to proper decompositions of $\beta\gamma = \alpha$, and so we should count them. This will lead to $a_\beta b_\gamma = b_\gamma$ new decompositions. However, if $a_\beta = 0$, then none of the decompositions of γ will lead to proper decompositions of $\beta\gamma = \alpha$, and so we should not count them. This will lead to $a_\beta b_\gamma = 0$ new decompositions. Finally, summing over all $\beta\gamma = \alpha$ with $|\beta| \geq 1$ gives us the final result. Since we've counted all possible decompositions and only those possible decompositions, it follows that b_α counts the number of ways you can decompose α into subwords $\alpha = \alpha_1\alpha_2\cdots\alpha_i$ such that $a_{\alpha_1} = a_{\alpha_2} = \cdots = a_{\alpha_i} = 1$, as desired. ■

Example 10 *Let* $f = \sum_{i=1}^{n}\left(X_i + \sum_{j=1}^{n} X_{ij}\right)$*. Then* $b_\alpha = F_{|\alpha|+1}$*, where* F_m *is the mth Fibonacci number.*

Proof. Clearly $b_0 = 1 = F_1$ and $b_{g_i} = 1 = F_2$ for all $i \leq n$. Assume $b_\alpha = F_{|\alpha|+1}$for all $|\alpha| < m$. Let α be given such that $|\alpha| = m$. It then follows that

$b_\alpha = \sum_{\substack{\beta\gamma=\alpha \\ |\beta|\geq 1}} a_\beta b_\gamma = a_\rho b_\sigma + a_\tau b_\phi$, where $\rho\sigma = \alpha = \tau\phi$ with $|\rho| = 1$ and $|\tau| = 2$. Thus, $b_\alpha = a_\rho b_\sigma + a_\tau b_\phi = F_{|\sigma|+1} + F_{|\tau|+1} = F_{|\alpha|} + F_{|\alpha|-1} = F_{|\alpha|+1}$ as desired. ■

While the finite case is quite pleasant, an example illustrates the potential difficulties encountered when transitioning to the infinite case:

Example 11 *Look at f and g such that*

$$a_\alpha^f = \begin{cases} 1 & if \quad |\alpha| = 1 \\ 2^k & if \quad \alpha = 12^k 1 \\ 0 & \quad else \end{cases} \qquad a_\alpha^g = \begin{cases} 1 & if \quad |\alpha| = 1 \\ 4^k & if \quad \begin{array}{c} \alpha = 12^k 11 \\ \alpha = 12^k 12 \end{array} \\ 0 & \quad else \end{cases}$$

Look at $\varepsilon : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$. Is this bounded?

$$\begin{aligned} \|\varepsilon\|^2 &= \sup_{\alpha\in\mathbb{F}_n^+} \frac{b_\alpha^f}{b_\alpha^g} \geq \frac{b_{12^k1}^f}{b_{12^k1}^g} = \frac{\sum\limits_{\substack{\gamma\beta=12^k1 \\ |\gamma|\geq 1}} a_\gamma^f b_\beta^f}{\sum\limits_{\substack{\pi\sigma=12^k1 \\ |\pi|\geq 1}} a_\pi^g b_\sigma^g} \\ &= \frac{a_1^f \left(a_2^f\right)^k a_1^f + a_{12^k1}^f}{a_1^g \left(a_2^g\right)^k a_1^g} = \frac{1+2^k}{1} \to \infty \end{aligned}$$

So ε is unbounded. Now look at $Y : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$ by $Y\delta_0 = \delta_1$. Is this bounded? Let α be given. We can decompose α as $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ such that $\alpha_i \in \left\{1, 2, 12^k1\right\}$. This may yield several decompositions. However, it is easy to show that there exists a minimal decomposition in the sense that for every other decomposition $\alpha = \alpha_1'\alpha_2'\cdots\alpha_m'$, $m \geq n$. Next, note that for every i, if $|\alpha_i| = 1$, then $b^f_{\alpha_1\alpha_2\cdots\alpha_{i-1}\alpha_i\alpha_{i+1}\cdots\alpha_n} = b^f_{\alpha_1\alpha_2\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_n}$, and $b^g_{\alpha_1\alpha_2\cdots\alpha_{i-1}\alpha_i\alpha_{i+1}\cdots\alpha_n} \geq b^g_{\alpha_1\alpha_2\cdots\alpha_{i-1}\alpha_{i+1}\cdots\alpha_n}$. Thus, we only need look at the case when $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ with $\alpha_i = 12^{k_i}1$. Thus:

$$\begin{aligned} \frac{b_\alpha^f}{b_{\alpha g_1}^g} &= \frac{\sum\limits_{\substack{\gamma\beta=\alpha \\ |\gamma|\geq 1}} a_\gamma^f b_\beta^f}{\sum\limits_{\substack{\pi\sigma=\alpha 1 \\ |\pi|\geq 1}} a_\pi^g b_\sigma^g} \\ &= \frac{1 + (a_{\alpha_1}^f + \cdots + a_{\alpha_n}^f) + (a_{\alpha_1}^f a_{\alpha_2}^f + \cdots a_{\alpha_{n-1}}^f a_{\alpha_n}^f) + \cdots + (a_{\alpha_1}^f a_{\alpha_2}^f \cdots a_{\alpha_n}^f)}{1 + (a_{\alpha_1}^g + \cdots + a_{\alpha_n}^g) + (a_{\alpha_1}^g a_{\alpha_3}^g + a_{\alpha_1}^g a_{\alpha_4}^g + \cdots a_{\alpha_{n-2}}^g a_{\alpha_n}^g) + \cdots + (a_{\alpha_1}^g a_{\alpha_3}^g \cdots a_{\alpha_n}^g)} \\ &= \frac{1 + (2^{k_1} + \cdots + 2^{k_n}) + (2^{k_1}2^{k_2} + \cdots + 2^{k_{n-1}}2^{k_n}) + \cdots + (2^{k_1}2^{k_2}\cdots 2^{k_n})}{1 + (4^{k_1} + \cdots + 4^{k_n}) + (4^{k_1}4^{k_3} + 4^{k_1}4^{k_4} + \cdots 4^{k_{n-2}}4^{k_n}) + \cdots + (4^{k_1}4^{k_3}\cdots 4^{k_n})} \end{aligned}$$

It isn't difficult to notice that the bottom is always greater than or equal to the top. This is because if the top has n factors, the bottom can have at most $\left\lceil \frac{n}{2} \right\rceil$

factors. Thus, it follows that

$$\sup_{\alpha \in \mathbb{F}_n^+} \frac{b_\alpha^f}{b_\alpha^g} = \frac{b_1^f}{b_1^g} = \frac{1}{1} = 1$$

Thus, Y is bounded. This produces an example where f has infinitely many terms, ε is unbounded, but a nontrivial map is bounded.

4 Lifting Module Maps

If f is a positive regular free holomorphic function, it induces $\{b_\alpha : \alpha \in \mathbb{F}_n^+\}$ with the property that $b_\alpha > 0$ for every $\alpha \in \mathbb{F}_n^+$. We will now generalize this concept by looking at functions $f : \mathbb{F}_n^+ \longrightarrow (0, \infty)$ with $f(\alpha) = b_\alpha^f$ and a Hilbert space $\mathcal{F}^2(f)$ with orthonormal basis $\|\delta_\alpha\| = \sqrt{\frac{1}{b_\alpha^f}}$. It is easy to see that f induces a set $\left\{a_\alpha^f : \alpha \in \mathbb{F}_n^+\right\}$ satisfying:

$$\begin{aligned} b_\gamma^f &= \sum_{\substack{\beta\alpha=\gamma \\ |\beta|\geq 1}} a_\beta^f b_\alpha^f \text{ if } |\gamma| \geq 1 \\ b_\gamma^f &= \sum_{\substack{\alpha\beta=\gamma \\ |\beta|\geq 1}} a_\beta^f b_\alpha^f \text{ if } |\gamma| \geq 1 \end{aligned}$$

However, unlike positive regular free holomorphic functions, a_α^f need not be non-negative. In addition, the left and right creation operators may no longer be bounded, since the norm of the left and right creation operators will be given by:

$$\begin{aligned} \|L_i\|^2 &= \sup_{\alpha \in \mathbb{F}_n^+} \frac{\|\delta_{g_i\alpha}\|^2}{\|\delta_\alpha\|^2} = \sup_{\alpha \in \mathbb{F}_n^+} \frac{b_\alpha^f}{b_{g_i\alpha}^f} \\ \|R_i\|^2 &= \sup_{\alpha \in \mathbb{F}_n^+} \frac{\|\delta_{\alpha g_i}\|^2}{\|\delta_\alpha\|^2} = \sup_{\alpha \in \mathbb{F}_n^+} \frac{b_\alpha^f}{b_{\alpha g_i}^f} \end{aligned}$$

This leads immediately to the following definition:

Definition 12 *A function $f : \mathbb{F}_n^+ \longrightarrow (0, \infty)$ is called L-bounded if $\|L_i\| < \infty$ for every $i \leq n$. Similarly, A function $f : \mathbb{F}_n^+ \longrightarrow (0, \infty)$ is called R-bounded if $\|R_i\| < \infty$ for every $i \leq n$.*

We will look at the weighted Fock spaces generated by L-bounded and R-bounded functions. As with positive regular free holomorphic functions, the weighted Fock space generated by a L-bounded or R-bounded function f will be denoted $\mathcal{F}^2(f)$. Due to the two different definitions we now have for $\mathcal{F}^2(f)$, we will assume that f is a positive regular free holomorphic function unless explicitly stated.

Using this more general idea of L-bounded and R-bounded maps, we are now in the position to modify the proof of the commutant lifting theorem for module maps from [8], which was based on [1] and [4]:

Theorem 13 *Let $f : \mathbb{F}_n^+ \longrightarrow (0, \infty)$ be an L-bounded map and $g = \sum_{|\alpha|\geq 1} a_\alpha^g X_\alpha$ be a positive regular free holomorphic function with $\|\varepsilon(f, g)\| \leq 1$. In addition, let $(\mathcal{M}; V_1^f, V_2^f, \ldots, V_n^f)$ and $(\mathcal{N}; V_1^g, V_2^g, \ldots, V_n^g)$ be $*$-submodules of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \ldots, L_n \otimes I_{\mathcal{H}_1})$ and $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; L_1 \otimes I_{\mathcal{H}_2}, L_2 \otimes I_{\mathcal{H}_2}, \ldots, L_n \otimes I_{\mathcal{H}_2})$, respectively. If $X : \mathcal{M} \longrightarrow \mathcal{N}$ is a module map, then there exists a module map $\widehat{X} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \longrightarrow \mathcal{F}^2(g) \otimes \mathcal{H}_2$ such that $\|X\| = \left\|\widehat{X}\right\|$ and $XP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{X}$, i.e., the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{F}^2(f) \otimes \mathcal{H}_1 \arrow[r, two heads, "P_{\mathcal{M}}"] \arrow[d, "\exists", swap] \arrow[d, "\widehat{X}"] & \mathcal{M} \arrow[r] \arrow[d, "X"] & 0 \\
\mathcal{F}^2(g) \otimes \mathcal{H}_2 \arrow[r, two heads, "P_{\mathcal{N}}"] & \mathcal{N} \arrow[r] & 0
\end{tikzcd}$$

Proof. Let $(\mathcal{M}; V_1^f, V_2^f, \ldots, V_n^f)$ be a $*$-L-submodule of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \ldots, L_n \otimes I_{\mathcal{H}_1})$ and $(\mathcal{N}; V_1^g, V_2^g, \ldots, V_n^g)$ be a $*$-submodule of $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; L_1 \otimes I_{\mathcal{H}_2}, L_2 \otimes I_{\mathcal{H}_2}, \ldots, L_n \otimes I_{\mathcal{H}_2})$. Assume $X : \mathcal{M} \longrightarrow \mathcal{N}$ is a module map with $\|X\| = 1$. Let $Y = XP_{\mathcal{M}}$. Then $Y : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \longrightarrow \mathcal{N}$ is also a module map with $\|Y\| = \|X\| = 1$. Now look at $\delta_0 \otimes \mathcal{H}_2$. If $\delta_0 \otimes \mathcal{H}_2 \not\subseteq \mathcal{N}$, let $\mathcal{N}_1 = \overline{\delta_0 \otimes \mathcal{H}_2 + \mathcal{N}}$. If $\delta_0 \otimes \mathcal{H}_2 \subset \mathcal{N}$, find $\alpha \in \mathbb{F}_n^+$ such that $\delta_\alpha \otimes \mathcal{H}_2 \not\subseteq N$ but if $\alpha = \beta\gamma$ with $|\beta| \geq 1$ then $\delta_\gamma \otimes \mathcal{H}_2 \subset \mathcal{N}$. Then let $\mathcal{N}_1 = \overline{\delta_\alpha \otimes \mathcal{H}_2 + \mathcal{N}}$. Note that in either case,

$$(L_\alpha \otimes I_{\mathcal{H}_2})^* \mathcal{N}_1 \subset \mathcal{N} \qquad \text{for every } |\alpha| \geq 1 \tag{1}$$

Let $T_i^g = P_{\mathcal{N}_1} (L_i^g \otimes I_{\mathcal{H}_2}) P_{\mathcal{N}_1}$ for $i \leq n$. Then $(\mathcal{N}_1; T_1^g, T_2^g, \ldots, T_n^g)$ is a $*$-L-submodule of $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; L_1 \otimes I_{\mathcal{H}_2}, L_2 \otimes I_{\mathcal{H}_2}, \ldots, L_n \otimes I_{\mathcal{H}_2})$, and $\mathcal{N}$ is a $*$-L-submodule of $\mathcal{N}_1$. Now if $|\alpha| \geq 1$, it follows that:

$$T_\alpha^g P_{\mathcal{N}_1 \ominus \mathcal{N}} = 0 \tag{2}$$

Using this identity, a quick computation will reveal that for $|\alpha| \geq 1$

$$T_{\alpha\beta}^g P_{\mathcal{N}} = T_\alpha^g V_\beta^g P_{\mathcal{N}} \tag{3}$$

Now we need to find a module map $\widehat{Y} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \longrightarrow \mathcal{N}_1$ with $\left\|\widehat{Y}\right\| = \|X\|$ and $P_{\mathcal{N}}\widehat{Y} = Y$. For every $\alpha \in \mathbb{F}_n^+$, define $Y_\alpha : \mathcal{H}_1 \longrightarrow \mathcal{N}$ by $Y_\alpha(x) = Y(\delta_\alpha \otimes x)$.

Similarly, define $\widehat{Y}_\alpha : \mathcal{H}_1 \longrightarrow \mathcal{N}_1$ by $\widehat{Y}_\alpha(x) = \widehat{Y}\,(\delta_\alpha \otimes x)$. Now if $\widehat{Y}$ is to be a module map, each $\widehat{Y}_\alpha$, and thus $\widehat{Y}$, will be determined by $\widehat{Y}_0$:

$$\widehat{Y}_\alpha(x) = \widehat{Y}\,(\delta_\alpha \otimes x) = \widehat{Y}\left(\left(L^f_\alpha \otimes I_{\mathcal{H}_1}\right)(\delta_0 \otimes x)\right) = T^g_\alpha \widehat{Y}(\delta_0 \otimes x) = T^g_\alpha \widehat{Y}_0(x)$$

However, if $|\alpha| \geq 1$, then:

$$\widehat{Y}_\alpha(x) \quad = \quad T^g_\alpha \widehat{Y}_0(x) = T^g_\alpha P_{\mathcal{N}_1 \ominus \mathcal{N}} \widehat{Y}_0(x) + T^g_\alpha P_{\mathcal{N}} \widehat{Y}_0(x) = T^g_\alpha P_{\mathcal{N}} \widehat{Y}_0(x) = T^g_\alpha Y_0(x)$$

We can decompose $\mathcal{F}^2(f) \otimes \mathcal{H}_1$ as $\mathcal{F}^2(f) \otimes \mathcal{H}_1 = [\delta_0 \otimes \mathcal{H}_1] \oplus [\delta_0 \otimes \mathcal{H}_1]^\perp$ and $\mathcal{N}_1$ as $\mathcal{N}_1 = [\mathcal{N}_1 \ominus \mathcal{N}] \oplus \mathcal{N}$ and write $\widehat{Y}$ as a block matrix with respect to this decomposition:

$$\widehat{Y} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : [\delta_0 \otimes \mathcal{H}_1] \oplus [\delta_0 \otimes \mathcal{H}_1]^\perp \longrightarrow [\mathcal{N}_1 \ominus \mathcal{N}] \oplus [\mathcal{N}] \tag{4}$$

Now $Y = P_{\mathcal{N}} \widehat{Y} = \begin{bmatrix} c & d \end{bmatrix}$, so the second row of $\widehat{Y}$ is already determined, and $\left\| \begin{bmatrix} c & d \end{bmatrix} \right\| \leq 1$. Similarly, the second column of $\widehat{Y}$ is already determined, since

$$\begin{bmatrix} b \\ d \end{bmatrix}^* = \left(\widehat{Y}|_{[\delta_0 \otimes \mathcal{H}_1]^\perp}\right)^* : \mathcal{N}_1 \longrightarrow \bigoplus_{|\alpha| \geq 1} \delta_\alpha \otimes \mathcal{H}_1$$

is given by the following equation for $x \in \mathcal{N}_1$:

$$\left(\widehat{Y}|_{[\delta_0 \otimes \mathcal{H}_1]^\perp}\right)^* (x) = \sum_{|\alpha| \geq 1} b^f_\alpha \delta_\alpha \otimes \widehat{Y}^*_\alpha(x)$$

Next we need $\left\| \begin{bmatrix} b \\ d \end{bmatrix} \right\| \leq 1$.

$$\begin{aligned} \left\| \begin{bmatrix} b \\ d \end{bmatrix} \right\|^2 &= \left\| \left(\widehat{Y}|_{[\delta_0 \otimes \mathcal{H}_1]^\perp}\right)^* \right\|^2 = \sup_{\|x\|=1} \left\langle \left(\widehat{Y}|_{[\delta_0 \otimes \mathcal{H}_1]^\perp}\right)^* x, \left(\widehat{Y}|_{[\delta_0 \otimes \mathcal{H}_1]^\perp}\right)^* x \right\rangle \\ &= \sup_{\|x\|=1} \left\langle \sum_{|\alpha| \geq 1} b^f_\alpha \delta_\alpha \otimes \widehat{Y}^*_\alpha(x), \sum_{|\beta| \geq 1} b^f_\beta \delta_\beta \otimes \widehat{Y}^*_\beta(x) \right\rangle \\ &= \sup_{\|x\|=1} \sum_{|\alpha| \geq 1} \left(b^f_\alpha\right)^2 \langle \delta_\alpha, \delta_\alpha \rangle \left\langle \widehat{Y}^*_\alpha(x), \widehat{Y}^*_\alpha(x) \right\rangle \\ &= \sup_{\|x\|=1} \left\langle x, \left(\sum_{|\alpha| \geq 1} b^f_\alpha \widehat{Y}_\alpha \widehat{Y}^*_\alpha\right) x \right\rangle \end{aligned}$$

Thus it suffices to show that $\sum_{|\alpha| \geq 1} b^f_\alpha \widehat{Y}_\alpha \widehat{Y}^*_\alpha \leq I$. Notice that

$$\left[\sum_{\gamma \in \mathbb{F}^+_n} b^g_\gamma V^g_\gamma Y_0 Y^*_0 V^{g*}_\gamma\right] \leq \left[\sum_{\gamma \in \mathbb{F}^+_n} b^g_\gamma V^g_\gamma V^{g*}_\gamma\right] \leq I$$

since Y is a contraction. In addition, since $\varepsilon(f,g)$ is a contraction, $\sup\limits_{\alpha\in\mathbb{F}_n^+}\frac{b_\alpha^f}{b_\alpha^g}\leq 1$ by Corollary 8. Therefore, $b_\alpha^f\leq b_\alpha^g$ for every $\alpha\in\mathbb{F}_n^+$. Thus:

$$\begin{aligned}
\sum_{|\alpha|\geq 1} b_\alpha^f \widehat{Y}_\alpha \widehat{Y}_\alpha^* &= \sum_{|\alpha|\geq 1} b_\alpha^f T_\alpha^g Y_0 Y_0^* T_\alpha^{g*} \leq \sum_{|\alpha|\geq 1} b_\alpha^g T_\alpha^g Y_0 Y_0^* T_\alpha^{g*} \\
&= \sum_{|\alpha|\geq 1}\left[\sum_{\substack{\beta\gamma=\alpha\\ |\beta|\geq 1}} a_\beta^g b_\gamma^g\right] T_\alpha^g Y_0 Y_0^* T_\alpha^{g*} \\
&= \sum_{|\beta|\geq 1}\sum_{\gamma\in\mathbb{F}_n^+} a_\beta^g b_\gamma^g T_{\beta\gamma}^g Y_0 Y_0^* T_{\beta\gamma}^{g*} \\
&= \sum_{|\beta|\geq 1}\sum_{\gamma\in\mathbb{F}_n^+} a_\beta^g b_\gamma^g T_\beta^g V_\gamma^g Y_0 Y_0^* V_\gamma^{g*} T_\beta^{g*} \\
&= \sum_{|\beta|\geq 1} a_\beta^g T_\beta^g\left[\sum_{\gamma\in\mathbb{F}_n^+} b_\gamma^g V_\gamma^g Y_0 Y_0^* V_\gamma^{g*}\right] T_\beta^{g*} \\
&\leq \sum_{|\beta|\geq 1} a_\beta^g T_\beta^g T_\beta^{g*} \leq \sum_{|\beta|\geq 1} a_\beta^g \left(L_\beta^g\otimes I_{\mathcal{H}_2}\right)\left(L_\beta^g\otimes I_{\mathcal{H}_2}\right)^* \leq I
\end{aligned}$$

As desired. The last inequality follows from Proposition 4. Thus, $\left\|\begin{bmatrix} b \\ d\end{bmatrix}\right\|\leq 1$ and $\left\|\begin{bmatrix} c & d\end{bmatrix}\right\|\leq 1$. As in most commutant lifting theorems, we now apply Parrott's Lemma [7] to find a such that $\left\|\widehat{Y}\right\| = \left\|\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right\| = 1$. Thus, $\widehat{Y}:\mathcal{F}^2(f)\otimes\mathcal{H}_1\longrightarrow\mathcal{N}_1$ is a module map with $\left\|\widehat{Y}\right\| = \|X\|$ and $P_{\mathcal{N}}\widehat{Y}=Y$, as desired. By iterating this process, it follows that we can find a module map $\widehat{X}:\mathcal{F}^2(f)\otimes\mathcal{H}_1\longrightarrow\mathcal{F}^2(g)\otimes\mathcal{H}_2$ such that $\|X\| = \left\|\widehat{X}\right\|$ and $XP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{X}$, completing the proof. ■

We can also prove a similar result for the right creation operators R_i, as shown in the following theorem:

Theorem 14 *Let $f:\mathbb{F}_n^+\longrightarrow(0,\infty)$ be an R-bounded map and $g=\sum\limits_{|\alpha|\geq 1} a_\alpha^g X_\alpha$ be a positive regular free holomorphic function with $\|\varepsilon(f,g)\|\leq 1$. In addition, let $(\mathcal{M};V_1^f,V_2^f,\ldots,V_n^f)$ and $(\mathcal{N};V_1^g,V_2^g,\ldots,V_n^g)$ be $*$-submodules of $(\mathcal{F}^2(f)\otimes\mathcal{H}_1;R_1\otimes I_{\mathcal{H}_1},R_2\otimes I_{\mathcal{H}_1},\ldots,R_n\otimes I_{\mathcal{H}_1})$ and $(\mathcal{F}^2(g)\otimes\mathcal{H}_2;R_1\otimes I_{\mathcal{H}_2},R_2\otimes I_{\mathcal{H}_2},\ldots,R_n\otimes I_{\mathcal{H}_2})$, respectively. If $X:\mathcal{M}\longrightarrow\mathcal{N}$ is a module map, then there exists a module map $\widehat{X}:\mathcal{F}^2(f)\otimes\mathcal{H}_1\longrightarrow\mathcal{F}^2(g)\otimes\mathcal{H}_2$ such that $\|X\|=\left\|\widehat{X}\right\|$ and $XP_{\mathcal{M}}=P_{\mathcal{N}}\widehat{X}$, i.e., the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{F}^2(f)\otimes\mathcal{H}_1 \arrow[r, two heads, "P_{\mathcal{M}}"] \arrow[d, "\exists \mid \widehat{X}"'] & \mathcal{M} \arrow[r] \arrow[d, "X"] & 0 \\
\mathcal{F}^2(g)\otimes\mathcal{H}_2 \arrow[r, two heads, "P_{\mathcal{N}}"] & \mathcal{N} \arrow[r] & 0
\end{tikzcd}$$

Proof. The proof of this theorem is very similar to the proof of the previous theorem. Thus, we will only provide a sketch of the proof. Let $(\mathcal{M}; V_1^f, V_2^f, \ldots, V_n^f)$ be a $*$-R-submodule of $(\mathcal{F}^2(f)\otimes\mathcal{H}_1; R_1\otimes I_{\mathcal{H}_1}, R_2\otimes I_{\mathcal{H}_1}, \ldots, R_n\otimes I_{\mathcal{H}_1})$ and $(\mathcal{N}; V_1^g, V_2^g, \ldots, V_n^g)$ be a $*$-submodule of $(\mathcal{F}^2(g)\otimes\mathcal{H}_2; R_1\otimes I_{\mathcal{H}_2}, R_2\otimes I_{\mathcal{H}_2}, \ldots, R_n\otimes I_{\mathcal{H}_2})$. Assume $X:\mathcal{M}\longrightarrow\mathcal{N}$ is a module map with $\|X\|=1$. Let $Y=XP_{\mathcal{M}}$. For every $\alpha\in\mathbb{F}_n^+$, define $Y_\alpha:\mathcal{H}_1\longrightarrow\mathcal{N}$ by $Y_\alpha(x)=Y(\delta_\alpha\otimes x)$. Similarly, define $\widehat{Y}_\alpha:\mathcal{H}_1\longrightarrow\mathcal{N}_1$ by $\widehat{Y}_\alpha(x)=\widehat{Y}(\delta_\alpha\otimes x)$.

If $\delta_0\otimes\mathcal{H}_2\not\subseteq\mathcal{N}$, let $\mathcal{N}_1=\overline{\delta_0\otimes\mathcal{H}_2+\mathcal{N}}$. If $\delta_0\otimes\mathcal{H}_2\subset\mathcal{N}$, find $\alpha\in\mathbb{F}_n^+$ such that $\delta_\alpha\otimes\mathcal{H}_2\not\subseteq N$ but if $\alpha=\beta\gamma$ with $|\beta|\geq 1$ then $\delta_\gamma\otimes\mathcal{H}_2\subset\mathcal{N}$. Then let $\mathcal{N}_1=\overline{\delta_\alpha\otimes\mathcal{H}_2+\mathcal{N}}$. It follows that (1), (2), and (3) of theorem 13 hold.

Now we need to find a module map $\widehat{Y}:\mathcal{F}^2(f)\otimes\mathcal{H}_1\longrightarrow\mathcal{N}_1$ with $\left\|\widehat{Y}\right\|=\|X\|$ and $P_{\mathcal{N}}\widehat{Y}=Y$. As before, $\widehat{Y}_\alpha$, and thus $\widehat{Y}$, will be determined by $\widehat{Y}_0$, but this time we get that $\widehat{Y}_\alpha(x)=T^g_{\widetilde{\alpha}}\widehat{Y}_0(x)$. Furthermore, if $|\alpha|\geq 1$, it follows that $\widehat{Y}_\alpha(x)=T^g_{\widetilde{\alpha}}Y_0(x)$. Let $T_i^g=P_{\mathcal{N}_1}(R_i^g\otimes I_{\mathcal{H}_2})P_{\mathcal{N}_1}$ for $i\leq n$. Then $(\mathcal{N}_1; T_1^g, T_2^g, \ldots, T_n^g)$ is a $*$-R-submodule of $(\mathcal{F}^2(g)\otimes\mathcal{H}_2; R_1\otimes I_{\mathcal{H}_2}, R_2\otimes I_{\mathcal{H}_2}, \ldots, R_n\otimes I_{\mathcal{H}_2})$, and $\mathcal{N}$ is a $*$-R-submodule of $\mathcal{N}_1$. We now decompose $\widehat{Y}$ into the same 2×2 matrix given by in theorem 13.

As in theorem 13, $\left\|\begin{bmatrix} c & d\end{bmatrix}\right\|\leq 1$, and to prove that $\left\|\begin{bmatrix} c \\ d\end{bmatrix}\right\|\leq 1$, it suffices to prove that $\sum_{|\alpha|\geq 1} b_\alpha^f\widehat{Y}_\alpha\widehat{Y}_\alpha^*=\sum_{|\alpha|\geq 1} b_{\widetilde{\alpha}}^f\widehat{Y}_{\widetilde{\alpha}}\widehat{Y}_{\widetilde{\alpha}}^*\leq I$:

$$
\begin{aligned}
\sum_{|\alpha|\geq 1} b^f_{\widetilde{\alpha}} \widehat{Y}_{\widetilde{\alpha}} \widehat{Y}^*_{\widetilde{\alpha}} &= \sum_{|\alpha|\geq 1} b^f_{\widetilde{\alpha}} T^g_\alpha Y_0 Y_0^* T^{g*}_\alpha \leq \sum_{|\alpha|\geq 1} b^g_{\widetilde{\alpha}} T^g_\alpha Y_0 Y_0^* T^{g*}_\alpha \\
&= \sum_{|\alpha|\geq 1} \left[\sum_{\substack{\widetilde{\gamma}\widetilde{\beta}=\widetilde{\alpha} \\ |\widetilde{\beta}|\geq 1}} a^g_{\widetilde{\beta}} b^g_{\widetilde{\gamma}} \right] T^g_\alpha Y_0 Y_0^* T^{g*}_\alpha \\
&= \sum_{|\beta|\geq 1} \sum_{\gamma\in\mathbb{F}_n^+} a^g_{\widetilde{\beta}} b^g_{\widetilde{\gamma}} T^g_{\beta\gamma} Y_0 Y_0^* T^{g*}_{\beta\gamma} \\
&= \sum_{|\beta|\geq 1} \sum_{\gamma\in\mathbb{F}_n^+} a^g_{\widetilde{\beta}} b^g_{\widetilde{\gamma}} T^g_\beta V^g_\gamma Y_0 Y_0^* V^{g*}_\gamma T^{g*}_\beta \\
&= \sum_{|\beta|\geq 1} a^g_{\widetilde{\beta}} T^g_\beta \left[\sum_{\gamma\in\mathbb{F}_n^+} b^g_{\widetilde{\gamma}} V^g_\gamma Y_0 Y_0^* V^{g*}_\gamma \right] T^{g*}_\beta \\
&\leq \sum_{|\beta|\geq 1} a^g_{\widetilde{\beta}} T^g_\beta T^{g*}_\beta \leq \sum_{|\beta|\geq 1} a^g_{\widetilde{\beta}} \left(R^g_\beta \otimes I_{\mathcal{H}_2}\right)\left(R^g_\beta \otimes I_{\mathcal{H}_2}\right)^* \leq I
\end{aligned}
$$

As desired. The last inequality follows from the second part of Proposition 4. Finally, iterating finishes the proof. ■

We can actually say quite a bit more when $\mathcal{F}^2(g)$ is the full Fock space. In this case, any module map between $\mathcal{F}^2(f)$ and $\mathcal{F}^2(g)$ ends up being a contraction, as shown in the following proposition:

Proposition 15 *Let $f = \sum_{|\alpha|\geq 1} a^f_\alpha X_\alpha$, where f has only finitely many terms, and g be the full Fock space. Let $(\mathcal{M}; V^f_1, V^f_2, \ldots, V^f_n)$ and $(\mathcal{N}; V^g_1, V^g_2, \ldots, V^g_n)$ be $*$-submodules of $(\mathcal{F}^2(f)\otimes\mathcal{H}_1; L_1\otimes I_{\mathcal{H}_1}, L_2\otimes I_{\mathcal{H}_1}, \ldots, L_n\otimes I_{\mathcal{H}_1})$ and $(\mathcal{F}^2(g)\otimes \mathcal{H}_2; L_1\otimes I_{\mathcal{H}_2}, L_2\otimes I_{\mathcal{H}_2}, \ldots, L_n\otimes I_{\mathcal{H}_2})$, respectively. If $\varepsilon(f,g)$ is bounded, then $\varepsilon(f,g)$ is a contraction, and therefore X can be lifted to $\widehat{X}$ such that $\left\|\widehat{X}\right\| = \|X\|$ and $XP_{\mathcal{M}} = P_{\mathcal{N}}\widehat{X}$.*

Proof. Popescu proved in [8] that for all $\alpha \in \mathbb{F}_n^+$ $b_{\alpha\alpha} \geq b_\alpha b_\alpha = b^2_\alpha$. Thus, if $b^f_\alpha > 1$ for any α, $\lim_{k\to\infty} b^f_{\alpha^k} \leq \lim_{k\to\infty} (b^f_\alpha)^k = \infty$. Therefore, if $\varepsilon(f,g)$ is bounded, $\sup_{\alpha\in\mathbb{F}_n^+} \frac{b^f_\alpha}{b^g_\alpha} = \sup_{\alpha\in\mathbb{F}_n^+} b^f_\alpha < \infty$ and so $b_\alpha \leq 1$ for all $\alpha \in \mathbb{F}_n^+$. It follows that $\|\varepsilon\| = \sup_{\alpha\in\mathbb{F}_n^+} \frac{b^f_\alpha}{b^g_\alpha} \leq 1$ and thus we can apply theorem 13 to obtain our $\widehat{X}$, as desired. ■

Now if $\mathcal{M}$ is a submodule of $(\mathcal{F}^2(f)\otimes\mathcal{H}_1; L_1\otimes I_{\mathcal{H}_1}, L_2\otimes I_{\mathcal{H}_1}, \ldots, L_n\otimes I_{\mathcal{H}_1})$ and $\varepsilon : \mathcal{F}^2(f) \longrightarrow \mathcal{F}^2(g)$ is bounded, then $\varepsilon(\mathcal{M})$ is a submodule of $\mathcal{F}^2(g)$. This follows immediately due to the nice property that $L_i\delta_\alpha = \delta_{g_i\alpha}$. However, it is important to note that the L_i^*'s do not behave nearly so nicely. It is easy to verify that $L_i^*\delta_{g_i\alpha} = \frac{b_\alpha}{b_{g_i\alpha}}\delta_\alpha$. In particular, if $(\mathcal{M}; V_1^f, V_2^f, \ldots, V_n^f)$ is a $*$-submodule of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \ldots, L_n \otimes I_{\mathcal{H}_1})$ and $\varepsilon(f,g)$ is bounded, $(\epsilon(\mathcal{M}); V_1^g, V_2^g, \ldots, V_n^g)$ need not be a $*$-submodule of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \ldots, L_n \otimes I_{\mathcal{H}_1})$.

However, when $\mathcal{F}^2(f) \cong \mathcal{F}^2(g)$, there is a nice correspondence between $*$-submodules of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \ldots, L_n \otimes I_{\mathcal{H}_1})$ and those of $(\mathcal{F}^2(g) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \ldots, L_n \otimes I_{\mathcal{H}_1})$, as shown in the following lemma:

Lemma 16 *Let f and g be positive regular free holomorphic functions with $\mathcal{F}^2(f) \cong \mathcal{F}^2(g)$. Then if $(\mathcal{M}; V_1^f, V_2^f, \ldots, V_n^f)$ is a $*$-submodule of $\mathcal{F}^2(f)$, there exists a $*$-submodule $(\mathcal{N}; V_1^g, V_2^g, \ldots, V_n^g)$ of $\mathcal{F}^2(g)$ such that $\mathcal{M} \cong \mathcal{N}$. Furthermore, there exists a module map $\widehat{\varepsilon}(\mathcal{M}, \mathcal{N}) : \mathcal{M} \longrightarrow \mathcal{N}$ with $\|\widehat{\varepsilon}\| \leq \|\varepsilon\|$ such that $\widehat{\varepsilon} P_{\mathcal{M}} = P_{\mathcal{N}} \varepsilon$.*

Proof. Let the $*$-submodule $(\mathcal{M}; V_1^f, V_2^f, \ldots, V_n^f)$ be given. We can decompose $\mathcal{F}^2(f)$ as $\mathcal{F}^2(f) = \mathcal{M} \oplus \mathcal{L}$. This means that $\mathcal{M} = \mathcal{F}^2(f)/\mathcal{L}$, where $\mathcal{L}$ is a submodule of $\mathcal{F}^2(f)$. Now since $\mathcal{F}^2(f) \cong \mathcal{F}^2(g)$, we can look at $\mathcal{N} = \mathcal{F}^2(g)/\varepsilon(\mathcal{L})$. Define $V_i^g : \mathcal{N} \longrightarrow \mathcal{N}$ by $V_i^g(x + \varepsilon(\mathcal{L})) = L_i x + \varepsilon(\mathcal{L})$. It is easy to see that V_i^g is well defined. By proposition 5.9 of Douglas and Paulsen [5], this induces a unique $\widehat{\varepsilon}$ with $\|\widehat{\varepsilon}\| \leq \|\varepsilon\|$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L} \arrow[r, hook] \arrow[d, "\varepsilon|_{\mathcal{L}}"] & \mathcal{F}^2(f) \otimes \mathcal{H}_1 \arrow[r, two heads] \arrow[d, "\varepsilon"] & \mathcal{M} \arrow[r] \arrow[d, "\widehat{\varepsilon}"] & 0 \\
0 \arrow[r] & \varepsilon(\mathcal{L}) \arrow[r, hook] & \mathcal{F}^2(g) \otimes \mathcal{H}_2 \arrow[r, two heads] & \mathcal{N} \arrow[r] & 0
\end{tikzcd}$$

It immediately follows that $(\mathcal{N}; V_1^g, V_2^g, \ldots, V_n^g)$ is a $*$-submodule as desired.
■

Now Theorems 13 and 14 only apply to cases where ε is a contraction. However, it turns out that as long as ε is bounded, we can lift all module maps with only a small penalty, as shown in the main theorem of this paper:

Theorem 17 *Let f and g be positive regular free holomorphic functions with $\|\varepsilon(f,g)\| \leq C$. Let $(\mathcal{M}; V_1^f, V_2^f, \ldots, V_n^f)$ and $(\mathcal{N}; V_1^g, V_2^g, \ldots, V_n^g)$ be $*$-submodules of $(\mathcal{F}^2(f) \otimes \mathcal{H}_1; L_1 \otimes I_{\mathcal{H}_1}, L_2 \otimes I_{\mathcal{H}_1}, \ldots, L_n \otimes I_{\mathcal{H}_1})$ and $(\mathcal{F}^2(g) \otimes \mathcal{H}_2; L_1 \otimes I_{\mathcal{H}_2}, L_2 \otimes I_{\mathcal{H}_2}, \ldots, L_n \otimes I_{\mathcal{H}_2})$, respectively. If $X : \mathcal{M} \longrightarrow \mathcal{N}$ is a module map, then there exists a module map $\widehat{X} : \mathcal{F}^2(f) \otimes \mathcal{H}_1 \longrightarrow \mathcal{F}^2(g) \otimes \mathcal{H}_2$ such that $\left\|\widehat{X}\right\| \leq C \|X\|$ and $X P_{\mathcal{M}} = P_{\mathcal{N}} \widehat{X}$, i.e., the following diagram commutes:*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}_{\mathcal{M}} \arrow[r, hook] \arrow[d, "\widehat{X}|_{\mathcal{L}_{\mathcal{M}}}", "\exists"'] & \mathcal{F}^2(f) \otimes \mathcal{H}_1 \arrow[r, two heads, "P_{\mathcal{M}}"] \arrow[d, "\widehat{X}", "\exists"'] & \mathcal{M} \arrow[r] \arrow[d, "X"] & 0 \\
0 \arrow[r] & \mathcal{L}_{\mathcal{N}} \arrow[r, hook] & \mathcal{F}^2(g) \otimes \mathcal{H}_2 \arrow[r, two heads, "P_{\mathcal{N}}"] & \mathcal{N} \arrow[r] & 0
\end{tikzcd}$$

Proof. Assume without loss of generality that $\|\varepsilon(f,g)\| = C > 1$. Define an L-bounded function $h : \mathbb{F}_n^+ \longrightarrow (0,\infty)$ as follows:

$$
\begin{aligned}
b_0^h &= 1 \\
b_\alpha^h &= \frac{b_\alpha^f}{C^2} \qquad \text{for every } \alpha \in \mathbb{F}_n^+, |\alpha| \geq 1
\end{aligned}
$$

Since f is a positive regular free holomorphic function, it follows immediately that h is indeed L-bounded. This gives us the weighted Fock space $\mathcal{F}^2(h)$. In addition, $\mathcal{F}^2(f) \cong \mathcal{F}^2(h)$. This is because for $|\alpha| \geq 1$:

$$
\left\|\delta_\alpha^h\right\| = \frac{1}{\sqrt{b_\alpha^h}} = \sqrt{\frac{C^2}{b_\alpha^f}} = C\left\|\delta_\alpha^f\right\|
$$

Note that since $\left\|\delta_0^h\right\| = \frac{1}{\sqrt{b_0^h}} = 1 = \frac{1}{\sqrt{b_0^f}} = \left\|\delta_0^f\right\|$, $\mathcal{F}^2(g)$ and $\mathcal{F}^2(h)$ are not isometric.

Now let $\varepsilon(h,f) : \mathcal{F}^2(h) \otimes \mathcal{H}_2 \longrightarrow \mathcal{F}^2(f) \otimes \mathcal{H}_2$ be defined in the typical way. It follows that $\|\varepsilon(h,f)\| = 1$ and $\left\|\varepsilon(h,f)^{-1}\right\| = C$. Let $\widetilde{\mathcal{M}}$ be the $*$-submodule isomorphic to $\mathcal{M}$ given by Lemma 16.

We get the following diagram:

$$
\begin{tikzcd}
\mathcal{F}^2(f) \otimes \mathcal{H}_1 \arrow[rr, two heads, "P_{\mathcal{M}}"] \arrow[dr, "\varepsilon(h,f)^{-1}"] \arrow[ddr, dashed] & & \mathcal{M} \arrow[d, "\widehat{\varepsilon}(h,f)^{-1}"'] \arrow[dd, bend left, "X"] \\
& \mathcal{F}^2(h) \otimes \mathcal{H}_1 \arrow[r, two heads, "P_{\widetilde{\mathcal{M}}}"] \arrow[d, dashed] & \widetilde{\mathcal{M}} \arrow[d, "X \circ \widehat{\varepsilon}(h,f)"'] \\
& \mathcal{F}^2(g) \otimes \mathcal{H}_2 \arrow[r, two heads, "P_{\mathcal{N}}"] & \mathcal{N}
\end{tikzcd}
$$

If we focus only on the bottom two rows, we obtain the following diagram:

$$
\begin{tikzcd}
\mathcal{F}^2(h) \otimes \mathcal{H}_1 \arrow[r, two heads, "P_{\widetilde{\mathcal{M}}}"] \arrow[d, dashed] & \widetilde{\mathcal{M}} \arrow[d, "X \circ \widehat{\varepsilon}(h,f)"] \\
\mathcal{F}^2(g) \otimes \mathcal{H}_2 \arrow[r, two heads, "P_{\mathcal{N}}"] & \mathcal{N}
\end{tikzcd}
$$

A brief calculation gives us

$$\begin{aligned}
\|\varepsilon(h,g)\|^2 &= \sup_{\alpha\in\mathbb{F}_n^+}\frac{b_\alpha^h}{b_\alpha^g} = \sup_{\alpha\in\mathbb{F}_n^+}\frac{b_\alpha^f}{C^2 b_\alpha^g} \\
&= \frac{1}{C^2}\sup_{\alpha\in\mathbb{F}_n^+}\frac{b_\alpha^f}{b_\alpha^g} = \frac{1}{C^2}\|\varepsilon(f,g)\|^2 \\
&= 1
\end{aligned}$$

We are thus in a position to apply Theorem 13. This gives us $\widehat{Y}:\mathcal{F}^2(h)\otimes\mathcal{H}_1\to\mathcal{F}^2(g)\otimes\mathcal{H}_2$ such that $\left\|\widehat{Y}\right\|\le\|X\|$ and $(X\circ\widehat{\varepsilon}(h,f))P_{\widetilde{\mathcal{M}}}=P_{\mathcal{N}}\widehat{Y}$. The inequality is verified by a quick calculation:

$$\begin{aligned}
\left\|\widehat{Y}\right\| &= \|X\circ\widehat{\varepsilon}(h,f)\| \\
&\le \|X\|\,\|\widehat{\varepsilon}(h,f)\| \\
&\le \|X\|\,\|\varepsilon(h,f)\| \\
&= \|X\|
\end{aligned}$$

Finally, define $\widehat{X}:=\widehat{Y}\circ\varepsilon(h,f)^{-1}$. Then

$$\begin{aligned}
\left\|\widehat{X}\right\| &= \left\|\widehat{Y}\circ\varepsilon(h,f)^{-1}\right\| \\
&\le \left\|\widehat{Y}\right\|\,\left\|\varepsilon(h,f)^{-1}\right\| \\
&\le C\,\|X\|
\end{aligned}$$

All that is left is to verify that $XP_{\mathcal{M}}=P_{\mathcal{N}}\widehat{X}$:

$$\begin{aligned}
XP_{\mathcal{M}} &= XP_{\mathcal{M}}\varepsilon(h,f)\varepsilon(h,f)^{-1} \\
&= X\widehat{\varepsilon}(h,f)P_{\widetilde{\mathcal{M}}}\varepsilon(h,f)^{-1} \\
&= P_{\mathcal{N}}\widehat{Y}\varepsilon(h,f)^{-1} \\
&= P_{\mathcal{N}}\widehat{X}
\end{aligned}$$

as desired. ■

Theorem 18 *Let f and g be positive regular free holomorphic functions with $\|\varepsilon(f,g)\|\le C$. Let $(\mathcal{M};V_1^f,V_2^f,\ldots,V_n^f)$ and $(\mathcal{N};V_1^g,V_2^g,\ldots,V_n^g)$ be $*$-submodules of $(\mathcal{F}^2(f)\otimes\mathcal{H}_1;R_1\otimes I_{\mathcal{H}_1},R_2\otimes I_{\mathcal{H}_1},\ldots,R_n\otimes I_{\mathcal{H}_1})$ and $(\mathcal{F}^2(g)\otimes\mathcal{H}_2;R_1\otimes I_{\mathcal{H}_2},R_2\otimes I_{\mathcal{H}_2},\ldots,R_n\otimes I_{\mathcal{H}_2})$, respectively. If $X:\mathcal{M}\longrightarrow\mathcal{N}$ is a module map, then there exists a module map $\widehat{X}:\mathcal{F}^2(f)\otimes\mathcal{H}_1\longrightarrow\mathcal{F}^2(g)\otimes\mathcal{H}_2$ such that $\left\|\widehat{X}\right\|\le C\,\|X\|$ and $XP_{\mathcal{M}}=P_{\mathcal{N}}\widehat{X}$, i.e., the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{F}^2(f)\otimes\mathcal{H}_1 \arrow[r, two heads, "P_{\mathcal{M}}"] \arrow[d, "\exists\ \widehat{X}"] & \mathcal{M} \arrow[r] \arrow[d, "X"] & 0 \\
\mathcal{F}^2(g)\otimes\mathcal{H}_2 \arrow[r, two heads, "P_{\mathcal{N}}"] & \mathcal{N} \arrow[r] & 0
\end{tikzcd}$$

Proof. Identical to the previous proof. ■

Using theorems 17 and 18, we can build projective resolutions. To build such resolutions, we must first recall, with a slight reformulation, the construction of Poisson kernels developed by Popescu in [8] based on [3]:

Lemma 19 *(G. Popescu, [8]) Let f be a positive regular free holomorphic function, and let $\mathcal{N}\subset\mathcal{F}^2(f)\otimes\mathcal{H}$ be semi-invariant under the maps $L_i\otimes I_{\mathcal{H}}$ for $i\leq n$, and let $W_i=P_{\mathcal{N}}(L_i\otimes I_{\mathcal{H}})|_{\mathcal{N}}$ for $i\leq n$. Then:*

(i) The sequence $\Delta_N=\sum_{|\alpha|\leq N} a_\alpha W_\alpha W_\alpha^*$ is non-negative and non-increasing

(ii) $\lim_{N\to\infty}\sum_{|\gamma|>N}\left[\sum_{\substack{\alpha\beta=\gamma\\|\alpha|\leq N}} b_\alpha a_\beta\right]W_\gamma W_\gamma^*=0$

(iii) $\Delta=\lim_{N\to\infty}\Delta_N$ exists, $0\leq\Delta\leq I$, and Δ is not equal to zero, and

(iv) $\sum_{\alpha\in\mathbb{F}_n^+} b_\alpha W_\alpha\Delta W_\alpha^*=I_{\mathcal{N}}$.

Theorem 20 *(G. Popescu, [8] - Poisson kernels) Let f be a positive regular free holomorphic function and let $\mathcal{N}\subset\mathcal{F}^2(f)\otimes\mathcal{H}$ be semi-invariant under $L_i\otimes I_{\mathcal{H}}$ for $i\leq n$, and let $W_i=P_{\mathcal{N}}(L_i\otimes I_{\mathcal{H}})|_{\mathcal{N}}$ for $i\leq n$. Define $D=\Delta^{\frac{1}{2}}$, where Δ is the non-negative operator given by the previous lemma. Define $K:\mathcal{N}\longrightarrow\mathcal{F}^2(f)\otimes\mathcal{N}$ by*

$$K(x)=\sum_{\alpha\in F_n^+} b_\alpha\delta_\alpha\otimes DW_\alpha^* x.$$

Then K is an isometry and K^* is a module map. Moreover, $\mathcal{N}$ is isomorphic to $K(\mathcal{N})$ and $K(\mathcal{N})$ is a $*$-invariant submodule of $\mathcal{F}^2(f)\otimes\mathcal{H}$. We call K the Poisson Kernel of $\mathcal{N}$.

We use the Poisson kernel to construct resolutions.

Proposition 21 *Let f be a positive regular free holomorphic function and let $\mathcal{M}\subset\mathcal{F}^2(f)\otimes\mathcal{H}$ be semi-invariant under $L_i\otimes I_{\mathcal{H}}$ for $i\leq n$. Then there exists Hilbert spaces $\mathcal{H}_1,\mathcal{H}_2,\dots$, partial isometric module maps $\Phi_i:\mathcal{F}^2(f)\otimes\mathcal{H}_{i+1}\longrightarrow\mathcal{F}^2(f)\otimes\mathcal{H}_i$, $i=1,2,\dots$, and a partial isometric module map $\Phi_0:\mathcal{F}^2(f)\otimes\mathcal{H}_1\longrightarrow\mathcal{M}$ such that the following sequence is exact:*

$$\cdots \xrightarrow{\Phi_3} \mathcal{F}^2(f)\otimes\mathcal{H}_3 \xrightarrow{\Phi_2} \mathcal{F}^2(f)\otimes\mathcal{H}_2 \xrightarrow{\Phi_1} \mathcal{F}^2(f)\otimes\mathcal{H}_1 \xrightarrow{\Phi_0} \mathcal{M} \longrightarrow 0$$

Proof. Since $\mathcal{M}$ is a semi-invariant subspace of $\mathcal{F}^2(f)\otimes\mathcal{H}$, it has a Poisson kernel $K_0 : \mathcal{M} \longrightarrow \mathcal{F}^2(f)\otimes\mathcal{H}_1$, for some Hilbert space $\mathcal{H}_1$. Set $\Phi_0 = K_0^*$. Notice that $\ker\Phi_0$ is a submodule of $\mathcal{F}^2(f)\otimes\mathcal{H}_1$, and hence it has a Poisson kernel $K_1 : \ker\Phi_0 \longrightarrow \mathcal{F}^2(f)\otimes\mathcal{H}_2$, for some Hilbert space $\mathcal{H}_2$. Define $\Phi_1 : \mathcal{F}^2(f)\otimes\mathcal{H}_2 \longrightarrow \mathcal{F}^2(f)\otimes\mathcal{H}_1$ by $\Phi_1 = \iota\circ K_1^*$, as illustrated in the following diagram:

$$\begin{tikzcd}
\mathcal{F}^2(f)\otimes\mathcal{H}_2 \arrow[rr, "\Phi_1"] \arrow[dr, "K_1^*"'] & & \mathcal{F}^2(f)\otimes\mathcal{H}_1 \arrow[r, "\Phi_0"] & \mathcal{M} \\
& \ker\Phi_0 \arrow[ur, hook, "\iota"'] & & \\
0 \arrow[ur] & & &
\end{tikzcd}$$

The other maps are constructed similarly: $\ker\Phi_1$ is a submodule of $\mathcal{F}^2(f)\otimes\mathcal{H}_2$, and hence it has a Poisson kernel $K_2 : \ker\Phi_1 \longrightarrow \mathcal{F}^2(f)\otimes\mathcal{H}_3$, for some Hilbert space $\mathcal{H}_3$, $\Phi_2 = \iota\circ K_2^*$. ∎

This Proposition together with 17 and 18 give the following result:

Theorem 22 *Let f and g be positive regular free holomorphic functions. Let $(\mathcal{M}; V_1^f, V_2^f, \ldots, V_n^f)$ and $(\mathcal{N}; V_1^g, V_2^g, \ldots, V_n^g)$ be subquotients of $(\mathcal{F}^2(f)\otimes\mathcal{H}; L_1\otimes I_{\mathcal{H}}, L_2\otimes I_{\mathcal{H}}, \ldots, L_n\otimes I_{\mathcal{H}})$ and $(\mathcal{F}^2(g)\otimes\widehat{\mathcal{H}}; L_1\otimes I_{\widehat{\mathcal{H}}}, L_2\otimes I_{\widehat{\mathcal{H}}}, \ldots, L_n\otimes I_{\widehat{\mathcal{H}}})$, respectively. Assume there exist a module map $T : \mathcal{M} \longrightarrow \mathcal{N}$. Then if $\|\epsilon(f,g)\| = C$, there exists module maps $T_1, T_2, T_3\ldots$ with $\|T_i\| \le C^i\,\|T\|$, such that the following diagram commutes:*

$$\begin{tikzcd}
\cdots \arrow[r, "\Phi_3"] & \mathcal{F}^2(f)\otimes\mathcal{H}_3 \arrow[r, "\Phi_2"] \arrow[d, "\exists"', "T_3"] & \mathcal{F}^2(f)\otimes\mathcal{H}_2 \arrow[r, "\Phi_1"] \arrow[d, "\exists"', "T_2"] & \mathcal{F}^2(f)\otimes\mathcal{H}_1 \arrow[r, "\Phi_0"] \arrow[d, "\exists"', "T_1"] & \mathcal{M} \arrow[r] \arrow[d, "T"] & 0 \\
\cdots \arrow[r, "\Psi_3"] & \mathcal{F}^2(g)\otimes\widehat{\mathcal{H}}_3 \arrow[r, "\Psi_2"] & \mathcal{F}^2(g)\otimes\widehat{\mathcal{H}}_2 \arrow[r, "\Psi_1"] & \mathcal{F}^2(g)\otimes\widehat{\mathcal{H}}_1 \arrow[r, "\Psi_0"] & \mathcal{N} \arrow[r] & 0
\end{tikzcd}$$

Proof. Clearly we have that T_1 exists via theorems 17 and 18. $\Phi_1 = \iota\circ K_1^*$ where $K_1 : \ker\Phi_0 \longrightarrow \mathcal{F}^2(f)\otimes\mathcal{H}_2$ is the Poisson kernel of $\ker\Phi_0$, and $\Psi_1 = \iota\circ\widehat{K}_1^*$ where $\widehat{K}_1 : \ker\Psi_0 \longrightarrow \mathcal{F}^2(f)\otimes\widehat{\mathcal{H}}_2$ is the Poisson kernel of $\ker\Psi_0$, as shown below:

$$\begin{tikzcd}
\mathcal{F}^2(f)\otimes\mathcal{H}_2 \arrow[r, "K_1^*"] \arrow[rr, bend left, "\Phi_1"] & \ker\Phi_0 \arrow[r, hook, "\iota"] \arrow[d, "T_1"] & \mathcal{F}^2(f)\otimes\mathcal{H}_1 \arrow[r, "\Phi_0"] \arrow[d, "T_1"] & \mathcal{M} \arrow[r] \arrow[d, "T"] & 0 \\
\mathcal{F}^2(g)\otimes\widehat{\mathcal{H}}_2 \arrow[r, "\widehat{K}_1^*"] \arrow[rr, bend right, "\Psi_1"'] & \ker\Psi_0 \arrow[r, hook, "\iota"] & \mathcal{F}^2(g)\otimes\widehat{\mathcal{H}}_1 \arrow[r, "\Psi_0"] & \mathcal{N} \arrow[r] & 0
\end{tikzcd}$$

Since K_1 and $\widehat{K}_1$ are isometries, by theorem 20, the Poisson kernels induce a module map $\widehat{T}_1 : K_1(\ker\Phi_0) \longrightarrow \widehat{K}_1(\ker\Psi_0)$ with $\left\|\widehat{T}_1\right\| = \|T_1\|$. Therefore, theorems 17 and 18 give us T_2 with $\|T_2\| \leq C\|T_1\|$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}^2(f)\otimes\mathcal{H}_2 \arrow[r, two heads, "P"] \arrow[d, "T_2", "\exists"'] & K_1(\ker\Phi_0) \arrow[d, "\widehat{T}_1"] \\
\mathcal{F}^2(g)\otimes\widehat{\mathcal{H}}_2 \arrow[r, two heads, "P"] & \widehat{K}_1(\ker\Psi_0)
\end{tikzcd}$$

Iterating this process gives us

$$\begin{tikzcd}
\cdots \arrow[r, "\Phi_3"] & \mathcal{F}^2(f)\otimes\mathcal{H}_3 \arrow[r, "\Phi_2"] \arrow[d, "T_3"] & \mathcal{F}^2(f)\otimes\mathcal{H}_2 \arrow[r, "\Phi_1"] \arrow[d, "T_2"] & \mathcal{F}^2(f)\otimes\mathcal{H}_1 \arrow[r, "\Phi_0"] \arrow[d, "T_1"] & \mathcal{M} \arrow[r] \arrow[d, "T"] & 0 \\
\cdots \arrow[r, "\Psi_3"] & \mathcal{F}^2(g)\otimes\widehat{\mathcal{H}}_3 \arrow[r, "\Psi_2"] & \mathcal{F}^2(g)\otimes\widehat{\mathcal{H}}_2 \arrow[r, "\Psi_1"] & \mathcal{F}^2(g)\otimes\widehat{\mathcal{H}}_1 \arrow[r, "\Psi_0"] & \mathcal{N} \arrow[r] & 0
\end{tikzcd}$$

with $\|T_i\| \leq C^i\|T\|$, as desired. ∎

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