CONTINUOUS COFINAL MAPS ON ULTRAFILTERS

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ABSTRACT. An ultrafilter $\mathcal{U}$ on a countable base set B has *continuous Tukey reductions* if whenever an ultrafilter $\mathcal{V}$ is Tukey reducible to $\mathcal{U}$, then every monotone cofinal map $f : \mathcal{U} \to \mathcal{V}$ is continuous with respect to the Cantor topology, when restricted to some cofinal subset of $\mathcal{U}$. We show that the slightly stronger property of having basic Tukey reductions is inherited under Tukey reducibility. It follows that the class of ultrafilters Tukey reducible to any p-point has continuous Tukey reductions. We also prove that every countable iterate of Fubini products of p-points has finitary Tukey reductions. The proof makes use of the association between countable iterates of Fubini products of p-points and ultrafilters generated by $\vec{\mathcal{U}}$-trees on some front B. The finitary Tukey reductions are in fact continuous when viewed on the space of $\mathcal{P}(\hat{B})$ with the Cantor topology, where $\hat{B}$ is the tree of all initial segments of members of the front B.

1. INTRODUCTION

Let D and E be partial orderings. We say that a function $f : E \to D$ is *cofinal* if the image of each cofinal subset of E is cofinal in D. We say that D is *Tukey reducible* to E, and write $D \leq_T E$, if there is a cofinal map from E to D. An equivalent formulation of Tukey reducibility was noticed by Schmidt in [12]. Given partial orderings D and E, a map $g : D \to E$ such that the image of each unbounded subset of D is an unbounded subset of E is called a *Tukey map* or an *unbounded map*. $D \leq_T E$ iff there is a Tukey map from D into E. If both $D \leq_T E$ and $E \leq_T D$, then we write $D \equiv_T E$ and say that D and E are Tukey equivalent. It is clear that $\equiv_T$ is an equivalence relation, and $\leq_T$ on the equivalence classes forms a partial ordering. The equivalence classes can be called *Tukey types*.

The notion of Tukey reducibility between two directed partial orderings was first introduced by Tukey in [17] to study the Moore-Smith theory of net convergence in topology. This naturally led to investigations of Tukey types of more general partial orderings, directed and later non-directed. These investigations often reveal useful information for the comparison of different partial orderings. For example, Tukey reducibility preserves calibre-like properties, such as the countable chain condition, property K, precalibre $\aleph_1$, σ-linked, and σ-centered (see [15]). For more on classification theories of Tukey types for certain classes of ordered sets, we refer the reader to [17], [2], [7], [14], and [15].

In this paper we continue a recent line of research into the structure of the Tukey types of ultrafilters on ω ordered by reverse inclusion. (See [10], [4], [11], and [5].) For any ultrafilter $\mathcal{U}$ on a countable base set, $(\mathcal{U}, \supseteq)$ is a directed partial ordering. We remark that for any two directed partial orderings D and E, $D \equiv_T E$ iff D and

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E are *cofinally similar*; that is, there is a partial ordering into which both D and E embed as cofinal subsets (see [17]). Thus, for ultrafilters, Tukey equivalence is the same as cofinal similarity.

For ultrafilters, we may restrict our attention to monontone cofinal maps. We say that a map $f : \mathcal{U} \to \mathcal{V}$ is *monotone* if for any $X, Y \in \mathcal{U}$, $X \supseteq Y$ implies $f(X) \supseteq f(Y)$. It is not hard to show that whenever $\mathcal{U} \geq_T \mathcal{V}$, then there is a *monotone* cofinal map witnessing this (see Fact 6 of [4]). Thus, we shall assume throughout this paper that each cofinal map under consideration is monotone.

Another motivation for this study is that Tukey reducibility is a generalization of Rudin-Keisler reducibility. Recall that $\mathcal{U} \geq_{RK} \mathcal{V}$ iff there is a function $f : \omega \to \omega$ such that the ultrafilter generated by the collection $\{f(U) : U \in \mathcal{U}\}$ is equal to $\mathcal{V}$. Whenever $\mathcal{U} \geq_{RK} \mathcal{V}$, then also $\mathcal{U} \geq_T \mathcal{V}$ (see Fact 1 in [4]). In general, Tukey and Rudin-Keisler reducibility are quite distinct. Various instances of this can be seen in [4], [5], [6], [11], and in the following.

Theorem 1 ([7], [8])**.** *There is an ultrafilter $\mathcal{U}_{\mathrm{top}}$ on ω realizing the maximal cofinal type among all directed sets of cardinality continuum; that is, $\mathcal{U}_{\mathrm{top}} \equiv_T [\mathfrak{c}]^{<\omega}$.*

Note that there are $2^{\mathfrak{c}}$ many ultrafilters of maximal Tukey type, since any collection of independent sets can be used in a canonical way to construct an ultrafilter with maximal type. Thus the top Tukey type has cardinality $2^{\mathfrak{c}}$. In contrast, every Rudin-Keisler equivalence class has cardinality $\mathfrak{c}$. Moreover, there is no maximal equivalence class in the Rudin-Keisler ordering. So the maximal Tukey class contains $2^{\mathfrak{c}}$ many Rudin-Keisler equivalence classes, none of which is maximal in the Rudin-Keisler sense.

We now turn our attention to p-points.

Definition 2. An ultrafilter $\mathcal{U}$ on ω is a *p-point* iff for each decreasing sequence $A_0 \supseteq A_1 \supseteq \dots$ of elements of $\mathcal{U}$, there is an $A \in \mathcal{U}$ such that $A \subseteq^* A_n$, for all $n < \omega$.

Isbell's Problem [7], whether there is an ultrafilter with Tukey type strictly below the maximal type, is consistently still open. It was shown in [4] that countable iterations of Fubini products of p-points (and in fact the more general class of so-called "basically generated" ultrafilters) are strictly below the maximal Tukey type. However, it is open whether there is a model of ZFC with no p-points in which all non-principal ultrafilters have the maximal Tukey type.

It follows from work in [13] that p-points have the following special property: If $\mathcal{U}$ is a p-point and $\mathcal{V} \leq_T \mathcal{U}$, then there is a definable monotone cofinal map from $\mathcal{U}$ into $\mathcal{V}$. Hence every p-point has Tukey type of cardinality $\mathfrak{c}$. In fact, p-points have even stronger properties in terms of cofinal maps. Identify $\mathcal{P}(\omega)$ with 2^ω, the set of characteristic functions of subsets of ω, and endow $\mathcal{P}(\omega)$ with the corresponding topology. A sequence $(X_n)_{n<\omega}$ of elements of $\mathcal{P}(\omega)$ converges to an element $X \in \mathcal{P}(\omega)$ iff for each $k < \omega$ there is an $N < \omega$ such that for each $n \geq N$, $X_n \cap k = X \cap k$. A function $f : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is *continuous* if and only if whenever $X_n \to X$, then also $f(X_n) \to f(X)$. Given $\mathcal{D} \subseteq \mathcal{P}(\omega)$, a function $f : \mathcal{D} \to \mathcal{P}(\omega)$ is said to be continuous if it is continuous on $\mathcal{D}$ considered as a topological subspace of $\mathcal{P}(\omega)$.

Definition 3. Let $\mathcal{U}$ be an ultrafilter on ω.

(1) $\mathcal{U}$ *has continuous Tukey reductions* if whenever $f : \mathcal{U} \to \mathcal{V}$ is a monotone cofinal map, there is a cofinal subset $\mathcal{D} \subseteq \mathcal{U}$ such that $f \restriction \mathcal{D}$ is continuous.

(2) $\mathcal{U}$ *has finitary Tukey reductions* if whenever $f : \mathcal{U} \to \mathcal{V}$ is a monotone cofinal map, there is a cofinal subset $\mathcal{D} \subseteq \mathcal{U}$ and a function $\hat{g} : [\omega]^{<\omega} \to [\omega]^{<\omega}$, such that

(1) $\hat{g}$ is monotone: $s \subseteq t \to \hat{g}(s) \subseteq \hat{g}(t)$; and
(2) $\hat{g}$ generates f on $\mathcal{D}$: For each $X \in \mathcal{D}$, $f(X) = \bigcup_{k<\omega} \hat{g}(X \cap k)$.

It is easy to see that (1) implies (2). We point out that any ultrafilter which has finitary Tukey reductions has Tukey type of cardinality $\mathfrak{c}$.

The following [Theorem 20 in [4]] provided a fundamental tool for all subsequent research on the classification of Tukey types of p-points. We state the theorem in the language used in this paper.

Theorem 4 (Dobrinen/Todorcevic [4]). *Suppose $\mathcal{U}$ is a p-point on ω. Then $\mathcal{U}$ has continuous Tukey reductions. In fact, $\mathcal{U}$ has the stronger property of having basic Tukey reductions (see Definition 7).*

Continuous cofinal maps are used in the analysis of the structure of the Tukey types of p-points in [4]. They are crucial to the work in [11], [5], and [6]. In those papers, continuous cofinal maps provide the key to being able to apply information from Ramsey-classification theorems on barriers to classify the Rudin-Keisler structure within the Tukey types of selective ultrafilters ([11]), and furthermore, a large class of rapid p-points ([5] and [6]). Continuous cofinal maps are also used in the following theorem, which reveals the surprising fact that the Tukey and Rudin-Keisler orders sometimes coincide. Recall that $\le_{RB}$ is the Rudin-Blass ordering, which implies $\le_{RK}$.

Theorem 5 (Raghavan [11]). *Let $\mathcal{U}$ be any ultrafilter and let $\mathcal{V}$ be a q-point. Suppose $f : \mathcal{U} \to \mathcal{V}$ is continuous, monotone, and cofinal in $\mathcal{V}$. Then $\mathcal{V} \le_{RB} \mathcal{U}$.*

In Section 2, we define the property of an ultrafilter on ω having basic Tukey reductions. This property is possessed by all p-points, and implies having continuous Tukey reductions. We show in Theorem 9 that the property of having basic Tukey reductions is inherited under Tukey reducibility. Thus, assuming the existence of p-points, there is a large class of ultrafilters, closed under Tukey reducibility, which have continuous Tukey reductions.

Theorem 10. *If $\mathcal{U}$ is Tukey reducible to a p-point, then $\mathcal{U}$ has basic, hence continuous, Tukey reductions.*

Thorems 5 and 10 yield the following corollary.

Corollary 11. *Suppose $\mathcal{W}$ is Tukey reducible to a p-point. Then every ultrafilter Tukey reducible to $\mathcal{W}$ is in fact Rudin-Blass reducible to $\mathcal{W}$.*

Remark. The property of having basic Tukey reductions is the only property yet known to be inherited under Tukey reducibility, whereas many standard properties, such as being a p-point or selective, are inherited under Rudin-Keisler reducibility but not under Tukey reducibility.

In Section 3, we extend Theorem 4 to all countable iterations of Fubini products of p-points in as strong a manner as possible. In general, Fubini products of p-points simply do not have continuous Tukey reductions. However, it follows from Theorem 21 that all countable iterations of Fubini products of p-points have finitary Tukey reductions. Moreover, they are continuous in a sense which we make precise. Toward this end, we introduce the notions of $\vec{\mathcal{U}}$-trees, which are trees on a front with ultrafilter branching, and *basic* Tukey reductions on $\vec{\mathcal{U}}$-trees (see Definition 19), which are the analogues of basic Tukey reductions for ultrafilters on ω. In Facts 15 and 16, we point out how countable iterations of Fubini products of ultrafilters can be represented as ultrafilters generated by $\vec{\mathcal{U}}$-trees on so-called flat-top fronts. Then we prove the main theorem of Section 3.

Theorem 21. *Let $\mathcal{W}$ be an ultrafilter on a base B, which is a flat-top front, generated by $\vec{\mathcal{U}}$-trees of p-points, and let $\hat{B}$ denote the tree of all initial segments of members of B. Then for each monotone map f from $\mathcal{W}$ into $\mathcal{P}(\omega)$, there is a cofinal subset of $\mathcal{W}$ on which f is generated by a monotone, initial segment and level preserving, finitary map $\hat{f}$, defined on $[\hat{B}]^{<\omega}$. This map $\hat{f}$ is continuous on the space $2^{\hat{B}}$ with the Cantor topology.*

Thus, every countable iteration of Fubini products of p-points has basic Tukey reductions, and therefore, finitary Tukey reductions.

That countable iterations of Fubini products of p-points might have Tukey reductions with nice properties was forshadowed in theorem of Todorcevic in [11], where he proved that the Tukey type of a selective ultrafilter consists (up to isomorphism) of exactly the countable Fubini iterates of that ultrafilter. Recently, similar results were obtained for weakly Ramsey ultrafilters and the more general class of ultrafilters $\mathcal{U}_\alpha$ ($\alpha < \omega_1$) introduced and investigated by Laflamme in [9]. See [5] and [6] for more details.

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2. Basic Tukey reductions are preserved under Tukey reducibility

As was discussed in the Introduction, continuity of cofinal maps provides a key tool in the study of Tukey types of ultrafilters. We prove in Theorem 9 that the (possibly stronger) property of having basic Tukey reductions is inherited under Tukey reducibility. Since, as was shown in [5], all p-points have basic Tukey reductions, it will follow that every ultrafilter Tukey reducible to a p-point has continuous Tukey reductions. Moreover, all basic Tukey reductions on some cofinal subset of an ultrafilter extend to a continuous map on $\mathcal{P}(\omega)$. This is shown in Theorem 8, which is employed in the proof of Theorem 9. It is not known whether the property of having continuous Tukey reductions equivalent to having basic Tukey reductions.

We use $2^{<\omega}$ to denote the collection of finite sequences $s : n \to 2$, for $n < \omega$. For $s, t \in 2^{<\omega}$, we write $s \sqsubseteq t$ to denote that s is an initial segment of t; that is, $\mathrm{dom}\,(s) \subseteq \mathrm{dom}\,(t)$ and $t \restriction \mathrm{dom}\,(s) = s$. We also use $a \sqsubseteq X$ for sets $a, X \subseteq \omega$ to denote that, given their strictly increasing enumerations, a is an initial segment of X.

We would like to identify subsets of ω with their characteristic functions. Of course, since the same finite set determines different characteristic functions on different domains, we take the slightly tedious but unambiguous path of distinguishing between a set and its characteristic function on a given domain. Thus, for $X \subseteq \omega$, we let χ_X denote the characteristic function of X with domain ω; and given $m < \omega$, we let $\chi_X \restriction m$ denote the characteristic function of $X \cap m$ with domain m. For $s \in 2^m$, we shall let $d(s)$ denote $s^{-1}(\{1\})$, the subset of m for which s is the characteristic function.

Definition 6. Given a subset D of $2^{<\omega}$, we shall call a map $\hat{f} : D \to 2^{<\omega}$ *level preserving* if there is a strictly increasing sequence $(k_m)_{m<\omega}$ such that for each $s \in D \cap 2^{k_m}$, we have that $\hat{f}(s) \in 2^m$. A level preserving map $\hat{f}$ is *initial segment preserving* if whenever $m < m'$, $s \in D \cap 2^{k_m}$, and $s' \in D \cap 2^{k_{m'}}$, then $s \sqsubseteq s'$ implies $\hat{f}(s) \sqsubseteq \hat{f}(s')$. $\hat{f}$ is *monotone* if for each $s, t \in D$, $d(s) \subseteq d(t)$ implies $d(\hat{f}(s)) \subseteq d(\hat{f}(t))$.

Definition 7. A monotone map f on a subset $\mathcal{D} \subseteq \mathcal{P}(\omega)$ is said to be *basic* if f is generated by a monotone, level and initial segment preserving map in the following manner: There is some strictly increasing sequence $(k_m)_{m<\omega}$ such that, letting

$$D = \{\chi_X \restriction k_m : X \in \mathcal{D},\ m < \omega\}, \tag{1}$$

there is a level and initial segment preserving map $\hat{f} : D \to 2^{<\omega}$ such that for each $X \in \mathcal{D}$,

$$f(X) = \bigcup_{m<\omega} d(\hat{f}(\chi_X \restriction k_m)). \tag{2}$$

In this case, we say that $\hat{f}$ *generates* f.

We say that an ultrafilter $\mathcal{U}$ *has basic Tukey reductions* if for every monotone cofinal map $f : \mathcal{U} \to \mathcal{V}$, f is basic on some cofinal subset $\mathcal{D} \subseteq \mathcal{U}$.

Remark. It follows from the definition that a basic map f on a subset $\mathcal{D}$ of $\mathcal{P}(\omega)$ is continuous on $\mathcal{D}$. If $\hat{f}$ generates f, then for each $X \in \mathcal{D}$ and $m < \omega$, $f(X) \cap m = d(\hat{f}(\chi_X \restriction k_m))$. Moreover, $\hat{f}$ generates a continuous map on $\overline{\mathcal{D}}$, the closure of $\mathcal{D}$ in $\mathcal{P}(\omega)$, extending f.

The next theorem shows that any basic cofinal map from some cofinal subset of an ultrafilter $\mathcal{U}$ into another ultrafilter $\mathcal{V}$ can be extended to a basic map on the whole space $\mathcal{P}(\omega)$ in such a way that its restriction to $\mathcal{U}$ is a continuous cofinal map.

Theorem 8 (Extension Theorem)**.** *Suppose $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters, $f : \mathcal{U} \to \mathcal{V}$ is a monotone cofinal map, and there is a cofinal subset $\mathcal{D} \subseteq \mathcal{U}$ such that $f \restriction \mathcal{D}$ is basic. Then there is a continuous, monotone $\tilde{f} : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that*

1. *$\tilde{f}$ is basic on $\mathcal{P}(\omega)$;*
2. *$\tilde{f} \restriction \mathcal{D} = f \restriction \mathcal{D}$; and*
3. *$\tilde{f} \restriction \mathcal{U} : \mathcal{U} \to \mathcal{V}$ is a cofinal map.*

Thus, $\mathcal{U}$ has basic Tukey reductions if and only if for every monotone cofinal map $f : \mathcal{U} \to \mathcal{V}$ there is some cofinal $\mathcal{D} \subseteq \mathcal{U}$ for which $f \restriction \mathcal{D}$ is basic.

Proof. We first extend the basic map $f \restriction \mathcal{D}$ to a map on all of $\mathcal{U}$. For $U \in \mathcal{U}$, define

$$f'(U) = \bigcup\{f(X) : X \in \mathcal{D} \text{ and } X \subseteq U\}. \tag{3}$$

Claim 1. f' is a monotone cofinal map from $\mathcal{U}$ into $\mathcal{V}$, and $f' \restriction \mathcal{D} = f \restriction \mathcal{D}$.

Proof. Let $U \in \mathcal{U}$. Then $f'(U)$ is a union of elements in $\mathcal{V}$, hence is itself in $\mathcal{V}$. It is easy to see that f' is monotone, by its definition. Let $X \in \mathcal{D}$. By definition, $f'(X) \supseteq f(X)$. Since f is monotone, for each $X' \in \mathcal{D}$ such that $X' \subseteq X$, we have $f(X') \subseteq f(X)$. Thus, $f'(X) \subseteq f(X)$. Hence, $f' \restriction \mathcal{D} = f \restriction \mathcal{D}$. Since the image of $\mathcal{U}$ under f' contains the image of $\mathcal{D}$ under f, which is cofinal in $\mathcal{V}$, it follows that f' is a monotone cofinal map from $\mathcal{U}$ into $\mathcal{V}$. □

Let $\hat{f}$ be a monotone initial segment and level preserving map witnessing that $f \restriction \mathcal{D}$ is basic, and let $(k_m)_{m<\omega}$ be the levels on which $\hat{f}$ is defined. Thus, the domain of $\hat{f}$ is $D = \{\chi_X \restriction k_m : X \in \mathcal{D},\ m < \omega\}$; and for each $s \in D \cap 2^{k_m}$, $\hat{f}(s) \in 2^m$. Recall that $\hat{f}$ being initial segment preserving implies that for each $m < n$ and $s \in D \cap 2^{k_n}$, $\hat{f}(s \restriction k_m) = \hat{f}(s) \restriction m$.

Claim 2. There is a monotone, level and initial segment preserving map $\hat{g}$ which generates a function $\tilde{f} : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $\tilde{f} \restriction \mathcal{U} = f'$.

Proof. Since $\mathcal{D}$ is cofinal in $\mathcal{U}$, for each $m < \omega$, the finite sequence of zeros of length 2^{k_m} is in D. Thus, for each $m < \omega$,

$$\{t \in 2^{k_m} : \exists s \in D \cap 2^{k_m} (d(s) \subseteq d(t))\} = 2^{k_m}. \tag{4}$$

Let $C = \bigcup_{m<\omega} 2^{k_m}$. Define $\hat{g}$ on C as follows: For $t \in 2^{k_m}$, define $\hat{g}(t)$ to be the characteristic function with domain m so that

$$d(\hat{g}(t)) = \bigcup \{d(\hat{f}(s)) : s \in D \cap \bigcup_{n \le m} 2^{k_n} \text{ and } d(s) \subseteq d(t)\}. \tag{5}$$

Essentially, $\hat{g}(t)$ is the union of all $\hat{f}$-images of sets contained within t.

By its definition, $\hat{g}$ is monotone and level preserving. To see that $\hat{g}$ is initial segment preserving, suppose $t \sqsubset t'$, where $t \in 2^{k_m}$ and $t' \in 2^{k_{m'}}$ for some $m < m'$. We shall show that $\hat{g}(t) \sqsubset \hat{g}(t')$. Note that for each $s \in D \cap \bigcup_{n \le m'} 2^{k_n}$, if $d(s) \subseteq d(t')$ then $d(s \restriction k_m) \subseteq d(t)$. Further, note that for all $s \in D \cap \bigcup_{n \le m'} 2^{k_n}$, we have $d(\hat{f}(s)) \cap m = d(\hat{f}(s) \restriction m) = d(\hat{f}(s \restriction k_m))$. Thus,

$$d(\hat{g}(t') \restriction m) = d(\hat{g}(t')) \cap m \tag{6}$$

$$= \bigcup \{d(\hat{f}(s)) \cap m : s \in D \cap \bigcup_{n \le m'} 2^{k_n} \text{ and } d(s) \subseteq d(t')\} \tag{7}$$

$$= \bigcup \{d(\hat{f}(s \restriction k_m)) : s \in D \cap \bigcup_{n \le m'} 2^{k_n} \text{ and } d(s) \subseteq d(t')\} \tag{8}$$

$$= \bigcup \{d(\hat{f}(s)) : s \in D \cap \bigcup_{n \le m} 2^{k_n} \text{ and } d(s) \subseteq d(t')\} \tag{9}$$

$$= d(\hat{g}(t)). \tag{10}$$

Since $m = \mathrm{dom}\,(\hat{g}(t))$, we have that $\hat{g}(t') \restriction m = \hat{g}(t)$. Therefore, $\hat{g}(t) \sqsubset \hat{g}(t')$.

Now define $\tilde{f} : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ by

$$\tilde{f}(Z) = \bigcup \{d(\hat{g}(\chi_Z \restriction k_m)) : m < \omega\} \tag{11}$$

$$= \bigcup \{d(\hat{g}(t)) : \exists m < \omega\, (t \in 2^{k_m} \text{ and } d(t) \sqsubseteq Z)\}. \tag{12}$$

By definition, $\tilde{f}$ is generated by the monotone level and initial segment preserving map $\hat{g}$. It follows that $\tilde{f}$ is monotone and continuous. In fact, since $\hat{g}$ is monotone, it follows that for each $Z \subseteq \omega$,

$$\tilde{f}(Z) = \bigcup \{d(\hat{g}(t)) : \exists m < \omega\, (t \in 2^{k_m} \text{ and } d(t) \subseteq Z)\}. \tag{13}$$

Lastly, we check that $\tilde{f} \upharpoonright \mathcal{U} = f'$. Let $U \in \mathcal{U}$.

$$\begin{aligned} f'(U) &= \bigcup \{f(X) : X \in \mathcal{D} \text{ and } X \subseteq U\} \\ &= \bigcup \{d(\hat{f}(\chi_X \upharpoonright k_m)) : X \in \mathcal{D},\ X \subseteq U, \text{ and } m < \omega\} \\ &= \bigcup \{d(\hat{f}(s)) : s \in D \text{ and } d(s) \subseteq U\}. \end{aligned} \tag{14}$$

At the same time, putting together (13) and (5) and simplifying the expression, we have

$$\tilde{f}(U) = \bigcup \{d(\hat{f}(s)) : s \in D \text{ and } d(s) \subseteq U\}. \tag{15}$$

Therefore, $\tilde{f}(U) = f'(U)$. □

By Claims 1 and 2, the theorem is proved. □

In the next theorem, which constitutes the main result of this section, we show that the property of having basic Tukey reductions is inherited under Tukey reducibiltiy.

Theorem 9. *Suppose that $\mathcal{U}$ has basic Tukey reductions. Then for every ultrafilter $\mathcal{W} \le_T \mathcal{U}$, $\mathcal{W}$ has basic Tukey reductions.*

Proof. Suppose $\mathcal{U}$ has basic Tukey reductions, and let $\mathcal{W} \le_T \mathcal{U}$. By Theorem 8, there is basic map $\tilde{f} : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ generated by a monotone level and initial segment preserving map $\hat{f} : \bigcup_{m<\omega} 2^{k_m} \to 2^{<\omega}$, for some increasing sequence $(k_m)_{m<\omega}$, such that $f : \mathcal{U} \to \mathcal{W}$ is a cofinal map, where $f := \tilde{f} \upharpoonright \mathcal{U}$. Suppose $\mathcal{V} \le_T \mathcal{W}$, and let $h : \mathcal{W} \to \mathcal{V}$ be a monotone cofinal map. Extend h to the monotone map $\tilde{h} : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ defined as follows: For each $X \in \mathcal{P}(\omega)$, let

$$\tilde{h}(X) = \bigcap \{h(W) : W \in \mathcal{W} \text{ and } W \supseteq X\}. \tag{16}$$

Then $\tilde{h}$ is monotone and $\tilde{h} \upharpoonright \mathcal{W} = h$.

Let $\tilde{g} = \tilde{h} \circ \tilde{f}$. Then $\tilde{g} : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ and is monotone. Letting $g = \tilde{g} \upharpoonright \mathcal{U}$, we see that $g = h \circ f$; hence $g : \mathcal{U} \to \mathcal{V}$ is a monotone cofinal map. Thus, there is a cofinal subset $\mathcal{D} \subseteq \mathcal{U}$ such that $g \upharpoonright \mathcal{D}$ is basic, generated by some monotone, level and initial segment preserving map $\hat{g}$. Without loss of generality, we may assume that $\hat{f}$ and $\hat{g}$ are defined on the same set of levels $\bigcup_{m<\omega} 2^{k_m}$: For if $\hat{g}$ is defined on $\bigcup_{m<\omega} 2^{j_m}$, we can take $l_m = \max(k_m, j_m)$ and define $\hat{f}'(s) = \hat{f}(s \upharpoonright k_m)$ and $\hat{g}'(s) = \hat{g}(s \upharpoonright j_m)$ for $s \in 2^{l_m}$. Let

$$D = \{\chi_X \upharpoonright k_m : X \in \mathcal{D},\ m < \omega\}. \tag{17}$$

Note that for each $s \in D \cap 2^{k_m}$, if $d(s) \sqsubseteq X \in \mathcal{D}$ then $d(\hat{f}(s)) = f(X) \cap m$ and $d(\hat{g}(s)) = g(X) \cap m$.

Let $\mathcal{C} = f''\mathcal{D}$. Then $\mathcal{C}$ is cofinal in $\mathcal{W}$. Let $\overline{\mathcal{D}}$ denote the closure of $\mathcal{D}$ in the topological space $\mathcal{P}(\omega)$. Since f is continuous on the compact space $\mathcal{P}(\omega)$, $\overline{\mathcal{C}} = \overline{f''\mathcal{D}} = f''\overline{\mathcal{D}}$. Define

$$C = \{\hat{f}(s) : s \in D\}. \tag{18}$$

Note that C is the collection of all characteristic functions of finite initial segments of elements of $\mathcal{C}$:

$$C = \bigcup_{m<\omega} \{t \in 2^m : \exists Y \in \mathcal{C}\,(d(t) = Y \cap m)\}. \tag{19}$$

Define $\hat{h} : C \to 2^{<\omega}$ as follows: For $t \in C \cap 2^m$, define $\hat{h}(t)$ to be the member of 2^m such that

$$d(\hat{h}(t)) = \bigcap\{d(\hat{g}(s)) : s \in D \cap 2^{k_m} \text{ and } \hat{f}(s) = t\}. \tag{20}$$

Note that $\hat{h}$ is level preserving, simply by its definition. In fact, $t \in C \cap 2^m$ implies $\hat{h}(t) \in 2^m$.

The idea behind the definition of $\hat{h}(t)$ is to take the intersection of all $\hat{g}$-images of all $\hat{f}$-preimages of t. This gives the smallest possible approach to approximating h by a finitary function, and we will show that it works. It will turn out that, in fact, any and all $\hat{g}$-images of $\hat{f}$-preimages of t will give us the correct information for h, provided we restrict to a set of good levels which we will determine below. Toward this end, we prove the following Claims 1 - 3 to find good levels to which to restrict $\hat{h}$, thus obtaining an $\hat{i}$ which we show in Claim 4 witnesses that h is basic on $\mathcal{C}$.

Claim 1. For each $Y \in \overline{\mathcal{C}}$ and each $m < \omega$, there is an $\tilde{m} \geq m$ satisfying the following: For each $Z \in \overline{\mathcal{D}}$ such that $\tilde{f}(Z) \cap \tilde{m} = Y \cap \tilde{m}$, there is an $X \in \overline{\mathcal{D}}$ such that $\tilde{g}(X) \cap m = \tilde{g}(Z) \cap m$ and $\tilde{f}(X) = Y$.

Proof. Let $Y \in \overline{\mathcal{C}}$ and suppose the claim fails. Then there is an m such that for each $n \geq m$, there is a $Z_n \in \overline{\mathcal{D}}$ such that $\tilde{f}(Z_n) \cap n = Y \cap n$, but for each $X \in \overline{\mathcal{D}}$ such that $\tilde{f}(X) = Y$, $\tilde{g}(X) \cap m \neq \tilde{g}(Z_n) \cap m$. $\overline{\mathcal{D}}$ is compact, so there is a subsequence $(Z_{n_i})_{i<\omega}$ which converges to some $Z \in \overline{\mathcal{D}}$. Since $\tilde{f}$ is continuous, $\tilde{f}(Z_{n_i})$ converges to $\tilde{f}(Z)$. Since $\tilde{f}(Z_{n_i}) \cap n_i = Y \cap n_i$ for each i, it follows that $\tilde{f}(Z_{n_i})$ converges to Y. Therefore, $\tilde{f}(Z) = Y$. Since $\tilde{g}$ is continuous, $\tilde{g}(Z_{n_i})$ converges to $\tilde{g}(Z)$. But that implies that for all sufficiently large values of i, $\tilde{g}(Z_{n_i}) \cap m = \tilde{g}(Z) \cap m$, contradicting that for all n, $\tilde{g}(Z_n) \cap m \neq \tilde{g}(Z) \cap m$. □

Claim 2. There is a strictly increasing sequence $(j_m)_{m<\omega}$ such that for each $m < \omega$, for all $Y \in \overline{\mathcal{C}}$ and $Z \in \overline{\mathcal{D}}$ with $\tilde{f}(Z) \cap j_m = Y \cap j_m$, there is an $X \in \overline{\mathcal{D}}$ such that $\tilde{g}(X) \cap m = \tilde{g}(Z) \cap m$ and $\tilde{f}(X) = Y$.

Proof. Let $j_0 = 0$. Then j_0 vacuously satisfies the claim. Now suppose that $m \geq 1$ and suppose we have chosen $j_0 < \cdots < j_{m-1}$ satisfying the claim. For each $Y \in \overline{\mathcal{C}}$, there is an $n(Y, m) \geq m$ satisfying Claim 1. The finite segments $Y \cap n(Y, m)$ determine basic open sets in $\mathcal{P}(\omega)$, and the union of these open sets (over all $Y \in \overline{\mathcal{C}}$) covers $\overline{\mathcal{C}}$. Since $\overline{\mathcal{C}}$ is compact, there is a finite subcover, determined by some $Y_0 \cap n(Y_0, m), \ldots, Y_l \cap n(Y_l, m)$. Take $j_m > \max\{j_{m-1}, n(Y_0, m), \ldots, n(Y_l, m)\}$. Then $(j_m)_{m<\omega}$ forms a strictly increasing sequence which satisfies the claim. □

Claim 3. Let $Y \in \overline{\mathcal{C}}$ and m be given, and let $t = \chi_Y \restriction j_m$. Then $\tilde{h}(Y) \cap m = d(\hat{h}(t)) \cap m$.

Proof. Let $Y \in \overline{\mathcal{C}}$ and m be given, and let $t = \chi_Y \restriction j_m$. By definition of $\hat{h}$,

$$d(\hat{h}(t)) = \bigcap\{d(\hat{g}(s)) : s \in D \cap 2^{k_{j_m}} \text{ and } \hat{f}(s) = t\}. \tag{21}$$

Let $s \in D \cap 2^{k_{j_m}}$ such that $\hat{f}(s) = t$. $s \in D$ implies there is a $Z \in \mathcal{D}$ such that $s = \chi_Z \restriction k_{j_m}$; hence $d(\hat{f}(s)) = \tilde{f}(Z) \cap j_m$. Thus, $\tilde{f}(Z) \cap j_m = d(\hat{f}(s)) = d(t) = Y \cap j_m$. By Claim 2, there is an $X \in \overline{\mathcal{D}}$ such that $\tilde{g}(X) \cap m = \tilde{g}(Z) \cap m$ and $\tilde{f}(X) = Y$. Note that $\tilde{g}(X) = \tilde{h} \circ \tilde{f}(X) = \tilde{h}(Y)$ since $\tilde{f}(X) = Y$. Since $s = \chi_Z \restriction k_{j_m}$, it follows that $d(\hat{g}(s)) = \tilde{g}(Z) \cap j_m$; hence, $d(\hat{g}(s)) \cap m = \tilde{g}(Z) \cap m$. Therefore,

$$\tilde{h}(Y) \cap m = \tilde{g}(X) \cap m = \tilde{g}(Z) \cap m = d(\hat{g}(s)) \cap m. \tag{22}$$

Thus, we have shown that for each $s \in D \cap 2^{k_{j_m}}$ such that $\hat{f}(s) = t$, $d(\hat{g}(s)) \cap m = \tilde{h}(Y) \cap m$. Therefore, $d(\hat{h}(t)) \cap m = \tilde{h}(Y) \cap m$. □

It follows from Claim 3 that, for each $Y \in \overline{\mathcal{C}}$,

$$\tilde{h}(Y) = \bigcup_{m<\omega} d(\hat{h}(\chi_Y \restriction j_m)) \cap m. \tag{23}$$

We now define $\hat{\imath}$ on domain $C \cap \bigcup_{m<\omega} 2^{j_m}$ as follows: For each $m < \omega$ and each $t \in C \cap 2^{j_m}$, define

$$\hat{\imath}(t) = \hat{h}(t) \restriction m. \tag{24}$$

Claim 4. $\hat{\imath}$ is a monotone, level and initial segment preserving map which generates $\tilde{h} \restriction \overline{\mathcal{C}}$, and hence generates $h \restriction \mathcal{C}$.

Proof. By its definition, $\hat{\imath}$ is level preserving, mapping members of 2^{j_m} into 2^m. $\hat{\imath}$ is initial segment preserving, since $\hat{h}$ is initial segment preserving. Let $Y \in \overline{\mathcal{C}}$. By Claim 3 and the definition of $\hat{\imath}$,

$$\tilde{h}(Y) = \bigcup_{m<\omega} d(\hat{h}(\chi_Y \restriction j_m)) \cap m = \bigcup_{m<\omega} d(\hat{\imath}(\chi_Y \restriction j_m)). \tag{25}$$

Thus, $\hat{\imath}$ generates $\tilde{h} \restriction \overline{\mathcal{C}}$. Since $\tilde{h}$ is monotone and $\hat{\imath}$ is initial segment preserving, it follows that $\hat{\imath}$ is monotone. □

Thus, $h \restriction \mathcal{C} = \tilde{h} \restriction \mathcal{C}$ is basic, generated by $\hat{\imath}$. By Theorem 8, $h \restriction \mathcal{C}$ extends to some basic map $h^* : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $h^* \restriction \mathcal{W} \to \mathcal{U}$ is cofinal. □

Every p-point satisfies the conditions of Theorem 9 as was shown in the proof of Theorem 20 of [4], where the cofinal set $\mathcal{D}$ there is of the simple form $\mathcal{P}(X) \cap \mathcal{U}$ for some $X \in \mathcal{U}$. From this along with Theorems 8 and 9, we obtain the main result of this section.

Theorem 10. *If $\mathcal{U}$ is Tukey reducible to a p-point then $\mathcal{U}$ has basic Tukey reductions.*

Recall that an ultrafilter $\mathcal{V}$ is Rudin-Blass reducible to an ultrafilter $\mathcal{W}$ if there is a finite-to-one map $h : \omega \to \omega$ such that $\mathcal{V} = h(\mathcal{W})$. Thus, Rudin-Blass reducibility implies Rudin-Keisler reducibility. Since basic Tukey reductions on ultrafilters on base ω are continuous, Theorem 10 along with Theorem 5 yield the following corollary.

Corollary 11. *Suppose $\mathcal{W}$ is Tukey reducible to a p-point. Then every ultrafilter Tukey reducible to $\mathcal{W}$ is in fact Rudin-Blass reducible to $\mathcal{W}$.*

Remark. There is a notion of ultrafilter on the base FIN $= [\omega]^{<\omega} \setminus \{\emptyset\}$ called a *stable ordered-union ultrafilter*, which is the analogue of a p-point for ultrafilters on the base set FIN (see [1]). In Theorems 71 and 72 of [4], it was shown that for each stable ordered union ultrafilter $\mathcal{U}$, both $\mathcal{U}$ and its projection $\mathcal{U}_{\min,\max}$ have basic Tukey reductions. It should be the case that by arguments similar to those in Theorem 9, one can prove that every ultrafilter Tukey reducible to some stable ordered union ultrafilter also has basic Tukey reductions. We leave this open as part of Problem 5 in Section 4. It is of interest that the ultrafilter $\mathcal{U}_{\min,\max}$ is rapid, but is neither a p-point nor a q-point, and yet, by Theorem 10 has basic Tukey reductions. Rather than add many definitions here, we refer the interested reader to [1] and [4].

3. Basic cofinal maps on iterated Fubini products of p-points

In this section, we prove that every monotone map on an ultrafilter which is a countable iteration of Fubini products of p-points is represented by a finitary function on some cofinal subset; thus, countable iterations of Fubini products of p-points have finitary Tukey reductions. The representation is even better, though, than just being represented by a finitary function. Making use of the natural representation of Fubini iterates of p-points as ultrafilters generated by $\vec{\mathcal{U}}$-trees on some front B (see Facts 15 and 16), we show in Theorems 20 and 21 that every monotone Tukey reduction from some Fubini iterate of p-points is *basic* (see Definition 19). Hence, such Tukey reductions are continuous on the space $2^{\hat{B}}$ with the Cantor topology, where $\hat{B}$ is the tree consisting of all initial segments of members of the front B, where B is the base for the ultrafilter. Thus, the key properties for p-points obtained in Theorem 4, due to Dobrinen and Todorcevic, are extended to a larger class of ultrafilters.

The main theorem of this section, Theorem 21, is proved by induction on the rank of the front. The basis for the induction is proved in Theorem 20. All but one of the key concepts in the proof of Theorem 21 appear in the base case which is simpler, hence its inclusion.

We begin by reviewing Fubini products of ultrafilters, and then explicate how they can be precisely represented by $\vec{\mathcal{U}}$-trees on fronts.

Notation. Let $\mathcal{U}$ and $\mathcal{V}_n$ $(n < \omega)$ be ultrafilters. The *Fubini product* of $\mathcal{V}_n$ over $\mathcal{U}$, denoted $\lim_{n\to\mathcal{U}} \mathcal{V}_n$, is defined as follows:

$$\lim_{n\to\mathcal{U}} \mathcal{V}_n = \{A \subseteq \omega \times \omega : \{n \in \omega : \{j \in \omega : (n, j) \in A\} \in \mathcal{V}_n\} \in \mathcal{U}\}. \tag{26}$$

When all $\mathcal{V}_n = \mathcal{V}$, then we let $\mathcal{U} \cdot \mathcal{V}$ denote $\lim_{n\to\mathcal{U}} \mathcal{V}_n$.

The Fubini product construction can be iterated countably many times, each time producing an ultrafilter. However, this construction is not precise at most limit stages. For example, given an ultrafilter $\mathcal{V}$, let $\mathcal{V}^1$ denote $\mathcal{V}$, and let $\mathcal{V}^{n+1}$ denote $\mathcal{V} \cdot \mathcal{V}^n$. Naturally, $\mathcal{V}^\omega$ denotes $\lim_{n\to\mathcal{V}} \mathcal{V}^n$. Continuing in this manner, we obtain $\mathcal{V}^\alpha$, for all $2 \le \alpha < \omega \cdot 2$. At this point, it is ambiguous what is meant by $\mathcal{V}^{\omega\cdot 2}$: It is standard practice to let $\mathcal{V}^{\omega\cdot 2}$ denote any ultrafilter constructed by choosing (arbitrarily) an increasing sequence $(\alpha_n)_{<\omega}$ cofinal in $\omega \cdot 2$ and defining $\mathcal{V}^{\omega\cdot 2}$ to be $\lim_{n\to\mathcal{V}} \mathcal{V}^{\alpha_n}$. Moreover, for all indecomposable $\omega < \alpha < \omega_1$, what exactly meant by $\mathcal{V}^\alpha$ is ambiguous.

However, each iteration of Fubini products of ultrafilters (including the choice of sequence at limit stages) can be represented as an ultrafilter generated by $\vec{\mathcal{U}}$-trees on a base set which is a front. This representation is unambiguous at limit stages. For this reason, our theorem showing that iterations of Fubini products of p-points have finitary Tukey reductions, will be carried out in the setting of $\vec{\mathcal{U}}$-trees.

At this point, we recall the definition of front and define the new notion of *flat-top front*, which is exactly the type of front on which iterated Fubini products of ultrafilters are represented. Then we shall define the notion of $\vec{\mathcal{U}}$-trees on a flat-top front on ω. (The reader desiring more background on fronts and $\vec{\mathcal{U}}$-trees than we present here is referred to [16], pages 12 and 190, respectively.) For sets a and b of natural numbers, recall that a is an *initial segment of* b, denoted $a \sqsubseteq b$, if and only if $a \subseteq b$ and $\min(b \setminus a)$ is greater than every member of a. We use $a \sqsubset b$ to denote that a is a proper initial segment of b; that is, $a \sqsubseteq b$ and $a \neq b$.

Definition 12. A family B of finite subsets of some infinite subset I of ω is called a *front* on I if

(1) $a \not\sqsubset b$ whenever a, b are in B; and
(2) For every infinite $X \subseteq I$ there exists $b \in B$ such that $b \sqsubset X$.

Recall the following standard set-theoretic notation: $[\omega]^k$ denotes the collection of k-element subsets of ω, $[\omega]^{<k}$ denotes the collection of subsets of ω of size less than k, and $[\omega]^{\le k} = [\omega]^{<k+1}$. It is easy to see that for each $k < \omega$, $[\omega]^k$ is a front.

Every front is lexicographically well-ordered, and hence has a unique rank associated with it, namely the ordinal length of its lexicographical well-ordering. For example, $\mathrm{rank}(\{\emptyset\}) = 1$, $\mathrm{rank}([\omega]^1) = \omega$, and $\mathrm{rank}([\omega]^2) = \omega \cdot \omega$.

Given a front B, for each $n \in \omega$, we define $B_n = \{b \in B : n = \min(b)\}$ and $B_{\{n\}} = \{b \setminus \{n\} : b \in B_n\}$. Then $B = \bigcup_{n \in \omega} B_n$, and each $B_n = \{\{n\} \cup a : a \in B_{\{n\}}\}$. Note that for each $n \in \omega$, $B_{\{n\}}$ is a front on $\omega \setminus (n+1)$ with rank strictly less than the rank of B. Conversely, given any collection of fronts $B_{\{n\}}$ on $\omega \setminus (n+1)$, the union $\bigcup_{n \in \omega} B_n$ is a front on ω, where B_n is defined as above to be $\{\{n\} \cup a : a \in B_{\{n\}}\}$.

Definition 13. We call a set $B \subseteq [\omega]^{<\omega}$ a *flat-top front* if B is a front on ω, $B \neq \{\emptyset\}$, and

(1) Either $B = [\omega]^1$; or
(2) $B \subseteq [\omega]^{\ge 2}$ and for each $b \in B$, letting $a = b \setminus \{\max(b)\}$, $\{c \setminus a : c \in B,\ c \sqsupset a\}$ is equal to $[\omega \setminus (\max(a) + 1)]^1$.

Flat-top fronts are exactly the fronts on which iterated Fubini products of ultrafilters are represented, as will be seen in Facts 15 and 16. For example, $[\omega]^2$ is the flat-top front on which a Fubini product of the form $\lim_{n \to \mathcal{U}} \mathcal{V}_n$ is represented. For each $k < \omega$, $[\omega]^k$ is a flat-top front. Moreover, flat-top fronts are preserved under the following recursive construction: Given flat-top fronts $B_{\{n\}}$ on $\omega \setminus (n+1)$, $n < \omega$, the union $\bigcup_{n \in \omega} B_n$ is a flat-top front on ω.

Given any (flat-top) front B, let $C = C(B)$ denote the collection of all proper initial segments of elements of B; that is, $C = \{c \in [\omega]^{<\omega} : \exists \in B\, (c \sqsubset b)\}$. Let $\hat{B} = B \cup C$, the collection of all initial segments of elements of B. Both C and $\hat{B}$ form trees under the partial ordering of initial segments.

Definition 14. Given a flat-top front B and a sequence $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in C(B))$ of nonprincipal ultrafilters $\mathcal{U}_c$ on ω, a $\vec{\mathcal{U}}$-*tree* is a tree $T \subseteq \hat{B}$ with the property that $\{n \in \omega : c \cup \{n\} \in T\} \in \mathcal{U}_c$ for all $c \in C$.

Notation. Given a flat-top front B and a sequence $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in C)$ of nonprincipal ultrafilters on ω, let $\mathfrak{T} = \mathfrak{T}(\vec{\mathcal{U}})$ denote the collection of all $\vec{\mathcal{U}}$-trees. For any $c \in C$ and $T \in \mathfrak{T}$, let $T_c = \{t \in T : t \sqsubseteq c \text{ or } t \sqsupset c\}$, the tree with stem c consisting of all nodes in T comparable with c. For any tree T, let $[T]$ denote the collection of maximal branches through T.

The following Facts 15 and 16 were pointed out to us by Todorcevic.

Fact 15. *The Fubini product* $\lim_{n\to\mathcal{U}} \mathcal{V}_n$ *of ultrafilters on* ω *is isomorphic to the ultrafilter on* $B = [\omega]^2$ *generated by* $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in [\omega]^{\le 1})$*-trees, where* $\mathcal{U}_\emptyset = \mathcal{U}$ *and for each* $n \in \omega$, $\mathcal{U}_{\{n\}} = \mathcal{V}_n$.

Proof. Suppose that $\mathcal{W} = \lim_{n\to\mathcal{U}} \mathcal{V}_n$. Define $\mathcal{U}_\emptyset = \mathcal{U}$, and $\mathcal{U}_{\{n\}} = \mathcal{V}_n$ for each $n < \omega$. Let $B = [\omega]^2$ and $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in [\omega]^{\le 1})$. Let Δ denote the upper triangle $\{(m,n) : m < n < \omega\}$ on $\omega \times \omega$. Let $\theta : \Delta \to [\omega]^2$ by $\theta((m,n)) = \{m,n\}$. Then θ witnesses that $\mathcal{W} \restriction \Delta = \{W \in \mathcal{W} : W \subseteq \Delta\}$ is isomorphic to $\{[S] : \exists T \in \mathfrak{T}(\vec{\mathcal{U}})\,(S \supseteq T)\}$. Since $\mathcal{W} \restriction \Delta$ is isomorphic to the original $\mathcal{W}$, we have that the ultrafilter on B generated by the set $\{[T] : T \in \mathfrak{T}(\vec{\mathcal{U}})\}$ is isomorphic to $\mathcal{W}$. In particular, $\{[T] : T \in \mathfrak{T}(\vec{\mathcal{U}})\}$ is isomorphic to a base for $\mathcal{W}$. □

Since we will be interested only in iterated Fubini products of p-points, we shall restrict our attention to these, as it makes the exposition of the identification between iterated Fubini products and ultrafilters on flat-top fronts more explicit. Let $\mathcal{P}_0$ denote the collection of all p-points on ω. Given $\alpha < \omega_1$, define $\mathcal{P}_{\alpha+1} = \{\lim_{n\to\mathcal{U}} \mathcal{V}_n : \mathcal{U} \in \mathcal{P}_0 \text{ and } \mathcal{V}_n \in \mathcal{P}_\alpha\}$. For each limit ordinal α, define $\mathcal{P}_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}_\beta$. Then $\mathcal{P}_{<\omega_1} := \bigcup\{\mathcal{P}_\alpha : \alpha < \omega_1\}$ is the collection of all iterated Fubini products of p-points. Since the Fubini product of p-points is never a p-point, each $\mathcal{W} \in \mathcal{P}_{<\omega_1}$ has a well-defined notion of rank, namely $\mathrm{rank}(\mathcal{W})$ is the least $\alpha < \omega_1$ for which it is a member of $\mathcal{P}_\alpha$.

Fact 16. *If* $\mathcal{W}$ *is a countable iteration of Fubini products of p-points, then there is a flat-top front* B *and p-points* $\mathcal{U}_c$, $c \in C(B)$ *such that* $\mathcal{W}$ *is isomorphic to the ultrafilter on* B *generated by the* $(\mathcal{U}_c : c \in C)$*-trees.*

Proof. We prove by induction on $\alpha < \omega_1$ that the fact holds for every ultrafilter in $\mathcal{P}_\alpha$. If $\mathcal{W} \in \mathcal{P}_0$, then $\mathcal{W}$ is a p-point and is represented on the flat-top front $B = [\omega]^1$ via the obvious isomorphism $n \mapsto \{n\}$. If $\mathcal{W} \in \mathcal{P}_1$, then Fact 15 proves our claim.

Let $2 \le \alpha < \omega_1$ and assume the fact holds for each ultrafilter in $\bigcup_{\gamma<\alpha} \mathcal{P}_\gamma$. If α is a limit ordinal, then there is nothing to prove, so assume $\alpha = \beta + 1$ for some $1 \le \beta < \omega_1$. Suppose that $\mathcal{W} \in \mathcal{P}_\alpha$. Then $\mathcal{W} = \lim_{n\to\mathcal{U}} \mathcal{W}_n$, where $\mathcal{U}$ is a p-point and for each n, $\mathcal{W}_n \in \mathcal{P}_\beta$. By the induction hypothesis, for each $n < \omega$ there is a flat-top front $B(n)$ on ω and there are p-points $\mathcal{U}_c(n) : c \in C(B(n))$ such that $\mathcal{W}_n$ is isomorphic to the ultrafilter generated by $(\mathcal{U}_c : c \in C(B(n)))$-trees on $B(n)$. In the standard way, we glue the fronts together to obtain a new flat-top front: Let $B_{\{n\}}$ be the front on $\omega \setminus (n+1)$ which is the isomorphic image of $B(n)$, via the isomorphism $\varphi_n : \omega \to \omega \setminus (n+1)$ by $\varphi_n(m) = n + 1 + m$. Then $B = \bigcup_{n<\omega}\{\{n\} \cup b : b \in B_{\{n\}}\}$ is a flat-top front.

Given $n < \omega$, for each $c \in C(B(n))$, $\mathcal{U}_c(n)$ is isomorphic to $\varphi_n(\mathcal{U}_c(n))$. Therefore, the ultrafilter generated by $(\mathcal{U}_c(n) : c \in C(B(n)))$-trees on $B(n)$ is isomorphic to the ultrafilter generated by $(\varphi_n(\mathcal{U}_{\varphi_n^{-1}(a)}(n)) : a \in C(B_{\{n\}}))$. For each $n < \omega$ and

$a \in C(B_{\{n\}})$, let $\mathcal{V}_{\{n\}\cup a}$ denote $\mathcal{U}_{\varphi_n^{-1}(a)}(n)$. Finally, let $\mathcal{V}_\emptyset = \mathcal{U}$. Then the ultrafilter on B generated by the $(\mathcal{V}_a : a \in C(B))$-trees is isomorphic to $\lim_{n\to\mathcal{U}} \mathcal{W}_n$. □

Definition 17. Let $\prec$ denote the following well-ordering on $[\omega]^{<\omega}$. Given any $a, b \in [\omega]^{<\omega}$ with $a \neq b$, enumerate their elements in increasing order as $a = \{a_1, \dots, a_m\}$ and $b = \{b_1, \dots, b_n\}$. Here m equals the cardinality of a and n equals the cardinality of b, and no comparison between m and n is assumed. Define $a \prec b$ iff

(1) $a = \emptyset$; or
(2) $\max(a) < \max(b)$; or
(3) $\max(a) = \max(b)$ and $a_i < b_i$, where i is the least such that $a_i \neq b_i$.

Thus, $\prec$ well-orders $[\omega]^{<\omega}$ in order type ω as follows: $\emptyset \prec \{0\} \prec \{0,1\} \prec \{1\} \prec \{0,1,2\} \prec \{0,2\} \prec \{1,2\} \prec \{2\} \prec \{0,1,2,3\} \prec \dots$. Moreover, for each $k < \omega$, the set $\{c \in [\omega]^{<\omega} : \max(c) = k\}$ forms a finite interval in $([\omega]^{<\omega}, \prec)$.

The following example illustrates why it is impossible for a Fubini product of p-points to have continuous Tukey reductions, with respect to the Cantor topology on 2^B, where B is the base for the ultrafilter. Let $\mathcal{U}$ and $\mathcal{V}$ be any ultrafilters, p-points or otherwise, and let $f : \omega \times \omega \to \omega$ be given by $f((n, j)) = n$. Then $f : \mathcal{U} \cdot \mathcal{V} \to \mathcal{U}$ is a monotone cofinal map, and there is no cofinal $\mathcal{X} \subseteq \mathcal{U} \cdot \mathcal{V}$ for which $f \restriction \mathcal{X}$ is basic on the topological space $2^{\omega\times\omega}$. However, we will soon show that each ultrafilter $\mathcal{W}$ which is an iterated Fubini product of p-points has finitary Tukey reductions which, moreover, are basic (hence continuous) on the topological space $2^{\hat{B}}$ (see Definition 19), where B is the front on which $\mathcal{W}$ is represented as in Fact 16. Toward this end, we proceed to give the definition of basic for this context, and then prove the main results of this section.

Notation. For any subset $A \subseteq [\omega]^{<\omega}$ and $k < \omega$, let $A \restriction k$ denote $\{a \in A : \max(a) < k\}$. For $A \subseteq \hat{B}$ and $k < \omega$, let $\chi_A \restriction k$ denote the characteristic function of $A \restriction k$ on domain $\hat{B} \restriction k$. For each $k < \omega$, let $2^{\hat{B}\restriction k}$ denote the collection of characteristic functions of subsets of $\hat{B} \restriction k$ on domain $\hat{B} \restriction k$.

Definition 18. Let B be a flat-top front on ω. Let $(n_k)_{k<\omega}$ be an increasing sequence. We say that a function $\hat{f} : \bigcup_{k<\omega} 2^{\hat{B}\restriction n_k} \to 2^{<\omega}$ is *level preserving* if $\hat{f} : 2^{\hat{B}\restriction n_k} \to 2^k$, for each $k < \omega$. $\hat{f}$ is *initial segment preserving* if for all $k < m$, $A \subseteq \hat{B} \restriction n_k$ and $A' \subseteq \hat{B} \restriction n_m$, if $A = A' \restriction n_k$ then $\hat{f}(\chi_A) = \hat{f}(\chi_{A'}) \restriction k$. $\hat{f}$ is *monotone* if whenever $A \subseteq A' \subseteq \hat{B}$ are finite, then $d(\hat{f}(s)) \subseteq d(\hat{f}(t))$.

Let $\mathcal{W}$ be an ultrafilter on B generated by $(\mathcal{U}_c : c \in C(B))$-trees, let $f : \mathcal{W} \to \mathcal{V}$ be a monotone cofinal map, where $\mathcal{V}$ is an ultrafilter on base ω, and let $\tilde{T} \in \mathfrak{T}(\vec{\mathcal{U}})$. We say that $\hat{f} : \bigcup_{k<\omega} 2^{\hat{B}\restriction n_k} \to 2^{<\omega}$ *generates* f on $\mathfrak{T} \restriction \tilde{T}$ if for each $T \subseteq \tilde{T}$ in $\mathfrak{T}$,

$$f([T]) = \bigcup_{k<\omega} d(\hat{f}(\chi_T \restriction n_k)). \tag{27}$$

Definition 19. Let B be a flat-top front and $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in C(B))$ be a sequence of ultrafilters. We say that the ultrafilter $\mathcal{W}$ on B generated by the $\vec{\mathcal{U}}$-trees has *basic Tukey reductions* if whenever $f : \mathcal{W} \to \mathcal{V}$ is a monotone cofinal map, then there is a $\tilde{T} \in \mathfrak{T}(\vec{\mathcal{U}})$ and a monotone, initial segment and level preserving map $\hat{f}$ which generates f on $\mathfrak{T} \restriction \tilde{T}$.

Remark. Note that if $\hat{f}$ witnesses that f is basic on $\mathfrak{T} \restriction \tilde{T}$, then $\hat{f}$ generates a continuous map on the collection of *trees* in $\mathfrak{T} \restriction \tilde{T}$, continuity being with respect to the Cantor topology on $2^{\hat{B}}$. Moreover, we may define a finitary map $\hat{g}$ on B as follows: For each finite subset $A \subseteq B$, define $\hat{g}(A) = \hat{f}(\hat{A})$, where $\hat{A}$ is the collection of all initial segments of members of A. Then $\hat{g}$ is finitary, but not necessarily continuous on 2^B, and generates f on $\mathcal{D}$. Thus, for ultrafilters generated by $\mathcal{U}$-trees, basic Tukey reductions imply finitary Tukey reductions.

The next theorem provides the base case for Theorem 21. We prove this theorem first, as almost all the key points of the general construction come to light in this simpler setting.

Theorem 20. *Suppose $\mathcal{U}_\emptyset$ and $\mathcal{U}_{\{n\}}$, $n \in \omega$, are p-points, and let $\vec{\mathcal{U}}$ denote $(\mathcal{U}_c : c \in [\omega]^{\le 1})$. Then the ultrafilter on base $[\omega]^2$ generated by the $\vec{\mathcal{U}}$-trees has basic Tukey reductions. That is, the Fubini product $\lim_{n\to\mathcal{U}_\emptyset} \mathcal{U}_{\{n\}}$ has basic Tukey reductions.*

Proof. We begin by setting up the relevant notation. Since $B = [\omega]^2$, we have $\hat{B} = [\omega]^{\le 2}$ and $C = [\omega]^{\le 1}$. We let $\mathfrak{T}$ denote $\mathfrak{T}(\mathcal{U}_c : c \in [\omega]^{\le 1})$. Fix an enumeration of the non-empty $\sqsubseteq$-closed subsets of $\hat{B}$ as $\langle A_i : i < \omega\rangle$ and a sequence $(p_k)_{k<\omega}$ such that for each k, the sequence $\langle A_i : i < p_k\rangle$ lists all non-empty $\sqsubseteq$-closed subsets of $[k]^{\le 2}$. (An example of such sequence is the following: $A_0 = \{\emptyset\}$, $A_1 = \{\emptyset, \{0\}\}$, $A_2 = \{\emptyset, \{1\}\}$, $A_3 = \{\emptyset, \{0\}, \{0, 1\}\}$, $A_4 = \{\emptyset, \{0\}, \{1\}\}$, $A_5 = \{\emptyset, \{0\}, \{0, 1\}, \{1\}\}$, $A_6 = \{\emptyset, \{0\}, \{2\}\}, \dots$. For this sequence, $p_0 = 1, p_1 = 2$, and $p_2 = 6$.) For all $k < \omega$ and $i < p_k$, define

$$\hat{B}^k_i = A_i \cup \{b \in [\omega]^{\le 2} : \exists a \in A_i (b \sqsupset a \text{ and } \min(b \setminus a) \ge k)\}. \tag{28}$$

Thus, $\hat{B}^k_i$ is the maximal tree in $\mathfrak{T}$ for which $T \restriction k = A_i$. For a tree $T \subseteq \hat{B}$ and $c \in T \cap C$, define

$$U_c(T) = \{l > \max(c) : c \cup \{l\} \in T\}, \tag{29}$$

the set of immediate extensions of c in T. Note that if $T \in \mathfrak{T}$, then for each $c \in T \cap C$, $U_c(T)$ is a member of $\mathcal{U}_c$.

Our goal is to construct a tree $\tilde{T} \in \mathfrak{T}$ and find a sequence $(n_k)_{k<\omega}$ of good cut-off points such that the following (⊛) holds.

(⊛) For each $T \subseteq \tilde{T}$ in $\mathfrak{T}$, $k < \omega$, and $i < p_{n_k}$ such that $A_i = T \restriction n_k$,

$$k \in f([T]) \iff k \in f([\tilde{T} \cap \hat{B}^{n_k}_i]).$$

This will suffice for constructing a monotone, initial segment and level preserving finitary map $\hat{f}$ on $\bigcup_{k<\omega} 2^{\hat{B}\restriction n_k}$ which represents f on $\mathfrak{T} \restriction \tilde{T}$, as we now show. Note that (⊛) implies that for each $T \in \mathfrak{T} \restriction \tilde{T}$ whether or not $k \in f([T])$ is decided by $R^{n_k}_i$, where i satisfies $A_i = T \restriction n_k$. Thus, we may define a monotone, level and initial segment preserving map $\hat{f}$ which generates f on $\mathfrak{T} \restriction \tilde{T}$ as follows. For each $k < \omega$ and $i < p_k$, define

$$\hat{f}(A_i) = \{j \le k : j \in f([R^{n_k}_i])\}. \tag{30}$$

Then $\hat{f}$ is monotone and level and initial segment preserving. Moreover, for each $T \in \mathfrak{T} \restriction \tilde{T}$, $f([T]) = \bigcup\{\hat{f}(T \restriction n_k) : k < \omega\}$.

We remark that defining $\hat{f}(A)$ for any $A \subseteq \hat{B}$ which is not closed under $\sqsubseteq$ is irrelevant, since such an A never is an initial segment of any tree. However, if one cares, for each $A \subseteq \tilde{T} \restriction n_k$, we may define $\hat{g}(A)$ to be the characteristic function on domain k of the set $\bigcap\{d(\hat{f}(A_i)) : i < p_{n_k}$ and $A \subseteq A_i\}$. Then it is clear that

$\hat{g}(A_i) = \hat{f}(A_i)$ for each $i < \omega$. Moreover, $\hat{g}$ is monotone, level and initial segment and level preserving, and generates f on $\mathfrak{T} \restriction \tilde{T}$.

The construction of $\tilde{T}$ takes place in three stages.

Stage 1. In the first stage toward the construction of $\tilde{T}$, we choose some $R_i^k \in \mathfrak{T}$ with the property that for all $k < \omega$, $(*)_k$ holds:

$(*)_k$ For all $i < p_k$ and $T \subseteq R_i^k$ in $\mathfrak{T}$ with $T \restriction k = A_i$, for each $j \le k$, $j \in f([T]) \iff j \in f([R_i^k])$.

Then we construct trees T_c^k, $k > \max(c)$, to be used in Stage 2.

Step 0. Note that for each $R \in \mathfrak{T}$, $R \restriction 0 = \{\emptyset\}$, which is exactly A_0. If there is an $R \in \mathfrak{T}$ such that and $0 \notin f([R])$, then let R_0^0 be such an R. If there is no such R, then let $R_0^0 = \hat{B}$.

Step 1. Recall that $p_1 = 2$. If there is an $R \subseteq R_0^0$ in $\mathfrak{T}$ such that $R \restriction 1 = A_0$ and $1 \notin f([R])$, then let R_0^1 be such an R. If not, let $R_0^1 = R_0^0 \cap \hat{B}_0^1$. If there is an $R \in \mathfrak{T}$ such that $R \restriction 1 = A_1$ and $0 \notin f([R])$, then let $R_{1,0}^1$ be such an R. If there is no such R, then let $R_{1,0}^1 = \hat{B}_1^1$. If there is an $R \in \mathfrak{T}$ such that $R \subseteq R_{1,0}^1$, $R \restriction 1 = A_1$, and $1 \notin f([R])$, then let R_1^1 be such an R. If not, then let $R_1^1 = R_{1,0}^1$.

Step k. Having completed Step $k-1$, for each $i < p_{k-1}$, if there is an $R \subseteq R_i^{k-1}$ in $\mathfrak{T}$ such that $R \restriction k = A_i$ and $k \notin f([R])$, then let R_i^k be such an R. If not, let $R_i^k = R_i^{k-1} \cap \hat{B}_i^k$.

For each $p_{k-1} \le i < p_k$, if there is an $R \in \mathfrak{T}$ such that $R \restriction k = A_i$ and $0 \notin f([R])$, then let $R_{i,0}^k$ be such an R. If there is no such R, then let $R_{i,0}^k = \hat{B}_i^k$. Given $R_{i,j}^k$ for $j < k$, if there is an $R \subseteq R_{i,j+1}^k$ in $\mathfrak{T}$ such that $R \restriction k = A_i$ and $j+1 \notin f([R])$, then let $R_{i,j+1}^k$ be such an R. If not, then let $R_{i,j+1}^k = R_{i,j}^k$. Finally, let $R_i^k = R_{i,k}^k$.

We check that $(*)_k$ holds for each k. Let $T \in \mathfrak{T}$ such that $T \subseteq R_i^k$ and $T \restriction k = A_i$. Let $j \le k$. If $j \notin f([R_i^k])$, then $j \notin f([T])$, since f is monotone. On the other hand, if $j \in f([R_i^k])$, then j must be in $f(T)$, since in this case, there was no $R \subseteq R_i^k$ in $\mathfrak{T}$ with $R \restriction k = A_i$ and $j \notin f([R])$.

For $c \in [\omega]^{\le 1}$, recall that $\hat{B}_c$ denotes $\{a \in \hat{B} : a \sqsubseteq c \text{ or } a \sqsupset c\}$. Define $T_\emptyset^0 = R_0^0$. Define $T_\emptyset^1 = T_\emptyset^0 \cap R_0^1 \cap R_1^1$. If $\{0\} \in T_\emptyset^0$, then define $T_{\{0\}}^1 = \hat{B}_{\{0\}} \cap R_1^1 \cap T_\emptyset^0$. If $\{0\} \notin T_\emptyset^0$, then define $T_{\{0\}}^1 = \hat{B}_{\{0\}} \cap R_1^1$. Now suppose $k \ge 2$. For $c \in [k-1]^{\le 1}$, define

$$T_c^k = T_c^{k-1} \cap \bigcap \{R_i^k : i < p_k \text{ and } c \in A_i\}. \tag{31}$$

If there is a $j < k$ such that $\{k-1\} \in T_\emptyset^j$, then let l denote the maximal such j and define

$$T_{\{k-1\}}^k = \hat{B}_c \cap \bigcap \{R_i^k : i < p_k \text{ and } c \in A_i\} \cap T_\emptyset^l. \tag{32}$$

Otherwise, for all $j < k$, $\{k-1\} \notin T_\emptyset^j$, and we define

$$T_{\{k-1\}}^k = \hat{B}_c \cap \bigcap \{R_i^k : i < p_k \text{ and } c \in A_i\}. \tag{33}$$

Note the following properties of the trees T_c^k, which will be subsequently useful. For each $c \in [\omega]^{\le 1}$, $k > \max(c)$ implies T_c^k is defined. By our construction, $T_\emptyset^0 \supseteq$

$T^1_\emptyset \supseteq \dots$, and for all $k < \omega$, $T^{k+1}_{\{k\}} \supseteq T^{k+2}_{\{k\}} \supseteq \dots$. Further, whenever $c \in [k]^{\le 1}$ and $c \in A_i$, then $T^k_c \subseteq R^k_i$.

Stage 2. For each $c \in [\omega]^{\le 1}$ and $\max(c) < k < \omega$, let $U^k_c := U^k_c(T^k_c)$, the collection of immediate extensions of c in T^k_c. In the second stage, we diagonalize through the U^k_c using some strictly increasing functions $m(c, \cdot) : \omega \to \omega$ which will often line up, or *mesh*, as follows.

(†) For each $k < \omega$, for all $c \in [k+1]^{\le 1}$ there is an i_c such that

$$m(\emptyset, 2i_\emptyset) = m(\{0\}, 2i_{\{0\}}) = \dots = m(\{k\}, 2i_{\{k\}}).$$

The meshing functions of (†) will help us obtain a tree $T^* \in \mathfrak{T}$ with the following properties.

(‡) (a) $U_\emptyset(T^*) \subseteq U^0_\emptyset$ and for all $k < \omega$, $U_{\{k\}}(T^*) \subseteq U^{k+1}_{\{k\}}$;
For all $c \in T^* \cap [\omega]^{\le 1}$ and $i < \omega$,
(b) $U_c(T^*) \setminus m(c, i+1) \subseteq U^{m(c,i)}_c$; and
(c) $U_c(T^*) \cap [m(c, 2i), m(c, 2i+1)) = \emptyset$.

Since $\mathcal{U}_\emptyset$ is a p-point, there is a $U^*_\emptyset \in \mathcal{U}_\emptyset$ such that $U^*_\emptyset \subseteq^* U^k_\emptyset$ for each k. Let $g_\emptyset : \omega \to \omega$ be a strictly increasing function such that for each k, $U^*_\emptyset \setminus g_\emptyset(k+1) \subseteq U^{g_\emptyset(k)}_\emptyset$. If $\bigcup_{i\in\omega}[g_\emptyset(2i), g_\emptyset(2i+1)) \in \mathcal{U}_\emptyset$, then define $m(\emptyset, k) = g_\emptyset(k+1)$. Otherwise, $\bigcup_{i\in\omega}[g_\emptyset(2i+1), g_\emptyset(2i+2)) \in \mathcal{U}_\emptyset$, and we define $m(\emptyset, k) = g_\emptyset(k)$. Let $Y_\emptyset = \bigcup_{i\in\omega}[m(\emptyset, 2i+1), m(\emptyset, 2i+2))$, and define $U_\emptyset = U^0_\emptyset \cap U^*_\emptyset \cap Y_\emptyset$. Note that for each k, $U_\emptyset \setminus m(\emptyset, k+1) \subseteq U^{m(\emptyset,k)}_\emptyset$.

Given $k < \omega$, suppose we have defined $m(b, \cdot)$ and U_b for all $b \in [k]^{\le 1}$. Let c denote $\{k\}$. If $k \ge 1$, let a denote $\{k-1\}$; otherwise, $k = 0$ and we let a denote $\emptyset$. That is, a denotes the immediate $\prec$-predecessor of c in $[\omega]^{\le 1}$. Since $\mathcal{U}_c$ is a p-point, there is a $U^*_c \in \mathcal{U}^*_c$ for which $U^*_c \subseteq^* U^l_c$, for all $l > k$. Let $g_c : \omega \to \omega$ be a strictly increasing function such that $g_c(0) > k$ and

(1) For each i, $U^*_c \setminus g_c(i+1) \subseteq U^{g_c(i)}_c$; and
(2) For each j, there is an i such that $g_c(j) = m(a, 2i)$.

If $\bigcup_{i\in\omega}[g_c(2i), g_c(2i+1)) \in \mathcal{U}_c$, then define $m(c, i) = g_c(i+1)$. If $\bigcup_{i\in\omega}[g_c(2i+1), g_c(2i+2)) \in \mathcal{U}_c$, then define $m(c, i) = g_c(i)$. Let $Y_c = \bigcup_{i<\omega}[m(c, 2i+1), m(c, 2i+2))$. Then $Y_c \in \mathcal{U}_c$. Let $U_c = U^{k+1}_c \cap U^*_c \cap Y_c$.

Now define $T^* \in \mathfrak{T}$ as follows: Let $U_\emptyset(T^*) = U_\emptyset$, and for each $k \in U_\emptyset$, let $U_{\{k\}}(T^*) = U_{\{k\}}$. (†) follows from (2) holding at each step in the recursive construction. It is easy to check from (1) and the definitions of $m(c, \cdot)$ and T^* that (‡) holds.

Stage 3. In this final stage, we find an increasing sequence $(n_k)_{k<\omega}$ where many of the $m(c, 2i)$ are equal, and thin through $U_\emptyset(T^*)$, deleting key intervals just below some of the $m(c, 2i)$, to obtain $\tilde{T}$. The deletions provide the space needed to show that the n_k are good cut-off points so that (⊛) holds.

Define a strictly increasing function $h_\emptyset : \omega \to \omega$ as follows. Let $h_\emptyset(0) = m(\emptyset, 0)$. Given $h_\emptyset(k)$ take $h_\emptyset(k+1)$ such that for each $c \in [h_\emptyset(k)]^{\le 1}$, there is an i_c such that

$$h_\emptyset(k) < m(c, 2i_c - 1) < m(c, 2i_c) = h_\emptyset(k+1). \tag{34}$$

This is possible by (†). If $\bigcup_{i<\omega}[h_\emptyset(2i), h_\emptyset(2i+1)) \in \mathcal{U}_\emptyset$, then define $n(\emptyset, i) = h_\emptyset(i+2)$; if $\bigcup_{i<\omega}[h_\emptyset(2i+1), h_\emptyset(2i+2)) \in \mathcal{U}_\emptyset$, then define $n(\emptyset, i) = h_\emptyset(i+1)$. Thus,

$Z_\emptyset := \bigcup_{i<\omega}[n(\emptyset, 2i), n(\emptyset, 2i+1))$ is in $\mathcal{U}_\emptyset$. Let $\tilde{T}$ be obtained from T^* simply by thinning the first level of T^* through $Z_\emptyset$. Thus, $U_\emptyset(\tilde{T}) = U_\emptyset(T^*) \cap Z_\emptyset$, and for each $l \in U_\emptyset(\tilde{T})$, $U_{\{l\}}(\tilde{T}) = U_{\{l\}}(T^*)$. Define $n_k = n(\emptyset, 2k+2)$. Then $n_k > k$, for every $k < \omega$. This completes the construction of $\tilde{T}$ and $(n_k)_{k<\omega}$.

By the definition of n_k and (34), we have that for each k there is an $r_\emptyset$ such that $m(\emptyset, 2r_\emptyset) = n_k$. Since $U_\emptyset(\tilde{T}) \subseteq Z_\emptyset$, it follows that if $l \in U_\emptyset(\tilde{T}) \cap n_k$, then in fact, $l < n(\emptyset, 2k+1)$. By (34), we have

$(**)_k$ For all $c \in [n_k]^{\le 1} \cap \tilde{T}$, there is an r_c such that

$$m(c, 2r_c) = n_k.$$

Thus, we have finished Stage 3.

Finally, we check that (⊛) holds. Let $k < \omega$ and $i < p_{n_k}$ be given. We show that $\tilde{T} \cap \hat{B}_i^{n_k} \subseteq R_i^{n_k}$. This will imply that for each $T \in \mathfrak{T} \restriction \tilde{T}$ with $T \restriction n_k = A_i$, we in fact have $T \subseteq R_i^{n_k}$. Thus, (⊛) will follow from $(*)_{n_k}$, for all $k < \omega$.

Let $k < \omega$ be given and $i < p_k$. Let S denote $\tilde{T} \cap \hat{B}_i^{n_k}$. We shall that for each $c \in S \cap [\omega]^{\le 1}$, $U_c(S) \setminus n_k \subseteq U_c(R_i^{n_k})$. Since $S \restriction n_k = A_i = R_i^{n_k} \restriction n_k$, it will follow immediately that $S \subseteq R_i^{n_k}$. Recall that $S \subseteq \tilde{T} \subseteq T^*$. Thus, for all $c \in S \cap [\omega]^{\le 1}$, $U_c(S) \setminus n_k \subseteq U_c(\tilde{T}) \setminus n_k \subseteq U_c(T^*) \setminus n_k$. We have two cases.

Case 1. $c \in S \cap [n_k]^{\le 1}$. Then by $(**)_k$, there is an r_c such that $m(c, 2r_c) = n_k$. Since (‡) (c) gives that $U_c(T^*) \cap [m(c, 2r_c), m(c, 2r_c+1)) = \emptyset$, we have that $U_c(T^*) \setminus n_k = U_c(T^*) \setminus m(c, 2r_c+1)$. By (‡) (b), $U_c(T^*) \setminus m(c, 2r_c+1) \subseteq U_c^{m(c,2r_c)}$, which is exactly $U_c^{n_k}$. Note that $c \in S \cap [n_k]^{\le 1}$ implies $c \in A_i$, which implies that $c \in R_i^{n_k}$. Thus, by our construction, $T_c^{n_k} \subseteq R_i^{n_k}$. Therefore, $U_c^{n_k} := U_c(T_c^{n_k}) \subseteq U_c(R_i^{n_k})$. Hence, $U_c(S) \setminus n_k \subseteq U_c(R_i^{n_k})$.

Case 2. $c = \{l\} \in S$ and $l \ge n_k$. Then $U_c(T^*) \setminus n_k = U_c(T^*)$. By (‡) (a), $U_c(T^*) \subseteq U_c^{l+1}$, which by definition is exactly $U_c(T_c^{l+1})$. Since $l \in U_\emptyset(S) \setminus n_k$, which by Case 1 is contained in $U_\emptyset(R_i^{n_k})$, we have that $c \in R_i^{n_k}$. Therefore, $T_c^{l+1} \subseteq R_i^{n_k}$. Hence, $U_c(S) \subseteq U_c(T^*) \subseteq U_c(R_i^{n_k})$. By Cases 1 and 2, $S \subseteq R_i^{n_k}$.

Now suppose $T \in \mathfrak{T} \restriction \tilde{T}$ and $T \restriction n_k = A_i$. Then $T \subseteq S$; so by $(*)_{n_k}$, $k \in f([T]) \iff k \in f([R_i^{n_k}]) \iff k \in f([S])$. This completes the proof of (⊛), and thus, the proof of this theorem. □

Now we prove the main theorem of this section.

Theorem 21. *Let B be any flat-top front and $\vec{\mathcal{U}} = (\mathcal{U}_c : c \in C(B))$ be a sequence of p-points. Then the ultrafilter on base B generated by the $\vec{\mathcal{U}}$-trees has basic Tukey reductions. Therefore, every countable iteration of Fubini products of p-points has basic Tukey reductions.*

Proof. The proof closely follows the proof of Theorem 20, the main differences being that now our induction arguments are over arbitrary flat-top fronts, and the construction for Stage 3 requires much more, in particular Lemma 22. Enumerate the non-empty $\sqsubseteq$-closed subsets of $\hat{B}$ as $\langle A_i : i < \omega\rangle$ in such a way that there is a sequence $(p_k)_{k<\omega}$ so that for each k, the sequence $\langle A_i : i < p_k\rangle$ lists all $\sqsubseteq$-closed subsets of $\hat{B} \restriction k$. For $i < p_k$, let

$$\hat{B}_i^k = A_i \cup \{b \in \hat{B} : \exists a \in A_i (b \sqsupset a \text{ and } \min(b \setminus a) > k)\}. \tag{35}$$

Our goal is to construct a tree $\tilde{T} \in \mathfrak{T}$ and find a sequence $(n_k)_{k<\omega}$ of good cut-off points such that the following (⊛) holds.

(⊛) For each $T \subseteq \tilde{T}$ in $\mathfrak{T}$, $k < \omega$, and $i < p_{n_k}$ such that $A_i = T \restriction n_k$,

$$k \in f([T]) \Longleftrightarrow k \in f([\tilde{T} \cap \hat{B}_i^{n_k}]).$$

As before, this will suffice for constructing a monotone, initial segment and level preserving finitary map $\hat{f}$ on $\bigcup_{k<\omega} 2^{\hat{B}\restriction n_k}$ which represents f on $\mathfrak{T} \restriction \tilde{T}$, as follows: For each $k < \omega$ and $i < p_k$, define

$$\hat{f}(A_i) = \{j \leq k : j \in f([R_i^{n_k}])\}. \tag{36}$$

Then $\hat{f}$ is monotone and level and initial segment preserving, and for each $T \in \mathfrak{T} \restriction \tilde{T}$, $f([T]) = \bigcup\{\hat{f}(T \restriction n_k) : k < \omega\}$.

The construction of $\tilde{T}$ and $(n_k)_{k<\omega}$ takes place in three stages. The first two stages proceed almost identically as in the proof of Theorem 20.

Stage 1. In the first stage toward the construction of $\tilde{T}$, we choose some $R_i^k \in \mathfrak{T}$ with the property that for all $k < \omega$, $(*)_k$ holds:

$(*)_k$ For all $i < p_k$ and $T \subseteq R_i^k$ in $\mathfrak{T}$ with $T \restriction k = A_i$, for each $j \leq k$, $j \in f([T]) \Longleftrightarrow j \in f([R_i^k])$.

Then we construct trees T_c^k, $k > \max(c)$, to be used in Stage 2.

For each flat-top front, it is always the case that $A_0 = \{\emptyset\}$ and $p_0 = 1$. Thus, we choose R_0^0 exactly as in Step 0 of the proof of Theorem 20. For $k > 0$, Step k also proceeds exactly as in the proof of Theorem 20. For each $i < p_{k-1}$, if there is an $R \subseteq R_i^{k-1}$ in $\mathfrak{T}$ such that $R \restriction k = A_i$ and $k \notin f([R])$, then let R_i^k be such an R. If not, let $R_i^k = R_i^{k-1} \cap \hat{B}_i^k$. For each $p_{k-1} \leq i < p_k$, if there is an $R \in \mathfrak{T}$ such that $R \restriction k = A_i$ and $0 \notin f([R])$, then let $R_{i,0}^k$ be such an R. If there is no such R, then let $R_{i,0}^k = \hat{B}_i^k$. Given $R_{i,j}^k$ for $j < k$, if there is an $R \in \mathfrak{T}$ such that $R \subseteq R_{i,j+1}^k$, $R \restriction k = A_i$, and $j + 1 \notin f([R])$, then let $R_{i,j+1}^k$ be such an R. If not, then let $R_{i,j+1}^k = R_{i,j}^k$. Finally, let $R_i^k = R_{i,k}^k$.

Define $T_\emptyset^0 = R_0^0$. Now suppose $k \geq 1$. For $c \in C \restriction (k-1)$, define

$$T_c^k = T_c^{k-1} \cap \bigcap\{R_i^k : i < p_k \text{ and } c \in A_i\}. \tag{37}$$

Now suppose $c \in C \restriction k$ with $\max(c) = k - 1$. For each proper initial segment $a \sqsubset c$, if $c \in T_a^l$ for some $l > \max(a)$, let $l(c, a)$ be the maximal such l and let $S(c, a) = T_a^{l(c,a)}$; if $c \notin T_a^l$ for any $l > \max(a)$, then let $S(c, a) = \hat{B}$. (Here we make the convention of considering 0 as greater than $\max(\emptyset)$.) Define

$$T_c^k = \hat{B}_c \cap \bigcap\{R_i^k : i < p_k \text{ and } c \in A_i\} \cap \bigcap\{S(c, a) : a \sqsubset c\}. \tag{38}$$

It follows from our construction that $(*)_k$ holds for all $k < \omega$.

Stage 2. We proceed similarly as in Stage 2 of the proof of Theorem 20. For each $c \in C$ and $k < \omega$, let U_c^k denote $U_c(T_c^k) := \{l > \max(c) : c \cup \{l\} \in T_c^k\}$. We construct a tree T^* which diagonalizes through the U_c^k, and some meshing functions $m(c, \cdot) : \omega \to \omega$ which will often line up to help us in Stage 3 to find good cut-off points n_k. In particular, the construction will ensure the following properties (†) and (‡).

(†) For each $k < \omega$, for all $c \in C \restriction k$ there is an i_c such that all $m(c, 2i_c)$, $c \in C \restriction k$, are equal.

The meshing functions of (†) will help us obtain a tree $T^* \in \mathfrak{T}$ with the following properties.

(‡) For all $c \in T^* \cap C$,
  (a) $U_c(T^*) \subseteq U_c^{\max(c)+1}$; and
  For all $i < \omega$,
  (b) $U_c(T^*) \setminus m(c, i+1) \subseteq U_c^{m(c,i)}$; and
  (c) $U_c(T^*) \cap [m(c, 2i), m(c, 2i+1)) = \emptyset$.

The construction proceeds by recursion on the well-ordering $(C, \prec)$. Since $\emptyset$ is $\prec$-minimal in C, we start by choosing $g_\emptyset$, $m(\emptyset, \cdot)$ and $Y_\emptyset$ exactly as in Stage 2 of the proof of Theorem 20. Now suppose $c \in C$ and for all $b \prec c$ in C, g_b and $m(b, \cdot)$ have been defined. Since $\mathcal{U}_c$ is a p-point, there is a $U_c^* \in \mathcal{U}_c^*$ for which $U_c^* \subseteq^* U_c^k$, for all $k > \max(c)$. Let a denote the immediate $\prec$-predecessor of c in C. Let $g_c : \omega \to \omega$ be a strictly increasing function such that $g_c(0) > \max(c)$ and

(1) For each i, $U_c^* \setminus g_c(i+1) \subseteq U_c^{g_c(i)}$; and
(2) For each j, there is an i such that $g_c(j) = m(a, 2i)$.

Let Y_c denote the one of the two sets $\bigcup_{i\in\omega}[g_c(2i+2), g_c(2i+3))$ or $\bigcup_{i\in\omega}[g_c(2i+1), g_c(2i+2))$ which is in $\mathcal{U}_c$. In the first case, define $m(c, i) = g_c(i+1)$. In the second case, define $m(c, i) = g_c(i)$. Then $Y_c = \bigcup_{i<\omega}[m(c, 2i+1), m(c, 2i+2))$ and is in $\mathcal{U}_c$. Let $U_c = U_c^{\max(c)+1} \cap U_c^* \cap Y_c$.

Define $T^* \in \mathfrak{T}$ to be the tree such that $U_\emptyset(T^*) = U_\emptyset$, and if $c \in T^* \cap C$, then $U_c(T^*) = U_c$. The properties (†) and (‡) follow from the construction.

Stage 3. For the general case, Stage 3 is more intricate than in the proof of Theorem 20. Since B is a front, $\hat{B}$ is a well-founded tree. Nevertheless, the height of $\hat{B}$ may be infinite. To find good cut-off points n_k, the following lemma will be useful, which is proved by induction on the rank of flat-top fronts.

Given a front B and $C = C(B)$, we let C_* denote the set of c in C which are not $\sqsubseteq$-maximal in C. That is, $C_* = \{c \setminus \{\max(c)\} : c \in C\}$. To get one's bearings, note that for $B = [\omega]^2$, $C_* = \{\emptyset\}$, and in the proof of Theorem 20 we only constructed one n-function, namely $n(\emptyset, \cdot)$. If $B = [\omega]^3$ then $C_* = [\omega]^{\le 1}$; if $B = [\omega]^4$ then $C_* = [\omega]^2$, and so forth.

Lemma 22. *Suppose B is a flat-top front with rank at least $\omega{\cdot}\omega$ and $(\mathcal{U}_c : c \in C(B))$ is a sequence of p-points. Suppose that for $c \in C$, we have functions $m(c, \cdot) : \omega \to \omega$ satisfying (†). Then there are functions $n(c, \cdot) : \omega \to \omega$ and sets $Z_c \in \mathcal{U}_c$, $c \in C_*$, which satisfy the following.*

(i) *There is an increasing sequence $(j_i)_{i<\omega}$ such that for each $c \in C \restriction j_i$, there is an r such that $m(c, 2r) = j_{i+1}$ and $m(c, 2r-1) > j_i$.*
(ii) *For each l, $n(c, l) = j_i$ for some i.*
(iii) *If c is not $\sqsubseteq$-maximal in C_*, then for each $q \ge 1$ and each $l \in U_c(T^*) \cap n(c, q-1)$, there is a q' such that $n(c, q) = n(c \cup \{l\}, 2q')$, and $n(c, q-1) < n(c \cup \{l\}, 2q'-1)$.*
(iv) *For each $c \in C_*$, $Z_c := \bigcup_{i<\omega}[n(c, 2i), n(c, 2i+1)) \in \mathcal{U}_c$.*

Proof. The proof is by induction on the rank of the flat-top front B. The base case is when $B = [\omega]^2$; that is $\text{rank}(B) = \omega \cdot \omega$, for this is the smallest rank of a flat-top front which can be a base for a Fubini product of p-points. Stage 3 in the proof of Theorem 20 gives the lemma for $[\omega]^2$.

Now suppose that B is a flat-top front of rank $\alpha > \omega{\cdot}\omega$ and that the lemma holds for all flat-top fronts of smaller rank. First, go through Stages 1 and 2 to construct functions $m(c, \cdot)$, $c \in C$, satisfying (†). Then choose an increasing sequence $(j_i)_{i<\omega}$

as follows. Let $j_0 = m(\emptyset, 0)$. In general, take $j_{i+1} > j_i$ such that for each $c \in C \restriction j_i$, there is a q such that $j_i < m(c, 2q-1)$ and $m(c, 2q) = j_{i+1}$. This is possible by (†). The sequence $(j_i)_{i<\omega}$ satisfies (i).

For each $l < \omega$, let $B_l = \{b \in B : \min(b) = l\}$. Note that B_l is isomorphic to $B_{\{l\}} := \{b \setminus \{l\} : b \in B_l\}$, which is a flat-top front on $\omega \setminus (l+1)$ of rank less than α. Thus, the induction hypothesis applies to each B_l. Define C_l to be $\{c \in C : c \sqsupseteq \{l\}\}$. Use the induction hypothesis on B_0 with the sequence $(j_i)_{i<\omega}$ to find meshing functions $n(c, \cdot) : \omega \to \omega$ and Z_c, $c \in C_0 \cap C_*$, which satisfy (ii) - (iv). Define $j^1_i = n(\{0\}, 2i)$, for each $i < \omega$. For $l \geq 1$, given the sequence $(j^l_i)_{i<\omega}$, use the induction hypothesis on B_l to find meshing functions $n(c, \cdot) : \omega \to \omega$ and Z_c, $c \in C_l \cap C_*$, which satisfy (ii) - (iv) with regard to $(j^l_i)_{i<\omega}$. Define $j^{l+1}_i = n(\{l\}, 2i)$, for each $i < \omega$. Continuing in this manner, we obtain for all $l < \omega$ functions $n(\{l\}, \cdot)$, sequences $(j^l_i)_{i<\omega}$, and $Z_{\{l\}} \in \mathcal{U}_{\{l\}}$ satisfying (ii) - (iv). Moreover, we also have that for each $l < \omega$, for all $i < \omega$, $j^{l+1}_i = n(\{l\}, 2i)$, and the functions mesh: For all $l < l'$ and all q', there is a q such that $n(c \cup \{l'\}, q') = n(c \cup \{l\}, 2q)$. This will be important in the construction of $n(\emptyset, \cdot)$.

Finally, we construct $n(\emptyset, \cdot) : \omega \to \omega$ to mesh with all the $n(\{l\}, \cdot)$, $l < \omega$. By the work in the previous paragraph, this will guarantee that $n(\emptyset, \cdot)$ meshes with all $n(c, \cdot)$, $c \in C_*$; in particular, (ii) holds. Let $h_\emptyset(0) = n(\{0\}, 2)$. Given $h_\emptyset(i)$, take $h_\emptyset(i+1)$ to be $j^{h_\emptyset(i)}_p > h_\emptyset(i)$ for some p such that for each $c \in C_* \setminus \{\emptyset\}$ with $\max(c) < h_\emptyset(i)$, there is a q_c such that $n(c, 2q_c) = h_\emptyset(i+1)$ and $n(c, 2q_c - 1) > h_\emptyset(i)$. If $\bigcup_{i<\omega}[h_\emptyset(2i), h_\emptyset(2i+1)) \in \mathcal{U}_\emptyset$, then let $n(\emptyset, i) = h_\emptyset(i+2)$. If $\bigcup_{i<\omega}[h_\emptyset(2i+1), h_\emptyset(2i+2)) \in \mathcal{U}_\emptyset$, then let $n(\emptyset, i) = h_\emptyset(i+1)$. Then $Z_\emptyset := \bigcup_{i<\omega}[n(\emptyset, 2i), n(\emptyset, 2i+1))$ is in $\mathcal{U}_\emptyset$. This, along with the way we chose the $n(\{l\}, \cdot)$ for $l < \omega$, ensures that (ii) - (iv) hold. □

With Lemma 22, we are prepared to construct $\tilde{T}$. Define $\tilde{T}$ to be T^* thinned through the Z_c, $c \in C_*$, from Lemma 22. That is, $U_\emptyset(\tilde{T}) = U_\emptyset \cap Z_\emptyset$; if $c \in C_* \cap \tilde{T}$, then $U_c(\tilde{T}) = U_c$; and if $c \in (C \setminus C_*) \cap \tilde{T}$, then $U_c(\tilde{T}) = U_c(T^*)$. For each $c \in C \cap \tilde{T}$, let $\tilde{U}_c$ denote $U_c(\tilde{T})$. For each $k < \omega$, define $n_k = n(\emptyset, 2k+2)$. Since the sequence $(n_k)_{k<\omega}$ is a subsequence of $(i_j)_{j<\omega}$, it follows from (i) of Lemma 22 that

$(**)_k$ For all $c \in \tilde{T} \cap C \restriction n_k$, there is an r_c such that $m(c, 2r_c) = n_k$.

This finishes Stage 3 of the construction.

Finally, we check that (⊛) holds. We show that for all $k < \omega$ and $i < p_{n_k}$, $\tilde{T} \cap \hat{B}^{n_k}_i \subseteq R^{n_k}_i$. This will imply that for each $T \in \mathfrak{T} \restriction \tilde{T}$ with $T \restriction n_k = A_i$, we in fact have $T \subseteq R^{n_k}_i$. Thus, (⊛) will follow from $(*)_{n_k}$, for all $k < \omega$.

Let $k < \omega$ be given and $i < p_k$, and let S denote $\tilde{T} \cap \hat{B}^{n_k}_i$. We shall that for each $c \in S \cap C$, $U_c(S) \setminus n_k \subseteq U_c(R^{n_k}_i)$. Since $S \restriction n_k = A_i = R^{n_k}_i \restriction n_k$, it will follow immediately that $S \subseteq R^{n_k}_i$. Recall that $S \subseteq \tilde{T} \subseteq T^*$. Thus, for all $c \in S \cap C$, $U_c(S) \setminus n_k \subseteq \tilde{U}_c \setminus n_k \subseteq U_c(T^*) \setminus n_k$. The next two claims handle the cases $c \in S \cap C \restriction n_k$ and $c \in (S \cap C) \setminus (C \restriction n_k)$, respectively.

Claim 3. For all $c \in S \cap C \restriction n_k$, we have $U_c(S) \setminus n_k \subseteq \tilde{U}_c \setminus n_k \subseteq U_c(T^{n_k}_c) \subseteq U_c(R^{n_k}_i)$ and moreover, $T^{n_k}_c \subseteq R^{n_k}_i$.

Proof. The properties (i) - (iv) of Lemma 22 are essential in the proof. By (ii), there is some i^* such that $j_{i^*+1} = n_k$. Let $c = \{l_0, \dots, l_r\} \in S \cap C \restriction n_k$. For each

$i \le r+1$, let a_i denote $\{l_j : j < i\}$; in particular, $a_0 = \emptyset$ and $a_{r+1} = c$. Note that $a_r \in C_*$.

We proceed by induction on $i \le r$. Recall that $n_k = n(\emptyset, 2k+2)$. By (iv), $\tilde{U}_\emptyset \cap [n(\emptyset, 2k+1), n(\emptyset, 2k+2)) = \emptyset$; so $l_0 < n(\emptyset, 2k+1)$. Thus, (iii) implies there is a q_1 such that $n(a_1, 2q_1) = n_k$. For the induction step, suppose $i < r$ and there is q_i such that $n(a_i, 2q_i) = n_k$. By (iv) we have that $\tilde{U}_{a_i} \cap [n(a_i, 2q_i - 1), n(a_i, 2q_i)) = \emptyset$. Thus, $l_i < n(a_i, 2q_i - 1)$. From (iii) it follows that there is a q_{i+1} such that $n(a_{i+1}, 2q_{i+1}) = n_k$.

At the end of this induction, we have q_r such that $n(a_r, 2q_r) = n_k$. By (iv), $\tilde{U}_{a_r} \cap [n(a_r, 2q_r - 1), n(a_r, 2q_r)) = \emptyset$; so $l_r < n(a_r, 2q_r - 1)$. Note that $n(a_r, 2q_r - 1) \le j_{i^*}$ (by construction, the intervals between the $n(a, \cdot)$ are at least as large as the intervals between the j_i), so (i) implies there is a p_c such that $m(c, 2p_c) = j_{i^*+1}$, which is n_k. Now by (‡) (b), we have

$$\tilde{U}_c \setminus n_k \subseteq U^*_c \setminus m(c, 2p_c + 1) \subseteq U_c(T_c^{m(c,2p_c)}) = U_c(T_c^{n_k}). \tag{39}$$

Moreover, since $c \in A_i \subseteq R_i^{n_k}$, we have that $T_c^{n_k} \subseteq R_i^{n_k}$. Thus, $U_c(T_c^{n_k}) \subseteq U_c(R_i^{n_k})$. Therefore, $U_c(S) \setminus n_k \subseteq U_c(R_i^{n_k})$. □

Claim 4. For all $c \in S \cap C$ such that $\max(c) \ge n_k$, we have $U_c(S) \subseteq \tilde{U}_c \subseteq U_c(T_c^{\max(c)+1}) \subseteq U_c(R_i^{n_k})$ and moreover, $T_c^{\max(c)+1} \subseteq R_i^{n_k}$.

Proof. Let $c \in S \cap C$ such that $\max(c) \ge n_k$. Then $U_c(T^*) \setminus n_k = U_c(T^*)$. By (‡) (a), $U_c(T^*) \subseteq U_c(T_c^{\max(c)+1})$.

The rest of the proof is by induction on the cardinality of $c \setminus n_k$. Suppose $|c \setminus n_k| = 1$. Then $a := c \setminus \{\max(c)\}$ is in $S \cap C \restriction n_k$. Since $\max(c) \in U_a(S) \setminus n_k$, which by Claim 3, is contained in $U_a(R_i^{n_k})$, we have that $c \in R_i^{n_k}$. Therefore, $T_c^{\max(c)+1} \subseteq R_i^{n_k}$. Hence, $U_c(S) \subseteq U_c(T^*) \subseteq U_c(T_c^{\max(c)+1}) \subseteq U_c(R_i^{n_k})$. Assume that for all $c \in S \cap C$ with $|c \setminus n_k| = m$, the Claim holds. Let $c \in S \cap C$ with $|c \setminus n_k| = m+1$. Letting $a = c \setminus \{\max(c)\}$, the induction hypothesis implies that $U_a(S) \subseteq U_a(R_i^{n_k})$; thus, $c \in R_i^{n_k}$. Therefore, $T_c^{\max(c)+1} \subseteq R_i^{n_k}$. Hence, we again have $U_c(S) \subseteq U_c(T^*) \subseteq U_c(T_c^{\max(c)+1}) \subseteq U_c(R_i^{n_k})$. □

By Claims 3 and 4, for all $c \in S \cap C$, we have $U_c(S) \subseteq U_c(R_i^{n_k})$. Therefore, $S \subseteq R_i^{n_k}$.

Now suppose $T \in \mathfrak{T} \restriction \tilde{T}$ and $T \restriction n_k = A_i$. Then $T \subseteq S$; so by $(*)_{n_k}$, $k \in f([T]) \iff k \in f([R_i^{n_k}]) \iff k \in f([S])$. This completes the proof of (⊛), and thus, the proof of this theorem. □

4. Open problems

We conclude this paper with some open problems. We proved that every ultrafilter Tukey below a p-point has basic, and hence continuous, Tukey reductions. Are there (consistently) any others?

Problem 1. Determine the class of all ultrafilters which have continuous Tukey reductions.

More generally, we would like to know the following.

Problem 2. Determine the class of all ultrafilters which have finitary Tukey reductions.

In particular, it is likely, though it does not seem to be immediately clear, that the analogue of Theorem 9 should hold for all ultrafilters Tukey reducible to some iterated Fubini product of p-points.

Problem 3. Suppose $\mathcal{W}$ is Tukey reducible to some iterated Fubini product of p-points, and suppose $\mathcal{V}$ is an ultrafilter on a countable base set. Is every monotone cofinal map from $\mathcal{W}$ into $\mathcal{V}$ finitely represented on some cofinal subset of $\mathcal{W}$?

A positive answer to Problem 3 would most likely involve answering the next problem.

Problem 4. What is the correct analogue of the Extension Theorem 8 to the setting of Fubini iterates of p-points?

It is likely that those ultrafilters which have some p-point-like property (in the sense that for some suitably defined analogue of $\supseteq^*$, any decreasing sequence of elements of the ultrafilter will have some sort of pseudointersection) will have basic, and hence finitary Tukey reductions. In each setting, basic is to be interpreted on the base in a way that is analogous to Section 3. In particular, we conjecture the following.

Conjecture 23. All ultrafilters forced by $\mathcal{P}(B)/\mathcal{I}$, where B is a countable base set and $\mathcal{I}$ is some definable ideal on B, have basic, and hence finitary, Tukey reductions.

We point out that in recent work [3], we show that the family of ultrafilters on base ω^n, $n < \omega$, forced by $\mathcal{P}(\omega^n)/\mathrm{Fin}^{\otimes n}$ have basic Tukey reductions.

It seems very likely that the same construction as in Theorem 21 can be carried out if some or all of the $\mathcal{U}_c$ are stable ordered union ultrafilters on the base set $\mathrm{FIN} = [\omega]^{<\omega} \setminus \{\emptyset\}$, with the usual notion of front associated with $\mathrm{FIN}^{[\infty]}$ considered as a topological Ramsey space. Theorems 71 and 72 in [5] give the correct notion of basic maps for this setting.

Problem 5. Prove the analogues of Theorems 8, 9 and 21 for stable ordered union ultrafilters and their iterated Fubini products.

Recall that Theorem 1 implies that the top Tukey type has cardinality $2^{\mathfrak{c}}$. On the other hand, all currently considered ultrafilters with Tukey type strictly below $\mathcal{U}_{\mathrm{top}}$ have Tukey type of cardinality $\mathfrak{c}$. (This follows from work of Raghavan in [11] for basically generated ultrafilters, work in [5] for stable ordered union ultrafilters, and work of Dobrinen in [3] for ultrafilters forced by $\mathcal{P}(\omega^n)/\mathrm{Fin}^{\otimes n}$.) Where exactly is the line delineating those ultrafilters with Tukey type of size $\mathfrak{c}$ and those of size $2^{\mathfrak{c}}$?

Problem 5. Does $\mathcal{U} <_T \mathcal{U}_{\mathrm{top}}$ imply that the Tukey type of $\mathcal{U}$ has size $\mathfrak{c}$?

We point out that it is still unknown whether the class of basically generated ultrafilters is equal to the class of iterated Fubini products of p-points.

Problem 6. Is there a basically generated ultrafilter which is not isomorphic to some iterated Fubini product of p-points?

To finish, we remind the reader that for ultrafilters on base set ω, the property of having basic Tukey reductions implies having continuous Tukey reductions. Are they equivalent?

Problem 7. Is there an ultrafilter on base ω having continuous Tukey reductions but not basic Tukey reductions?

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